one side is twice the other; this is what is responsible for our poor prediction of the kaon splitting.

Our estimate of the nontadpole contributions to the baryon self-masses depends on the input experimental form factors and on the method chosen to calculate the strange baryon form factors. Judging by the differences between Tables I and II, and also by several model calculations we have done, we feel that the sensitivity is such that our final results are not trustworthy to within more than one MeV. Any accuracy greater than this displayed by Table III is probably only coincidence.

We believe that the agreement we have obtained between theory and experiment offers considerable support both to the notion of tadpole dominance and to our policy of neglecting the first two possible sources of error cited in Sec. I, and that methods similar to those used here should give results of similar accuracy for other electromagnetic mass splittings (e.g., those within the Y_1^* multiplet) and mass-like electromagnetic transition matrix elements (e.g., that for the two pion decay of the ω).

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Threshold Regge Poles and the Effective-Range Expansion*

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It is shown, without any reference to potentials, that an infinite number of Regge poles approach $l=-\frac{1}{2}$ in the limit of zero energy. Unitarity and the assumption of a Sommerfeld-Watson transform play the crucial role. As a by-product an improved effective-range expansion is obtained.

IT is well known in potential theory that an infinite number of Regge poles arrive at $l=-\frac{1}{2}$ as $\nu\rightarrow 0$, ν being the square of the c.m. momentum.[1] We shall show below that even without any reference to potentials this result will hold provided: (i) Partial wave amplitudes for a fixed energy are analytic except for a finite number of poles and branch cuts in the region to the right of $\mathrm{Re}l=-\frac{1}{2}-\epsilon$, where ϵ is an arbitrarily small positive number. (ii) There exists a Sommerfield-Watson transform in the same region. As a by-product we obtain an improved effective-range expansion.

For simplicity we shall consider the Mandelstam representation to hold even though it is possible that our results may be true also for a more complicated singularity structure.[2] The reduced partial wave amplitude $A(\lambda,\nu)$ satisfies the generalized unitarity relation[3]

$$A^{-1}(\lambda,\nu)-A^{*-1}(\lambda,\nu)=-2i\nu^{\lambda};\quad \lambda=l+\tfrac{1}{2},\quad \nu\geq 0. \quad (1)$$

For all real λ, A is real in the region between threshold and the branch point of the left-hand cut. One can, therefore, write[3,4]

$$A^{-1}(\lambda,\nu)=R(\lambda,\nu)+(\nu^{\lambda}e^{-i\pi\lambda}/\sin\pi\lambda)\,, \quad (2)$$

where $R(\lambda,\nu)$ has the left-hand cut and also the right-hand cuts beginning at the thresholds of inelastic channels.

Now, in the absence of cancellation from the first term, the second term in the right-hand side of (2) would make $A(\lambda,\nu)$ vanish at integral values of λ *independently* of the value of ν. This would be a rather extraordinary situation. In fact it follows from Carlson's theorem[5] and the assumption (ii) above that $A(\lambda,\nu)$ must then vanish identically. The only way to avoid this is for $R(\lambda,\nu)$ to supply the necessary terms that cancel the poles coming from $\sin\pi\lambda$. Separating out the pole parts, we have for small ν,[6]

$$R(\lambda,\nu)=\bar{R}(\lambda,\nu)-\sum_{n=0}\frac{\nu^{n}}{\pi(\lambda-n)}\,, \quad (2)$$

$$A^{-1}(\lambda,\nu)=\bar{R}(\lambda,\nu)-\sum_{n=0}\frac{\nu^{n}}{\pi(\lambda-n)}+\frac{\nu^{\lambda}e^{-i\pi\lambda}}{\sin\pi\lambda}. \quad (4)$$

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[1] B. R. Desai and R. G. Newton, Phys. Rev. **129**, 1445 (1963); and **130**, 2109 (1963); V. N. Gribov and I. Ya. Pomeranchuk, Phys. Rev. Letters **9**, 238 (1962), had independently indicated the existence of these poles.

[2] The information we really need is that $R(\lambda,\nu)$ in (2) can be expanded in a power series in ν, for $\mathrm{Re}\lambda>-\epsilon$.

[3] For the relativistic case, where one considers the invariant amplitude, the right-hand side should be multiplied by an appropriate energy-dependent factor. If the masses are equal $(=m)$ this factor would be $(\nu+m^2)^{-\frac{1}{2}}$.

[4] See, for instance, A. O. Barut and D. E. Zwanziger, Phys. Rev. **127**, 974 (1962).

[5] See E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, London, 1939), 2nd ed., p. 186.

[6] One actually requires all but a finite number of poles to be canceled so that some of poles coming from $\sin\pi\lambda$ may remain. Such cases, however, should be considered accidental.

Since we have assumed the S-W transform to exist only to the right of $\text{Re}\lambda=-\epsilon$, the sum of the pole terms above do not include negative integral values. However, statements about the negative λ region can be made if the so called Mandelstam symmetry, i.e., the relation

$$S(-m,\nu)=S(m,\nu) \quad \text{or} \quad \nu^m A^{-1}(-m,\nu)=\nu^{-m}A^{-1}(m,\nu) \quad (5)$$

holds, m being an integer. This relation implies that $A^{-1}(-m,\nu)$ must also be finite. Therefore, one can decompose $\bar{R}(\lambda,\nu)$ further as

$$\bar{R}(\lambda,\nu)=\bar{R}'(\lambda,\nu)-\sum_{n=1}\frac{\nu^{-n}}{\pi(\lambda+n)}. \quad (6)$$

It must, of course, be kept in mind that for a fixed $\lambda\neq -n$, as $\nu\rightarrow 0$, $\bar{R}(\lambda,\nu)$ is well-behaved as a function of ν (this being the basis of effective-range expansions) and, therefore, $\bar{R}'(\lambda,\nu)$ must cancel the second term in (6) in this limit. The second term will, however, dominate as $\lambda\rightarrow -n$ for a fixed ν.[7] We are, at present, interested only in the region to the right of $\text{Re}\lambda=-\epsilon$, and, therefore, in what follows we shall use (3) instead of (6).

In general, $\bar{R}$ will not have any poles at integral λ and can, therefore, be expanded near $\lambda=n$ as follows[8]:

$$\bar{R}(\lambda,\nu)=p(\nu)+(\lambda-n)q(\nu)+\cdots, \quad (7)$$

where p, q, etc., can be expanded in a power series in ν. Substituting (7) in (4), we observe that if $\lambda=n(\neq 0)$ is one of the zeroes of (4) at $\nu=0$, then its behavior for small ν is given by

$$\lambda(\nu)=n+a\nu+\cdots+b\nu^{n-1}-(c/\pi)\nu^n(\ln\nu-i\pi). \quad (8)$$

The zeroes near $\lambda=0$ for small ν are given by

$$\nu^\lambda e^{-i\pi\lambda}=1+A\lambda+B\lambda^2 \quad \text{as} \quad \nu\rightarrow 0. \quad (9)$$

This equation is the same as the one obtained by Desai and Newton for the case of potential scattering and will have, therefore, the same solutions:[1] $\lambda_i^{(n)}\simeq 2n\pi/|\ln\nu|$, $\lambda_r^{(n)}\simeq 2n\pi^2/|\ln\nu|^2$ as $\nu\rightarrow 0+$ and $\lambda_i^{(n)}\simeq 2n\pi/|\ln(-\nu)|$, $\lambda_r^{(n)}\simeq -2(A^2-2B)n^2\pi/|\ln(-\nu)|^3$ as $\nu\rightarrow 0-$, where $n=\pm 1, \pm 2$, with $|n|\ll|\ln(|\nu|)|/2\pi$. These solutions show that in the limit of zero energy, $l=-\frac{1}{2}$ is the end point of an infinitely many Regge trajectories. For a fixed energy, of course, there are only a finite number of these poles to the right of $\text{Re}l=-\frac{1}{2}-\epsilon$, consistent with the assumption (i).

We have, therefore, shown that the existence of the threshold poles can be proved without any reference to potentials. Unitarity and the assumption of a Sommerfeld-Watson transform play the crucial role. We have also shown that because of the presence of the term $\sum[\nu^n/\pi(\lambda-n)]$, (4) is a better expression for the effective-range-type expansions than (2).[9] It correctly reproduces the threshold poles and also the logarithmic behavior in (8) for the zero-energy poles at integral λ.

We have profited from our discussions with Professor C. J. Goebel.

[7] To illustrate these points consider potential scattering where (5) is known to be true for reasonably well-behaved potentials. For simplicity we keep only the first power in the potential in the Fredholm expansion of the numerator and the denominator functions of $A(\lambda,\nu)$. We then obtain,

$$R(\lambda,\nu)=-\nu^\lambda\left[1+\frac{\pi}{2\sin\pi\lambda}\int_0^\infty dr\,rV(r)J_\lambda(dr)J_{-\lambda}(kr)\right]\Bigg/\frac{\pi}{2}\int_0^\infty dr\,rV(r)J_\lambda^2(kr),$$

where $\nu=k^2$ and $J_{\pm\lambda}$ are the Bessel functions $[J_{-n}(z)=(-1)^nJ_n(z)$ for integral $n]$. Consider a single Yukawa potential, for which the S-W transform is known to exist. By expanding $J_{\pm\lambda}$ in a power series in (kr) and carrying out the integrations we obtain

$$R(\lambda,\nu)=-\left[1+\frac{\gamma}{2\lambda}\left(1-\frac{\nu}{1-\lambda^2}+\frac{3\nu^2}{2(1-\lambda^2)}+\cdots\right)\right]\Bigg/\left\{\frac{\pi\gamma}{2}\frac{\Gamma(1+2\lambda)}{2^{2\lambda}\Gamma^2(1+\lambda)}\times\left[1-\nu(1+2\lambda)+\frac{\nu^2(2+\lambda)(3+2\lambda)(1+2\lambda)}{4(1+\lambda)}+\cdots\right]\right\}, \quad (A)$$

where γ is the strength of the potential and the range is taken to be unity. It is now easy to check that $R(\lambda,\nu)$ has poles at $\lambda=0$ and at positive and negative integrals values of λ. Furthermore, for negative integers, we have the canceling terms $\bar{R}'(\lambda,\nu)$. For instance, near $\lambda=-1$ and small ν, (A) has terms of the form

$$\nu^{-1}(1+b\nu)/\pi(\lambda+1+c\nu^2)-\nu^{-1}/\pi(\lambda+1),$$

where b and c are constants. The first term is $\bar{R}'(\lambda,\nu)$ and as expected it cancels the second term in the limit $\nu\rightarrow 0$ and $\lambda\neq -1$. For a fixed ν as $\lambda\rightarrow -1$, the second term, of course, dominates. A similar situation will hold for other negative integers.

[8] Again it could accidentally happen that cuts pass through some of the integral values of λ. In that case, of course, the expansion will not hold.

[9] This fact has been known to Professor Marc Ross (private communication) but his results have not been published.