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An Iteration Method in the S-Matrix Theory

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An iteration method is formulated for the determination of the partial-wave scattering amplitude on the basis of analyticity and unitarity postulates. The analytic properties in the physical and unphysical sheets are considered simultaneously in a study of the logarithmic S function $\ln S_l(s)$. The usual N/D approach and some of its associated drawbacks are avoided. A relationship between the total number of composite particles and the phase change of $S_l(s)$ along the left-hand cut is derived; this may be regarded as a generalization of Levinson's theorem. The use of this relationship in the iteration method is discussed.

I. INTRODUCTION

So far in the development of the analytic S-matrix theory, calculations of the partial-wave scattering amplitudes have been based almost exclusively on the N/D method.[1] Its advantage lies in the fact that the nonlinear integral equation of a scattering amplitude satisfying analyticity and unitarity postulates can be reduced by this method to a set of two coupled linear integral equations. However, it is marred by the disadvantages associated inherently with the definition of a function as a quotient of two functions. Given a particular left-hand cut representing the dynamical force operating in a channel, it is not impossible that the D function has zeros in the complex energy plane. Since causality forbids the amplitude to have poles in the complex plane of the physical sheet, this implies either that the N function must have zeros there also or that the input force is unrealistic. In either case some remedy seems necessary, which is to be imposed to meet an extra condition not already contained in the postulates of analyticity and unitarity, contrary to the philosophy of the S-matrix theory. It is therefore desirable to have a method which is free of this shortcoming, that is, a method in which analyticity and unitarity automatically guarantee that all the complex poles of the amplitude are in the unphysical sheet.

Another drawback of the N/D method is that the analytic property which is to be assigned to D is not unambiguous. It can have the entire right-hand unitarity cut or just the elastic section of this cut or the entire right-hand cut with only the elastic discontinuity. One must examine whether this freedom is consistent with the one-to-one correspondence between a pole of the scattering amplitude and a zero of D, which is generally assumed unless proven inadmissible *a posteriori* on the grounds of other consistency requirements. If D is required to have only the elastic cut, then special care must be taken to ensure that the amplitude does not acquire an artificial singularity at the inelastic threshold.[2] In so doing an integral equation of the Wiener-Hopf form must be solved. It is not apparent, however, that the complications involved in solving that equation all have physical content.

Finally, we note that the N and D functions are associated with the scattering amplitude defined on the physical sheet only. Although it is not difficult to construct the amplitude on the unphysical sheet (reached by continuation across the elastic unitarity cut) in terms of N and D, there is no reason to prejudice one sheet against the other, when resonances and bound states are regarded as generically the same.

We propose here a method for determining the partial-wave scattering amplitude without recourse to the factorization of the amplitude into two analytic functions, thus avoiding some of the drawbacks of the N/D procedure. In our approach the physical and unphysical[3] sheets are explicitly put on the same footing. This is accomplished by utilizing the fact that the S function on the unphysical sheet S_u is the inverse of S on the physical sheet; thus, the function $\ln S(s)$ is singular at all the positions in the complex s plane where either $S(s)$ or $S_u(s)$ is singular. Our principal dynamical equation is a dispersion relation of this logarithmic function. It is supplemented by a number of subsidiary equations. This system of equations is then to be solved by an iteration procedure.

We shall derive a generalized form of the Levinson's theorem, which relates the phase change of $S(s)$ along the left-hand cut to the total number of composite particles—resonances and bound states—in the channel under consideration. The iteration method shows how the pole positions of these composite states move as a result of the unitarity correction, which, for potentials not too singular, never increases the number of such states. Thus, even before a calculation is attempted, one can predict on the basis of the nature of the input dynamical force whether a certain number of composite states in a particular partial wave is possible.

The movements of the poles in the complex s plane can also be studied as a function of the interaction strength or the angular momentum. The pole positions can be complex only in the unphysical sheet; any one emerging into the physical sheet through the elastic

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

[2] G. F. Chew, Phys. Rev. **130**, 1264 (1963).

[3] Here and in the following, the unphysical sheet shall always refer to the one reached by continuation across the elastic unitarity cut.

cut must stay on the real axis below the elastic threshold. An inversion of the dependence of the pole positions on angular momentum gives, of course, the Regge trajectories.

In the iteration method there are no integral equations to be solved. One simply evaluates integrals over known integrands at each stage of the iteration. The only reservation one may have about such a procedure is that in the initial stage of the iteration the results may oscillate so much as to render the method difficult. However, such difficulty, if it exists, can easily be eliminated by proper numerical programming, which turns on the interaction adiabatically.

Section II contains the description of the iteration method for physical partial waves; the consideration needed for the extension to nonintegral values of l is discussed in Sec. IV. In Sec. III is given a generalization of Levinson's theorem and its application.

II. DYNAMICAL EQUATIONS

We consider the scattering of two neutral spinless particles of equal mass μ. Let s be the total c.m. energy squared, and the S-matrix element for a given partial wave l be written as

$$S_l(s)=1+2i\rho(s)A_l(s)\,, \tag{2.1}$$

where

$$\rho(s)=k/s^{1/2}=[(s-4\mu^2)/4s]^{1/2}. \tag{2.2}$$

We assume that $A_l(s)$ satisfies the analyticity and unitarity postulates so that it is a meromorphic function in the cut s plane. The branch cuts are on the real axis running from $s=-\infty$ to 0 and from $s_1=4\mu^2$ to $+\infty$. If s_2 (assumed to be greater than s_1) is the inelastic threshold, then by means of the unitarity condition on $S_l(s)$ between s_1 and s_2 plus real analyticity—i.e., $A_l(s)=A_l^*(s^*)$—or on the basis of the discontinuity equation for the two-particle branch cut, the scattering amplitude can be continued[4] across the elastic unitarity cut into the unphysical sheet u, and one obtains

$$S_u(s)=S^{-1}(s)\,. \tag{2.3}$$

Here and in the following the partial-wave index l will be suppressed until Sec. IV, where the problem for noninteger l will be considered.

It is clear from (2.3) that the elastic cut connects only two sheets. The zeros of $S(s)$ correspond to the poles of $S_u(s)$. Thus, the singularities of the S function on both sheets are present in the logarithmic function

$$K(s)\equiv \ln S(s)\,. \tag{2.4}$$

Poles of $S(s)$ and $S_u(s)$ both appear as logarithmic singularities of $K(s)$, differing only in the sign factor.

We assume in this work that $S(s)$ tends to unity as $s\to\infty$. This has been shown by Omnes[5] to be true if the asymptotic behavior is dominated by one Regge pole in the cross channel, which has the properties that its trajectory in the complex angular-momentum plane satisfies the Froissart limit and that it loops back to the left of $\mathrm{Re}\,l=1$ at large momentum transfer. The same result holds if a finite number of Regge poles of such character contributes to the asymptotic behavior, but it is not yet known whether the conclusion is to be altered when an infinite number of poles or a cut in the l plane governs the asymptotic behavior. With $S(s)$ tending to unity at infinity, $K(s)$ approaches a constant asymptotically, and the dispersion relation for $K(s)$ which we shall consider exists without subtraction.

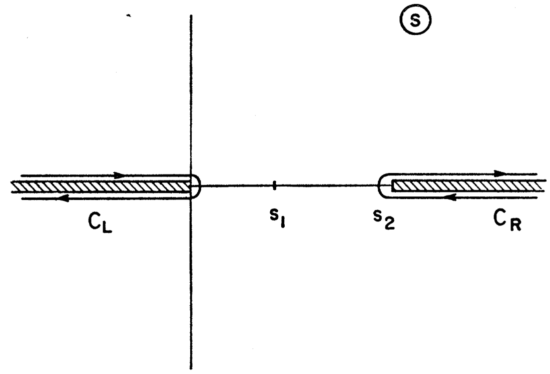

FIG. 1. Contours C_L and C_R in the s plane.

Let us consider first the situation in which $S(s)$ has neither zeros nor poles; this can always be made possible by letting the interaction strength be weak enough. In this case $K(s)$ is analytic in the s plane cut from $-\infty$ to 0 and from s_1 to $+\infty$. We choose the branch of the logarithm in which $K(s)$ is pure real on the real axis between 0 and s_1. Because of unitarity, $K(s)$ is pure imaginary between s_1 and s_2, having opposite signs on the two sides of the real axis. Thus, if $(s-s_1)^{1/2}$ is defined in the s plane cut from s_1 to $+\infty$, then $K(s)/(s-s_1)^{1/2}$ is regular at $s=s_1$ and has cuts in the s plane from $-\infty$ to 0 and from s_2 to $+\infty$. By Cauchy's theorem we have

$$\frac{K(s)}{(s-s_1)^{1/2}}=\frac{1}{2\pi i}\left[\int_{C_L}+\int_{C_L}\right]\frac{K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}\,, \tag{2.5}$$

where C_L and C_R are contours shown in Fig. 1. Along the left-hand cut it is the imaginary part of K that contributes to the discontinuity, and $\mathrm{Im}K(s)$ is just the phase of $S(s)$. In the integral over the inelastic cut[6] the contribution comes from $\mathrm{Re}K(s)$, which is $\ln\eta(s)$, where η is the absorption coefficient defined by

$$S(s)=\eta(s)e^{2i\delta(s)}\,,\quad s>s_1\,. \tag{2.6}$$

[4] R. C. Hwa and D. Feldman, Ann. Phys. (N. Y.) **21**, 453 (1963); for earlier work see references cited therein.
[5] R. Omnes, Phys. Rev. **133**, B1543 (1964).
[6] Cf. M. Froissart, Nuovo Cimento **22**, 191 (1961).

We thus obtain[7]

$$K(s)=\frac{(s-s_1)^{1/2}}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}+\frac{(s-s_1)^{1/2}}{i\pi}\times\int_{s_2}^{\infty}\frac{\ln\eta(s')ds'}{(s'-s)(s'-s_1)^{1/2}}.\quad(2.7)$$

From (2.1) and (2.4) we have

$$\mathrm{Im}K(s)=\cos^{-1}\{e^{-\mathrm{Re}K(s)}[1-2\rho(s)\,\mathrm{Im}A(s)]\}.\quad(2.8)$$

In solving the present problem the inelasticity function $\eta(s)$, $s>s_2$, and the left-hand discontinuity $2i\,\mathrm{Im}A(s)$, $s<0$, are the input information that is assumed known. Thus (2.7) and (2.8) constitute a closed set of equations which can be solved by successive iteration. Let $A^B(s)$ denote the Born term that gives rise to the left-hand cut and the right-hand inelastic cut, and $K^B(s)\equiv\ln(1+2i\rho A^B)$. Then the iteration procedure involves first putting $\mathrm{Im}K^B$ and $\ln\eta$ in the first and second integrals of (2.7), evaluating the two integrals, and obtaining the once-iterated $K(s)$ for any value of s in the entire cut plane. The real part of this result along the negative real axis is used in (2.8) to give an improved $\mathrm{Im}K(s)$, and the iteration is repeated. An important point to note is that in this method the iteration is done only along the left-hand cut where $\mathrm{Re}S(s)$ is known exactly, so the results at each iterative step are constrained to be partially correct at all times. The solution is expected to converge rapidly if the input force is weak and is such that $S(s)$ has no zeros or poles in the cut s plane.

Consider now the situation that $S(s)$ can have zeros or poles. We shall show in the next section how the total number of poles in the two-sheeted Riemann surface is related to the phase change of $S(s)$ along the boundary of this surface. It suffices to remark here that if when the interaction strength is initially weak $S(s)$ has no zeros or poles, then as the interaction is strengthened, zeros of $S(s)$ may emerge into the complex plane of the physical sheet from the left-hand cut

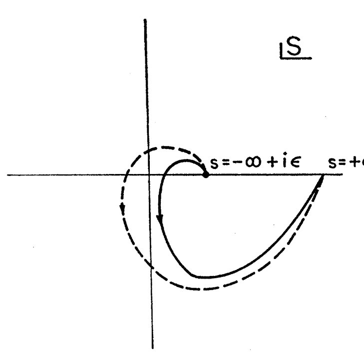

FIG. 2. Example of the images of the upper half of C_L under the mapping $S=S(s)$.

[7] A discussion of the measures to guarantee proper threshold behavior is deferred until the end of this section.

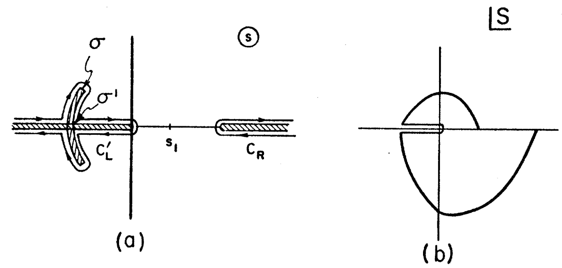

FIG. 3. Distorted contour C_L' (a) and its image in the S plane (b).

or the right-hand inelastic cut. So long as none of these zeros crosses the elastic cut at $s=s_1$, no poles of $S(s)$ can enter into the complex s plane; not from infinity, since $S(s)$ is constrained to unity there at all times; not from the right-hand cut, on account of the restriction $\eta(s)<1$; and not from the left-hand cut because, for any negative value of s, $\mathrm{Re}S(s)=1-2\rho(s)\,\mathrm{Im}A(s)$ is a continuous and monotonic function of the interaction strength—a property which forbids the emergence of a pole of $S(s)$ through this cut.

To see how a zero of $S(s)$ can enter into the complex plane, let us suppose that for some weak coupling the image of the upper half contour of C_L under the mapping $S=S(s)$ is as shown in Fig. 2 by the solid line. As the interaction strength is increased, the image may move to the dashed line in the same figure. If such is the case, then in the process of the change a zero of $S(s)$ moves through C_L as it emerges from the left-hand cut. Notice that the phase difference of $S(s\approx 0)$ in the two cases is 2π. Since a zero of $S(s)$ corresponds to a logarithmic branch point of $K(s)$, the contour C_L of the Cauchy integral in (2.5) must be distorted to avoid the advancing singularity of the integrand. If we place the logarithmic branch cut of $K(s)$ along the image of the negative real axis of the S plane under the inverse mapping $s=S^{-1}(S)$, the distorted contour C_L' may appear as shown in Fig. 3(a); the mapping of the upper half of C_L' into the S plane is then as indicated in Fig. 3(b).

Similar considerations can be made for zeros coming out from the right-hand inelastic cut. This occurs when the coupling to other channels is strong enough that resonances in those channels induce poles in $S_u(s)$.

Consider the modification needed for the dispersion relation for $K(s)$ when the left-hand cut is such as to provide a pair of zeros of $S(s)$ in the complex s plane. The Cauchy integral along C_L' may be separated into several terms:

$$\int_{C_L'}\frac{K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}=\int_{\sigma'}^{\sigma}\frac{\Delta K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}+\left[\int_{-\infty}^{\sigma'-\epsilon}+\int_{\sigma'+\epsilon}^{0}\right]\frac{K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}+\cdots,\quad(2.9)$$

where the dots symbolize similar terms corresponding to integration along the lower half of C_L'. The limits of integration, σ and σ', are defined by $S(\sigma)=0$ and $\mathrm{Im}S(\sigma')=0$, $\mathrm{Re}S(\sigma')<0$. The discontinuity ΔK across the complex logarithmic cut is just $2\pi i$, since $S(s)$ is assumed to have only a simple zero at $s=\sigma$. Thus the first term on the right of (2.9) gives

$$\frac{2\pi i}{(s-s_1)^{1/2}}\ln\frac{(\sigma-s_1)^{1/2}-(s-s_1)^{1/2}}{(\sigma-s_1)^{1/2}+(s-s_1)^{1/2}}-[\sigma\to\sigma']. \quad (2.10)$$

The logarithm term in the square bracket cancels a similar term in (2.9) coming from integrations ending at $\sigma'-\epsilon$ and $\sigma'+\epsilon$. Hence the dispersion relation for $K(s)$ has the form[8]

$$K(s)=f(s)+\frac{(s-s_1)^{1/2}}{\pi}\int_{-\infty}^{0}\frac{\mathrm{Im}K(s')ds'}{(s'-s)(s'-s_1)^{1/2}}+\frac{(s-s_1)^{1/2}}{i\pi}\int_{s_2}^{\infty}\frac{\ln\eta(s')ds'}{(s'-s)(s'-s_1)^{1/2}}, \quad (2.11)$$

where

$$f(s)=\sum_i \ln\frac{(\sigma_i-s_1)^{1/2}-(s-s_1)^{1/2}}{(\sigma_i-s_1)^{1/2}+(s-s_1)^{1/2}}. \quad (2.12)$$

The summation is over the two zeros of $S(s)$ at complex conjugate positions σ_i, and should clearly extend to all the zeros if there are more than one pair of them. In (2.11) the function $\mathrm{Im}K(s')$ inside the integral over the the left-hand cut is now the continuous function $\arg S(s')$ for s' running from $-\infty$ to 0 just above the real axis, and should not contain a discontinuity $2\pi i$ at σ', which has been removed by the cancellation mentioned above. In other words we have, in deriving (2.11), moved the complex branch cuts of $K(s)$, originally between σ_i and σ', to positions connecting σ_i and the threshold s_1, as is evidenced by the logarithm terms in (2.12).

We note that each term in (2.12) has the properties that the argument of the logarithm has a zero at $s=\sigma_i$, but that if σ_i goes across the unitarity cut beginning at s_1, then the argument has a pole at $s=\sigma_i$. This is, of course, what is expected as a resonance becomes a bound state. The companion pole in the pair originally in the complex conjugate position remains[9] in the unphysical sheet and gives rise to the virtual state. Because of symmetry in reflection across the real axis, these poles must be on the real axis below s_1.

Since $f(s)$ depends only on the positions of the poles of the S function on the two sheets, (2.11) provides a formula ideally suited for the parametrization of the phase shift, which is $K(s)/2i$, $s\geqslant s_1$. The last integral can be evaluated, since $\eta(s')$ is determined by experiment, while the integral over the left-hand cut can be approximated by some poles.

To proceed with the formulation of the iteration method when $S(s)$ has zeros, we note that (2.8) can be used to improve the first integrand of (2.11) at successive stages of the iteration, but we need another equation to improve also the values of σ_i, lest the iteration not converge. This equation is supplied by the dispersion relation for $S(s)$ itself. Since $\mathrm{Re}S(s)=1-2\rho(s)\,\mathrm{Im}A(s)$ is a known function for s real and negative, we apply the Cauchy theorem to $S(s)/s^{1/2}$ and obtain

$$S(s)=\frac{s^{1/2}}{i\pi}\int_{-\infty}^{0}\frac{\mathrm{Re}S(s')ds'}{(s'-s)(s')^{1/2}}+\frac{s^{1/2}}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Im}S(s')ds'}{(s'-s)(s')^{1/2}}. \quad (2.13)$$

In the second integral $\mathrm{Im}S(s')$ is provided by the output of (2.11) at each stage of the iteration, so (2.13) can be used to determine the values of σ_i where $S(s)$ vanishes. The numerical procedure involves simply the determination of the direction, at each point, in which $d|S(s)|/d|s|$ is greatest and the successive progression along the path of steepest descent toward the point where $|S(s)|=0$. When σ_i are found, they are then substituted in (2.12) for the next iteration. Thus Eqs. (2.11) and (2.12), supplemented by (2.8) and (2.13), form a closed system of equations from which a unique solution can be sought, provided that the interaction is such that there can be no stable particles, elementary or composite.

To eliminate this last restriction we must have a final equation to determine the positions of the poles of $S(s)$. When there are poles [i.e., when σ_i in (2.12) moves to a different branch of $(\sigma_i-s_1)^{1/2}$], (2.13) must first be augmented by a term

$$\sum_i\left(\frac{s}{\sigma_i^p}\right)^{1/2}\frac{\lambda_i^p}{\sigma_i^p-s} \quad (2.13')$$

on the right-hand side. The pole position and residue are determined by the zero position and the derivative there of the inverse function $S^{-1}(s)$ given by the dispersion relation

$$S^{-1}(s)=\sum_i\left(\frac{s}{\sigma_i^0}\right)^{1/2}\frac{\lambda_i^0}{\sigma_i^0-s}+\frac{s^{1/2}}{i\pi}\int_{-\infty}^{0}\frac{\mathrm{Re}S^{-1}(s')ds'}{(s'-s)(s')^{1/2}}+\frac{s^{1/2}}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Im}S^{-1}(s')ds'}{(s'-s)(s')^{1/2}}, \quad (2.14)$$

where σ_i^0 and λ_i^0 are obtained from (2.13) plus (2.13′). For every set of discontinuities along the right- and left-hand cuts, these two equations are iterated to give

[8] Cf. C. H. Albright and W. D. McGlinn, Nuovo Cimento **25**, 193 (1962); T. Ogimoto, Progr. Theoret. Phys. (Kyoto) **27**, 396 (1962); A. M. Bincer and B. Sakita, Phys. Rev. **129**, 1905 (1963).

[9] P. V. Landshoff, Nuovo Cimento **28**, 123 (1963).

the best σ_i^p and σ_i^0, which are used in (2.12) for the next iteration of (2.11).

The numerical work involved in this iteration procedure should not be complicated, since all integrals are straightforward evaluations. There are no difficulties regarding the possibility of any artificial singularity at $s=s_2$, and there are no integral equations to be solved. The stability of the iterated solution can be controlled by adiabatic variation of the coupling strength.

As a final remark of this section, we note that, for $l\geqslant 1$, (2.7) and (2.11) do not guarantee the threshold behavior $K(s)\propto(s-s_1)^{l+\frac{1}{2}}$ as $s\rightarrow s_1$. This can be corrected if we consider the dispersion relation for $K(s)/(s-s_1)^{l+\frac{1}{2}}$. The only changes that are entailed in (2.7) and (2.11) are that all the integrands should be multiplied by the factor $[(s-s_1)/(s'-s_1)]^l$ and that $f(s)$ should be replaced by the function

$$f(s)=\sum_i (s-s_1)^{l+\frac{1}{2}}\int^{\sigma_i}\frac{ds'}{(s'-s)(s'-s_1)^{l+\frac{1}{2}}}. \quad (2.15)$$

All other considerations proceed as before without alteration.

III. NUMBER OF COMPOSITE STATES

In the preceding section, we have anticipated the emergence of a zero of $S(s)$ into the complex plane from the left-hand cut of the physical sheet, thus changing the phase of $S(s)$ along C_L. We now derive this result, which may be regarded as a generalization of Levinson's theorem.[10]

Consider the integral

$$I=\int_C [S'(s)/S(s)]ds, \quad (3.1)$$

where $S'(s)$ is the first derivative of $S(s)$ and C is the contour shown in Fig. 4. If n_0 and n_p are, respectively, the total number of zeros and poles of $S(s)$ inside C, then we have the identity

$$I=2\pi i(n_0-n_p). \quad (3.2)$$

Now, since $S(s)$ tends to a constant at infinity, the contribution to the integral from the integration along the infinite part of C vanishes. Relating the integration around the right-hand cut to the phase shift, we thus have

$$K(s)|_{C_L}+4i[\delta(\infty)-\delta(s_1)]=2\pi i(n_0-n_p), \quad (3.3)$$

where the notation $|_{C_L}$ implies the difference experienced as s is taken along the contour C_L. If there is no inelastic contribution to the unitarity cut, the usual Levinson's theorem states

$$\delta(s_1)-\delta(\infty)=(n_p-n_e)\pi, \quad (3.4)$$

where n_p is the total number of stable particles [and therefore poles of $S(s)$] and n_e is the number of elementary particles, or equivalently the number of CDD (Castillejo-Dalitz-Dyson) poles. Combining (3.3) and (3.4), we obtain

$$K(s)|_{C_L}=2\pi i(n_0+n_p-2n_e). \quad (3.5)$$

Since $S(s)$ is constrained by kinematics to be unity at $s=s_1$, and is unbounded at $s=0$, we see that if we introduce a pole in the gap between $s=0$ and s_1, representing an elementary particle interacting weakly with the system, then there must be a zero of $S(s)$ in the neighborhood of this elementary particle pole. As the coupling strength is increased, or as the position of the pole is varied, this zero moves away, either staying on the real axis in the gap, or moving into the complex plane together with another zero at the complex conjugate position, but it never disappears except through the boundaries of the Riemann surface. We therefore have the general formula $n_0+n_p=2n_e+n_c$, where n_c is the total number of poles of $S(s)$ and $S_u(s)$ corresponding to composite particles (whether or not these poles are near the physical region to give a particle interpretation). We thus have

$$K(s)|_{C_L}=2\pi i n_c. \quad (3.6)$$

The left-hand side of this equation is just the change in phase of $S(s)$ as s is taken along the contour C_L. Since the phase difference may be different if some other path is followed, adherence to C_L is to be noted explicitly. This formalizes our earlier surmise that all the "resonance," virtual, and bound-state poles are fed into the two-sheeted Riemann surface through the left-hand cut of the unphysical sheet in the case of no inelasticity.

If there is coupling to other channels through unitarity, (3.4) must be modified and (3.6) is therefore not valid in general. However, if the coupled channels do not contribute to any resonance poles in $S(s)$, as is assumed in the $\pi\pi$ problem in the strip approximation,[11]

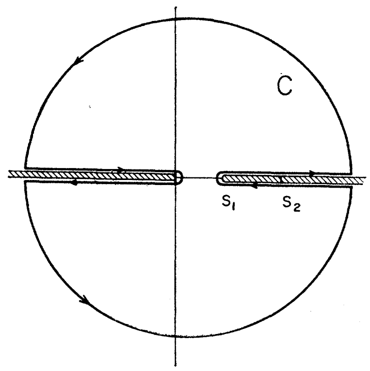

FIG. 4. Contour C in the s plane.

[10] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949); S. C. Frautschi, *Regge Poles and S-Matrix Theory* (W. A. Benjamin, Inc., New York, 1963); R. L. Warnock, Phys. Rev. **131**, 1320 (1963).

[11] G. F. Chew, Phys. Rev. **129**, 2363 (1963); G. F. Chew and C. E. Jones, *ibid.* **135**, B208 (1964).

then (3.6) can of course still be used to determine the total number of composite states. We have not succeeded in generalizing (3.6) to the case in which inelastic unitarity is the source of some resonance poles, but in such a case the wisdom of restricting ones considerations to the study of a single channel is questionable.[11a]

In the remainder of this section we illustrate how n_c can be estimated from an examination of the Born term $A^B(s)$, which gives rise to the left-hand cut. We have already observed from (3.6) that n_c is the number of times S goes around the origin, where S is the image of C_L under the mapping $S=S(s)$. Since S is unity at $s=-\infty$, n_c is therefore the number of times S crosses the negative real axis with negative $d(\mathrm{Im}S)/ds$ minus the times it crosses with positive $d(\mathrm{Im}S)/ds$, where s has the sense of C_L. In order that $S(s)$ be real along the negative real s axis, $\mathrm{Re}A(s)$ must vanish there. The contribution to $A(s)$ from the unitarity integral (and bound-state pole if any) for negative s is always real and positive; let us postpone for the moment the discussion of its effects. What remains is just the "potential" term $A^B(s)$, which is presumed known.

Ignoring any particles exchanged in the u channel for the convenience of the present discussion, we have[12]

$$A_l(s)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{2dt}{s-s_1}A_t(s,t)Q_l\left(1+\frac{2t}{s-s_1}\right), \tag{3.7}$$

where $2iA_t(s,t)$ is the discontinuity of $A(s,t)$ across its t cut. Consider the force arising from the exchange of a single particle of mass m and spin j in the t channel. Then $A_t(s,t)$ has the form

$$A_t^B(s,t)=\lambda_jP_j(1+2s/(m^2-t_1))\delta(t-m^2), \tag{3.8}$$

where λ_j is a real constant proportional to the strength of interaction, and t_1 is the elastic threshold of the t channel. Equation (3.8) is, of course, incorrect in the asymptotic region of s, where a proper Regge formula should be used to ensure that $A_l(s)$ is damped out logarithmically.[5] For our purpose here we assume that in the finite part of the left-hand s cut $A_l^B(s)$ is determined by (3.7) and (3.8), and that some damping factor is introduced in the asymptotic region to reduce $A_l^B(s)$ to zero. Thus except in the asymptotic region we have

$$A_l^B(s)=\frac{2\lambda_jP_j(1+2s/(m^2-t_1))}{\pi(s-s_1)}Q_l\left(1+\frac{2m^2}{s-s_1}\right). \tag{3.9}$$

Now, $S_l^B(s)$ is real along the negative real s axis if $\mathrm{Re}A_l^B(s)$ vanishes; this occurs at the zeros of $P_j(1+2s/(m^2-t_1))$ and of $\mathrm{Re}Q_l(z)$ for $-1<z<+1$, where $z=1+2m^2/(s-s_1)$.

In view of the relationship

$$Q_l(-z\mp i\epsilon)=(-1)^{l+1}Q_l(z\pm i\epsilon) \tag{3.10}$$

for integral l and z in the interval $[-1,+1]$, we see that $\mathrm{Re}Q_l(z)$ is symmetric (antisymmetric) if l is odd (even); in fact, $\mathrm{Re}Q_l(z)$ has $l+1$ zeros in this interval. Among all the zeros the ones corresponding to $S_l^B(s)$ being negative satisfy the requirement $\mathrm{Im}A_l^B(s+i\epsilon)>1/2\rho(s+i\epsilon)=1/2|\rho(s)|$; this puts a lower bound on $|\lambda_j|$ if use is made of the property

$$\mathrm{Im}Q_l(z\pm i\epsilon)=\mp\tfrac{1}{2}\pi P_l(z), \quad z\epsilon(-1,+1).$$

Take, for example, the case of $j=1$ and $m^2>t_1=s_1$, and consider only the p-wave amplitude. It can be established that, if $\lambda_1>0$, then the only values of s on C_L at which $S_{l=1}{}^B$ is real and negative are where $\mathrm{Re}Q_1(z\pm i\epsilon)=0$, i.e., $z=+0.83$, provided that

$$\lambda_1>\frac{s-s_1}{1+2s/(m^2-t_1)}[2z|\rho(s)|]^{-1}. \tag{3.11}$$

At these values of s, i.e., $(s_1-2m^2/0.17)\pm i\epsilon$, the derivative $d(\mathrm{Im}S)/ds$ along C_L is negative. Thus, if the interaction is attractive and strong enough that (3.11) is satisfied, $S_1^B(s)$ for $s\epsilon C_L$ turns counterclockwise around the origin twice, so it has two zeros in the cut s plane, corresponding to one resonance state. We therefore see from this kind of consideration that it is possible to attribute the existence of ρ in the $\pi\pi$ system to the force arising from the exchange of ρ.

If λ_1 is negative, i.e., a repulsive potential, then one can show by using (3.9) that S_1^B is real and negative on C_L at $s=+\epsilon$, $\epsilon>0$, and at $z=-0.83\pm i\epsilon$, i.e., $s=(s_1-2m^2/1.83)\pm i\epsilon$, provided that

$$\lambda_1<\frac{s-s_1}{1+2s/(m^2-t_1)}[-2|z\rho(s)|]^{-1}. \tag{3.12}$$

In this case $S_1^B(s)$ has three zeros, one of which must be on the real axis.

Thus far the considerations are based only on the potential term without unitarity correction. Let $A_l(s)$ be written as $A_l^B(s)+\bar{A}_l(s)$, where $\bar{A}_l(s)$ is the unitarity integral

$$\bar{A}_l(s)=\frac{1}{\pi}\int_{s_1}^{\infty}\frac{\mathrm{Im}A(s')}{s'-s}ds'.$$

For $s<s_1$, $\bar{A}_l(s)$ is real and positive, and decreases monotonically as s varies from s_1 to $-\infty$. Thus, the zeros of $\mathrm{Re}A_l(s)$ along the left-hand cut occur at the values of s where $\mathrm{Re}A_l^B(s)=-\bar{A}_l(s)$. Clearly if $\mathrm{Re}A_l^B(s)$ oscillates around zero, not around some value such that two adjacent maximum and minimum values both have the same sign (a property generally satisfied by forces due to particles exchanged in crossed channels), then the monotonic behavior of $\bar{A}_l(s)$ implies that $\mathrm{Re}A_l(s)$ cannot have more zeros than

[11a] *Note added in proof*. We have subsequently generalized (3.6) to the many-channel case with two particles in each channel. It is necessary to consider all sheets connected by all sections of the unitarity cut. See R. C. Hwa, Lawrence Radiation Laboratory Report UCRL-11625 (to be published).
[12] M. Froissart, La Jolla Conference on the Theory of Weak and Strong Interactions, 1961 (unpublished).

$\mathrm{Re}A_l^B(s)$. The unitarity correction to $S_l^B(s)$, as is calculated by the iteration procedure, therefore does not introduce any additional zeros into the complex s plane. It only moves the positions of the zeros of $S_l(s)$ away from the original positions associated with $S_l^B(s)=0$. Thus, if $A_l^B(s)$ has the property described above, the number of composite states determined by a consideration of the Born term alone is the maximum number possible when the unitarity condition is fully taken into account. In the cases where $\mathrm{Re}A_l^B(s)$ does have more than one extremum between two adjacent zeros for $s<0$, then additional zeros of $S_l(s)$ can emerge from the left-hand cut. However, the upper bound of n_c is still determined by the oscillatory nature of $A_l^B(s)$. It is taken as understood that the above statements apply only to the problems in which the inelastic unitarity does not introduce any resonance poles.

Although the unitarity correction does not generally introduce any resonance poles, it can make some zeros of $S_l^B(s)$ retreat to the left-hand cut. That occurs when the minimum requirement on the interaction strength, such as (3.11) or (3.12), is no longer satisfied, as the value of s, where $\mathrm{Re}A_l(s)=0$, is shifted. The odd zero of $S_l(s)$ on the real axis, which we have encountered in the above example for $\lambda_1<0$, will always remain in the interval between $s=0$ and $s=s_1$, so long as the interaction strength is nonzero. This is because in those cases in which an odd zero occurs, $S_l(+\epsilon)$ is large and negative; since $S_l(s_1)=1$, $S_l(s)$ must vanish on the real axis in the interval $(0,s_1)$ unless it has a pole of positive residue (a bound state) in the same interval.[13] This pole may be regarded as having moved into the physical sheet from the unphysical one. Whichever sheet it is on, it is an odd pole unaccompanied by any other.

Our considerations in this section not only have led to results of interest in their own right, viz., Eq. (3.6) and its usefulness in giving a quick estimate of n_c on the basis of $A_l^B(s)$, but are also helpful to practical calculations using the iteration method. It is important that the iteration program should start off with the proper number of poles corresponding to the particular $A_l^B(s)$ that is used, if it is decided for the sake of computing speed that the interaction is not to be turned on adiabatically. The results of this section then indicate how many are to be found. By inspecting $A_l^B(s)$ along the left-hand cut and remembering that the unitarity contribution, $\bar{A}_l(s)$, is always positive there, one knows before the iteration process is started, whether the unitarity correction will cause the zeros of $S_l(s)$ to move toward the left-hand cut or away from it. The movement is toward the left-hand cut if $d[\mathrm{Im}A_l^B(s)]/d[\mathrm{Re}A_l^B(s)]>0$ at s where $\mathrm{Re}A_l^B(s)=0$ and $\mathrm{Im}A_l^B(s)>0$; otherwise, they move away. Such qualitative knowledge is useful in giving more stability to the iteration method.

[13] R. Blankenbecler, M. L. Goldberger, S. W. MacDowell, and S. B. Treiman, Phys. Rev. **123**, 692 (1961).

IV. COMPLEX ANGULAR MOMENTUM

Our interest in the analytic properties of $K_l(s)$, defined as $\ln S_l(s)$, is based on the fact that in the unphysical sheet the S function is $S_l^{-1}(s)$, so that a pole in this sheet results in a singularity of $K_l(s)$. However, such a relationship between the physical and unphysical sheets has been shown to be true only for integral values of l. An invalidation of this relationship for nonintegral l would necessitate the search for a new logarithmic function $K(l,s)$ which can put the physical and the first unphysical sheets on the same footing explicitly.

For the purpose of continuation in l the partial-wave amplitude is first expressed in the form

$$A(l,s)=h_1(l,s)+(-1)^l h_2(l,s)\,, \tag{4.1}$$

where

$$h_1(l,s)=\frac{1}{\pi}\int_{t_1}^{\infty}\frac{2dt'}{s-s_1}A_t(s,t')Q_l\left(1+\frac{2t'}{s-s_1}\right), \tag{4.2}$$

$$h_2(l,s)=\frac{1}{\pi}\int_{u_1}^{\infty}\frac{2du'}{s-s_1}A_u(s,u')Q_l\left(1+\frac{2u'}{s-s_1}\right). \tag{4.3}$$

The j-parity amplitude is then defined in terms of $h_1(l,s)$ and $h_2(l,s)$ as

$$F^{\pm}(l,s)=(k/s^{1/2})[h_1(l,s)\pm h_2(l,s)]\,. \tag{4.4}$$

In the following we omit the signature symbol $\pm$ for the sake of convenience.

Now, the unitarity condition, when generalized to complex l, has the form[14]

$$F(l,s+)-F^*(l^*,s+)=2iF(l,s+)F^*(l^*,s+)\,, \tag{4.5}$$

where $s+$ implies $s+i\epsilon$. Writing $F^*(l^*,s+)$ as $F^*(l^*,s^*-)$, we obtain from (4.5)

$$F(l,s+)=\frac{F^*(l^*,s^*-)}{1-2iF^*(l^*,s^*-)}\,. \tag{4.6}$$

Among the infinite number of sheets connected by the cut between s_1 and s_2 when l is not an integer, let the first unphysical sheet be the one reached directly from the physical sheet by a clockwise continuation around s_1. Thus, by definition $F_u(l,s-)=F(l,s+)$. Continuing the right-hand side of (4.6) to the complex s^* plane simultaneously as $F_u(l,s-)$ is continued to the complex s plane, we obtain

$$F_u(l,s)=\frac{F^*(l^*,s^*)}{1-2iF^*(l^*,s^*)}\,. \tag{4.7}$$

It has been shown by Okubo[15] that the reality of the

[14] E. J. Squires, Nuovo Cimento **25**, 242 (1962).
[15] S. Okubo, University of Rochester, NYO-10239 (to be published).

double spectral functions implies the following reflection relationship:

$$F^*(l^*,s^*) = -F(l,s)\exp(-2\pi il). \tag{4.8}$$

We now define the generalized S function to be

$$S(l,s) = 1 + 2iF(l,s)e^{-2\pi il}. \tag{4.9}$$

Its continuation to the first unphysical sheet satisfies the property

$$S_u(l, s-) = S(l, s+). \tag{4.10}$$

From (4.7) and (4.8) we thus obtain

$$S_u(l,s) = (1-e^{-2\pi il}) + S^{-1}(l,s)e^{-2\pi il}. \tag{4.11}$$

Clearly, when l is an integer, we regain (2.3). When l is not an integer, the definition of $S(l,s)$ has made possible the association of a pole in the unphysical sheet with a zero in $S(l,s)$. It therefore follows that the logarithmic function which we should be interested in is

$$K(l,s) = \ln S(l,s). \tag{4.12}$$

To eliminate the elastic cut for nonintegral l, it is the dispersion relation for $K(l,s)/(s-s_1)^{l+\frac{1}{2}}$ which we must consider. The remarks at the end of Sec. II are therefore especially relevant in the use of the iteration method.

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Representations and Mass Formulas for SU(4)

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A hierarchial scheme of SU(n) symmetries among the strong interactions is proposed with the $n-1$ additive quantum numbers exactly conserved. Simple dynamics as well as mass-splitting considerations are shown to favor the assignment of the mesons to the 15-dimensional representations and the baryons and $\frac{3}{2}^+$ isobars to two inequivalent 20-dimensional representations of SU(4). The predictions of new particles are discussed.

THE experimental studies of the boson mass spectrum in the region 400–1600 MeV[1] suggest a structure far too complicated for a simple SU(3) model to cope with. It seems worthwhile therefore to look for supersymmetries that would have SU(3) as a subgroup and that would also have large representations suitable for containing, e.g., all the known or suggested vector mesons. SU(4) suggests itself as the most obvious candidate for such a supersymmetry.[2] In this scheme it is also easy to formulate a baryon-lepton symmetry[3] of the Cabibbo type.[4]

In an earlier paper[2] it was suggested that SU(4) would be physically relevant only for the vector mesons, the reason being their property of bootstrapping themselves. For other multiplets the breakdown of the symmetry would be catastrophic and the conservation of the new additive quantum number Z, supercharge (also called hyperstrangeness), would lose all meaning.

In this paper we adopt the point of view of an exactly conserved supercharge. The mesons, both pseudoscalar (M) and vector (V), belong to the regular (adjoint) 15-dimensional representation $\psi_i\bar{\psi}_j-\delta_{ij}$, where ψ_i ($i=1\cdots4$) is the basic four-component quark field of SU(4). The baryons (B) are included in $\psi\psi\bar{\psi}$ and the $\frac{3}{2}^+$ isobars (B^*) in some representation of the baryon-meson system. We characterize the irreducible representations by the combination ($\lambda\mu\nu$) in analogy to the common SU(3) usage. In Table I we give the dimensionalities

$$d = (\lambda+1)(\mu+1)(\nu+1)(\lambda+\mu+2) \times(\mu+\nu+2)(\lambda+\mu+\nu+3)/12$$

and SU(3) multiplet contents of some of the representations. Notice that there are three inequivalent 20-dimensional representations. The most interesting

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[2] P. Tarjanne and V. L. Teplitz, Phys. Rev. Letters **11**, 447 (1963).

[3] Y. Hara, Phys. Rev. **134**, B701 (1964).

[4] N. Cabibbo, Phys. Rev. Letters **10**, 531 (1963); **12**, 62 (1964).