Relativistic Calculation of the Deuteron Electromagnetic Form Factor. II*

FRANZ GROSS
Laboratory of Nuclear Studies, Cornell University, Ithaca, New York
(Received 25 May 1964)

The deuteron form factor is calculated in a one-pion-exchange (OPE) approximation using single-variable unsubtracted dispersion relations in the squared-momentum-transfer variable. As a first step, the imaginary parts of the form factors are presented in terms of the four invariants of the deuteron-nucleon (d–N) vertex. The resulting equations are compared in detail with similar expressions obtained from the Jankus potential theory, and a clear understanding of the precise way in which the d–N invariants play the role of the deuteron wave function emerges, enabling us to obtain relativistic wave functions. This comparison also shows the presence of new relativistic terms not included in the Jankus results (in particular, a new term appears in the magnetic moment). Then, the imaginary parts of the d–N vertex invariants are calculated numerically. In this calculation the OPE contribution in its anomalous region is calculated exactly, while the contributions above the normal threshold are obtained by an educated guess based on sum rules which the invariants are assumed to satisfy. Finally, using these results, the form factors are calculated numerically. With the assumption of unsubtracted dispersion relations, it is necessary to assume only the charge and mass of the deuteron, and the pion-nucleon coupling constant, in order to completely determine the form factors. The numerical results for the magnetic and quadrupole moments agree with experiment to within 2%; the D-state probability is 5.5% but is not very reliably determined, and the other results, while less good, are still quite reasonable.

1. INTRODUCTION

THIS paper presents the numerical results of a relativistic calculation of the deuteron electromagnetic form factor using single variable unsubtracted dispersion relations and coupled unitarity equations. The philosophy and foundations of such a calculation have been discussed in an earlier paper[1] (which we shall hereafter refer to as I), and the basic equations which we employ were derived there.

The calculation is based on the three diagrams shown in Fig. 2. Our central approximation is to assume that the imaginary parts of the deuteron form factors are dominated by the discontinuities calculated from these diagrams in their anomalous regions (region between the anomalous and normal thresholds). Then, to allow for the effect of higher thresholds approximately we employ certain sum rules [Eqs. (3.28) and (3.29)] which can be derived on the assumption that the deuteron form factors satisfy unsubtracted dispersion relations, and which are suggested by a comparison with potential theory. In addition, we assume that the pion-nucleon coupling constant is given by

$$g^2/4\pi = 14. \quad (1.1)$$

When all of these ideas are put together, we are left with only one free parameter, which is adjusted to yield the correct deuteron charge. The D-state probability, magnetic moment, quadrupole moment, and momentum transfer dependence of the three form factors are thereby completely determined. The numerical results are in reasonable agreement with experiment. We conclude that if one is careful the calculation contains uncertainties of not more than 5% at low momentum transfer, and the agreement with experiment is well within these uncertainties. Furthermore, it seems that even if one is relatively crude the diagrams in Fig. 2 yield results within 10 or 15% of the experimental values. These remarks are explained in greater detail in Sec. 5, and the specific numerical results are given in Tables III and IV.

It has become customary to assume that dispersion theory calculations meet with little numerical success. Our experience suggests that for certain problems (i.e., those dominated by anomalous thresholds) one need not be so pessimistic. While the results are still a long way from the 1% accuracy that one dreams of, there seems to be reason to entertain a feeble hope that additional work may place one close to this goal. The next job to undertake is an estimate of the explicit three-pion contribution,[2] and to calculate the deuteron-nucleon vertex more carefully. In addition one needs to understand more about some of the other contributions discussed in I. Many of these can be explicitly calculated; it may be possible to prove that many are small at low momentum transfer.

The partial success of dispersion theory in this calculation is perhaps not surprising. First, it should be borne in mind that the objective is not actually to calculate the deuteron form factor completely, but only to express it in terms of the isoscalar nucleon form factor, a simpler job. In addition, the loosely bound structure of the deuteron means that much of its structure is largely kinematic or due to one- or two-particle exchange and hence is amenable to direct attack.

In spite of this limited success the theory is not yet

* Parts of this paper are based on a thesis submitted to Princeton University in partial fulfillment of the requirements for the Ph.D. degree. Research supported in part by the U. S. Office of Naval Research.

[1] F. Gross, Phys. Rev. **134**, B405 (1964) (hereafter referred to as I). This reference contains some other relevant references. For errata in I see Appendix E of this paper.

[2] B. M. Casper (private communication).

strong enough to make very meaningful estimates of the neutron form factor from the experimental data. The reason for this is simple enough; in order to extract anything from the experimental data a very accurate theory is needed.

Let us review this point briefly. One recalls that the relativistic differential cross section for elastic electron-deuteron scattering is[3]

$$\frac{d\sigma}{d\Omega}=\frac{d\sigma}{d\Omega}\bigg|_{\text{N.S.}}\bigg\{G_C^2(s)+\frac{s^2}{18M^4}G_Q^2(s) -\frac{s}{6M^2}\bigg[1+2\bigg(1-\frac{s}{4M^2}\bigg)\tan^2(\theta/2)\bigg]G_M^2(s)\bigg\}, \quad (1.2)$$

where $s=q^2$ is the square of the four-momentum transfer; M is the deuteron mass; and G_C, G_Q, and G_M are the charge, quadrupole moment, and magnetic-moment form factors of the deuteron. The differential scattering cross section for point particles is (in the laboratory system):

$$\frac{d\sigma}{d\Omega}\bigg|_{\text{N.S.}}=\bigg(\frac{e^2}{2E_0}\bigg)^2\frac{\cos^2(\theta/2)}{\sin^4(\theta/2)} \times\frac{1}{[1+(2E_0/M)\sin^2(\theta/2)]}, \quad (1.3)$$

where E_0 is the energy of the incoming electron, θ the laboratory scattering angle of the electron.

Experimentally, one measures the ratio

$$r^2=\frac{G_C^2(s)+(s^2/18M^4)G_Q^2(s)}{F_p^2(s)}, \quad (1.4)$$

which is compared with a theoretical ratio

$$r=[1+F_n(s)/F_p(s)]D_C(s)+R(s), \quad (1.5)$$

where F_n and F_p are the neutron and proton form factors and R is a "nonadditive" correction which one is not in the habit of including but which can be expected to be, say, of the order of 5% of D_C. If, in addition, D_C is not to be better than 5% (as is the case in this paper but probably not the case with a good potential theory—see Sec. 5), and r is known to only 2% experimentally, then we have

$$1+F_n(s)/F_p(s)\cong(r/D_C(s))(1\pm0.02\pm0.10), \quad (1.6)$$

where the second error is the combined theoretical error. However, knowledge of $1+F_n/F_p$ to only 12% cannot be expected to yield much information about F_n, when, for small s, $F_n\approx0.1F_p$. At larger momentum transfer, where $F_n\approx F_p$ we are no better off, because then the contribution R is more important and at the moment we have no knowledge of its size. For these reasons we shall refrain from presenting our own calculations of the neutron charge form factor at this time.

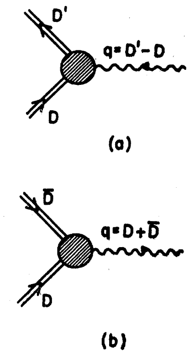

FIG. 1. The deuteron-photon vertex, or the deuteron form factor in (a) the scattering channel ($s<0$) and (b) the annihilation channel ($s>4M^2$).

The form factors that we discuss in this paper, G_C, G_M, and G_Q are defined in terms of three other form factors, G_1, G_2, and G_3. The deuteron scattering form factor is

$$G^\mu(s)=(2D^0 2D'^0)^{1/2}\langle D'|j^\mu|D\rangle =-e\bigg\{G_1(s)(\xi'^*\cdot\xi)d^\mu+G_2(s)[\xi^\mu(\xi'^*\cdot q)-\xi'^{*\mu}(\xi\cdot q)] -G_3(s)\frac{(\xi\cdot q)(\xi'^*\cdot q)}{2M^2}d^\mu\bigg\}, \quad (1.7)$$

where ξ and ξ' are polarization vectors for the incoming and outgoing deuterons of momenta D and D', respectively, (see Fig. 1)

$$\xi'\cdot D'=0 \quad \xi'^2=-1$$
$$\xi\cdot D=0 \quad \xi^2=-1$$

and $d^\mu=D'^\mu+D^\mu$, $q^\mu=D'^\mu-D^\mu$. The form factors G_C, G_M, and G_Q are defined as:

$$G_C(s)=G_1(s)-(s/6M^2)G_Q(s),$$
$$G_M(s)=G_2(s), \quad (1.8a)$$
$$G_Q(s)=G_1(s)-G_2(s)+(1-s/4M^2)G_3(s).$$

These form-factor combinations are introduced because they can be shown[3,4] to reduce in their nonrelativistic limits to the nonrelativistic form factors conventionally used and first introduced by Jankus.[5,6] Their names derive from this fact, and their static values correspond to the charge, magnetic moment, and quadrupole moment of the deuteron. More specifically, we have[3,4]

$$G_C(0)=1,$$
$$G_M(0)=2M\mu_d=\mu, \quad (1.8b)$$
$$G_Q(0)=M^2Q_d=Q,$$

[3] M. Gourdin, Nuovo Cimento **28**, 533 (1963). Note however that the form factor $D_M{}^M$ presented in this reference differs from that presented in Refs. 5 and 6 and in this paper [Eq. (2.8)].

[4] F. Gross, Ph.D. thesis, Princeton University, 1963 (unpublished).

[5] V. Z. Jankus, Phys. Rev. **102**, 1586 (1956).

[6] N. K. Glendenning and G. Kramer, Phys. Rev. **126**, 2159 (1962).

where μ_d and Q_d are the deuteron magnetic and quadrupole moments, and μ and Q are these moments in units of $e/2M$ and e/M^2, respectively.

For calculational purposes it is more convenient to work with the annihilation form factor which can be obtained from the crossed reaction shown in Fig. 1(b). To obtain this we let $D'\to -\bar{D}$, $\xi'^*\to\eta^*$ and change the sign of the over-all amplitude. Hence, we obtain:

$$\begin{aligned}G^\mu(s)&=(2D^0 2\bar{D}^0)^{1/2}\langle 0|j^\mu|D\bar{D}\rangle\\&=e\{G_1(s)(\eta^*\cdot\xi)d^\mu-G_2(s)[\xi^\mu(\eta^*\cdot q)-\eta^{\mu*}(\xi\cdot q)]\\&\quad-G_3(s)[(\eta^*\cdot q)(\xi\cdot q)/2M^2]d^\mu\},\end{aligned}\quad(1.9)$$

where $d=D-\bar{D}$ and now $q=D+\bar{D}$. The form factors describe the annihilation channel when $s>4M^2$ and the scattering channel when $s<0$.

In Sec. 2 we review the Jankus[5] nonrelativistic results for the deuteron form factor, and cast these into a form which facilitates comparison with the relativistic theory. This comparison is extremely instructive, and provides motivation for the use of the sum rules we shall introduce, but is in no way essential to the calculation. Then, in Sec. 3 we present the results of the calculation of the form factors based on the diagrams shown in Fig. 2. We express these results in terms of the four invariant functions of the deuteron-nucleon (d–N) vertex, first calculated by Blankenbecler and Cook.[7] It is observed that when one writes the equations in terms of certain combinations of these d-N vertex functions, the results have a very similar structure to those of Sec. 2. This enables one to interpret the various terms present and to observe in detail how the d-N vertex functions play the role of the deuteron wave function. We then discuss in some detail the relationship between a potential theory and a relativistic theory, including the uniqueness of our identification of combinations of d–N vertex invariants with the deuteron wave function. Because of the importance of the d–N vertex, Sec. 4 is devoted to a calculation of it in the one-pion-exchange approximation. At this time we also impose certain sum-rule conditions on the vertex functions. Finally in Sec. 5 we present detailed results of numerical calculations, and summarize our conclusions. The reader who is not interested in the details of the calculation may still find some useful remarks in Sec. 5.

We have included five appendixes. Appendix A contains a summary of conventions. Appendixes B and D contain detailed results of the calculations of the form factors and the d–N vertex functions, respectively, while Appendix C contains mathematical material to establish a uniqueness argument developed in Sec. 3 and discussed in Sec. 5. Appendix E contains a list of errata in I.

[7] R. Blankenbecler and L. F. Cook, Jr., Phys. Rev. **119**, 1745 (1960).

2. POTENTIAL THEORY

In this section we will cast the Jankus potential theory into a form which will be suitable for a detailed comparison with the results of dispersion theory obtained in the next sections.

The potential-theory results are well known. We can write the deuteron wave function in terms of nucleon Pauli spinors:

$$\Phi_d(\mathbf{r})=\chi_n{}^\dagger\psi_d(\mathbf{r})\chi_p\,,\qquad(2.1)$$

where χ_n and χ_p are nucleon spinors. The matrix $\psi_d(\mathbf{r})$ is

$$\psi_d(\mathbf{r})=\frac{1}{(4\pi)^{1/2}}\left\{\frac{u(r)}{r}\boldsymbol{\sigma}\cdot\boldsymbol{\xi}-\frac{1}{\sqrt{2}}\frac{w(r)}{r}\times\left(\frac{3(\boldsymbol{\sigma}\cdot\mathbf{r})(\mathbf{r}\cdot\boldsymbol{\xi})}{r^2}-\boldsymbol{\sigma}\cdot\boldsymbol{\xi}\right)\right\}\frac{i\sigma_2}{\sqrt{2}},\quad(2.2)$$

where $\boldsymbol{\sigma}$ are the Pauli spin matrixes and $\boldsymbol{\xi}$ is the deuteron polarization vector

$$\begin{aligned}\xi^{\pm1}&=(\mp1/\sqrt{2},\,-i/\sqrt{2},\,0)\,,\\\xi^0&=(0,0,1)\,.\end{aligned}\qquad(2.3)$$

The S- and D-state radial wave functions are u and w, respectively. The constants have been chosen so that

$$\begin{aligned}\int_{-\infty}^{+\infty}|\Phi_d(\mathbf{r})|^2d^3r&=\int_{-\infty}^{+\infty}\text{trace}\{\psi_d{}^\dagger(\mathbf{r})\psi_d(\mathbf{r})\}d^3r\\&=\int_0^\infty[u^2(r)+w^2(r)]dr\,.\end{aligned}\qquad(2.4)$$

The deuteron form factors can be expressed in terms of the radial functions u and w. Introducing the proton and neutron charge and magnetic form factors[8] $F_C{}^p(\mathbf{q}^2)$, $F_M{}^p(\mathbf{q}^2)$, $F_C{}^n(\mathbf{q}^2)$, $F_M{}^n(\mathbf{q}^2)$, the first-order current density is commonly written (in the Breit frame) as

$$\begin{aligned}j^0&=e\{F_C{}^p(\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{r}/2}+F_C{}^n(\mathbf{q}^2)e^{-i\mathbf{q}\cdot\mathbf{r}/2}\}\,,\\j^k&=-ei/m\{F_C{}^p(\mathbf{q}^2)e^{i\mathbf{q}\cdot\mathbf{r}/2}\boldsymbol{\nabla}_k-F_C{}^n(\mathbf{q}^2)e^{-i\mathbf{q}\cdot\mathbf{r}/2}\boldsymbol{\nabla}_k\}\\&\quad+ei/2m\{F_M{}^p(\mathbf{q}^2)(\boldsymbol{\sigma}\times\mathbf{q})^ke^{i\mathbf{q}\cdot\mathbf{r}/2}\\&\quad+F_M{}^n(\mathbf{q}^2)(\boldsymbol{\sigma}\times\mathbf{q})^ke^{-i\mathbf{q}\cdot\mathbf{r}/2}\}\,,\quad k=1,2\\j^3&=0\,.\end{aligned}\qquad(2.5)$$

Here m is the nucleon mass. The form factor is then

$$G^\mu{}_d(\mathbf{q})=\langle\Phi_{d'}|j^\mu|\Phi_d\rangle\qquad(2.6)$$

and becomes

$$\begin{aligned}G^0{}_d(\mathbf{q})&=e\{(\boldsymbol{\xi}'\cdot\boldsymbol{\xi})G_C(\mathbf{q}^2)\\&\quad+[(\boldsymbol{\xi}'\cdot\mathbf{q})(\boldsymbol{\xi}\cdot\mathbf{q})-\tfrac{1}{3}\mathbf{q}^2(\boldsymbol{\xi}\cdot\boldsymbol{\xi}')]G_Q(\mathbf{q}^2)/2M^2\}\,,\end{aligned}\qquad(2.7)$$

$$G^k{}_d(\mathbf{q})=\frac{e}{4m}[\xi^k(\boldsymbol{\xi}'\cdot\mathbf{q})-\xi'^k(\boldsymbol{\xi}\cdot\mathbf{q})]G_M(\mathbf{q}^2)\,,$$

[8] The quantities $F(\mathbf{q}^2)$ are assumed to be the nonrelativistic counterparts of $F(s)$ introduced in Eq. (3.3).

where

$$G_C(\mathbf{q}^2)=F_C(\mathbf{q}^2)\int_0^\infty [u^2+w^2]j_0\left(\frac{qr}{2}\right)dr\,,$$

$$G_Q(\mathbf{q}^2)=F_C(\mathbf{q}^2)\frac{6\sqrt{2}M^2}{\mathbf{q}^2}\int_0^\infty \left(uw-\frac{w^2}{\sqrt{8}}\right)j_2\left(\frac{qr}{2}\right)dr\,;$$

$$G_M(\mathbf{q}^2)=F_C(\mathbf{q}^2)D_M{}^C(\mathbf{q}^2)+F_M(\mathbf{q}^2)D_M{}^M(\mathbf{q}^2)\,,$$

$$D_M{}^C(\mathbf{q}^2)=\frac{3}{2}\int_0^\infty w^2\left[j_0\left(\frac{qr}{2}\right)+j_2\left(\frac{qr}{2}\right)\right]dr\,, \tag{2.8}$$

$$D_M{}^M(\mathbf{q}^2)=2\int_0^\infty (u^2-\tfrac{1}{2}w^2)j_0\left(\frac{qr}{2}\right)dr+\sqrt{2}\int_0^\infty \left(uw+\frac{w^2}{\sqrt{2}}\right)j_2\left(\frac{qr}{2}\right)dr\,;$$

here j_0 and j_2 are the usual spherical Bessel functions, $q=|\mathbf{q}|$, and

$$F_C=F_C{}^p+F_C{}^n\,,\quad F_M=F_M{}^p+F_M{}^n\,.$$

We wish to cast these well-known results into a different form. To this end we recall that if the potential is a superposition of Yukawa wells, then the wave functions can be written in the following form:

$$u(r)=\frac{1}{(2\pi)^{1/2}}\int_\alpha^\infty f_1(\sigma)e^{-\sigma r}d\sigma\,, \tag{2.9a}$$

$$w(r)=\frac{1}{(2\pi)^{1/2}}\int_\alpha^\infty \sigma^2 g_1(\sigma)e^{-\sigma r}\left[1+\frac{3}{\sigma r}+\frac{3}{(\sigma r)^2}\right]d\sigma\,, \tag{2.9b}$$

$$=\frac{1}{(2\pi)^{1/2}}\int_\alpha^\infty \lambda(\sigma)e^{-\sigma r}d\sigma\,, \tag{2.9c}$$

where $\alpha^2=m\epsilon$, and ϵ is the deuteron binding energy.

Integration by parts gives

$$\lambda(\sigma)=\sigma^2 g_1(\sigma)+3\sigma\int_\alpha^\sigma g_1(y)dy\,,$$

$$g_1(\sigma)=\frac{\lambda(\sigma)}{\sigma^2}-\frac{3}{\sigma^4}\int_\alpha^\sigma y\lambda(y)dy\,. \tag{2.10}$$

As it turns out, the choice of g_1 as the D-state weight function is more appropriate than λ to a comparison with the results of Sec. 3.

If the representations (2.9) are substituted into Eqs. (2.8), an interesting form for the form factors emerges. To obtain this, it is necessary only to make use of the following identity valid for integral n:

$$\int_0^\infty dr\int_\alpha^\infty d\sigma\int_\alpha^\infty d\sigma' e^{-(\sigma+\sigma')r}j_{2n}\left(\frac{qr}{2}\right)\rho(\sigma)\rho(\sigma')$$

$$=(-1)^n\int_{16\alpha^2}^\infty \frac{d\chi}{(\mathbf{q}^2+\chi)}\frac{1}{\chi^{1/2}}\int_\alpha^\infty d\sigma\int_\alpha^\infty d\sigma'\,\theta\left[\frac{\chi^{1/2}}{2}-\sigma-\sigma'\right]P_{2n}\left(\frac{2(\sigma+\sigma')}{\chi^{1/2}}\right)\rho(\sigma)\rho(\sigma')\,, \tag{2.11}$$

where j_{2n} and P_{2n} are spherical Bessel functions and Legendre polynomials of order $2n$. With the use of this identity for $n=0$ and 1, and successive integrations by parts, we eventually obtain the following integral results for the form factors:

$$G_{C,Q}(\mathbf{q}^2)=\frac{F_C(\mathbf{q}^2)}{\pi}\int_{16\alpha^2}^\infty \frac{d\chi}{\mathbf{q}^2+\chi}A_{C,Q}(\chi)\,,$$

$$D_M{}^{C,M}(\mathbf{q}^2)=\frac{1}{\pi}\int_{16\alpha^2}^\infty \frac{d\chi}{\mathbf{q}^2+\chi}A_M{}^{C,M}(\chi)\,, \tag{2.12}$$

where if

$$\Sigma=\frac{1}{2(\chi)^{1/2}}\int_{\alpha^2}^\infty d\eta\int_{\alpha^2}^\infty d\bar{\eta}\,\theta\left[\frac{\chi^{1/2}}{2}-\eta^{1/2}-\bar{\eta}^{1/2}\right], \tag{2.13}$$

and introducing

$$f(\eta)=\frac{f_1(\eta^{1/2})}{2\eta^{1/2}}\,,\quad g(\eta)=\frac{g_1(\eta^{1/2})}{2\eta^{1/2}}\,, \tag{2.14}$$

we have

$$A_C(\chi)=\Sigma\left\{f(\eta)f(\bar{\eta})+\frac{3}{128}g(\eta)g(\bar{\eta})\left[\chi^2-8\chi(\eta+\bar{\eta})+16(\eta^2+\bar{\eta}^2)+\frac{32}{3}\eta\bar{\eta}\right]\right\},$$

$$A_Q(\chi)=M^2\Sigma\left\{\frac{3\sqrt{2}}{16}f(\eta)g(\bar{\eta})\left[3+\frac{8\bar{\eta}}{\chi}-\frac{24\eta}{\chi}+48\frac{(\eta-\bar{\eta})^2}{\chi^2}\right]-\frac{9}{256}g(\eta)g(\bar{\eta})\left[\chi-4(\eta+\bar{\eta})-16\left(\frac{\eta^2+\bar{\eta}^2}{\chi}\right)-\frac{32}{3}\frac{\eta\bar{\eta}}{\chi}+64\frac{(\eta+\bar{\eta})(\eta-\bar{\eta})^2}{\chi^2}\right]\right\},$$

$$A_M{}^C(\chi)=\Sigma\,\frac{9}{512}g(\eta)g(\bar{\eta})\left[\chi^2-12\chi(\eta+\bar{\eta})+48(\eta^2+\bar{\eta}^2)+32\eta\bar{\eta}-64\frac{(\eta+\bar{\eta})(\eta-\bar{\eta})^2}{\chi}\right],$$

$$A_M{}^M(\chi)=\Sigma\left\{2f(\eta)f(\bar{\eta})-\frac{3\sqrt{2}}{32}f(\eta)g(\bar{\eta})\left[\chi+\frac{8}{3}\bar{\eta}-8\eta+16\frac{(\eta-\bar{\eta})^2}{\chi}\right]-\frac{9}{256}g(\eta)g(\bar{\eta})\left[\chi^2-\frac{20}{3}\chi(\eta+\bar{\eta})+\frac{16}{3}(\eta^2+\bar{\eta}^2)+\frac{32}{9}\eta\bar{\eta}+\frac{64}{3}\frac{(\eta+\bar{\eta})(\eta-\bar{\eta})^2}{\chi}\right]\right\}. \tag{2.15}$$

Observe that the above expressions bear a remarkable resemblance to dispersion integrals, with the imaginary parts given by the A's. This will enable us to make an easy comparison of these potential theory results with the results obtained in Sec. 3.

Note that in order that these integrals be finite it is necessary and sufficient that, for arbitrarily large N,

$$\int_{\alpha^2}^{N^2} f(\eta)d\eta < \text{const}N^{1/2}, \qquad \int_{\alpha^2}^{N^2} g(\eta)d\eta < \frac{\text{const}}{N^{3/2}}. \tag{2.16}$$

These minimal conditions will be referred to as the (A) conditions. The physical significance of these conditions is apparent from Eqs. (2.8) and (2.11); they are the necessary and sufficient conditions that the S and D wave functions be normalizable. Note that the condition on $g(\eta)$ is relatively stringent, a reflection of the fact that the D state has been written as a superposition of wave functions each of which is itself not normalizable.

In practice, we may wish to require that the S- and D-state wave functions be zero at the origin. This means that

$$\int_{\alpha^2}^{\infty} f(\eta)d\eta = 0, \qquad \int_{\alpha^2}^{\infty} \eta g(\eta)d\eta = \int_{\alpha^2}^{\infty} g(\eta)d\eta = 0. \tag{2.17}$$

These will be referred to as the (B) conditions.

The (B) conditions represent the simplest of a general class of restrictions which we would impose on the wave functions if we believed that they should demonstrate a repulsive core behavior. A repulsive core would express itself in the requirement that the first L derivatives of the wave function at $r=0$ are zero. In terms of f and g, this becomes

$$\int_{\alpha^2}^{\infty} \eta^{n/2} f(\eta)d\eta = 0 \qquad n=1,0,\cdots L$$
$$\int_{\alpha^2}^{\infty} g(\eta)d\eta = \int_{\alpha^2}^{\infty} \eta^{(n/2+1)} g(\eta)d\eta = 0 \quad n\neq 1. \tag{2.18}$$

Note that the first derivative of the D-state wave function is automatically zero.

To facilitate the comparison of expressions (2.12) with relativistic theory and also to focus attention on the significant parts of these expressions at low momentum transfer we isolate that part of the expression which is zero at $\mathbf{q}^2=0$. These parts cannot contribute to the deuteron static moments, and furthermore will give a low contribution for small $\mathbf{q}^2$.

By integrating by parts it is possible to show that a number of the terms in (2.12) are zero at $\mathbf{q}^2=0$, independent of the structure of the weight functions f and g, so long as the (A) conditions hold. To show this it is sufficient to consider the integrals:

$$\int_{16\alpha^2}^{\infty} \frac{d\chi}{\chi^{5/2}} \int_{\alpha^2}^{\infty} d\eta \int_{\alpha^2}^{\infty} d\bar{\eta}\; \Lambda(\eta,\bar{\eta}) P(\chi;\eta\bar{\eta})\theta\left[\frac{\chi^{1/2}}{2} - \eta^{1/2} - \bar{\eta}^{1/2}\right],$$
$$\int_{16\alpha^2}^{\infty} \frac{d\chi}{\chi^{7/2}} \int_{\alpha^2}^{\infty} d\eta \int_{\alpha^2}^{\infty} d\bar{\eta}\; \Lambda(\eta,\bar{\eta}) Q(\chi;\eta\bar{\eta})\theta\left[\frac{\chi^{1/2}}{2} - \eta^{1/2} - \bar{\eta}^{1/2}\right], \tag{2.19}$$

where P and Q are third-order polynomials in χ, η, $\bar{\eta}$, and $\Lambda(\eta,\bar{\eta})$ is an arbitrary function subject to the condition

$$\int_{\alpha^2}^{\infty} d\eta\, \Lambda(\eta,\bar{\eta}) = 0.$$

The Λ represents the products $g(\eta)f(\bar{\eta})$ or $g(\eta)g(\bar{\eta})$. By integrating (2.19) by parts one can find the most general polynomials P and Q which will always guarantee that the integrals (2.19) are zero. For P we need only the class of polynomials symmetric in η and $\bar{\eta}$:

$$P = a\chi^3 + 2b(\eta+\bar{\eta})\chi^2 + (16a+16b+2c)(\eta^2+\bar{\eta}^2)\chi + ((160/3)a + 32b + \tfrac{4}{3}c)\eta\bar{\eta}\chi - 32(4a+3b+\tfrac{1}{4}c)(\eta+\bar{\eta})(\eta-\bar{\eta})^2, \tag{2.20}$$

where a, b and c are arbitrary constants. The polynomial Q is

$$Q = a\chi^3 + 2b(\eta+\bar{\eta})\chi^2 + 2c(\eta-\bar{\eta})\chi^2 + (144a - 48b + \tfrac{3}{5}d)(\eta^2+\bar{\eta}^2)\chi - (48c + \tfrac{2}{5}d)(\eta^2-\bar{\eta}^2)\chi + (480a - 96b + 2d)\eta\bar{\eta}\chi + (160b - 640a - (8/3)d)(\eta+\bar{\eta})(\eta-\bar{\eta})^2 + (10/3)(48c + \tfrac{2}{5}d)(\eta^3 - \bar{\eta}^3 + 3\eta\bar{\eta}^2 - 3\eta^2\bar{\eta}), \tag{2.21}$$

where again a, b, c, and d are arbitrary constants. We wish to call attention to five special cases of these polynomials:

$$P_1 = \chi[\chi^2 - 8(\eta+\bar{\eta})\chi + 16(\eta^2+\bar{\eta}^2) - 32\eta\bar{\eta}],$$
$$P_2 = \chi[\chi^2 - 12(\eta+\bar{\eta})\chi + 48(\eta^2+\bar{\eta}^2) - (160/3)\eta\bar{\eta} - 64(\eta+\bar{\eta})(\eta-\bar{\eta}^2)/\chi],$$
$$P_3 = \chi[\chi^2 - (20/3)(\eta+\bar{\eta})\chi + (16/3)(\eta^2+\bar{\eta}^2) - (224/9)\eta\bar{\eta} + (64/3)(\eta+\bar{\eta})(\eta-\bar{\eta})^2/\chi],$$
$$Q_1 = \chi^2[\chi + (4/15)(\eta+\bar{\eta}) - 16(\eta^2+\bar{\eta}^2)/\chi - 32(7/5)\eta\bar{\eta}/\chi + 64(\eta+\bar{\eta})(\eta-\bar{\eta})^2/\chi^2],$$
$$Q_2 = \chi^2[\chi + (8/3)\eta - 8\bar{\eta} + 16(\eta^2+\bar{\eta}^2)/\chi - 32\eta\bar{\eta}/\chi].$$

In terms of these, the form factors can be written in another form. Making use of the identity

$$\int_{16\alpha^2}^{\infty}\frac{\chi d\chi}{\mathbf{q}^2+\chi}f(\chi)=\int_{16\alpha^2}^{\infty}d\chi f(\chi)-\mathbf{q}^2\int_{16\alpha^2}^{\infty}\frac{d\chi f(\chi)}{\mathbf{q}^2+\chi},$$

we have

$$\begin{aligned}G_{C,Q}(\mathbf{q}^2)&=\frac{F_C(\mathbf{q}^2)}{\pi}\int_{16\alpha^2}^{\infty}\frac{d\chi}{\mathbf{q}^2+\chi}a_{C,Q}(\chi)\\&\quad-\frac{\mathbf{q}^2F_C(\mathbf{q}^2)}{\pi}\int_{16\alpha^2}^{\infty}\frac{d\chi}{\mathbf{q}^2+\chi}b_{C,Q}(\chi),\\D_M{}^{C,M}(\mathbf{q}^2)&=\frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{d\chi}{\mathbf{q}^2+\chi}a_M{}^{C,M}(\chi)\\&\quad-\frac{\mathbf{q}^2}{\pi}\int_{16\alpha^2}^{\infty}\frac{d\chi}{\mathbf{q}^2+\chi}b_M{}^{C,M}(\chi),\end{aligned}\tag{2.22}$$

where now [recalling Eq. (2.13)]

$$\begin{aligned}a_C(\chi)&=\sum\{f(\bar\eta)f(\eta)+g(\bar\eta)g(\eta)\eta\bar\eta\},\\b_C(\chi)&=\sum\frac{3}{128\chi}g(\bar\eta)g(\eta)P_1,\\a_M{}^C(\chi)&=\sum\tfrac{3}{2}g(\bar\eta)g(\eta)\eta\bar\eta,\\b_M{}^C(\chi)&=\sum\frac{9}{512\chi}g(\bar\eta)g(\eta)P_2,\\a_M{}^M(\chi)&=\sum\{2f(\bar\eta)f(\eta)-g(\bar\eta)g(\eta)\eta\bar\eta\},\\b_M{}^M(\chi)&=\sum\left\{-\frac{3\sqrt{2}}{32\chi^2}f(\bar\eta)g(\eta)Q_2-\frac{9}{256}\frac{g(\bar\eta)g(\eta)}{\chi}P_3\right\},\\a_Q(\chi)&=\sum\left\{\frac{3M^2\sqrt{2}}{16}f(\bar\eta)g(\eta)\left[3-\frac{24\bar\eta}{\chi}+\frac{8\eta}{\chi}+48\frac{(\eta-\bar\eta)^2}{\chi^2}\right]\right.\\&\quad\left.+\tfrac{1}{4}M^2g(\bar\eta)g(\eta)\left[\tfrac{3}{5}(\eta+\bar\eta)-(24/5)\frac{\eta\bar\eta}{\chi}\right]\right\},\\b_Q(\chi)&=\sum-\frac{9M^2}{256\chi^2}g(\bar\eta)g(\eta)Q_1.\end{aligned}\tag{2.23}$$

Let us turn now to the relativistic case.

3. THE RELATIVISTIC FORM FACTORS

In this section we turn our attention to a calculation of the relativistic deuteron form factors as determined from the diagrams shown in Fig. 2. We shall rely on the results of I, and write down the contributions from the three diagrams by inspection. First, however, we must introduce our notation for the nucleon form factor and d-N vertex.

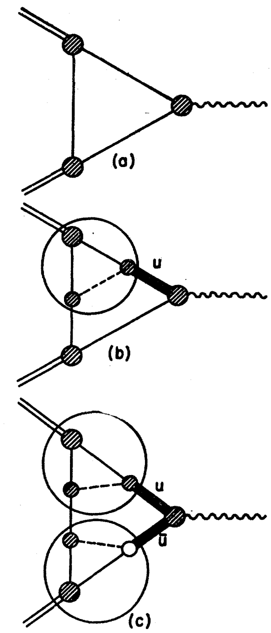

FIG. 2. The three diagrams on which this calculation is based. The double solid lines represent deuterons or antideuterons, the solid lines are nucleons or antinucleons and the dashed lines are pions. The heavy solid lines represent off mass shell nucleons or antinucleons of mass u or $\bar{u}$ respectively. Parts of (b) and (c) are circled to emphasize the intimate role of the deuteron-nucleon vertex in the problem. The same key is employed in Fig. 3.

Since the deuteron has isotopic spin-zero, only the isotopic-scalar nucleon form factor can contribute to the amplitude, and this can be written[9] in terms of the conventional charge and anomalous magnetic-moment form factors as

$$\begin{aligned}F^\mu(s)&=(4n^0\bar{n}^0)^{1/2}\langle 0|j^\mu|n\bar{n}\rangle\\&=e\bar{v}(\bar{n})[F_1(s)\gamma^\mu-(iF_2(s)/2m)\sigma^{\mu\nu}(n+\bar{n})_\nu]u(n),\end{aligned}\tag{3.1}$$

where $\sigma^{\mu\nu}=\frac{1}{2}i(\gamma^\mu\gamma^\nu-\gamma^\nu\gamma^\mu)$ and m is the mass of the nucleon. The static limits of these invariants are well known:

$$F_1(0)=1,$$
$$F_2(0)=-0.12.$$

A more convenient form of this amplitude for calculational purposes is

$$F^\mu(s)=e\bar{v}(\bar{n})[F_M(s)\gamma^\mu-(F_2(s)/2m)P^\mu]u(n),\tag{3.2}$$

where $P=n-\bar{n}$. The invariant F_M, the magnetic-moment form factor, is

$$F_M(s)=F_1(s)+F_2(s).\tag{3.3a}$$

Finally, we shall ultimately be interested in expressing our results in terms of the charge and magnetic-moment form factors. The charge form factor is

$$\begin{aligned}F_C(s)&=F_1(s)+(s/4m^2)F_2(s)\\&=F_M-(1-s/4m^2)F_2.\end{aligned}\tag{3.3b}$$

The deuteron-nucleon vertex for an off-mass-shell nucleon of momentum p and a mass-shell nucleon of

[9] See, for example, S. D. Drell and F. Zachariasen, *Electromagnetic Structure of Nucleons* (Oxford University Press, New York, 1961).

momentum N (see Fig. 3) can be written in the form:

$$\Gamma(u,pN)=(4D^0N^0)^{1/2}\bar{u}(p)\langle N|f_p|D\rangle$$
$$=-\bar{u}(p)\Gamma^\alpha(u,pN)\mathcal{C}\bar{u}(N)\xi_\alpha,$$
$$\Gamma^\alpha(u,pN)=F'(u)\gamma^\alpha+(G'(u)/m)\tfrac{1}{2}(N-p)^\alpha+(p-m)/m \times\{H'(u)\gamma^\alpha+(I'(u)/m)\tfrac{1}{2}(N-p)^\alpha\}, \quad (3.4)$$

where $p=D-N$ and $u=p^2$, and our conventions pertaining to the Dirac equation are given in the Appendix A. We shall use the notation $F_0=F'(m^2)$ and $G_0=G'(m^2)$. This vertex plays the roll of the deuteron wave function as discussed by Blankenbecler and Cook[7] and in I. In the present section we are interested only in displaying the form factors in terms of the four invariants F', G', H', and I', and demonstrating how they play the role of the weight functions for the wave functions. In Sec. 4 we shall calculate the invariants in order to obtain a relativistic description of the deuteron wave function.

It is convenient to obtain the expression for Eq. (3.4) in an alternative form:

$$(4D^0N^0)^{1/2}\langle N|f_p^\dagger|D\rangle\mathcal{C}\bar{u}(p)$$
$$=-\bar{u}(N)\Lambda^\alpha(u,Np)\mathcal{C}\bar{u}(p)\xi_\alpha, \quad (3.5)$$

where now the off-mass-shell nucleon is sitting to the right. To obtain Λ^α in terms of Γ^α we take the transpose and observe that

$$\bar{u}(p)\mathcal{C}\Lambda^{\alpha T}(u,Np)\bar{u}(N)=-\bar{u}(p)\Gamma^\alpha(u,pN)\mathcal{C}\bar{u}(N),$$

where we have used $\mathcal{C}^T=-\mathcal{C}$. Hence

$$\Lambda^\alpha(u,Np)=-\mathcal{C}\Gamma^{\alpha T}(u,pN)\mathcal{C}^{-1}$$
$$=F'(u)\gamma^\alpha+(G'(u)/m)\tfrac{1}{2}(p-N)^\alpha-[H'(u)\gamma^\alpha+(I'(u)/m)\tfrac{1}{2}(p-N)^\alpha](p+m)/m. \quad (3.6)$$

Note that the terms involving F' and G' are symmetric, while those in H' and I' are not.

Let us also obtain the expression for the charge conjugate amplitude

$$(4\bar{D}^0\bar{N}^0)^{1/2}\langle\bar{N}|f_p^\dagger|\bar{D}\rangle v(\bar{p})$$
$$=-v(\bar{N})\mathcal{C}O^\alpha(u,\bar{N}\bar{p})v(\bar{p})\eta_\alpha^*, \quad (3.7)$$

where η is the 4-polarization of the antideuteron. Charge conjugation requires that

$$v(\bar{N})\mathcal{C}O^\alpha(u,\bar{N}\bar{p})v(\bar{p})\eta_\alpha^*$$
$$=\bar{u}_C(\bar{N})\Lambda^\alpha(u,\bar{N}\bar{p})\mathcal{C}\bar{u}_C(\bar{p})\xi_\alpha{}^C. \quad (3.8)$$

But because

$$\bar{u}_C(\bar{N})=-\mathcal{C}v(\bar{N}),$$
$$\xi^C=\eta^*,$$

we have

$$O^\alpha(u,\bar{N}\bar{p})=\Lambda^\alpha(u,\bar{N}\bar{p}). \quad (3.9)$$

What we want is the amplitude T for an antideuteron and nucleon coming in, and a virtual antinucleon of mass $\bar{u}$ going out. This can be obtained from Eq. (3.7) by an application of the substitution law, justified by crossing symmetry. We replace $\bar{N}$ by $-N$, and interpret $v(-\bar{N})$ as $u(N)$. Hence,

$$T=\eta_\alpha{}^*u(N)\mathcal{C}\Lambda^\alpha(\bar{u},-N\bar{p})v(\bar{p}). \quad (3.10)$$

The imaginary part of the deuteron form factor calculated from the diagrams shown in Fig. 2 can now be written down as a trace over products of γ matrices. Our expression is an immediate generalization of Eqs. (3.22) and (3.26) in Ref. 1.

$$\mathrm{Im}G^\mu(s)=\frac{e}{8[s(4M^2-s)]^{1/2}}\int_{m^2}du\int_{m^2}d\bar{u}W(s,u,\bar{u})$$
$$\times\theta\{s-4[(\alpha^2+\tfrac{1}{2}(u-m^2))^{1/2}+(\alpha^2+\tfrac{1}{2}(\bar{u}-m^2))^{1/2}]^2\}, \quad (3.11)$$

where

$$W(s,u,\bar{u})=(1/2\pi)^{1/2}\int_0^{2\pi}d\varphi$$
$$\times\mathrm{trace}\{(p+m)^T\mathcal{C}\Lambda^\alpha(\bar{u},-p,\bar{n})(\bar{n}-m)$$
$$\times(F_M\gamma^\mu-(F_2/2m)P^\mu)$$
$$\times(n+m)\Gamma^\beta(u,np)\mathcal{C}\}\eta_\alpha{}^*\xi_\beta \quad (3.12)$$

and Λ^α, Γ^β have already been introduced in Eqs. (3.4) and (3.6) except that it now must be understood that just as in Eqs. (3.26c) of Ref. 1 the actual combinations of the d-N invariants which appear are

$$F'(u)\rightarrow F(u)=F_0\delta(u-m^2)-\frac{\mathrm{Im}F'(u)}{\pi(u-m^2)}\theta(u-u_0),$$
$$G'(u)\rightarrow G(u)=G_0\delta(u-m^2)-\frac{\mathrm{Im}G'(u)}{\pi(u-m^2)}\theta(u-u_0), \quad (3.13)$$
$$H'(u)\rightarrow H(u)=-(\mathrm{Im}H'(u)/\pi(u-m^2))\theta(u-u_0),$$
$$I'(u)\rightarrow I(u)=-(\mathrm{Im}I'(u)/\pi(u-m^2))\theta(u-u_0),$$

and

$$u_0=m^2+2\mu(\mu+2\alpha). \quad (3.14)$$

The reader is cautioned not to confuse the new F, G, H, and I which enter Eq. (3.12) and are given by (3.13) with the original d-N invariants introduced in Eq. (3.4). This substitution has the advantage of saving us the trouble of redefining Λ^α and Γ^β.

Although these equations were only shown in I to be true in the anomalous region of the d–N vertex functions (hence the upper limits of $(m+\mu^2)$ on the u and $\bar{u}$ integrations in I) it is a trivial matter to see that Eq. (3.11) is valid for all u and $\bar{u}$ provided only that $s<(u^{1/2}+\bar{u}^{1/2})^2$; $u\bar{u}<M^2-m^2$ (i.e., in the anomalous region of the energy variable s). Of course, if u or $\bar{u}>(m+\mu)^2$, the imaginary parts introduced in Eq. (3.13) no longer refer to the imaginary parts in the anomalous regions of the d–N vertex. In what follows we shall assume the validity of Eqs. (3.11) and (3.12) for all u, $\bar{u}$, and s. Since we will ultimately approximate the imaginary parts for large s anyway, this approximation has no additional effect on the results.

In the scalar case, the ϕ integrations were trivial. Here they must be carefully done, and it must be recalled that the integrations are in the over-all center-of-mass system with $\cos\theta = \mathbf{D}\cdot\mathbf{n}/|\mathbf{D}||\mathbf{n}|$ evaluated from the equation $(D-n)^2 = m^2$. (See I for details.) The ϕ integrations are discussed in Appendix B.

The trace W can be immediately simplified to

$$W(s,u,\bar{u}) = \frac{1}{2\pi}\int d\phi\, \mathrm{trace}\{(\not{p}-m)\Lambda^{\alpha}(\bar{\not{n}}-m) \times (F_M\gamma^{\mu} - (F_2/2m)P^{\mu})(\not{n}+m)\Gamma^{\beta}\}\eta_{\alpha}{}^{*}\xi_{\beta} \quad (3.15)$$

and the traces taken. When the algebra is performed, and angular integrations completed, one can separate the results into the three invariants G_1, G_2, G_3 and then form the linear combinations G_C, G_M, and G_Q. This rather tedious algebra is sketeched in Appendix B.

The results can be expressed in terms of five functions of s:

$$\begin{aligned} G_C(s) &= F_C(s)D_C(s),\\ G_M(s) &= F_C(s)D_M{}^C(s) + F_M(s)D_M{}^M(s), \quad (3.16)\\ G_Q(s) &= F_C(s)D_Q{}^C(s) + F_M(s)D_Q{}^M(s). \end{aligned}$$

It is convenient to separate the D functions into parts which are explicitly zero at $s=0$ and other terms of special significance in the static limit:

$$\begin{aligned} D_C(s) &= P_S(s) + P_D(s) + \frac{s}{\pi}\int_{16\alpha^2}^{\infty} \frac{b_C(s')ds'}{s'(s'-s-i\epsilon)},\\ D_M{}^C(s) &= \tfrac{3}{2}P_D(s) + \frac{s}{\pi}\int_{16\alpha^2}^{\infty}\frac{b_M{}^C(s')ds'}{s'(s'-s-i\epsilon)},\\ D_M{}^M(s) &= 2P_S(s) - P_D(s) + R_M(s) + \frac{s}{\pi}\int_{16\alpha^2}^{\infty}\frac{b_M{}^M(s')ds'}{s'(s'-s-i\epsilon)}, \quad (3.17)\\ D_Q{}^C(s) &= P_Q(s) + R_Q(s) + \frac{s}{\pi}\int_{16\alpha^2}^{\infty}\frac{b_Q(s')ds'}{s'(s'-s-i\epsilon)},\\ D_Q{}^M(s) &= \frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{a_Q{}^M(s)}{s'-s-i\epsilon}, \end{aligned}$$

where P_S and P_D are the S- and D-state probabilities, P_Q is the contribution to the quadrupole moment which has the same form as that given in Sec. 2, and R_M and R_Q are additional (relativistic) terms not found in the potential theory results of Sec. 2:

$$\begin{aligned} P_{S,D,Q}(s) &= \frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{a^{S,D,Q}(s')ds'}{s'-s-i\epsilon},\\ R_{M,Q}(s) &= \frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{a_R{}^{M,Q}(s')ds'}{s'-s-i\epsilon}. \end{aligned} \quad (3.18)$$

Our results for the a's and b's are more directly comparable with Sec. 2 if we define the new invariants

$$\begin{aligned} \mathcal{F}'(u) &= F'(u) + \mathcal{H}'(u) - ((u-m^2+2\alpha^2)/6m^2)\mathcal{G}'(u),\\ \mathcal{G}'(u) &= G'(u) - \tfrac{1}{2}F'(u) + \tfrac{1}{6}\mathcal{H}'(u) + \tfrac{1}{3}\mathcal{I}'(u),\\ \mathcal{H}'(u) &= ((u-m^2)/2m^2)H'(u),\\ \mathcal{I}'(u) &= ((u-m^2)/2m^2)I'(u), \end{aligned} \quad (3.19)$$

and, as we did in Eq. (3.13), we introduce

$$\begin{aligned} \mathcal{F}(u) &= \mathcal{F}_0\delta(u-m^2) - (\mathrm{Im}\mathcal{F}'(u)/\pi(u-m^2))\theta(u-u_0),\\ \mathcal{H}(u) &= -(\mathrm{Im}H'(u)/2\pi m^2)\theta(u-u_0), \end{aligned} \quad (3.20)$$

and similarly $\mathcal{G}$ and $\mathcal{I}$, respectively. Finally, we wish to have it understood that if these functions appear with η or $\bar{\eta}$ as argument, we really mean

$$\mathcal{F}(\eta) \equiv \mathcal{F}(m^2 + 2(\eta-\alpha^2)).$$

There will be no confusion with this convention because the functions (3.20) will always appear as functions of u or η only. Finally, the a's and b's can be written in a compact form if we introduce:

$$\begin{aligned} \Sigma &= \frac{16m^2}{8[s(4M^2-s)]^{1/2}}\int_{\alpha^2}^{\infty} d\eta \int_{\alpha^2}^{\infty} d\bar{\eta}\,\theta[\tfrac{1}{2}s^{1/2} - \eta^{1/2} - \bar{\eta}^{1/2}],\\ u - m^2 &= 2(\eta-\alpha^2),\\ \bar{u} - m^2 &= 2(\bar{\eta}-\alpha^2). \end{aligned} \quad (3.21)$$

Hence,

$$\begin{aligned} a^S(s) &= \Sigma\, \mathcal{F}(\bar{\eta})\mathcal{F}(\eta),\\ a^D(s) &= \Sigma(2/9m^4)\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)\eta\bar{\eta},\\ a^Q(s) &= \Sigma\{\mathcal{F}(\bar{\eta})\mathcal{G}(\eta)(\tfrac{3}{2} + 4\eta/s - 12\bar{\eta}/s + 24(\eta-\bar{\eta})^2/s^2)\\ &\quad + (2/15m^2)\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)(\eta+\bar{\eta}-8\eta\bar{\eta}/s)\},\\ a_R{}^M(s) &= \Sigma\,[\mathcal{G}(\bar{\eta})(\bar{\eta}/m^2)(\mathcal{F}(\eta) + \tfrac{2}{3}\mathcal{I}(\eta) - \tfrac{4}{3}\mathcal{H}(\eta))\\ &\quad - (\mathcal{H}(\bar{\eta}) + \mathcal{I}(\bar{\eta}))2\mathcal{F}(\eta)](1 - 4(\eta-\bar{\eta})/s) - \tfrac{3}{2}a^D(s),\\ a_R{}^Q(s) &= \Sigma(\mathcal{H}(\bar{\eta})\mathcal{H}(\eta) - \mathcal{I}(\bar{\eta})\mathcal{I}(\eta))\\ &\quad \times(\tfrac{1}{12} + \tfrac{4}{3}\bar{\eta}/s - 4(\eta-\bar{\eta})^2/s^2) - \tfrac{2}{3}a_Q{}^M(s),\\ a_Q{}^M(s) &= \Sigma\{(\mathcal{I}(\bar{\eta}) - \mathcal{H}(\bar{\eta}))(\mathcal{F}(\eta) + \mathcal{G}(\eta)\eta/3m^2) \quad (3.22)\\ &\quad \times(3 + 16\bar{\eta}/s - 48(\eta-\bar{\eta})^2/s^2)\\ &\quad + (\mathcal{H}(\bar{\eta})\mathcal{H}(\eta) - \mathcal{I}(\bar{\eta})\mathcal{I}(\eta))\\ &\quad \times(1 + 8(\eta+\bar{\eta})/s - 48(\eta-\bar{\eta})^2/s^2)\},\\ b_C(s) &= \Sigma\, \tfrac{4}{3}(\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)/(16m^2)^2 s)P_1,\\ b_M{}^C(s) &= \Sigma(\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)/(16m^2)^2 s)P_2,\\ b_M{}^M(s) &= \Sigma\{-(\mathcal{G}(\bar{\eta})/8m^2 s^2)(\mathcal{F}(\eta) - 2\mathcal{H}(\eta) + \mathcal{I}(\eta)\\ &\quad + \mathcal{G}(\eta)\eta/3m^2)Q_2 - 2(\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)/(16m^2)^2 s)P_4\},\\ b_Q(s) &= \Sigma - (\mathcal{G}(\bar{\eta})\mathcal{G}(\eta)/32m^2 s^2)Q_1, \end{aligned}$$

where P_4 is a member of the class of polynomial (2.20).

$$P_4 = s(s^2 - 10(\eta+\bar{\eta})s + 32(\eta^2+\bar{\eta}^2) - (128/3)\eta\bar{\eta} - 32(\eta+\bar{\eta})(\eta-\bar{\eta})^2/s).$$

In order to obtain the decomposition of the Eqs. (3.17) into parts which are zero at $s=0$ (as we did in Sec. 2) it was necessary to approximate $(4M^2-s)^{1/2}$ by $4m$, an approximation which is extremely good in the region where the dominate contributions to the integrals arises. Hence we have taken:

$$\sum\cong\frac{m}{2s^{1/2}}\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\,\theta[\tfrac{1}{2}s^{1/2}-\eta^{1/2}-\bar{\eta}^{1/2}]. \quad (3.21a)$$

In addition to the above rather trivial approximation, we have retained only the leading terms in expressions (3.22). [See Appendix B following Eq. (B7) for a detailed discussion of this point.] This means essentially that we have neglected terms which are small when s is small and the d-N invariants are large, and large only when s is large and the d-N invariants are small.

We wish to call the readers attention to the term $D_Q{}^M(s)$, which has no counterpart in the Jankus potential theory. One might argue that $D_Q{}^M(0)=0$ because the quadrupole moment should depend only on the nucleon charge, and not its magnetic moment. Such a condition would be easy to impose by subtracting $D_Q{}^M(s)$ once at the origin. We have decided not to impose this condition here, and as it turns out it makes little difference since its effect is less than 1% anyway.

Now let us compare Eqs. (3.22) with our potential theory Eqs. (2.23). If we make the identifications

$$\begin{aligned} s&=-\mathbf{q}^2 &\text{(a)}\\ s'&=\chi &\text{(b)}\\ m^{1/2}\mathfrak{F}(\eta)&=f(\eta) &\text{(c)}\\ [(2m)^{1/2}/3m^2]\mathcal{G}(\eta)&=g(\eta) &\text{(d)}\end{aligned} \quad (3.23)$$

we note that the expressions are strikingly similar.[10] In fact the only terms not identical to their nonrelativistic counterparts are R_M, R_Q, and $D_Q{}^M$ for which there are no corresponding nonrelativistic terms and $b_M{}^M$ which is slightly more complicated than its potential theory counterpart. It is tempting to regard these extra terms as relativistic corrections, and the significance of this statement will be examined shortly. Of course, this close correspondence which is effected through Eq. (3.23) is not entirely an accident; we chose the particular invariants $\mathfrak{F}$ and $\mathcal{G}$ to make this as true as possible (see Appendix B). Nevertheless, it is significant that the three diagrams of Fig. 2 have a structure so similar to the Jankus potential theory.

At this point two questions beg for an answer. The first can be stated as follows: Are the deuteron wave functions defined uniquely by the relativistic theory? To be more specific: If one calculates D_C and $D_M{}^C$ relativistically using Eq. (3.22) and then chooses f and g (S- and D-state wave functions) to give the same answer, will the choice (3.23) be the only one possible?[11] The second question, closely related, is: Are the two theories really inconsistent? Is it true that we cannot find some f and g such that all of the nonrelativistic form factors agree with the relativistic ones? The answer to both of these questions is yes.

The proof of the uniqueness of the wave functions in the precise sense mentioned above has been relegated to Appendix C, and we will say no more about it here.

Once the uniqueness of the wave functions has been established, the answer to the second question follows trivially. Since (3.23) is the only identification which will make D_C and $D_M{}^C$ identical in the two theories, the rest of the nonrelativistic expressions can be calculated unambiguously using these f and g, and one observes that for example unless the term R_M is zero, the expressions for $D_M{}^M$ cannot agree. Hence, no matter how we chose f and g, we cannot in general produce a Jankus theory with the same predictions as (3.22) and the theories are fundamentally inconsistent (although the differences are very small at low momentum transfer). Among other things, this earns us the privilege of regarding terms like R_M as relativistic corrections, since these terms indicate the extent to which the two theories are incompatible.

Let us remind the reader that we have not yet said anything about the relationship of (3.22) to the predictions of an *arbitrary* potential theory. Our discussion has been limited only to the Jankus theory, defined by the choice of current density (2.5). In Sec. 5 we shall make some remarks about the more general problem.

Equations (3.23c) and (3.23d) for $\eta=\alpha^2$ [see Eq. (3.25) to follow] also follow from a comparison of the d-N vertex (3.4) with the Fourier transform of the deuteron wave function.[4,7] However, for values of u off the mass shell ($\eta\neq\alpha^2$) it is impossible to make such a comparison, for in this case all four of the invariants in (3.4) contribute and one does not know how to reduce these approximately to the two S- and D-state invariants which one has in the wave function. For this reason it has often been customary to neglect the H and I invariants in such discussions. Only by using (3.4) in a full calculation of the form factors, and then comparing the results with potential theory can one see how the four d–N invariants simulate the role of the S- and D-state wave functions. In particular, one observes that the $\mathcal{H}$ invariant is very important in the S-state wave function, and as we shall see in the next section it would have been disastrous to neglect it.

Let us return now to the discussion of the relativistic wave functions. From Eqs. (3.23), (2.9), and (2.14)

[10] Note that this identification differs from Ref. 1 by a factor of $m^{1/2}$ due to the presence of spin.

[11] There is an ambiguity of sign which is always present and would make the answer to this question trivially false. It is to be understood in subsequent discussion that we mean uniqueness up to a sign.

we have:

$$u(r)=N_S\left\{e^{-\alpha r}+2\int_{\eta_0}^{\infty}d\eta\mathfrak{F}(\eta)e^{-\eta^{1/2}r}\right\},$$

$$w(r)=N_D\left\{e^{-\alpha r}\left(1+\frac{3}{\alpha r}+\frac{3}{(\alpha r)^2}\right)+2\int_{\eta_0}^{\infty}d\eta\frac{\eta\mathcal{G}(\eta)}{\alpha^2\mathcal{G}_0}e^{-\eta^{1/2}r}\left(1+\frac{3}{\eta^{1/2}r}+\frac{3}{\eta r^2}\right)\right\},\tag{3.24}$$

where

$$N_S=\left(\frac{m}{8\pi}\right)^{1/2}\mathfrak{F}_0=\left(\frac{m}{8\pi}\right)^{1/2}\left(F_0-\frac{\alpha^2}{3m^2}\mathcal{G}_0\right),$$
$$N_D=N_S\frac{\sqrt{2}}{3}\frac{\epsilon}{m}\frac{\mathcal{G}_0}{\mathfrak{F}_0}=N_S\frac{\sqrt{2}\epsilon}{3m}\left(\frac{\mathcal{G}_0}{F_0-(\alpha^2/3m^2)\mathcal{G}_0}\right).\tag{3.25}$$

Introducing the deuteron effective range, $\rho(-\epsilon,-\epsilon)=\rho_r$, and the asymptotic D/S ratio; ρ_D, we have:

$$\mathfrak{F}_0=\left(\frac{16\pi\alpha}{m(1-\rho_r\alpha)}\right)^{1/2};\quad F_0=\mathfrak{F}_0\left(1+\frac{\rho_D}{\sqrt{2}}\right);$$
$$\gamma=\frac{\mathcal{G}_0}{F_0}=\frac{3\rho_D m}{\sqrt{2}\epsilon}\left(\frac{1}{1+\rho_D/\sqrt{2}}\right).\tag{3.26}$$

Equations quite similar to these have been given before,[7,12] although our results differ very slightly from these. Since there is some slight ambiguity in the choice of $\mathcal{G}$ (see discussion in Appendix B), this is not surprising.

In what follows, we will regard $\mathcal{G}_0$ and F_0 as undetermined parameters and hence shall determine ρ_r and ρ_D. In practice we have quite a fair knowledge of ρ_r and ρ_D however, and hence the closeness of our results for these parameters offers another check on the over-all validity of the theory. As a standard of comparison we use the results of Hamada, Johnston (HJ)[13] and Glendenning, Kramer (GK)[14] which quote values of ρ_D and ρ_r given in Table I. Also included in this table are $F_0{}^2$ and γ calculated from these values.

TABLE I. Values of the deuteron effective range $\rho(-\epsilon,-\epsilon)=\rho^r$ and the asymptotic D-to-S ratio ρ_D, and two quantities which depend on them, $F_0{}^2$ and γ [defined in Eq. (3.26)], obtained from Ref. 13 (HJ), Ref. 14 (GK), and this paper.

	$F_0{}^2$	γ	ρ_r (F)	ρ_D
HJ	4.31	22.9	1.77	0.02656
GK	4.31	22.9	1.76	0.02654
This paper	3.81	24.5	1.43	0.028

[12] R. Blankenbecler, M. L. Goldberger, and F. R. Halpern, Nucl. Phys. 12, 629 (1959).
[13] T. Hamada and I. D. Johnston, Nucl. Phys. 34, 382 (1962).
[14] Potential No. 8 of Ref. 6.

Now let us examine the structure of Eqs. (3.22) in a little more detail. One observes that in order to satisfy unsubtracted dispersion relations, it is necessary and sufficient that the following conditions hold (for arbitrarily large N):

$$\int_{\alpha^2}^{N^2}\mathfrak{F}(\eta)d\eta,\quad\int_{\alpha^2}^{N^2}\mathcal{H}(\eta)d\eta,\quad\int_{\alpha^2}^{N^2}\mathcal{I}(\eta)d\eta<\text{const}N^{1/2},\tag{3.27a}$$

$$\int_{\alpha^2}^{N^2}\mathcal{G}(\eta)d\eta<\frac{\text{const}}{N^{3/2}}.\tag{3.27b}$$

These conditions are the relativistic counterpart of the A conditions introduced in Sec. 2. The second equation (3.27b) means that the new invariant $\mathcal{G}'=G'-\frac{1}{2}F'+[(u-m^2)/6m^2](\frac{1}{2}H'+I')$ [recall Eqs. (3.19)] must satisfy an unsubtracted dispersion relation. In particular it is necessary that

$$\mathcal{G}_0=\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\text{Im}\mathcal{G}'(u')du'}{u'-m^2}.\tag{3.28}$$

Because of this condition it will be possible (and in fact necessary) to calculate the ratio γ introduced in (3.26). Hence we will be ultimately left with only one free parameter, F_0. This is discussed in considerable detail in the next section.

The conditions (3.27) and, in particular, (3.28) are crucial. It is just these conditions which make unsubtracted dispersion relations possible, and hence a fundamental determination of the deuteron parameters. Had we looked at the relativistic theory only, it would be natural to raise the following question: How do we know that such conditions are reasonable, and furthermore, how can we possibly expect an approximate calculation to satisfy the very stringent condition (3.28)?

Our fears can be considerably mitigated however by examining the results of Sec. 2, where we observed precisely the same "difficulty" with potential theory. Here we observed that the counterpart of (3.28) was just the requirement that the D-state wave function be normalizable, and its necessity arose because of our choice of the form (2.9b) instead of (2.9c) to describe the D-state wave function. The potential theory also suggests that this condition is one that must be *imposed* on any approximate calculation; that if it is not satisfied naturally we must arrange for it to be true, or else our theory cannot possibly give a satisfactory description of the D state. This is precisely the point of view we shall adopt in the next section; it is what will give us a determination of γ within the theory. It appears that this procedure provides a sound answer to the objections raised by Nuttall.[15]

[15] J. Nuttall, Nuovo Cimento 29, 841 (1963).

At this point it is also clear that in order to discuss the D-state wave function in an approximate manner, it is essential to treat at least the three diagrams shown in Fig. 2. This is because these three correspond to taking the simplest contribution to the dispersion integral in (3.28) (the one-pion-exchange approximation), and hence should be treated as a set. Any smaller set would consist of simply the basic triangle diagram, Fig. 2(a), and for this diagram it is impossible to satisfy (3.28) unless we take $\mathcal{G}_0=0$ (i.e., neglect the D state). Of course, we could neglect the polynomial s dependence associated with the $\mathcal{G}$ terms, as has been done by Jones,[16] but this does not lead to any readily identifiable D-state wave function. For some remarks about other diagrams which could be considered, see I.

In the calculation to follow we shall actually impose more than the minimum requirements (3.27). We shall require in addition to (3.28)

$$\mathfrak{F}_0=\frac{1}{\pi}\int_{u_0}^{\infty}\frac{\operatorname{Im}\mathfrak{F}'(u')du'}{u'-m^2},$$
$$\int_{u_0}^{\infty}\operatorname{Im}\mathcal{G}'(u')du'=\int_{u_0}^{\infty}\frac{\operatorname{Im}\mathcal{H}'(u')du'}{u'-m^2} \qquad (3.29)$$
$$=\int_{u_0}^{\infty}\frac{\operatorname{Im}\mathcal{J}'(u')du'}{u'-m^2}=0.$$

These requirements amount to a generalization of the nonrelativistic B conditions, and are sufficient to guarantee that the relativistic S- and D-wave functions of Eq. (3.24) will approach zero at the origin. They can be regarded as generalized sum rules, and we shall henceforth refer to them as such.

There is an important reason for requiring the sum-rule conditions (3.29). As we mentioned earlier, Eqs. (3.22) actually contain only the leading terms, and the more exact expressions for the form factors contain in addition many terms involving higher powers of s, η, or $\bar{\eta}$. The sum rules (3.29) are sufficient to guarantee that retention of the next two higher powers of s, η, or $\bar{\eta}$ would also lead to unsubtracted dispersion relations, and hence guarantee that neglect of these terms does not seriously alter our results. This reason alone is sufficient to introduce these sum rules, and the fact that they correspond nonrelativistically to the wave functions approaching zero at the origin merely provides an interpretation of the conditions.

In fact, one can generalize the above to the following result. If one retains polynomials in s, η, and $\bar{\eta}$ to order L, then the following conditions must be satisfied for each invariant:

$$\int_{\alpha^2}^{N^2}\mathfrak{F}(\eta)\eta^n d\eta<\frac{\text{const}}{N^{(L-\frac{1}{2}-2n)}}. \qquad (3.30)$$

[16] H. F. Jones, Nuovo Cimento **26**, 790 (1962).

Note that these conditions are equivalent nonrelativistically to requiring that the first few *even* derivatives of the wave functions be zero at the orgin [cf. Eq. (2.18)]. This is a curious fact, for if these were sufficient to imply that all of the first few derivatives of the wave function were zero at the origin, we would have a relativistic explanation for the so-called repulsive core. This is not the case, however, and thus the repulsive core appears to be of some other origin.

We now turn our attention to a calculation of the deuteron-nucleon vertex.

4. THE DEUTERON-NUCLEON VERTEX

We have seen in detail how the deuteron-nucleon (d–N) vertex plays the role of the wave function in the preceding section. In this section we shall calculate the four invariants of this vertex function in the one-pion-exchange (OPE) approximation. This will correspond to evaluating the form factor from the three diagrams shown in Fig. 2, as discussed in some detail in I. Recall that in order to include more diagrams of the type shown in Fig. 6 in I, it is only necessary to include the additional processes in a calculation of the d–N vertex and substitute the results into Eq. (3.22). Hence the results of Sec. 3 are actually valid for an infinite subclass of diagrams which pertain to the deuteron form factor.

Before we plunge into the details of a calculation of the d-N vertex, let us now discuss in some detail the approximations we shall make in this calculation and our general philosophy guiding it.

To begin with, we shall only calculate the imaginary part of the d–N invariants in the anomalous region of the OPE approximation. Outside of this region [in the normal region $u>(m+\mu)^2$] we shall estimate the imaginary parts with the aid of the B conditions discussed in Sec. 3. Our basic philosophy is that the really important contributions come from the anomalous region; beyond this the precise structure is not very important and can, and probably should, be estimated.

There are a number of arguments to justify this point of view. To begin with, experience with the scalar case in I suggests that the imaginary parts of the d-N invariants in the normal region are indeed small (see Fig. 10 in I), and hence a rough estimate can be expected to introduce only a small error. Furthermore, we have some idea how it behaves in this region, and hence we expect to do a credible job in estimating it. Secondly, the arguments in Sec. 3 suggest that the B conditions would arise naturally in an exact calculation and hence ought to be imposed on the invariants derived from an approximate calculation. The imposition of the B conditions is a way of estimating the effect of the higher mass states (u large), and hence the use of the B conditions to determine the d–N invariants in the normal region may even be more reliable than a straightforward calculation, which would, after all, be based only on the OPE approximation, when in the region of $u>(m+\mu)^2$ it is well known that many other diagrams begin to contribute.

The results of our calculation indicate to what extent this is true. We believe that detailed knowledge of the d–N invariants in the anomalous region alone is all that seems to be necessary for a 10% theory, and that this together with a rough estimate of these invariants in the normal region is sufficient to give a 5% theory. Nevertheless, it would be most interesting to calculate the absorptive parts of the d–N invariants in the normal region, and see how they compare with the estimates we will present later on in this section.

Let us turn now to the details of the calculation.

We will calculate the d–N vertex in the same manner as sketched for spinless particles in I. As a generalization of Eq. (3.15) in I we immediately obtain (see Fig. 3 for key to momenta):

$$\mathrm{Im}\Gamma^{\alpha}(u,pN)\mathcal{C} = \frac{1}{8\{[(M+m)^2-u][u-(M-m)^2]\}^{1/2}}\frac{1}{2\pi}\int_0^{2\pi}d\varphi \times\sum_i\{\gamma^5 g\mathbf{T}(i)\cdot\bar{\mu}(p)\boldsymbol{\sigma}\mu(n)(\boldsymbol{n}+m) \times[F_0\gamma^{\alpha}+(G_0/m)\tfrac{1}{2}(l-n)^{\alpha}] \times\mathcal{C}[g\gamma^5\mathbf{T}^*(i)\cdot\bar{\mu}(N)\boldsymbol{\sigma}\mu(l)(\boldsymbol{l}+m)]^T\}, \quad (4.1)$$

where we have introduced the pion-nucleon coupling as

$$(4p^0n^0)^{1/2}\langle p|f_\pi|n\rangle=\bar{u}(p)g\gamma^5u(n)\mathbf{T}^*\cdot\bar{\mu}(p)\boldsymbol{\sigma}\mu(n), \quad (4.2)$$

where $\mathbf{T}$ is the isospin of the pion and $\mu(p)$ and $\mu(n)$ the isospins of the nucleons. Throughout this paper we have assumed the pion-nucleon coupling constant is given by

$$g^2/4\pi=14. \quad (4.3)$$

The isotopic spin sums can be readily performed. We sum over the isospins of the exchanged pion, and average over the isospins of the emitted nucleons:

$$\begin{aligned}\mathcal{S}&=\textstyle\sum_i\mathbf{T}(i)\cdot\bar{\mu}(p)\boldsymbol{\sigma}\mu(n)\mathbf{T}^*(i)\cdot\bar{\mu}(N)\boldsymbol{\sigma}\mu(l)\\&=\textstyle\sum\bar{\mu}(n)\boldsymbol{\sigma}\mu(p)\cdot\bar{\mu}(N)\boldsymbol{\sigma}\mu(l)\\&=\tfrac{1}{2}\,\mathrm{trace}\textstyle\sum_k\{\sigma^k\sigma^k\},\end{aligned}$$

Hence

$$\mathcal{S}=-3 \quad (4.4)$$

and we obtain after some rearrangement of terms

$$\mathrm{Im}\Gamma^{\alpha}(u,pN) = \frac{3g^2}{8\{[(M+m)^2-u][u-(M-m)^2]\}^{1/2}}\frac{1}{2\pi}\int_0^{2\pi}d\varphi \times\{(\boldsymbol{n}-m)[F_0\gamma^{\alpha}-(G_0/m)\tfrac{1}{2}(l-n)^{\alpha}](\boldsymbol{l}+m)\}. \quad (4.5)$$

In Appendix D we present the integral identities which are needed to obtain the absorptive parts of the d–N vertex invariants. The results are finally

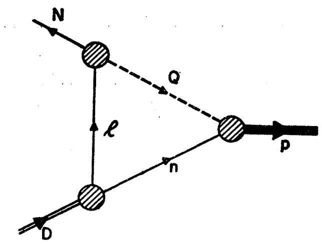

FIG. 3. The deuteron-nucleon vertex in the one-pion-exchange approximation.

$$\begin{aligned}\mathrm{Im}F'&=RF_0[\mu^2-(u-m^2)b-2\mu^2c],\\ \mathrm{Im}G'&=R\{F_0 2m^2(d-e)\\&\qquad+G_0[\mu^2+(a-b)\mu^2-(u-m^2)(b+e)]\},\\ \mathrm{Im}H'&=R\{-F_0m^2(a-b)-G_0\mu^2c\},\\ \mathrm{Im}I'&=R\{-F_0 2m^2(b+e)+G_0m^2(a+b+d+e)\},\\ R&=3g^2/8\{[(M+m)^2-u][u-(M-m)^2]\}^{1/2},\end{aligned} \quad (4.6)$$

where a, b, c, d, e are functions of u defined in Appendix D.[17]

Note that the invariants above depend on the two parameters F_0 and G_0, which we will regard as undetermined following the discussion in Sec. 3. Hence, letting the symbol Δ represent any of the invariants (4.6), we introduce the functions $\Delta_F{}^0$ and $\Delta_G{}^0$ according to

$$\mathrm{Im}\Delta=F_0\Delta_F{}^0+G_0\Delta_G{}^0, \quad (4.7)$$

where $F_G{}^0=0$. These seven functions are plotted in the anomalous region in Fig. 4 as a function of η where $u=m^2+2(\eta-\alpha^2)$.

Now, as outlined above, we assume the results are dominated by the anomalous contributions of the d–N invariants, and that the contribution of these invariants in the normal regions (which we shall hereafter refer to as the "tails" of the invariants) give only a slight modification of the basic results. But this cannot be true unless the integrals from the anomalous region alone are finite. In other words, we will impose the A conditions on the functions in (4.6) and then choose their tails so that the B conditions are also satisfied. The main advantage of this procedure is that it seems to emphasize the importance of the anomalous region, which clearly dominates the problem, and will enable us to determine the tails in an unambiguous fashion.

The imposition of the A conditions on (4.6) leads to the following requirement:

$$\mathcal{G}_0=\frac{1}{\pi}\int_{u_0}^{(m+\mu)^2}\frac{\mathrm{Im}\mathcal{G}'(u')du'}{u'-m^2}, \quad (4.8)$$

which in turn will enable us to specify the ratio γ and hence the asymptotic D/S ratio of the deuteron. This determination is essentially identical to that of Blankenbecler and Cook, and the success of our point of view

[17] The terms in Eq. (4.6) involving a factor of $u-m^2$ were omitted from Ref. 7.

is in part judged by the reasonableness of the ρ_D parameter thus determined.

It is found that the functions $\mathcal{G}_F{}^0$ and $\mathcal{G}_G{}^0$ are well approximated by

$$2\mathcal{G}_F{}^0=[70.2/\eta^2-3.36/\eta+0.71]\times\theta(\eta-1.77)\theta(7.33-\eta),\quad(4.9)$$

$$2\mathcal{G}_G{}^0\approx-0.03\theta[\eta-2.33]\theta[7.33-\eta].$$

Hence, (4.8) becomes

$$\mathcal{G}_0=G_0-\tfrac{1}{2}F_0=\int_{1.77}^{7.33}d\eta F_0\left(\frac{70.2}{\eta^2}-\frac{3.36}{\eta}+0.71\right)-\int_{2.33}^{7.33}d\eta G_0(0.33),\quad(4.10)$$

$$\gamma=\mathcal{G}_0/F_0=24.5,$$

$$\rho_D=0.028.$$

This value of ρ_D is in fairly satisfactory agreement with the results quoted in Sec. 3.

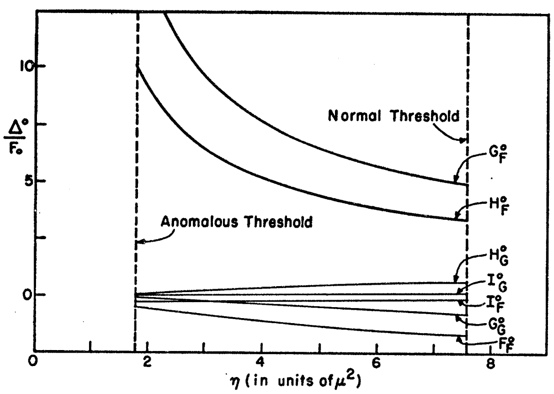

FIG. 4. A graph of the seven functions $\Delta_F{}^0$ and $\Delta_G{}^0$ in the one-pion-exchange anomalous region of the d-N vertex.

Now that the ratio γ is known, the invariants (3.20) are completely determined up to a constant F_0. These are shown in Figs. 5–9. In the same figures we have also included the tails, which were determined as discussed below.

Now, for the purposes of calculating the triple integrals we approximate the d–N invariants (3.20) in the anomalous region [i.e., $u_0\leq u\leq(m+\mu)^2$] by functions of the following form:

$$-\chi(\eta)=A_1/\eta^{5/2}+A_2\eta+A_3,\qquad(4.11)$$

where we have introduced $\chi(\eta)$ as a generic name for the functions defined in (3.20). These approximate forms do an extremely good job fitting the exact functions. The parameters A_i for the different invariants are given in Table II. For mathematical simplicity the coefficient A_2 in $\mathcal{G}$ was taken to be zero, and the combination $2\eta\mathcal{G}$

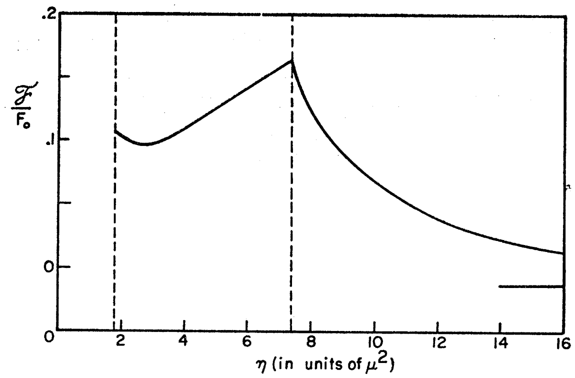

FIG. 5. A graph of the invariant $\mathcal{F}$ with γ determined by Eq. (4.10), and the tail chosen to satisfy the B conditions as discussed in the text. In the anomalous region the approximate form (4.11) agrees so well with the exact results obtained from (4.6) that no attempt has been made to draw them as separate curves. The short line to the lower right of the figure is the value of B_3, giving the asymptote of the curve as $\eta\to\eta_t=46.3\,\mu^2$. The two dashed lines show the positions of the anomalous and normal thresholds.

was introduced as a separate invariant to facilitate calculation of P_D. To insure maximum consistency, A_1 and A_3 of $\mathcal{G}$ were therefore chosen to give the same zeroth and first moment of $\mathcal{G}$ as that obtained from $2\eta\mathcal{G}$ directly.

TABLE II. Values of the A's and B's found to give a good fit to the d–N invariants (3.20) using the approximate forms (4.11) and (4.12).

	A_1	A_2	A_3	B_1	B_3
$\mathcal{F}$	0.187	0.0178	0.0320	25.7	−0.0125
$\mathcal{G}$	84.5	···	0.612	196.6	−0.156
$2\eta\mathcal{G}$	189.5	−3.54	36.6	2623.0	−6.10
$\mathcal{H}$	0.792	−0.0036	0.298	53.3	−0.089
$\mathcal{I}$	−0.143	0.030	−0.094	21.5	−0.023

The remaining job is to determine the tails of these d-N invariants. This was done by requiring, somewhat

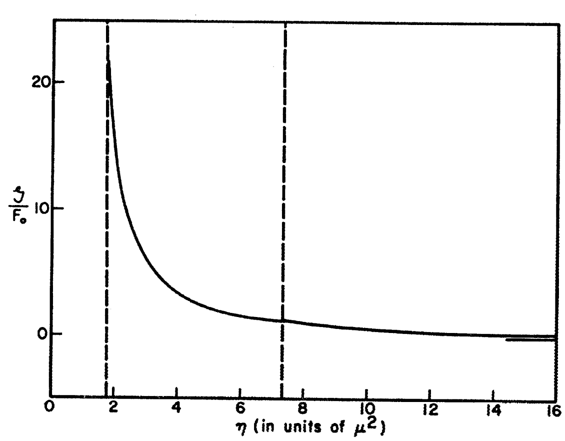

FIG. 6. A graph of the invariant $\mathcal{G}$. See caption to Fig. 5 for explanatory remarks.

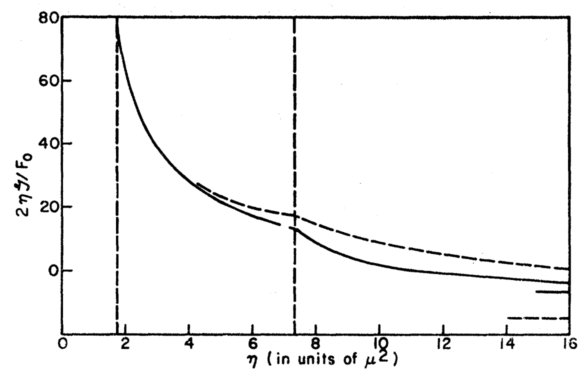

FIG. 7. A graph of the invariant $2\eta\mathcal{G}$ showing the tails as determined directly from the B conditions compared with that obtained from the tail of $\mathcal{G}$ (Fig. 6) multiplied by 2η (dashed line). See caption to Fig. 5 for additional explanatory remarks.

arbitrarily, that the tails have the functional form

$$(B_1/\eta^{5/2}+B_3)\theta(\eta-7.33)\theta(\eta_t-\eta)\,. \qquad (4.12)$$

The two parameters B_1 and B_3 were then chosen so that the functions would be continuous at the normal threshold, and so that the B conditions be satisfied. For the G invariant, we wished to retain Eq. (4.10) as sketched above, in addition to imposing the B conditions. This was possible only for a definite (unique) value of η_t, the parameter which determines the extent of the tails. In the case described above $\eta_t=46.3\ \mu^2$. The same value of η_t was required to hold for all of the tails.

Whatever the objection to the above procedures, they at least completely and uniquely determine the d-N invariants. The choice of the functional form (4.12) was motivated by the fact that they give shapes for the tails expected from experience with scalar theory. In addition, both (4.11) and (4.12) are easy to integrate. It should be emphasized that no effort has been made to adjust the shapes of the tails to improve the results; the above outlined procedure was decided on before the results were known.

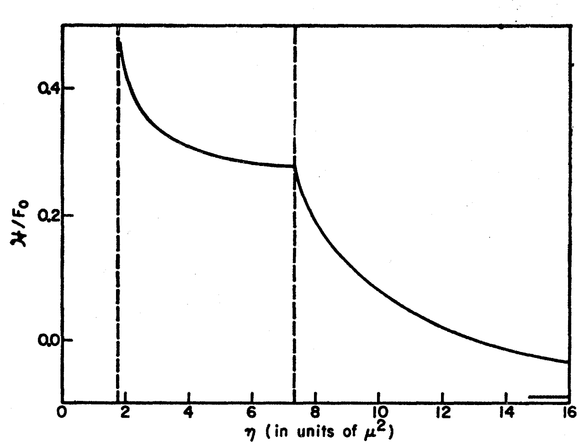

FIG. 8. A graph of the invariant $\mathcal{H}$. See caption to Fig. 5 for explanatory remarks.

Before we close this section it is amusing to calculate the deuteron wave functions implied by these invariants using Eq. (3.24). This can be easily done numerically, and the results are given in Fig. 10. Note that our wave functions show no repulsive core behavior.

5. NUMERICAL RESULTS AND CONCLUSIONS

Using the approximate functions defined in the preceding section, the integrals over η and $\bar{\eta}$ were done analytically and the absorptive parts of the form factors determined. It was also possible then to calculate the form factors themselves analytically, although this involves several hundred integrals. As a result, the final integration over s was done numerically.

The numerical results, obtained on the Cornell 1604 computer, are presented in Tables III and IV and Figs. 11–16. As an introduction to a review of the results, we mention two of the uncertainties in the present

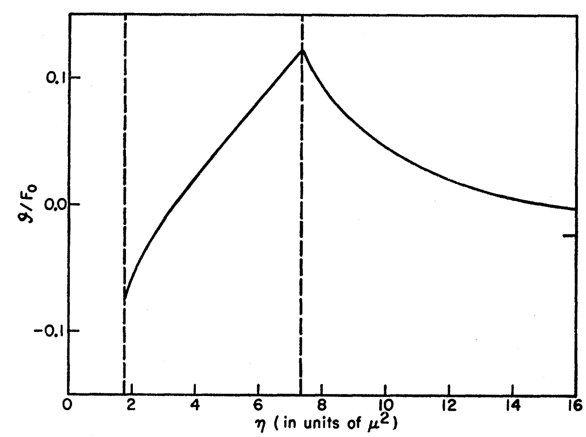

FIG. 9. A graph of the invariant $\mathcal{I}$. See caption to Fig. 5 for explanatory remarks.

calculation. Although there is little doubt that the tails of the d-N vertex invariants should be included in a careful calculation, and that the sum rules for the d-N invariants [Eqs. (3.28) and (3.29)] discussed in Secs. 3 and 4 should be retained, there is still some uncertainty associated with these tails for the obvious reason that we do not know their precise shape. This is probably most noticeable in our procedure for determining the ratio γ. Secondly, as the calculation stands, the sum rules guarantee that the form factors will be well defined if their absorptive parts are integrated to infinity, and that the contributions to the dispersion integrals for large s are small. However, for $s>181\ \mu^2$, the first normal threshold, the discontinuities of the form factor calculated from the diagrams of Fig. 2 begin to deviate from those assumed in (3.22). This effect is undoubtedly small, but it is aggravated by the fact that at these large values of s many other diagrams (some of them discussed in I) will begin to contribute. In other words, our calculation of the absorptive part above 181 μ^2 (say)

TABLE III. Values of the static (i.e., for $s=0$) integrals defined in Eq. (3.17) for the 4 cases discussed in the text. The case shown in *italics* is to be theoretically preferred, as discussed in the text. The definition of the magnetic moment μ and quadrupole moment Q of the deuteron is given in Eq. (1.8b).

	P_D	R_M	P_Q	$R_Q{}^C$	$R_Q{}^M$	μ	Q	$F_0{}^2$
With d–N tails								
Integration limit $181\ \mu^2$	7.3%	0.066	24.15	−0.45	0.50	1.735	24.4	3.79
Integration limit $736\ \mu^2$	*5.6%*	*0.060*	*24.79*	*−0.39*	*0.64*	*1.75*	*25.0*	*3.81*
Without d–N tails								
Integration limit $181\ \mu^2$	6.3%	0.042	23.03	−0.44	0.49	1.73	23.2	3.44
Integration limit $736\ \mu^2$	10.3%	−0.092	21.82	−0.46	0.77	1.56	22.0	3.27
Experimental	(6.5±1)%					1.715	25.5	

is clearly not any longer very meaningful, and it is perhaps unclear how far out we should integrate (3.22) in order to allow best for these unknown additional contributions.

Neither of these uncertainties is very serious. They both could be essentially eliminated by doing more work! However, here we will be content with presenting four different calculations to give an idea of the effect of these considerations.

To understand the importance of the d–N vertex tails we present calculations with (1) the tails as determined in Sec. 4, and (2) the tails arbitrarily set equal to zero, so that conditions B are no longer satisfied. In both cases conditions A are satisfied however.

To give a crude idea of the effect of the normal thresholds we present calculations in which (1) the imaginary parts have been integrated to infinity (infinity is actually $736\ \mu^2$ because with our choice of tails the B conditions guarantee the imaginary parts are zero above this[18]), and (2) the imaginary parts have been integrated to $181\ \mu^2$. The first case assumes that (3.22) gives a good estimate of the imaginary parts above $181\ \mu^2$; the second assumes that the unknown contributions somehow conspire to make the imaginary part zero above $181\ \mu^2$.

It is our feeling that the d–N vertex tails definitely should be included, and that the imaginary parts should be integrated to infinity. Unless specifically stated otherwise, all results refer to this case. The other cases are presented only to get an idea of how sensitive the calculation is to the uncertainties mentioned above, and are believed to represent extreme estimates on the overall uncertainties. Results for these four cases are shown in Table III for the static integrals, the magnetic and quadrupole moments, and the normalization constant $F_0{}^2$ related to the ρ's by Eq. (3.26). In all cases $F_0{}^2$ was chosen to give the correct charge, and ρ_D was given by Eq. (4.10).

Note that the effect of changing the upper limit is only a few percent when we include the tails in the d–N vertex functions, but may be fairly sensitive with the truncated functions. No matter what assumptions are made about the tails one seems guaranteed of at least a 10% theory; a reasonably careful inclusion of these tails seems to yield at least a 5% theory. Note how sensitive the relativistic correction to the magnetic moment, R_M, is to the upper limit with the use of truncated d–N vertex functions.

Table III points out clearly what the limitations of the present calculation are. Two of the most interesting terms, the D-state probability and the relativistic correction to the magnetic moment R_M are sensitive to the way in which they are calculated, and hence we must regard them as not very well determined. This is a disappointment, but is confirmed by all other attempts to determine these quantities! The reason for this uncertainty is shown in Figs. 11 and 12. Both of these functions depend strongly on the imaginary parts for large s, and hence are sensitive to our lack of information. At the same time, we feel the reasonable values obtained for these terms should be regarded as a success of this calculation.

It is unclear why the values for $F_0{}^2$ disagree with HJ and GK and with each other as much as they do. This is probably an indication of the fact that our ratio γ and our choice of d–N vertex tails is not what it should be.

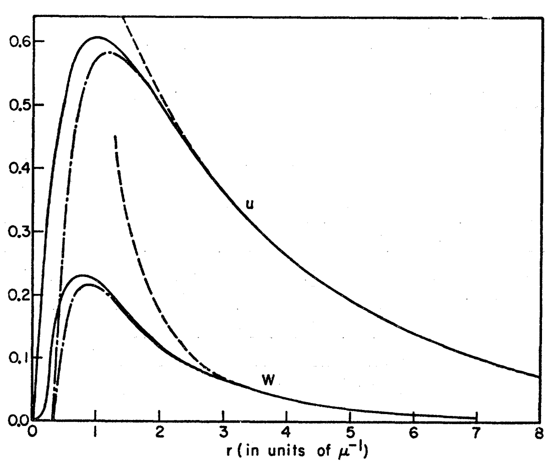

FIG. 10. The deuteron wave functions, normalized so that $\int_0^\infty (u^2+w^2)dr=C$. The solid line is this calculation with $C=1$. The dashed-dotted line is the Hamada-Johnson function (Ref. 13) with $C=0.9$. The dashed lines are the asymptotic forms [form (2.9b) is taken for w].

[18] Actually for a few of the small terms in Eq. (3.22) this is not true.

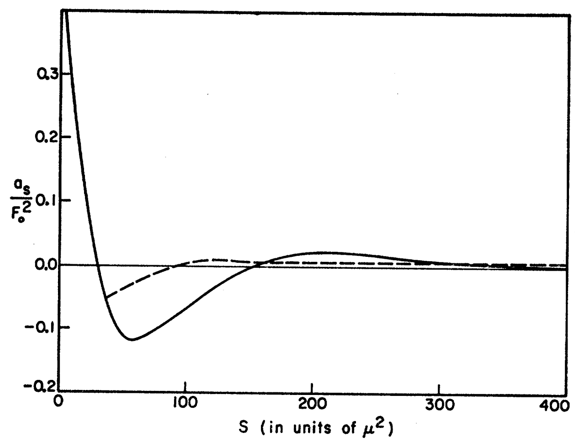

FIG. 11. A graph of the imaginary part a_R^M. See caption to Fig. 13 for explanatory remarks.

However, Fig. 13 would tend to suggest that F_0^2 should be relatively insensitive to this variation.

Finally, in Figs. 14–16 and Table IV we present results for the energy dependence of the form factors. For these results we have defined

$$Y_M=(s/6M^2)^{1/2}(G_M{}^C+0.880G_M{}^M),$$
$$Y_Q=(s^2/18M^4)^{1/2}(G_Q{}^C+0.880G_Q{}^M). \quad (5.1)$$

These Y would be the functions (except for a factor of F_C) which occur squared along with D_C in the cross section (1.2) *if* $F_C=F_M$. Note that the results differ from those obtained by GK.

It is natural at this point to make a few remarks about the possible significance of the explicit 3 pion contribution. The GK results (potential 8) are significantly lower than our results (Fig. 14). This can be understood from the fact that our wave functions have no hard-core behavior (Fig. 10). Now, since the neutron-charge form factor determined from the GK results has a distinct tendency to be negative, the results presented here

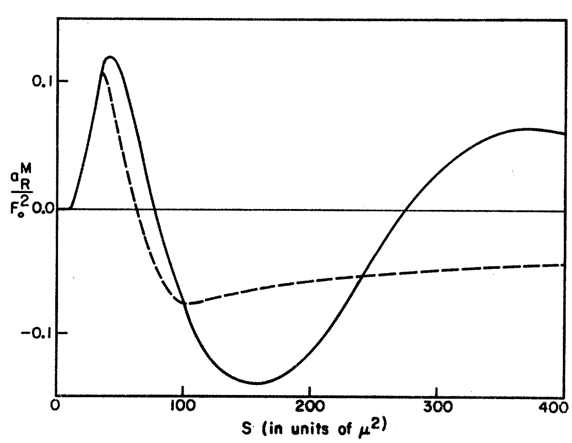

FIG. 12. A graph of the imaginary part a_D. See caption to Fig. 13 for explanatory remarks.

TABLE IV. Values of the deuteron form factors for a few representative values of momentum transfer squared. The Y's are defined in Eq. (5.1). In all of these the d–N vertex tails were included and the dispersion integrals were integrated to $736\,\mu^2$ (infinity).

$-s(\mu^2)$	D_C	$D_M{}^C$	$D_M{}^M$	Y_M	$D_Q{}^C$	$D_Q{}^M$	Y_Q
0.0	1.000	0.084	1.892	0.0000	24.41	0.64	0.0000
0.4	0.899	0.081	1.709	0.0304	21.70	0.63	0.0116
0.8	0.820	0.079	1.566	0.0396	19.56	0.62	0.0210
1.2	0.754	0.073	1.450	0.0450	17.83	0.61	0.0288
3.0	0.391	0.067	1.106	0.0547	12.73	0.57	0.0518
6.0	0.391	0.055	0.813	0.0573	8.50	0.52	0.0702
9.0	0.296	0.047	0.651	0.0565	6.24	0.48	0.0782
12.0	0.235	0.040	0.545	−0.0547	4.85	0.44	0.0821

would make it even more negative. However, one has a feeling that the form factor should be positive, to agree with the thermal neutron-scattering experiments and the inelastic electron-deuteron scattering experiments. The answer could lie in either of two directions.

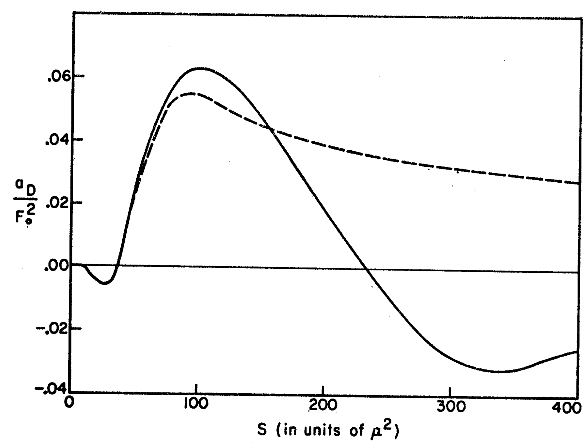

FIG. 13. A graph of the imaginary part a_s. The solid line includes d–N vertex tails as estimated by Eq. (4.12); the dashed line is the case when these tails have been set equal to zero.

The approximations and uncertainties in this calculation could produce this effect. If we were to impose additional sum rules of the type shown in Eq. (3.30), we would more nearly reproduce a hard-core wave function, which would cause the form factors to fall off more rapidly with energy. However, it is our feeling that a more probable explanation lies in the 3-pion contribution.

The explicit 3-pion contribution is yet to be calculated. To assume that this is negligible is almost certainly wrong, but it is reasonable to assume that it is small. If it were negative, it could easily reduce the form factor a significant amount, and hence account for the expected positive character of the neutron charge form factor. It is our guess that the 3-pion contribution will ultimately have this effect.

We summarize our principal conclusions. To begin with, one has some reason for optimism, but there are still many very important things left to do. Besides a

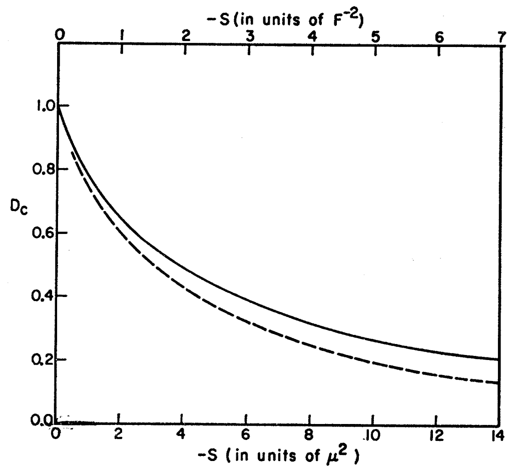

FIG. 14. A graph of D_C, solid line, compared with the equivalent quantity calculated by GK (Ref. 14), dashed line.

calculation of the 3-pion contribution and some of the extra diagrams sketched in I, it would be worthwhile to undertake a far more careful calculation of the d-N vertex functions, and at the same time retain all of the terms neglected in our formula in Sec. 3 (see Appendix B). With a better knowledge of the d–N vertex tails, which could be obtained in a relatively straightforward manner, one might hope to considerably reduce the uncertainties. Among other things, this could be expected to pin down the D-state probability, asymptotic D/S ratio, and the very interesting relativistic term R_M.

An amusing lesson learned from this work is that it would be unwise to neglect the d–N invariants H and I in any calculation in which one is attempting to treat the deuteron wave function relativistically. The H invariant turns out to give the principal contribution to the weight function of the S-state wave function; without it the radial wave function $u(r)$ shows no tendency to decrease as $r \rightarrow 0$ (in fact it increases faster than the asymptotic form $e^{-\alpha r}$). With the contribution from H, however, the radial wave function has a natural tendency to decrease toward zero at the origin.

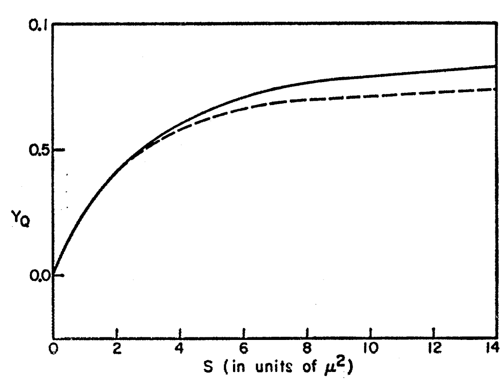

FIG. 15. A graph of Y_Q, solid line, compared with the equivalent quantity calculated by GK (Ref. 14), dashed line

We close this paper with some speculations about the relationship between a potential theory and a relativistic theory. To begin with, it is clear that this calculation differs in significant ways from the Jankus theory. As we showed in Sec. 3, no wave functions can be found which will make the two theories agree. This is due principally to the existence of the term R_M, and others like it. We also showed in Sec. 3 that this relativistic approach does provide an unambiguous way to determine relativistic wave functions for the deuteron. All of these results are interesting, but perhaps not very significant, and it is this we wish to discuss now.

It is well known that the introduction of $\mathbf{L}\cdot\mathbf{S}$ or $(\mathbf{L}\cdot\mathbf{S})^2$ terms into the potential will modify the effective magnetic moment operator.[19] In fact, any energy-dependent potential will have this effect, and it is not so surprising to speculate that for any relativistic calculation of the deuteron form factor, a sufficiently complicated energy-dependent potential could be found to simulate the relativistic results in a nonrelativistic manner. If this were true, then it is clear that a potential-

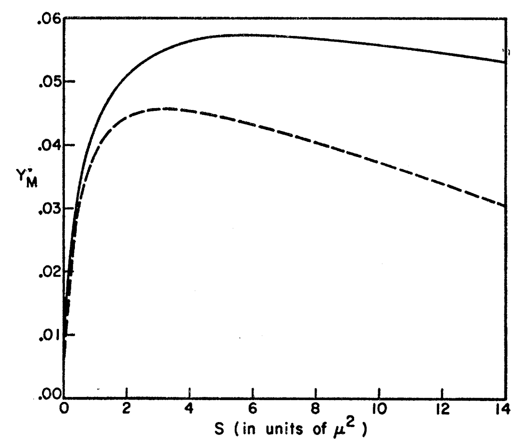

FIG. 16. A graph of Y_M, solid line, compared with the equivalent quantity calculated by GK (Ref. 14), dashed line.

theory approach could be just as correct or as fruitful as a relativistic one. Several cautioning remarks can be made, however.

The first is that it is pointless to talk about "relativistic corrections" without referring to some definite potential theory. Once we have committed ourselves to a definite potential theory [i.e., a choice of Hamiltonian or current density like Eq. (2.5) but not necessarily a definite choice of radial potential] then we can compare relativistic results with this theory, and introduce relativistic corrections. Even then, the only way to discuss these corrections is to isolate the diagrams which generate them, and since it would appear that we have no way of telling beforehand how to do this, the only way to discuss relativistic corrections is to do the entire

[19] See, for example, Blin-Stoyle, *Theories of Nuclear Moments* (Oxford University Press, New York, 1957).

problem relativistically and make a detailed comparison.

The second remark is that the failure of a potential theory to agree with a relativistic theory may not necessarily mean that the potential theory is incorrect. It could mean that certain terms have been left out of the relativistic calculation. The converse, of course, is also true. For example, the hard core usually introduced into phenomenological potentials could be the nonrelativistic way of allowing for the 3-pion contribution, and the failure of our wave functions to exhibit this behavior may be expected.

The final remark is that potential theories as viewed in this light are phenomenological. Since one chooses the potential to fit a wide variety of experimental data, it may be a fairly reliable tool for predicting the results of a new experiment. If one wants the feeling of explaining the experiments, however, a relativistic theory is necessary since adequate potentials are probably too complicated to appear very fundamental, and furthermore, must be constantly revised as more information accumulates (because they are nonrelativistic). In addition, there is no guarantee that a single potential (no matter how complicated) could explain a variety of experiments as well as a relativistic theory. *Note added in proof.* We thank Professor M. Gourdin for calling attention to the fact that J. Tran Thanh Van, using an approach similar to ours but treating the d-N invariants phenomenologically, has obtained numerical results for D_C comparable to ours [see Nuovo Cimento **30**, 1100 (1963)].

ACKNOWLEDGMENTS

I am especially indebted to Professor M. L. Goldberger and R. Blankenbecler for invaluable guidance during the formulative stages of this work. It also is a pleasure to acknowledge helpful discussions with Professor L. F. Cook, Jr., Professor S. B. Treiman, Professor J. S. Levinger, and Professor D. Harrington and with Dr. K. J. Barnes and B. M. Casper.

The helpfulness of the staff at the Cornell Computing Center is greatly appreciated.

APPENDIX A

In this paper, the following conventions are employed

$$m = \text{nucleon mass},$$
$$\mu = \text{pion mass},$$
$$M = \text{deuteron mass},$$
$$\hbar = c = 1,$$
$$m/\mu = 6.72,$$
$$\alpha^2 = m\epsilon = 0.108\,\mu^2.$$

Where ϵ is the deuteron binding energy. The metric is taken to be $k^2 = (k^0)^2 - \mathbf{k}^2$. For spin $\frac{1}{2}$ particles we write

$$\gamma^\mu\gamma^\nu + \gamma^\nu\gamma^\mu = 2g^{\mu\nu}, \quad \not{p} = \gamma^\mu p_\mu$$

and the Dirac equation is

$$(\not{p} - m)u(p) = 0 \qquad \bar{u} = u^*\gamma^0 \qquad \bar{u}u = 2m$$
$$(\not{p} + m)v(p) = 0 \qquad \bar{v} = v^*\gamma^0 \qquad \bar{v}v = -2m$$
$$u \otimes \bar{u} = \not{p} + m \quad v \otimes \bar{v} = \not{p} - m,$$

where $u(p)$ are the positive energy solutions and $v(p)$ the negative energy solutions. Our choice of normalization for the spinors has the advantage that the invariant matrix elements must contain a factor of $(2p_0)^{1/2}$ for each particle, whether bosons or fermions, instead of the usual $(2p_0)^{1/2}$ for bosons and $(p_0/m)^{1/2}$ for fermions. The charge conjugate solutions formed from the v's are

$$u_C(p) = \mathcal{C}\bar{v}(p),$$
$$\bar{u}_C(p) = -\mathcal{C}v(p),$$

where placement of $\bar{v}(p)$ to the right of $\mathcal{C}$ is understood to imply $\bar{v}^T$. The charge conjugation matrix satisfies

$$\mathcal{C}\gamma^\mu\mathcal{C}^{-1} = -\gamma^{\mu T},$$

and

$$\mathcal{C}^T = -\mathcal{C}.$$

For spin-1 particles of momentum p we introduce the polarization vector ξ subject to the requirements

$$\xi^2 = -1 \quad \xi \cdot p = 0.$$

The antiparticle states are described by a similar polarization vector η. The positive energy solution corresponding to η is

$$\xi_C = \eta^*.$$

APPENDIX B

In this Appendix we present the angular integrations necessary to obtain the results for the deuteron form factor quoted in Sec. 3 and the full expression for the from factors from which the approximate expressions (3.22) were obtained.

To simplify the computations of the trace W, Eq. (3.15), we write it as:

$$W(s,u,\bar{u}) = \frac{1}{2\pi}\int_0^{2\pi} d\varphi \times \text{trace}\left\{(\not{p} - m)\left[-2m\mathcal{K}(\bar{u})\boldsymbol{\eta}^* - K(\bar{u})(\eta^* \cdot p) + F(\bar{u})\boldsymbol{\eta}^*\bar{\boldsymbol{D}} + \frac{G(\bar{u})}{m}(p \cdot \eta^*)\bar{\boldsymbol{D}}\right]\left(F_M\gamma^\mu - \frac{F_2}{2m}P^\mu\right) \times \left[2m\mathcal{K}(u)\xi + K(u)(\xi \cdot p) + F(u)\boldsymbol{D}\xi + \frac{G(u)}{m}(p \cdot \xi)\boldsymbol{D}\right]\right\}, \quad \text{(B1)}$$

where

$$K(u) = 2G(u) - 2F(u) + 2\mathcal{G}(u),$$

and

$$\xi \equiv \gamma^\mu \xi_\mu,$$

The spin sums can be performed. The next step is to perform the nontrivial ϕ integrations. For this purpose one can prove the following identities:

$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\xi)d\varphi=(E-C)(\xi\cdot q)\,,$$
$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\eta^*)d\varphi=(E+C)(\eta^*\cdot q)\,,$$
$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\eta^*)(p\cdot\xi)d\varphi=(\eta^*\cdot\xi)A+(\eta\cdot q)(\xi\cdot q)B\,,$$
$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\xi)p^\mu d\varphi=\xi^\mu A+(\xi\cdot q)d^\mu\left(\frac{A}{d^2}-C^2+EC\right),\tag{B2}$$
$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\eta^*)p^\mu d\varphi=\eta^{\mu*}A-(\eta^*\cdot q)d^\mu\left(\frac{A}{d^2}-C^2-EC\right),$$
$$\frac{1}{2\pi}\int_0^{2\pi}(p\cdot\xi)(p\cdot\eta^*)p^\mu d\varphi=(\xi\cdot\eta^*)d^\mu AC+[\xi^\mu(\eta^*\cdot q)-\eta^{\mu*}(\xi\cdot q)]AC+(\xi\cdot q)(\eta^*\cdot q)d^\mu D,$$

where

$$\begin{aligned}E&=p\cdot q/s=-\tfrac{1}{2}(u-\bar{u}/s)\,,\\ C&=p\cdot d/d^2=(M+m^2-\tfrac{1}{2}(u+\bar{u}))/d^2\,,\\ A&=-\tfrac{1}{2}(\mathbf{p}^2+(p\cdot d)^2/d^2)\\ &=-(1/2d^2)[(M^2-m^2)^2+m^2s+(M^2/s)(u-\bar{u})^2-(u+\bar{u})(M^2+m^2)+u\bar{u}]\,,\\ B&=A/d^2-A/s-C^2+E^2\,,\\ D&=3AC/d^2-AC/s+E^2C-C^3\,,\\ d^2&=4M^2-s\,.\end{aligned}\tag{B3}$$

The last equation of (B2) includes only those terms which will ultimately contribute to the form factor, although there are some additional terms antisymmetric in u and $\bar{u}$ which do not contribute by virtue of the symmetrical integration to be ultimately performed over u and $\bar{u}$.

When these identities are employed on the results of the trace (B1), lengthy expressions emerge. As these exact expressions may be of significance in future work on the deuteron form factor, we present them below. They are written in terms of the invariants $F(u)$, $G(u)$, $\mathcal{H}(u)$ and $\mathcal{I}(u)$ defined in the text. For notational simplicity, the product $F(\bar{u})F(u)$ will be written simply as F^2 and $F(\bar{u})G(u)$ as FG. Note that $FG\neq GF$; the first term contains the $\bar{u}$ variable, the second the u variable. If we introduce W_1, W_2 and W_3 defined as

$$W=e4m^2\{W_1(\eta^*\cdot\xi)d^\mu-W_2[\xi^\mu(\eta^*\cdot q)-\eta^{\mu*}(\xi\cdot q)]-W_3((\xi\cdot q)(\eta^*\cdot q)/2M^2)d^\mu\}\,,\tag{B4}$$

we obtain

$$\begin{aligned}W_1={}&F^2F_M[(M^2/m^2)C+(2A/m^2)(1-2C)]-F^2F_2(1-2C)[M^2/2m^2-s/4m^2+2A/m^2]\\ &-FGF_M(4A/m^2)(1-2C)+FGF_2(4A/m^2)(1-2C)[(1-2C)d^2/8m^2-Es/4m^2+(1-M^2/4m^2)]\\ &+G^2F_M(2A/m^2)[1-2C-(M^2/2m^2)C]-G^2F_2(2A/m^2)(1-2C)[(1-4C)d^2/8m^2+(1-M^2/4m^2)]\\ &+F\mathcal{H}F_M(2)-F\mathcal{H}F_2(1-2C)[Cd^2/m^2+4A/m^2-Es/m^2]-G\mathcal{H}F_M(2A/m^2)+G\mathcal{H}F_2(4A/m^2)(1-2C)\\ &-F\mathcal{I}F_M(2A/m^2)(1-4C)+F\mathcal{I}F_2(4A/m^2)(1-2C)+G\mathcal{I}F_M(2A/m^2)(1-4C)\\ &-G\mathcal{I}F_2(4A/m^2)(1-2C)[1-Cd^2/4m^2+Es/4m^2]+\mathcal{H}^2F_M4C-\mathcal{H}^2F_22(1-2C)\\ &+\mathcal{H}\mathcal{I}F_2(4A/m^2)(1-2C)-\mathcal{I}^2F_M(4AC/m^2)-\mathcal{I}^2F_2(2A/m^2)(1-2C)\,;\end{aligned}\tag{B5}$$

$$\begin{aligned}W_2={}&F^2F_M[(M^2/m^2)C+C^2d^2/m^2+4AC/m^2+E^2s/m^2]-F^2F_2(4AC/m^2)\\ &+FGF_M[(C-E)((M^2/m^2)(1-4C)-(s/2m^2)(1-2C)+Es/m^2)-8AC/m^2]\\ &+FGF_2(8A/m^2)[(1-4C)(C-sC/16m^2+Es/16m^2)+4C^2(1-M^2/4m^2)]\\ &+G^2F_M(4AC/m^2)[s/8m^2+(1-M^2/4m^2)]-G^2F_2(4AC/m^2)[(1-4C)(2-s/8m^2)\\ &-(1-8C)(1-M^2/4m^2)]+F\mathcal{H}F_M[2+4(C+E)]+F\mathcal{H}F_2(2A/m^2)(1-4C)\\ &-G\mathcal{H}F_M[(C+E)(4-(4M^2/m^2)C+Cs/m^2+Es/m^2)+2A/m^2]-G\mathcal{H}F_2(2A/m^2)(1-4C)\\ &-F\mathcal{I}F_M[(C-E)(4(M^2/m^2)C-Cs/m^2-Es/m^2)+8AC/m^2]+F\mathcal{I}F_2(8AC/m^2)+G\mathcal{I}F_M(8AC/m^2)\\ &-G\mathcal{I}F_2(2AC/m^2)[4-(4M^2/m^2)C+Cs/m^2+Es/m^2]+\mathcal{H}^2F_M4C-\mathcal{H}\mathcal{I}F_M4(C-E)\\ &+\mathcal{H}\mathcal{I}F_2(8AC/m^2)+\mathcal{I}^2F_M(4AC/m^2)-\mathcal{I}^2F_2(4AC/m^2)\,;\end{aligned}\tag{B6}$$

$$\begin{aligned}(4m^2/M^2)W_3={}&-F^2F_M16(B-2D)+F^2F_2[16(B-2D)+4-8C]+FGF_M[32(B-2D)+8C-8E]\\ &-FGF_2[8(B-2D)(4+M^2/m^2-Cd^2/m^2-s/2m^2-Es/m^2)+8(Cd^2/2m^2-Es/2m^2)\\ &\times(C-2C^2+2EC-E+2A/d^2)]-G^2F_M[16B(1-(M^2/m^2)C)+(4s/m^2)(CB-D)\\ &-32D(1-(M^2/4m^2))]+G^2F_28(B-2D)[(1-4C)(2+M^2/2m^2-s/4m^2)+8C(1-M^2/4m^2)]\\ &+F\mathcal{H}F_2[32(B-2D)+16(E+C)-32(C^2+EC-A/d^2)]+G\mathcal{H}F_M[16B+32(C^2+EC-A/d^2)]\\ &-G\mathcal{H}F_2[32(B-2D)+16(E+C)-32(C^2+EC-A/d^2)]+F\mathcal{I}F_M16(B-4D)-F\mathcal{I}F_232(B-2D)\\ &-G\mathcal{I}F_M16(B-4D)+G\mathcal{I}F_232(B-2D)[1-(M^2/m^2)C+sC/4m^2+Es/4m^2]\\ &-\mathcal{H}\mathcal{I}F_232(B-2D)+\mathcal{I}^2F_M32D+\mathcal{I}^2F_216(B-2D)\,.\end{aligned}\tag{B7}$$

Finally, to obtain more manageable expressions we retain only the leading terms in the following sense. Examination of the structure of (B5)–(B7) reveals that they are essentially sums of products of the d–N invariants F, G, $\mathcal{H}$, and $\mathcal{J}$ multiplied by monomials in $s/4M^2$, $\eta/4M^2$, $\bar{\eta}/4M^2$. Since these variables are very small in the anomalous region where the d–N invariants are large (i.e., for $s<100\ \mu^2$) and become large only when the s is large and the d–N invariants are small, it is appropriate to regard these variables as second order compared with unity. In addition, examination of the exact imaginary parts above the normal threshold suggests that these polynomials do not grow as one might expect, but are cancelled out to a large extent. A notable example of this is that the singularity in s at $4M^2$ which is suggested by the d^{-2} terms does not seem to be present in the (correct) discontinuities above the normal threshold. In any case we do not wish to become embroiled in these details, but simply to observe that to treat these as second order variables seems to be a justifiable approximation.

Now, examination of the d–N invariants (Sec. 4) shows us that the G invariant is about 20 times as large as the F, $\mathcal{H}$, and $\mathcal{J}$ invariants, which are all about the same size.

Our approximation to retain only leading terms can now be precisely stated. We will retain only zeroth-order monomials (constants) in terms which do not involve G, up to first-order monomials in terms linear in G, and up to second-order monomials in terms quadratic in G. We have followed the same procedure with regard to the deuteron binding energy; α^2 is regarded as a second-order term and is thus handled in the same way as s, η and $\bar{\eta}$. Note that in W_1 and W_2 there are only terms of the form F^2, FGs, and G^2s^2 (or smaller) and none of the form FG, G^2, or G^2s (where F here represents F, $\mathcal{H}$, or $\mathcal{J}$ and s represents s, η, $\bar{\eta}$, or α^2). In W_3, however, the FG and G^2s terms are present. Hence we will retain in W_1 and W_2 only terms of the form F^2, FGs, and G^2s^2, while in W_3 we will retain F^2, FG, and G^2s. If our approximations are correct, we would expect W_3 to be large and dominated by the G terms. This is in fact the case, and we could have neglected the F^2 terms in W_3 without altering our results significantly.

Hence, to obtain these simplified expressions we introduce η and $\bar{\eta}$ [Eq. (3.19a)] and make the approximations:

$$\begin{aligned}
&M^2\cong 4m^2-4\alpha^2\,,\\
&E=-(\eta-\bar{\eta})/s\,,\\
&C\cong(1/d^2)[4m^2-(\eta+\bar{\eta})-2\alpha^2]\,,\\
&A\cong-(m^2/2sd^2)[s^2-8s(\eta+\bar{\eta})+16(\eta-\bar{\eta})^2]\,,\\
&B\cong-\tfrac{1}{32}(1+8(\eta+\bar{\eta})/s-48(\eta-\bar{\eta})^2/s^2)\,,\\
&D\cong\tfrac{1}{4}B\,,\\
&F_C\cong F_M-F_2\,.
\end{aligned}\tag{B8}$$

Substituting these expressions into the equations (B5)–(B7) yields

$$\begin{aligned}
W_1\cong{}&F_C\{(F+\mathcal{H})^2-(F+\mathcal{H})G(2A/m^2)-G^2(A/8m^4)(s-4(\eta+\bar{\eta}))\}\,,\\
W_2\cong{}&F_C\{-G^2(A/8m^4)(s-4(\eta+\bar{\eta}))\}+2F_M\{(F+\mathcal{H})^2-(F\mathcal{H}+\mathcal{H}^2+F\mathcal{J}+\mathcal{H}\mathcal{J}+FG(\alpha^2/2m^2)+G\mathcal{H}(\alpha^2/2m^2))\\
&\times(\tfrac{1}{2}+2(\eta-\bar{\eta})/s)-(FG/32m^2)(s-16\eta-8\bar{\eta}-16(\eta^2-\bar{\eta}^2)/s)+G\mathcal{J}(A/m^2)\\
&+(G\mathcal{H}/32m^2)(s-8\eta-16\bar{\eta}+16(\eta^2-\bar{\eta}^2)/s)+G^2(A/8m^4)(s-2(\eta+\bar{\eta}))\}\,,\\
W_3\cong{}&F_C\{-F^2(8B+2)+(FG+\mathcal{H}G-\mathcal{H}F)(16B+2+8(\eta-\bar{\eta})/s)+(F\mathcal{J}+\mathcal{H}\mathcal{J}-\tfrac{1}{2}\mathcal{J}^2)16B+(G^2/m^2)B(s-4(\eta+\bar{\eta}))\}\\
&+F_M\{2F^2+F\mathcal{H}(16B+2-8(\eta-\bar{\eta})/s)-(F\mathcal{J}+\mathcal{H}\mathcal{J}-\mathcal{J}^2)16B\}\,,\\
={}&F_CW_3{}^C+F_MW_{3M}\,.
\end{aligned}\tag{B9}$$

Finally, one obtains the results (3.22) by forming the combinations G_C, G_M, and G_Q [cf. Eq. (1.8)]. However, retaining only the leading terms means that the combinations we form in fact are

$$\begin{aligned}
&W_C=W_1-(s/24m^2)W_3\,,\\
&W_M=W_2\,,\\
&W_Q=W_1-W_2+W_3\,.
\end{aligned}\tag{B10}$$

To maintain our leading term approximation consistently we drop the terms involving G from W_1 and W_2 in W_Q, while in W_C we drop from W_3 the terms involving products of F, $\mathcal{H}$, and $\mathcal{J}$. Finally we emerge with Eqs. (3.22).

At this point we make a few additional technical remarks.

First, we would like to point out that the precise form of the invariants $\mathcal{G}$ and $\mathfrak{F}$ was chosen to make the expressions (3.22) agree as closely as possible with the potential theory results (2.23). We have said more about the significance of this step in Sec. 3 and Appendix C. At the moment, all we wish to observe is that there is a slight arbitrariness in our choice of $\mathcal{G}$ which is not present in our choice of $\mathfrak{F}$. This is because we may add a number of terms linear in F, $\mathcal{H}$, and $\mathcal{J}$ to G (to form $\mathcal{G}$) without changing the expressions W_C and W_M, because terms of this type have been neglected in W_C and W_M. Hence our only way of deciding which combination to take for $\mathcal{G}$ is to examine G_Q and make this agree as nearly as possible with the G_Q of Sec. 2. But it is impossible to find any $\mathcal{G}$ which will put both expressions in exactly the same form, and hence it is somewhat arbitrary as to what $\mathcal{G}$ we choose. The $\mathcal{G}$ we have chosen [Eq. (3.21a)] has the property that it makes the relativistic corrections to the quadrupole moment quite small.

Finally, we would like to observe that

$$\int_{16\alpha^2}^{\infty}\frac{ds}{s^{1/2}}\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\times\theta\left[\frac{s^{1/2}}{2}-\eta^{1/2}-\bar{\eta}^{1/2}\right]W_3{}^{C,M}(s,\eta,\bar{\eta})=0 \quad \text{(B11a)}$$

regardless of the detailed structure of the d-N invariants provided only that the A conditions (3.34) are satisfied. If we write $G_3=F_CD_3{}^C+F_MD_3{}^M$ this is the condition that

$$\int_{16\alpha^2}^{\infty}\mathrm{Im}D_3{}^{C,M}(s)ds=0\,, \quad \text{(B11b)}$$

which is necessary if we are to write unsubtracted dispersion relations for D_C, and $D_Q{}^{C,M}$ [cf. Eq. (B10)]. That this is so can be seen using the integrals introduced in Eq. (2.19). We must show that

$$s^3B;\; s^3+4(\eta-\bar{\eta})s^2$$

are polynomials of type Q, while

$$s^2B[s-4(\eta+\bar{\eta})]$$

is a polynomial of type P. This can be readily demonstrated, however, and we will not do it here.

APPENDIX C

In this Appendix we establish the uniqueness of the S- and D-state wave functions in the following precise sense.

Theorem. Regard $D_M{}^C(s)$ as a functional of g:

$$D_M{}^C\{g,s\}=\frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{ds'}{s'-s}\frac{1}{s'^{1/2}}\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\,\theta\left[\frac{s'^{1/2}}{2}-\eta^{1/2}-\bar{\eta}^{1/2}\right]g(\eta)g(\bar{\eta})\times[s'^2-12s'(\eta+\bar{\eta})+48(\eta^2+\bar{\eta}^2)+32\eta\bar{\eta}-64(\eta-\bar{\eta})^2(\eta+\bar{\eta})/s'], \quad \text{(C1)}$$

and similarly $D_C\{f,g;s\}$ functional of f and g. Further suppose that

$$\int_{\alpha^2}^{\infty}f(\eta)d\eta=\int_{\alpha^2}^{\infty}g(\eta)d\eta=0. \quad \text{(C2)}$$

Now suppose that we have two pairs of functions f_1, g_1 and f_2, g_2 such that

$$D_C\{f_1,g_1,s\}=D_C\{f_2,g_2,s\}\quad D_M{}^C\{g_1,s\}=D_M{}^C\{g_2,s\} \quad \text{(C3)}$$

for all s and each pair satisfies (C2). Then it is true that

$$f_1=\pm f_2\quad g_1=\pm g_2. \quad \text{(C4)}$$

Proof. It is straightforward to show by integration by parts that

$$D_M{}^C\{g;s\}=\frac{1}{\pi}\int_{16\alpha^2}^{\infty}\frac{ds'}{s'-s}\frac{1}{s'^{1/2}}\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\,\theta\left[\frac{s'^{1/2}}{2}-\eta^{1/2}-\bar{\eta}^{1/2}\right]\left(\eta g(\eta)+\frac{3}{2}\int_{\alpha^2}^{\eta}g(x)dx\right)\left(\bar{\eta}g(\bar{\eta})+\frac{3}{2}\int_{\alpha^2}^{\bar{\eta}}g(y)dy\right)\times((16)^2/3)\{P_0(2(\eta^{1/2}+\bar{\eta}^{1/2})/s^{1/2})-P_2(2(\eta^{1/2}+\bar{\eta}^{1/2})/s^{1/2})\}, \quad \text{(C5)}$$

where P_n is the Legendre polynomial of order n. Now, by applying (2.11), we obtain

$$D_M{}^C\{g,s\}=\frac{(16)^2}{3\pi}\int_0^{\infty}dr\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\,\exp[-(\eta^{1/2}+\bar{\eta}^{1/2})r]\left(\eta g(\eta)+\frac{3}{2}\int_{\alpha^2}^{\eta}g(x)dx\right)\left(\bar{\eta}g(\bar{\eta})+\frac{3}{2}\int_{\alpha^2}^{\bar{\eta}}g(y)dy\right)\times[j_0(s^{1/2}r/2)+j_2(s^{1/2}r/2)]. \quad \text{(C6)}$$

Hence

$$D_M{}^C\{g_1;s\}-D_M{}^C\{g_2;s\}=\frac{(16)^2}{3\pi}\int_0^{\infty}dr\int_{\alpha^2}^{\infty}d\eta\int_{\alpha^2}^{\infty}d\bar{\eta}\,\exp[-(\eta^{1/2}+\bar{\eta}^{1/2})r]\times\chi^+(\eta)\chi^-(\bar{\eta})[j_0(s^{1/2}r/2)+j_2(s^{1/2}r/2)]=0, \quad \text{(C7)}$$

where

$$\chi^{\pm}(\eta)=\eta(g_1\pm g_2)+\frac{3}{2}\int_{\alpha^2}^{\eta}[g_1(x)\pm g_2(x)]dx.$$

The theorem will be proved for the g's if we can show that either χ^+ or χ^- must be zero. This is because $\chi^\pm=0$ implies that $\int_{\alpha^2}^{\eta}[g(x)\pm g_2(x)]dx$ is a solution of a first order differential equation which is zero at $\eta=\alpha^2$ and hence

$$\int_{\alpha^2}^{\eta}[g_1(x)\pm g_2(x)]dx=0$$

and $g_1(\eta)=\pm g_2(\eta)$. Then an identical but algebracially simpler argument would give us the result for the f's.

To see that χ^+ or χ^- is zero we observe that

$$j_0(x)+j_2(x)=-(3/x)j_1(x)$$

and recalling the completeness of the Bessel functions $j_1(x)$ we have immediately

$$\int_{\alpha^2}^{\infty}d\eta[\exp(-\eta^{1/2}r)]\chi^+(\eta)\int_{\alpha^2}^{\infty}d\bar{\eta}[\exp(-\bar{\eta}^{1/2}r)]\chi^-(\bar{\eta})=0,$$

which implies that for either χ^+ or χ^-

$$\int_{\alpha^2}^{\infty}d\eta[\exp(-\eta^{1/2}r)]\chi(\eta)=0.$$

Taking successive derivatives at the origin implies

$$\int_{\alpha^2}^{\infty}\eta^{n/2}\chi(\eta)d\eta=0$$

for all integral n. Hence, $\chi(\eta)=0$, and the uniqueness is proved.

APPENDIX D

In this Appendix we list the angular integral identities sufficient to obtain the results (4.6) from (4.5). These are

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi\mathbf{Q}=a\mathbf{N}+b\mathbf{p},$$

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi Q^\alpha=(a-b)N^\alpha, \tag{D1}$$

$$\frac{1}{2\pi}\int_0^{2\pi}d\phi Q^\alpha\mathbf{Q}=\mu^2c\gamma^\alpha+dN^\alpha\mathbf{N}+eN^\alpha\mathbf{p},$$

where

$$\begin{aligned}
a&=(\mu^2+2N_0Q_0)/2N^2,\\
b&=(Q_0-N_0a)/u^{1/2},\\
c&=(N^2a^2-Q^2)/2\mu^2,\\
d&=(1+N_0/u^{1/2})(a^2+\mu^2c/N^2)-(Q_0/u^{1/2})a,\\
e&=-(N_0/u^{1/2})d+c\mu^2/u-Q_0^2/u\\
&\qquad+(Q_0a/u^{1/2})(1+N_0/u^{1/2}),
\end{aligned} \tag{D2}$$

and

$$\begin{aligned}
N_0&=-(1/2u^{1/2})(u+m^2-M^2),\\
Q_0&=(1/2u^{1/2})(u+\mu^2-m^2),\\
Q^2&=(1/4u)[u-(m+\mu)^2][u-(m-\mu)^2],\\
N^2&=(1/4u)[u-(m+M)^2][u-(m-M)^2].
\end{aligned} \tag{D3}$$

APPENDIX E

The following errata have been found in Ref. 1:

1. In Eq. (2.1) the sign of the term $(\Lambda_2^2-\Lambda_3^2)^2/4\Lambda_1^2$ should be changed to be positive.
2. In the discussion immediately following Eq. (3.3) the sentence "The variable z_v is the cosine of the angle on which v depends, and the angles are defined so that this integration is always in the rest system of the virtual particle which corresponds to v" should be changed to read: "The variable z_v is the cosine of the angle on which v depends, and the angles are always defined so that this integration is in the center of mass system of the a and c particles on which v depends."
3. In Eq. (3.6) on the right-hand side the v should be $v^{1/2}$.
4. In Eq. (3.7) a Γ_0 was omitted from the second term.
5. In Eq. (3.13) the factor $(2\pi)^2$ should be replaced by $2\pi^2$.
6. In Eq. (3.17) the quantity f^2-g' in the denominator of the arctan should be replaced by $f^2-|g'|$.
7. The theorem in the Appendix is stated too generally to follow from the proof given. We wish to change it to a less general form which is sufficient to include all of the cases treated in I and in this apper. To this end we require that $f(s)$ and $[g(s)]^{1/2}$ be real analytic functions, and that there exist no s_0 such that $f(s_0)=g(s_0)=0$. In addition we require that all s_1 such that $p(s_1)=\pm1$ are real. It can be shown that the cases treated in I (for suitable choices of M) satisfy these conditions, which are sufficient to allow us to conclude that all of the singularities are on the real s axis, and to exclude the troublesome special case when $g=f=0$. Then we may choose our cuts to lie on the real s axis, and the rest of the proof given in I follows in a straightforward manner.