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Interference Effects in Multimeson Resonances*

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The problem of interference effects in "cascade" decays of multimeson resonances is discussed; that is, decays of the form $X \to Y + (n \text{ mesons})$ followed by $Y \to (m \text{ mesons})$. Interference effects occur whenever there is more than one way to form the state $Y+n$ mesons out of the final $(m+n)$-meson state. For example, there will be interference between the two modes

$$A^0 \to \begin{Bmatrix} \rho^+ + \pi^- \\ \rho^- + \pi^+ \end{Bmatrix} \to \pi^+ + \pi^- + \pi^0.$$

This case is discussed in detail, and results are presented for the B-meson decay. The effect of experimental resolution is evaluated.

I. INTRODUCTION

IN recent publications, evidence has been reported for the existence of two new resonances tentatively called A and B.[1,2] Their principal decay modes are

$$\begin{aligned} A^+ &\to \pi^+ + \rho^0 \to \pi^+ + \pi^+ + \pi^-, \\ B^+ &\to \pi^+ + \omega \to \pi^+ + \pi^+ + \pi^0 + \pi^-. \end{aligned} \tag{1.1}$$

The determination of the quantum numbers of multipion resonances has been considered by many authors.[3] Practically all these tests are based on angular correlations among the outgoing pions. However, the observation that in both the decays in (1.1) there are two identical particles raises the possibility of observing interference effects which could be helpful in establishing the quantum numbers.[4]

More generally, we shall discuss in this paper the problem of interference effects in "cascade" decays of multipion resonances; that is, decays of the form

$$X \to Y + n\pi \tag{1.2}$$

followed by

$$Y \to m\pi.$$

Interference effects exist whenever there is more than one way to form the state $Y+n\pi$ out of the final $(m+n)\pi$ state; they are not confined to cases in which there are identical particles. For example, if the A has $I=2$, there will be interference between the three modes

$$A^0 \to \begin{Bmatrix} \rho^+ + \pi^- \\ \rho^- + \pi^+ \\ \rho^0 + \pi^0 \end{Bmatrix} \to \pi^+ + \pi^- + \pi^0.$$

(If the A has $I=0$, the last mode is forbidden.)

For the sake of clarity, we proceed directly in Sec. II to the discussion of the specific example $A^+ \to \pi^+ + \rho^0$. In Sec. III the influence of the experimental resolution on the observation of interference effects is considered. We show that it is impossible to observe these effects unless the resolution width is less than or comparable to the decay width of the particle Y. The width of particle X can be large, but in this case one must be careful to plot the data as a function of suitable variables. In Sec. IV we consider the decay $B^+ \to \pi^+ + \omega$. In the Appendix we discuss the general tensor methods used to write down correct amplitudes for the processes (1.2).

II. INTERFERENCE EFFECTS IN THE DECAY $A^+ \to \pi^+ + \rho^0$

In the decay $A^+ \to \pi^+ + \rho^0$ interference effects occur between the two possibilities $A^+ \to \pi_1^+ + \rho^0$, $\rho^0 \to \pi_2^+ + \pi^-$ and $A \to \pi_2^+ + \rho^0$, $\rho^0 \to \pi_1^+ + \pi^-$. We shall evaluate these effects for all spin and parity values through $J=2$, and for the two possible values of the isotopic spin $I=1, 2$. We follow the usual procedure of writing down the simplest decay amplitude consistent with the quantum numbers assumed, paying particular attention to the requirements of statistics. The results can conveniently be expressed in terms of the density distribution on the Dalitz plot of the A decay. The plot shows two ρ bands, with the interference effects occurring in the region where the bands cross.

A. The Isospin Matrix Element

A properly symmetrized decay matrix element can be formed from the sum of three terms represented diagrammatically in Fig. 1. The indices i, j, k are the isotopic spin indices of the pions in the Cartesian representation, and are related to states of given charge in the usual way:

$$\begin{aligned} -\sqrt{2}|+\rangle &= |1\rangle + i|2\rangle, \\ \sqrt{2}|-\rangle &= |1\rangle - i|2\rangle, \\ |0\rangle &= |3\rangle. \end{aligned} \tag{2.1}$$

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[1] M. Abolins, R. L. Lander, W. A. W. Mehlhop, N. H. Xuong, and P. M. Yager, Phys. Rev. Letters 11, 381 (1963).

[2] G. Goldhaber, J. L. Brown, S. Goldhaber, J. A. Kadyk, B. C. Shen, and G. H. Trilling, Phys. Rev. Letters 12, 336 (1964).

[3] C. Zemach, Phys. Rev. 133, B1201 (1964); F. R. Halpern, Phys. Rev. Letters 12, 252 (1964); M. Ademollo, R. Gatto, and G. Preparata, *ibid.* 12, 462 (1964).

[4] Interference effects involving unstable particles have been considered by C. Bouchiat and G. Flamand, Nuovo Cimento 23, 13 (1962), by R. H. Dalitz and D. H. Miller, Phys. Rev. Letters 6, 562 (1961); and by C. Zemach, Phys. Rev. 133, B1201 (1964).

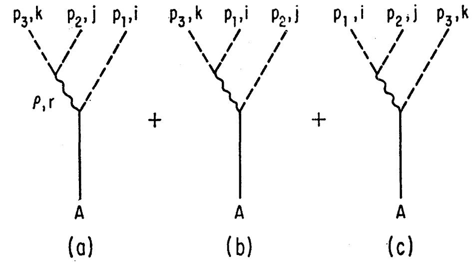

FIG. 1. The three possible modes of the decay $A \to \pi+\rho \to 3\pi$.

Consider first the case $I_A=1$. Then the isospin part of the matrix element for the diagram in Fig. 1(a) is formed by first combining pions 2 and 3 into a ρ meson with isospin index r, giving a factor $i\epsilon_{\bar{r}jk}$ (we place bars over indices of incoming particles); then combining the ρ with pion 3 to form the A (isospin index a), giving the additional factor $i\epsilon_{\bar{a}ri}$. The result for this diagram is then

$$P(s_1)M_{1,23}(\delta_{\bar{a}j}\delta_{ik}-\delta_{\bar{a}k}\delta_{ij})\,, \qquad (2.2)$$

where $M_{1,23}$ is a function of the momenta appropriate to whichever J^P is considered, and where $P(s)$ is the ρ propagator. Bose statistics require that $M_{1,32}=-M_{1,23}$. We use the notation

$$\begin{aligned} s_1&=(p_2+p_3)^2\,,\\ s_2&=(p_1+p_3)^2\,,\\ s_3&=(p_1+p_2)^2\,. \end{aligned} \qquad (2.3)$$

The complete matrix element of Fig. 1 is then

$$\begin{aligned} M=P(s_1)M_{1,23}&(\delta_{\bar{a}j}\delta_{ik}-\delta_{\bar{a}k}\delta_{ij})\\ &+P(s_2)M_{2,13}(\delta_{\bar{a}i}\delta_{jk}-\delta_{\bar{a}k}\delta_{ij})\\ &\quad+P(s_3)M_{3,21}(\delta_{\bar{a}j}\delta_{ik}-\delta_{\bar{a}i}\delta_{jk})\,. \end{aligned} \qquad (2.4)$$

Evaluating this for the charge configuration $A^+\to\pi_1{}^+ +\pi_2{}^+ +\pi^-$, we obtain the result

$$M=M_{1,23}P(s_1)+M_{2,13}P(s_2)\,, \qquad (2.5)$$

which is shown diagrammatically in Fig. 2.

Proceeding in the same manner for the case $I_A=2$, we find for Fig. 1(a) the result

$$[\tfrac{1}{2}(i\epsilon_{\bar{a}jk}\delta_{i\bar{b}}+i\epsilon_{\bar{b}jk}\delta_{i\bar{a}})-\tfrac{1}{3}\delta_{\bar{a}\bar{b}}i\epsilon_{ijk}]P(s_1)M_{1,23}\,, \qquad (2.6)$$

where a, b are the isospin indices of the A. The fact that this matrix element contains only $I_A=2$ follows from its being traceless and symmetric in a and b. The complete matrix element M is again the sum of the contributions of the three diagrams of Fig. 1, and the result for the charge configuration $A^+\to\pi_1{}^+ +\pi_2{}^+ +\pi^-$ is the same as the previous result, Eq. (2.5). This result could indeed have been written down without going through the preceding isospin formalism, since we know that regardless of the isospin of the A, the matrix element must be symmetric under the interchange of the two π^+ mesons.

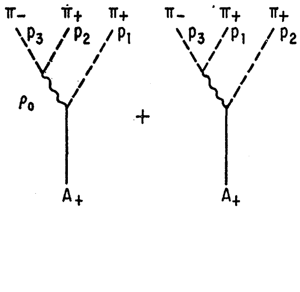

FIG. 2. The two possible modes of the decay $A^+\to\pi^+ +\rho^0\to\pi^+ +\pi^+ +\pi^-$.

More complicated cases can occur, however. Consider, for example, the decay $A^0\to\pi^+ +\pi^- +\pi^0$. Let the indices 1, 2, 3 refer to the π^+, π^-, and π^0, respectively. If $I_A=1$, then Eq. (2.4) gives the result

$$M=M_{1,23}P(s_1)+M_{2,13}P(s_2)\,. \qquad (2.7)$$

The absence of a term $M_{3,21}$ reflects the fact that the decay $A^0\to\rho^0+\pi^0$ is forbidden if $I_A=1$. This should provide a good test for I_A. If $I_A=2$, the result is

$$M=M_{1,23}P(s_1)-M_{2,13}P(s_2)+2M_{3,21}P(s_3)\,. \qquad (2.8)$$

Note that the sign of the interference between the first two terms is opposite to the sign for the case $I_A=1$.

B. The Spin Matrix Elements

For each possible J^P value of the A we shall use for $M_{1,23}$ the simplest expression consistent with the assumed quantum numbers. Pions 2 and 3 are combined to form a ρ, giving a factor $e_\rho\cdot(p_2-p_3)$. Then the appropriate expression is written down for the $A\to\rho+\pi$ vertex. Finally, the sum over polarization states of the ρ is done. The results are shown in Table I, where we have introduced the notation $A=p_1+p_2+p_3$. The quantities $P_{\mu\ldots}$ are tensors formed from the production variables, and satisfying the conditions of transversality $(A^\mu P_{\mu\ldots}=0)$, tracelessness, and symmetry, in all their indices.

For the ρ-meson propagator $P(s)$ we choose the following convenient normalization of the Breit-Wigner form:

$$P(s)=\frac{(\Gamma_\rho/\pi)^{1/2}}{s-m_\rho{}^2+i\Gamma_\rho}\,. \qquad (2.9)$$

C. The Dalitz Plot

The most convenient variables to use for displaying the interference between the two terms in Eq. (2.5) are s_1 and s_2, which are the standard Lorentz-invariant Dalitz plot variables. The physical region in the s_1s_2 plane is given by the inequality

$$s_1s_2s_3\geq m_\pi{}^2(W_A{}^2-m_\pi{}^2)^2\,, \qquad (2.10)$$

where s_3, defined in Eq. (2.3), is related to s_1 and s_2 by

$$s_1+s_2+s_3=W_A{}^2+3m_\pi{}^2\,. \qquad (2.11)$$

The quantity W_A is the invariant mass of the three pions. We reserve the notation m_A for the center of the A-meson peak. The density of events in this region is

TABLE I. The simplest matrix element for various values of J^P for the decay $A\to\pi_1+\rho$ followed by $\rho\to\pi_2+\pi_3$. The quantities $P_\mu\ldots$ are tensors formed from the production variables.

J^P	l	$A\to\rho+\pi$, vertex	$M_{1,23}$
0^-	1	$e_\rho\cdot p_1$	$(p_2-p_3)\cdot p_1=\frac{1}{2}(s_3-s_2)$
1^+	0	$e_\rho\cdot P$	$(p_2-p_3)\cdot P$
1^+	2	$(e_\rho\cdot p_1)(P\cdot p_1)$	$(p_2-p_3)\cdot p_1(P\cdot p_1)$
1^-	1	$\epsilon_{\mu\nu\lambda\sigma}P^\mu e_\rho{}^\nu p_1{}^\lambda A^\sigma$	$\epsilon_{\mu\nu\lambda\sigma}P^\mu p_1{}^\nu p_2{}^\lambda p_3{}^\sigma$
2^-	1	$p_1{}^\mu e_\rho{}^\nu P_{\mu\nu}$	$p_1{}^\mu(p_2-p_3)^\nu P_{\mu\nu}$
2^+	2	$p_{1\alpha}P^{\alpha\mu}\epsilon_{\mu\nu\lambda\sigma}e_\rho{}^\nu p_1{}^\lambda A^\sigma$	$p_{1\alpha}P^{\alpha\mu}\epsilon_{\mu\nu\lambda\sigma}p_1{}^\nu p_2{}^\lambda p_3{}^\sigma$

proportional to Σ, where

$$\Sigma=\int d^4p_1d^4p_2\delta(p_1{}^2-m_\pi{}^2)\delta(p_2{}^2-m_\pi{}^2)\times\delta[(A-p_1)^2-s_1]\delta[(A-p_2)^2-s_2]\times\delta[(A-p_1-p_2)^2-m_\pi{}^2]|M|^2. \quad (2.12)$$

Evaluating this expression in the rest frame of the A, one obtains

$$\Sigma=(\pi^2/4s_A)F(s_1,s_2)\,, \quad (2.13)$$

where

$$F(s_1,s_2)=\frac{1}{8\pi^2}\int d\Omega_1d\Omega_2\delta(Y-\cos\theta_{12})|M|^2, \quad (2.14)$$

$$\tau\equiv 2p_1p_2Y=s_1+s_2-W_A{}^2-m_\pi{}^2+2E_1E_2\,,$$

and

$$E_i=(W_A{}^2+m_\pi{}^2-s_i)/2W_A\,. \quad (2.15)$$

The quantity E_i is just the energy of pion i in the A rest frame. For $M=1$, one finds $F=1$, the familiar fact that phase space is uniform over the Dalitz plot.

Substituting Eqs. (2.5) and (2.9) in the definition of F, Eq. (2.14), we arrive at the following expression:

$$F(s_1,s_2)=D(s_1,s_2)\delta(s_1-m_\rho{}^2,\Gamma_\rho)+D(s_2,s_1)\delta(s_2-m_\rho{}^2,\Gamma_\rho)+2\pi\Gamma_\rho I(s_1,s_2)\delta(s_1-m_\rho{}^2,\Gamma_\rho)\times\delta(s_2-m_\rho{}^2,\Gamma_\rho)\Delta(s_1,s_2)\,, \quad (2.16)$$

where

$$D(s_1,s_2)=\frac{1}{8\pi^2}\int d\Omega_1d\Omega_2\delta(Y-\cos\theta_{12})(M_{1,23})^2, \quad (2.17)$$

$$I(s_1,s_2)=\frac{1}{8\pi^2}\int d\Omega_1d\Omega_2\delta(Y-\cos\theta_{12})(M_{1,23}M_{2,13}). \quad (2.18)$$

We have also defined the resonance function

$$\delta(X,\Gamma)=(\Gamma/\pi)/(X^2+\Gamma^2)\,, \quad (2.19)$$

$$\delta(X,0)=\delta(X)\,. \quad (2.20)$$

The correction factor $\Delta(s_1,s_2)$ is given by

$$\Delta(s_1,s_2)=1+(s_1-m_\rho{}^2)(s_2-m_\rho{}^2)/\Gamma_\rho{}^2. \quad (2.21)$$

The second term is negligible unless $D(s_1,s_2)$ or $I(s_1,s_2)$ changes by a large percent of its value over an interval of width Γ_ρ. Since D and I are in general slowly varying, we set $\Delta=1$ hereafter, and evaluate variables multiplied by a δ factor at $m_\rho{}^2$, obtaining

$$F(s_1,s_2)\approx D(m_\rho{}^2,s_2)\delta(s_1-m_\rho{}^2,\Gamma_\rho)+D(m_\rho{}^2,s_1)\times\delta(s_2-m_\rho{}^2,\Gamma_\rho)+2\pi\Gamma_\rho I(m_\rho{}^2,m_\rho{}^2)\times\delta(s_1-m_\rho{}^2,\Gamma_\rho)\delta(s_2-m_\rho{}^2,\Gamma_\rho)\,. \quad (2.22)$$

TABLE II. The direct and interference terms in A-meson decay, to be used in evaluating Eq. (2.22).

J^P	l	
0^-	1	$D(s_1,s_2)=(s_3-s_2)^2$ $I(s_1,s_2)=(s_3-s_2)(s_3-s_1)$
1^+	0	$D(s_1,s_2)=4\mathbf{p}_2{}^2+\mathbf{p}_1{}^2+2\tau$ $I(s_1,s_2)=2\mathbf{p}_2{}^2+2\mathbf{p}_1{}^2+\frac{5}{2}\tau$
1^+	2	$D(s_1,s_2)=2\mathbf{p}_1{}^2(s_3-s_2)^2$ $I(s_1,s_2)=\tau(s_3-s_2)(s_3-s_1)$
1^-	1	$D(s_1,s_2)=4\mathbf{p}_1{}^2\mathbf{p}_2{}^2-\tau^2$ $I(s_1,s_2)=-4\mathbf{p}_1{}^2\mathbf{p}_2{}^2+\tau^2$
2^-	1	$D(s_1,s_2)=2\mathbf{p}_1{}^4+4\tau\mathbf{p}_1{}^2+6\mathbf{p}_1{}^2\mathbf{p}_2{}^2+\frac{1}{2}\tau^2$ $I(s_1,s_2)=5\mathbf{p}_1{}^2\mathbf{p}_2{}^2+(5/4)\tau^2+2\tau(\mathbf{p}_1{}^2+\mathbf{p}_2{}^2)$
2^+	2	$D(s_1,s_2)=2\mathbf{p}_1{}^2(4\mathbf{p}_1{}^2\mathbf{p}_2{}^2-\tau^2)$ $I(s_1,s_2)=-\tau(4\mathbf{p}_1{}^2\mathbf{p}_2{}^2-\tau^2)$

This expression has a simple physical meaning: The first two terms give two ρ bands on the Dalitz plot, corresponding to the two decay modes $A^+\to\rho^++\pi_1{}^+$ and $A^+\to\rho^++\pi_2{}^+$; the last term, resulting from the interference between these modes, contributes significantly only in the region where the two bands cross. Outside this region the two π^+ are distinguishable, since it is possible to say which one came from the ρ decay. If the unstable particle is long-lived, the interference effect is important only over a small region of the Dalitz plot, and very good experimental resolution is required to observe it. We shall discuss this point in more detail in the next section.

In Table II the functions $D(s_1,s_2)$ and $I(s_1,s_2)$ are given for the various J^P considered. The quantity $\mathbf{p}_i{}^2$ is the square of the three-momentum of pion i in the A rest frame, $\mathbf{p}_i{}^2=E_i{}^2-m_\pi{}^2$. The normalization of D and I in Table II is arbitrary; i.e., constant factors common to D and I for a given J^P have been removed.

Before these results can be compared with the experimental data for a broad resonance such as the A, it is necessary to perform an integration over the mass spectrum of the A; i.e., to calculate the quantity

$$\int_{m_A-L_A}^{m_A+L_A}F(s_1,s_2,W_A)\delta(W_A-m_A,\Gamma_A)dW_A\,, \quad (2.23)$$

where L_A is the half-width of the region about m_A from which events are taken.

Note that the interference effects are maximal for the

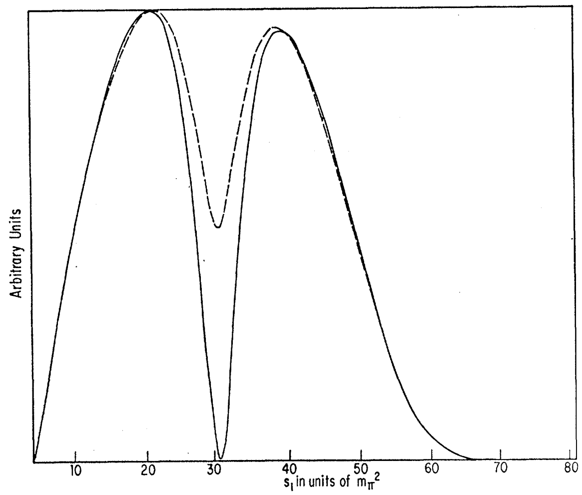

FIG. 3. Distribution of density of events expected along a ρ-meson band, $s_2=m_\rho^2$, in the Dalitz plot of the decay of the R meson (see Ref. 6), for the 1^- quantum number assignment. The solid line neglects the effect of experimental resolution, while the dashed line is the result for a resolution full width of 25 MeV (see Sec. III).

0^- and 1^- cases; constructive for the former, and destructive for the latter. The interference effects, as well as the significant variation of the matrix element, can be displayed by plotting the variation in density along a ρ band; i.e., $F(s_1,m_\rho^2)$. This plot, for the 1^- case, is shown by the solid line in Fig. 3. Graphs of all the J^P we have considered can be found in a recent paper by Lander *et al.*[5]

III. EXPERIMENTAL RESOLUTION

We have thus far idealized the problem by assuming that the experimental resolution width is small compared to the decay widths of the particles X and Y. We shall now investigate the effect of relaxing this restriction. Assume, for simplicity of subsequent calculations, that the resolution function is $\delta(x-x_0, R)$, defined by Eq. (2.19). That is, if the true value of some variable x is x_0, the observed variable x will be distributed according to the probability distribution $\delta(x-x_0, R)$, with a resolution width R. Then a given theoretical prediction $f(x)$ will be "smeared" by the experimental resolution into its "resolution transform" $\bar{f}(x)$, where

$$f(x)\equiv\int_{-\infty}^{\infty} dx'\delta(x'-x, R)f(x'). \tag{3.1}$$

By contour integration we find that the resolution transform of a Breit-Wigner function representing a resonance of width Γ is itself a Breit-Wigner function, but with width $\Gamma+R$; that is,

$$\bar{\delta}(x,\Gamma)=\delta(x, \Gamma+R). \tag{3.2}$$

[5] R. L. Lander, M. Abolins, D. D. Carmony, T. Hendricks, N. H. Xuong, and P. Yager (to be published).

The density of events $F(s_1,s_2)$ in the ρ bands of the Dalitz plot for A decay, given by Eq. (2.22), will be changed by the resolution into the function $\bar{F}(s_1,s_2)$, where

$$\bar{F}(s_1,s_2)=\int_{-\infty}^{\infty} ds_1'ds_2'\delta(s_1'-s_1, R) \times\delta(s_2'-s_2, R)F(s_1',s_2'). \tag{3.3}$$

If one then assumes that $D(s_1,s_2)$ varies slowly enough that it can be taken to be constant over a region of the size of the resolution width one obtains

$$\begin{aligned}\bar{F}(s_1,s_2)=&D(m_\rho^2,s_2)\delta(s_1-m_\rho^2, \Gamma_\rho+R)+D(m_\rho^2,s_1)\\ &\times\delta(s_2-m_\rho^2, \Gamma_\rho+R)+2\pi\Gamma_\rho I(m_\rho^2,m_\rho^2)\\ &\times\delta(s_1-m_\rho^2, \Gamma_\rho+R)\delta(s_2-m_\rho^2, \Gamma_\rho+R).\end{aligned}$$

In order to see the effect of the resolution on the observation of the interference term consider the density along a contour following one of the ρ bands,

$$\bar{F}(s_1,m_\rho^2)=\frac{1}{\pi\Gamma'}\left\{D(m_\rho^2,s_1)+\frac{\Gamma'^2}{(s_1-m_\rho^2)^2+\Gamma'^2}\times\left[D(m_\rho^2,m_\rho^2)+2\frac{\Gamma_\rho}{\Gamma'}I(m_\rho^2,m_\rho^2)\right]\right\}, \tag{3.4}$$

where $\Gamma'=\Gamma_\rho+R$. Note the factor Γ_ρ/Γ' which multiplies the interference term. If $R\gg\Gamma_\rho$, no interference effects can be observed, as one would expect. Figure 3 shows the effect of typical experimental resolution in the favorable case of $R\to\pi+\rho$. Figure 4 shows the effect on the unfavorable case $B\to\pi+\omega$ to be discussed in the next section.

Even in a favorable case such as $A\to\pi+\rho$ where

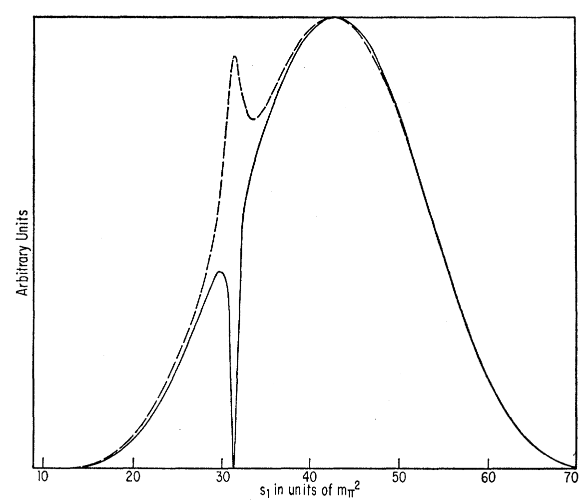

FIG. 4. Distribution of density of events expected along an ω band, $s_2=m_\omega^2$, in the decay of the B meson, for the 0^- quantum number assignment. The solid line neglects the effect of experimental resolution, while the dashed line is the result for a resolution full width of 20 MeV, and a band half-width L (see Sec. III) of 15 MeV.

$\Gamma_\rho \ll R$, the interference effects can be obscured if statistics force one to look at a broad band about $s_2 = m_\rho^2$. If we integrate Eq. (3.3) over a band of width $2L$ about $s_2 = m_\rho^2$, we find

$$\frac{1}{2L}\int_{m_\rho^2-L}^{m_\rho^2+L} F(s_1,s_2)ds_2$$
$$= U(L,\Gamma')D(m_\rho^2,s_1) + \delta(s_1 - m_\rho^2, \Gamma')$$
$$\times [D(m_\rho^2,m_\rho^2) + 2\pi\Gamma_\rho U(L,\Gamma')I(m_\rho^2,m_\rho^2)], \quad (3.5)$$

where

$$U(L,\Gamma') = (1/\pi L)\tan^{-1}(L/\Gamma'). \quad (3.6)$$

One can see from these equations that the interference term becomes small when $L \gg \Gamma_\rho$.

IV. CHOICE OF VARIABLES

We have seen how interference effects are smeared out when the resolution becomes comparable to the width of particle Y. The width of particle X has, on the other hand, no significant relation to the observability of the interference effects, as long as the Lorentz-invariant mass variables s_1 and s_2 are used. Often, however, one displays the data by means of the Dalitz plot in terms of the energies E_1, E_2. The purpose of this section is to point out that such a choice of variables obscures the interference effects unless X is a very narrow resonance. The reason is evident from Eq. (2.15) which gives the kinematical relation between s_i and E_i. The quantity W_A enters into this relation. Therefore, the width of the A will make the ρ peak in the variable E_1 appear broadened. To evaluate the effect quantitatively we must express $\delta(s_1 - m_\rho^2, \Gamma_\rho)$ in terms of E_1 and W_A, then integrate over the A mass spectrum. Evaluating slowly-varying terms at the peak of the δ function we find

$$\delta(s_1 - m_\rho^2, \Gamma_\rho) \approx \frac{1}{2E_\rho}\delta\left(W_A - W_1, \frac{\Gamma_\rho}{2E_\rho}\right), \quad (4.1)$$

where E_ρ is the energy of the ρ in the three-pion c.m. system,

$$E_\rho = (s_A + m_\rho^2 - m_\pi^2)/2W_A, \quad (4.2)$$

and where

$$W_1 = E_1 + (m_\rho^2 + E_1^2 - m_\pi^2)^{1/2}. \quad (4.3)$$

Integrating over the mass spectrum of the A we obtain

$$\int_{-\infty}^{\infty} dW_A \delta(W_A - m_A, \Gamma_A)\frac{1}{2E_\rho}\delta\left(W_A - W_1, \frac{\Gamma_\rho}{2E_\rho}\right)$$
$$\approx \frac{1}{2E_\rho}\delta\left(m_A - W_1, \Gamma_A + \frac{\Gamma_\rho}{2E_\rho}\right)$$
$$\approx \frac{1}{2m_A}\delta(E_1 - E_0, \gamma_\rho + \gamma_A), \quad (4.4)$$

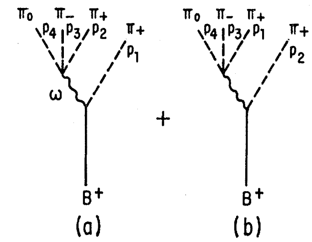

FIG. 5. The two possible modes of the decay $B^+ \to \pi^+ + \omega \to \pi^+ + \pi^+ + \pi^- + \pi^0$.

where

$$E_0 = (m_A^2 + m_\pi^2 - m_\rho^2)/2m_A, \quad (4.5)$$

and where

$$\gamma_\rho = \Gamma_\rho/2m_A, \quad \gamma_A = (1 - E_\rho/m_A)\Gamma_A. \quad (4.6)$$

The width of the ρ in the variable E_1 is thus $\gamma_\rho + \gamma_A$. For the case of an A meson of width ≈ 350 MeV, the term γ_A is about twice the "natural width" γ_ρ. The band structure of the Dalitz plot in E_1 and E_2 will be almost completely obscured.[6] One would expect that interference effects will also be obscured, and a detailed evaluation shows that this is indeed the case. We shall not reproduce this evaluation here, since the difficulty we are describing in this section need never occur as long as the Lorentz-invariant mass variables are used.

V. THE DECAY $B^+ \to \pi^+ + \omega$

The existence of two possible decay modes of the B^+, namely, $B^+ \to \pi_1^+ + \omega$; $\omega \to \pi_2^+ + \pi^- + \pi^0$ and $B^+ \to \pi_2^+ + \omega$; $\omega \to \pi_1^+ + \pi^- + \pi^0$; gives rise to the possibility of interference effects in a manner similar, in principle, to the one described in Sec. II for the decay of the A^+ meson.

The fact that one has to consider now four pions in the final state, however, makes the calculation of the interference effects a somewhat more difficult task. Also, the choice of the best set of variables in which to plot the data to show those effects is not straightforward.

In order to clarify the steps we followed in our calculation, we shall present first, in some detail, the case corresponding to the decay of the B^+ if it had quantum numbers $J^P = 1^+$. The generalization for the other possible J^P assignments is straightforward and the results, only, will be shown on Table 4.

Since the B^+ meson is an isotopic vector, the relevant isotopic spin vertex operators are simply $\delta_{\bar{i}i_2}$ for the diagram of Fig. 5(a) and $\delta_{\bar{i}i_2}$ for the case of Fig. 5(b). Therefore, the total matrix element for B^+ decay is

$$M = P(s_1)M_{1,234} + P(s_2)M_{2,134}, \quad (5.1)$$

where

$$s_1 = (p_2 + p_3 + p_4)^2, \quad s_2 = (p_1 + p_3 + p_4)^2. \quad (5.2)$$

[6] It appears that the A is not this broad, but is really two peaks: S. U. Chung, O. I. Dahl, L. M. Hardy, R. I. Hess, G. R. Kalbfleisch, J. Kirz, D. H. Miller, and G. A. Smith, University of California Lawrence Radiation Laboratory Report UCRL-11371, April 1964 (unpublished)

$P(s_i)$ are the ω meson propagators corresponding to each diagram and $M_{a,bcd}$ is a function of the momenta appropriate to the J^P considered. Then $|M|^2$ can be shown to have the same form as Eq. (2.16) exchanging Γ_ρ for Γ_ω everywhere. This fact makes it clear that a convenient set of variables to plot the data is the plane $(s_1; s_2)$, in complete similarity with the case of the A^+ decay. In this plane the interference effects will show in the region where the two ω bands cross.

The B^+ decay rate is proportional to

$$\Sigma=\int\prod_{i=1}^{4}d^4p_i\delta(p_i^2-m_\pi^2)\times\delta^4(p_1+p_2+p_3+p_4-B)|M|^2. \quad (5.3)$$

Following the procedure described in Sec. II we shall use for $M_{1,234}$ the simplest expression consistent with the assumed quantum numbers. Pions 2, 3, and 4 are combined to form the ω giving a factor

$$f_\alpha=\epsilon_{\alpha\beta\gamma\delta}p_2{}^\beta p_3{}^\gamma p_4{}^\delta. \quad (5.4)$$

Then for $J^P=1^+(l=0)$ we have:

$$M_{1,234}=P\cdot f, \quad (5.5)$$

where P is a vector associated with the production variables of the B^+ and having the property $P\cdot B=0$. Then

$$M=\delta(s_1-m_\omega^2,\Gamma_\omega)\epsilon_{\alpha\beta\gamma\delta}P^\alpha p_2{}^\beta p_3{}^\gamma p_4{}^\delta+\delta(s_2-m_\omega^2,\Gamma_\omega)\epsilon_{\alpha\beta\gamma\delta}P^\alpha p_1{}^\beta p_3{}^\gamma p_4{}^\delta. \quad (5.6)$$

Now the part of $|M|^2$ contributing to the direct term $D(s_1,s_2)$ is

$$|M_D|^2=\epsilon_{\alpha\beta\gamma\delta}P^\alpha p_2{}^\beta p_3{}^\gamma p_4{}^\delta\epsilon_{\mu\nu\lambda\sigma}P^\mu p_2{}^\nu p_3{}^\gamma p_4{}^\sigma. \quad (5.7)$$

Defining $Q=p_3+p_4$ and $K=\frac{1}{2}(p_3-p_4)$ one can integrate over d^4Q and d^4K obtaining

$$\Sigma_D=-\frac{\pi}{24}\int ds_{34}ds_1ds_2\frac{(s_{34}-4m_\pi^2)^{3/2}}{(s_{34})^{1/2}}G(s_1,s_2,s_{34}), \quad (5.8)$$

where

$$G(s_1,s_2,s_{34})=\int d^4p_1d^4p_2\delta(p_1^2-m_\pi^2)\delta(p_2^2-m_\pi^2)\times\delta[(B-p_1)^2-s_1]\delta[(B-p_2)^2-s_2]\times\delta[(B-p_1-p_2)^2-s_{34}]\times\Phi, \quad (5.9)$$

$$\Phi=\epsilon_{\alpha\beta\gamma\delta}\epsilon_{\mu\nu\gamma\sigma}P^\alpha P^\mu p_2{}^\beta p_2{}^\nu(B-p_1)^\delta\times(B-p_1)^\sigma. \quad (5.10)$$

Evaluating Φ in the rest frame of the B^+ one obtains

$$\Phi=-P^2m_\pi^2s_1+2(P\cdot p_1)(P\cdot p_2)(B-p_1)\cdot p_2+m_\pi^2(P-p_1)^2+s_1(P\cdot p_2)^2+P^2[(B-p_1)\cdot p_2]^2. \quad (5.11)$$

TABLE III. The simplest matrix elements for various values of J^P for decay $B\to\pi_1+\omega$ followed by $\omega\to\pi_2+\pi_3+\pi_4$. The quantities $P_\mu\ldots$ are tensors formed from the production variables. The vector B is the four-vector associated with the B and f_μ is defined in Eq. (5.4).

J^P	l	Matrix element
0^-	1	$p_1{}^\mu f_\mu$
1^-	1	$\epsilon_{\mu\nu\lambda\sigma}f^\mu P^\nu p_1{}^\lambda B^\sigma$
1^+	0	$P^\mu f_\mu$
1^+	2	$(P_\mu p_1{}^\mu)(p_1{}^\nu f_\nu)$
2^-	1	$p_1{}^\mu f^\nu P_{\mu\nu}$

The integral can now easily be reduced to

$$G(s_1,s_2,s_{34})=\frac{\pi^2}{s_B}\frac{1}{8\pi}\int d\Omega_1d\Omega_2\delta(\hat{p}_1\cdot\hat{p}_2-Y)\Phi, \quad (5.12)$$

where

$$z=2p_1p_2Y=s_1+s_2-W_B^2-s_{34}+2E_1E_2,\quad E_i=(W_B^2+m_\pi^2-s_i)/2W_B. \quad (5.13)$$

Finally one obtains:

$$D(s_1,s_2)=-\frac{96}{\pi^4P^2}\frac{\Sigma_D}{ds_Bds_1ds_2}=\frac{1}{s_B}\sum_{n=1}^{3}D_nX_n, \quad (5.14)$$

where[7]

$$X_n=\int_{L_-}^{L_+}dx\frac{(x-4m_\pi^2)^{3/2}}{x^{1/2}}x^{n-1}, \quad (5.15)$$

$$\begin{aligned}D_1&=-m_\pi^2s_1+E_2^2(W_B-E_1)^2+\tfrac{1}{3}(m_\pi^2p_1^2+s_1p_2^2)+\tfrac{4}{3}E_2(W_B-E_1)\sigma+(5/12)\sigma^2,\\D_2&=-\tfrac{4}{3}E_2(W_B-E_1)-\tfrac{5}{6}\sigma,\\D_3&=5/12,\end{aligned} \quad (5.16)$$

and

$$\sigma=s_1+s_2-s_B+2E_1E_2. \quad (5.17)$$

In a completely similar way one can calculate the interference term $I(s_1,s_2)$. Then the final expression is obtained simply using Eq. (2.22).

The physical interpretation of this formula is similar to the case of the A^+ meson as described in Sec. II. Of course, here also, due to the fact that the width of the

[7] The integral of Eq. (5.15) can be done analytically, obtaining

$$X_n=\sum_{s=0}^{n}(A_s{}^nm_\pi{}^{2(n-s)}x^s)\xi\Big|_{L_-}^{L_+}-4A_0{}^nm_\pi{}^{2n}\ln\eta,$$
$$\eta=[L_+{}^{1/2}-(L_+-4m_\pi^2)^{1/2}]/[L_-{}^{1/2}-(L_--4m_\pi^2)^{1/2}],$$
$$\xi=[x(x-4m_\pi^2)]^{1/2},$$
$$A_n{}^n=1/(n+1),$$
$$A_{n-1}{}^n=-2(2n+3)/n(n+1),$$
$$A_{n-2}{}^n=12/n(n+1)(n-1),$$
$$A_s{}^n=\frac{3\times4^{n-s-1}(s+\frac{3}{2})(s+\frac{5}{2})\cdots(n-\frac{3}{2})}{(n+1)n(n-1)\cdots(s+1)}\quad\text{for}\quad n>s+2.$$

TABLE IV. The functions $D_i(s_1,s_2)$ and $I_i(s_1,s_2)$ for the different values of J^P considered.

J^P	l	$D(s_1,s_2)$	$I(s_1,s_2)$
0^-	1	$D_1=\mathbf{p}_1^2\mathbf{p}_2^2-\frac{1}{4}\sigma^2$ $D_2=\frac{1}{2}\sigma$ $D_3=-\frac{1}{4}$	$I_1=-D_1$ $I_2=-D_2$ $I_3=-D_3$
1^-	1	$D_1=\frac{1}{3}\mathbf{p}_1^2\{s_1(\mathbf{p}_2^2-2m_\pi^2)+\frac{1}{2}[2E_2(W_B-E_1)+\sigma]^2\}-s_1\sigma^2/12$ $D_2=\frac{1}{6}\sigma s_1-\frac{1}{3}\mathbf{p}_1^2[2E_2(W_B-E_1)+\sigma]$ $D_3=\frac{1}{6}\mathbf{p}_1^2-\frac{1}{12}s_1$	$I_1=-\sigma^3/24+\sigma^2/12(E_1E_2-s_B)+\sigma[s_B(m_\pi^2-E_1E_2)+\mathbf{p}_1^2\mathbf{p}_2^2]/6-\mathbf{p}_1^2\mathbf{p}_2^2E_1E_2/3$ $I_2=\sigma^2/24+(s_B-m_\pi^2)\sigma/6+[s_B(E_1E_2-m_\pi^2)-\mathbf{p}_1^2\mathbf{p}_2^2]/6$ $I_3=[\sigma-2E_1E_2+4m^2-2s_B]/24$ $I_4=-1/24$
1^+	0	$D_1=-m_\pi^2s_1+E_2^2(W_B-E_1)^2+\frac{1}{3}(m_\pi^2\mathbf{p}_1^2+s_1\mathbf{p}_2^2)+\frac{4}{3}E_2(W_B-E_1)\sigma+(5/12)\sigma^2$ $D_2=-\frac{4}{3}E_2(W_B-E_1)-\frac{5}{6}\sigma$ $D_3=5/12$	$I_1=-\frac{1}{4}(s_B-s_1-s_2)(s_B-2m_\pi^2)+(W_BE_1-m_\pi^2)(W_BE_2-m_\pi^2)+\frac{1}{3}\mathbf{p}_1^2(W_BE_2-m_\pi^2)+\frac{1}{3}\mathbf{p}_2^2(W_BE_1-m_\pi^2)+\frac{1}{6}\sigma[W_B(E_1+E_2)-2m_\pi^2]$ $I_2=-\frac{1}{4}(2s_B-2m_\pi^2-s_1-s_2)+\frac{1}{6}[\sigma-W_B(E_1+E_2)+2m_\pi^2]$ $I_3=-5/12$
1^+	2	$D_1=\mathbf{p}_1^2(\mathbf{p}_1^2\mathbf{p}_2^2-\frac{1}{4}\sigma^2)/3$ $D_2=\sigma\mathbf{p}_1^2/6$ $D_3=-\mathbf{p}_1^2/12$	$I_1=\sigma(\sigma^2-4\mathbf{p}_1^2\mathbf{p}_2^2)/24$ $I_2=\mathbf{p}_1^2\mathbf{p}_2^2/6-\sigma^2/8$ $I_3=\sigma/8$ $I_4=-1/24$
2^-	1	$D_1=A_1\sigma^2+B_1\sigma+C_1$ $D_2=-2A_1\sigma-B_1$ $D_3=A_1$ $A_1=(18\mathbf{p}_1^2+s_1)/120$ $B_1=(7/15)\mathbf{p}_1^2E_2(W_B-E_1)$ $C_1=\mathbf{p}_1^2[4m_\pi^2\mathbf{p}_1^2+3s_1\mathbf{p}_2^2-10m_\pi^2s_1+10E_2^2(W_B-E_1)^2]/30$	$I_1=-\frac{\sigma^3}{240}+\sigma^2\left[\frac{1}{15}(s_B-2m_\pi^2)+\frac{E_1E_2}{120}\right]+\frac{\mathbf{p}_1^2\mathbf{p}_2^2E_1E_2}{10}-\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{30}(s_B-2m_\pi^2)+\sigma\left\{-\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{20}-\frac{E_1E_2}{12}(s_B-2m_\pi^2)+\frac{1}{6}(W_BE_1-m_\pi^2)(W_BE_2-m_\pi^2)+1/15[\mathbf{p}_1^2(W_BE_2-m_\pi^2)+\mathbf{p}_2^2(W_BE_1-m_\pi^2)]\right\}$ $I_2=\frac{19}{240}\sigma^2-\sigma\left[\frac{2(s_B-m_\pi^2)}{15}+\frac{E_1E_2}{10}\right]+\frac{E_1E_2}{12}(s_B-2m_\pi^2)-\frac{1}{15}[\mathbf{p}_1^2(W_BE_2-m_\pi^2)+\mathbf{p}_2^2(W_BE_1-m_\pi^2)]+\frac{\mathbf{p}_1^2\mathbf{p}_2^2}{60}-\frac{1}{6}(W_BE_2-m_\pi^2)(W_BE_1-m_\pi^2)$ $I_3=-\frac{7}{48}\sigma+\frac{s_B-2m_\pi^2}{15}+\frac{11}{120}E_1E_2$ $I_4=17/240$

B^+ is about 100 MeV one has to perform an integration similar to Eq. (2.23) in order to compare this result with the experimental data.

In Table III we give the form of the vertex operators corresponding to the different J^P values of the B^+ considered in this work. Finally, in Table IV we show the results $D(s_1,s_2)$ and $I(s_1,s_2)$ obtained for those quantum numbers.

As mentioned at the end of Sec. II, one can in this case, plot the variation of density along an ω band; i.e., $F(s_1,m_\omega^2)$, to compare with the experimental data. The results are shown in Fig. 4.

APPENDIX

In this appendix general methods are given for constructing the decay matrix elements for the cascade process

$$X\to Y+Z\,,$$
$$Y\to m\pi\,,$$
$$Z\to n\pi\,.$$

The particles X, Y, Z, and π have integral, buto therwise arbitrary spin and isotopic spin. Tensor methods will be used in the construction of the required invariants in space-time and in isospin space. The particles X, Y, and Z are assumed to have spins S_x, S_y, and S_z, parities θ_x, θ_y, and θ_z, and isotopic spins I_x, I_y, and I_z, respectively. First the decay of X is discussed and then the decays of Y and Z including symmetry effects.

The isotopic spin I of a particle will be described by an Ith rank tensor in isotopic spin space, symmetric and traceless in all indices. Such a tensor has 3^I components. However, if two components can be transformed into each other by a permutation of the indices, they must be equal by the symmetry requirement. Each index takes on only the values 1, 2, and 3 and so the components may be classified by the number of 1's, 2's, and 3's. Those with the same distribution of 1's, 2's, and 3's can be permuted into each other and are equal. Of the I indices a may be 1, where a is any integer from 0 to I; of the remaining $I-a$, b may be equal to 2, where b is an integer from 0 to $I-a$, and the remaining $I-a-b$ must be equal to 3. Thus there are

$$c=\sum_{a=0}^{I}\sum_{b=0}^{I-a}1$$

$$=(I+1)(I+2)/1\times 2,$$

independent components in a symmetric tensor of rank I. The trace of such a tensor with respect to any pair of indices is the same and is a symmetric tensor of rank $I-2$. The trace has $(I-1)I/1\times 2$ components and must vanish thus imposing $(I-1)I/1\times 2$ conditions on the original tensor. The total number of independent components is then

$$(I+1)(I+2)/1\times 2-(I-1)I/1\times 2=2I+1$$

as was to be expected.

In a similar way the spin S of a particle will be described by an Sth rank symmetric, traceless, transverse tensor. This tensor has 4^S components of which only $(S+1)(S+2)(S+3)/1\times 2\times 3$ are independent by symmetry. The requirement of tracelessness imposes $(s-1)s(s+1)/1\times 2\times 3$ additional conditions. This leaves $(s+1)^2$ independent components. Transversality is the requirement that the scalar product of the momentum vector of the particle with any index of the spin tensor vanish. This condition is obvious in the rest frame. This gives an additional S^2 conditions on the symmetric traceless tensor since the contraction leads to a symmetric traceless tensor of rank $S-1$. The total number of independent components is $(S+1)^2-S^2=2S+1$.

The spin tensors for X, Y and Z are denoted by ξ, η, and ζ and the isospin tensors by x, y, z. The indices will be suppressed as much as possible. The momenta of the particles X, Y, and Z are $p+q$, p, and q. The decay of X into Y and Z is described by constructing a scalar in isospin space from x, y, and z, the Levi-Civita antisymmetric three-index symbol ϵ_{ijk} and the Kronecker δ_{ij}. Since there is only one way to add three angular momenta to get 0, provided they satisfy the triangle inequality, the result is unique. The spatial part of the matrix elements is made by constructing a scalar or pseudoscalar from the tensors ξ, η, and ζ, the vectors p and q, antisymmetric four-index symbol ϵ and the metric tensor δ. There are now more materials to work with and the answer is no longer unique. Only the simplest scalars (fewest powers of p, q) will be exhibited.

The following standard terminology of tensor algebra is used: the tensor product c of two tensors $a_{i_1\cdots i_r}$ and $b_{q_1\cdots q_s}$ is

$$c_{i_1\cdots i_r j_1\cdots j_s}=a_{i_1\cdots i_r}b_{j_1\cdots j_s}.$$

Contraction of a pair of indices means equating them and summing from one to three or one to four with the appropriate minus sign.

The isotopic spins I_x, I_y, and I_z must satisfy the triangle inequality

$$|I_y-I_z|\leq I_x\leq I_y+I_z,$$

so that

$$I_y+I_z-I_x=\begin{cases}2\nu+1 & \text{case (i)}\\ 2\nu & \text{case (ii)}\end{cases},$$

where $\nu\geq 0$. In case (i) make the tensor product of y, z, and ϵ and contract an index of y and one of z with two from ϵ. In case (ii) make the tensor product of y and z. In both cases ν indices of y and ν indices of z are contracted. The remaining indices of y and z and, the ϵ in case (i), are contracted with the indices of x. If $I_y\geq I_z$, z must have enough indices to do all the contractions with since

$$I_z\geq I_y-I_x\geqq I_z+\begin{cases}2\nu+1 & \text{(i)}\\ 2\nu & \text{(ii)}\end{cases},$$

$$I_z\geq\begin{cases}\nu+\frac{1}{2} & \text{(i)}\\ \nu & \text{(ii)}\end{cases}.$$

Several examples are given below.

(a) $I_x=I_y=I_z=1$: $\epsilon_{ijk}x_iy_jz_k$,

(b) $I_x=2$, $I_y=1$, $I_z=1$: $x_{ij}y_iz_j$,

(c) $I_x=1$, $I_y=2$, $I_z=1$: $x_iy_{ij}z_j$,

(d) $I_x=1$, $I_y=I_z=2$: $x_i\epsilon_{ijk}y_{jl}z_{kl}$,

(e) $I_x=5$, $I_y=5$, $I_z=3$: $x_{abcde}y_{ijklm}z_{pqr}\epsilon_{ipa}\delta_{bj}\delta_{ck}\delta_{lq}\delta_{mr}$.

The spatial part is constructed similarly, except that S_x, S_y, and S_z do not necessarily satisfy the triangle inequality. If they fail to satisfy the triangle inequality the deficit may be made up by using the momentum vectors p and q. This is equivalent to derivative couplings in field theory. There are three cases to consider depending on the relative magnitudes of S_x, S_y, and S_z. These are

$$0\leq S_y-S_z\leq S_x\leq S_y+S_z\quad\text{case (i) (Triangle region)}$$

$$S_x<S_y-S_z\quad\text{case (ii)}$$

$$S_x>S_y+S_z\quad\text{case (iii)}.$$

In the triangle region the same procedure as was used with isotopic spin may be used. If an ϵ is required ($S_y+S_z-S_x$ odd) then the fourth index is contracted with $(q+p)$ which is just the X mass in its rest system. This result corresponds to no orbital angular momentum, and is only suitable if the intrinsic parity of X is the product of the intrinsic parities of Y and Z. If this is not the case a p-wave Y, Z matrix element may be constructed by introducing one factor of $q-p$. Since there is one unit of orbital angular momentum S_x, the total angular momentum of Y and Z may be found by combining the spins η and ζ to S_x-1, S_x or S_x+1, and then adding the unit of orbital angular momentum to give the required total S_x. Some of these three possibilities may fail if $S_x=0$ in which case only S_x+1 is possible, or if $S_x-1\leq|S_y-S_z|$, etc. The values S_x-1, S_x, and S_x+1 are constructed by the previous prescription. The resulting tensor in each case is symmetrized. In the case S_x-1 the tensor product with $(p-q)$ gives the required answer and the symmetrization is unnecessary since all indices are eventually contracted with ξ which is symmetric. For S_x the tensor product with ϵ, and $(p+q)$ is formed and one index of the symmetrized tensor, the indices of $p+q$, and $p-q$ are contracted with three indices of ϵ. In the case $S_x+1,(p-q)$ is contracted with an index of the symmetrized tensor. In each of the last two cases it is unnecessary to explicitly carry out the symmetrization. If η has a free indices and ζ has b free indices, any index in the symmetrized tensor appears $a/(a+b)$ times in η and $b/(a+b)$ times in ζ. For example, the symmetrized version of $\eta_{ij}\zeta_k$ is $\frac{1}{3}[\eta_{ij}\zeta_k+\eta_{ik}\zeta_j+\eta_{jk}\zeta_i]$. A contraction of an index of the symmetrized tensor T with the vector V is

$$T\cdot V=\frac{1}{(a+b)}[a(\eta\cdot V)\zeta+b\eta(\zeta\cdot V)],$$

where the dot product means contraction of the index of V with any index of the tensor. To illustrate the problem consider combining $S_y=3$, $S_z=2$ to an S_x of 4 with one unit of orbital angular momentum. The case when η and ζ are combined to give S_x is simple

$$\xi_{\alpha\beta\gamma\delta}\eta_{\lambda\mu\nu}\zeta_{\rho\sigma}(p-q)_\theta\delta^{\nu\sigma}\delta^{\lambda\alpha}\delta^{\beta\mu}\delta^{\gamma\rho}\delta^{\delta\theta}.$$

In the next case η and ζ are first combined to give $S_x=4$

$$\eta_{\lambda\mu\nu}\zeta_{\rho\sigma}\epsilon_{\nu\sigma\tau\varphi}(p+q)_\tau.$$

The free indices are $\lambda\ \mu\rho$ and φ and the tensor should be symmetrized in them. This would give rise to 24 terms. Next the tensor product with ϵ, $(p+q)$ and $(p-q)$ is formed

$$(1/24)(\eta_{\lambda\mu\nu}\zeta_{\rho\sigma}\epsilon_{\nu\sigma\tau\varphi}(p+q)_\tau+\cdots)\epsilon_{\lambda\theta\omega\kappa}(p+q)_\theta(p-q)_\omega.$$

Now the free indices are $\mu\rho\varphi$ and κ. The $+\cdots$ signifies the other 23 permutations. The λ will occur on the ζ and ϵ as well as on the η in these other permutations. Finally, the result is contracted with ξ.

$$\begin{aligned}(1/24)\xi_{\alpha\beta\gamma\delta}(\eta_{\lambda\alpha\nu}\zeta_{\rho\sigma}\epsilon_{\nu\sigma\tau\gamma}(p+q)_\tau+\cdots)&\\ \times\epsilon_{\lambda\theta\omega\delta}(p+q)_\theta(p-q)_\omega&\\ =\xi_{\alpha\beta\gamma\delta}[\tfrac{1}{4}\eta_{\alpha\beta\nu}\zeta_{\nu\sigma}\epsilon_{\nu\sigma\tau\lambda}+\tfrac{1}{4}\eta_{\alpha\beta\nu}\zeta_{\lambda\sigma}\epsilon_{\nu\sigma\tau\gamma}+\tfrac{1}{2}\eta_{\alpha\lambda\nu}\zeta_{\beta\sigma}\epsilon_{\nu\sigma\tau\gamma}]&\\ \times\epsilon_{\lambda\theta\omega\delta}(p+q)_\theta(p-q)_\omega(p+q)_\tau.&\end{aligned}$$

The last line follows since the indices $\alpha\beta\gamma$ are equivalent as they are the indices of the symmetric tensor ξ. The index λ occurs in the different terms in the ratio 2:1:1 as given by the lemma above. Finally, η and ζ are combined to give S_x+1 by simply forming the tensor product

$$\eta_{\lambda\mu\nu}\zeta_{\rho\sigma}.$$

This must be contracted with $(p-q)$ symmetrically thus the final result is

$$\xi_{\alpha\beta\gamma\delta}(3\eta_{\alpha\beta\lambda}\zeta_{\gamma\delta}(p-q)_\lambda+2\eta_{\alpha\beta\gamma}\zeta_{\delta\sigma}(p-q)_\sigma).$$

In case (ii) or (iii) $S_y-S_z-S_x$ or $S_x-S_y-S_z$ units of angular momentum are required to "close the triangle." In case (ii) the tensor product $\eta\zeta(p-q)\cdots(p-q)$ when there are $S_y-S_z-S_x$ factors $(p-q)$ and these factors are contracted with η and the usual process carried on the remaining tensor. Similarly, in case (iii) $S_x-S_y-S_z$ factors $(p-q)$ are added to the tensor product and then the contraction of all indices of η, ζ, and the $(p-q)$'s with the indices of ξ is done. These matrix elements are suitable for definite relative parity of X, Y, Z

$$\theta_x=\theta_y\theta_z(-1)^L,$$

$$L=\begin{cases}S_y-S_z-S_x & \text{case (ii)}\\ S_x-S_y-S_z & \text{case (iii)}\end{cases}.$$

It this relation is not satisfied another power of $(p-q)a(p+q)$ and an ϵ, and these together with an index of η or ζ are contracted against an ϵ. This completes the discussion of $X\to Y+Z$.

The matrix elements for $Y\to m\pi$ is constructed in a similar way as a product of an isotopic and spatial part. However, since the π mesons satisfy Bose statistics only certain combinations of space and isospace are permissible. The isospin of each pion is described by an isovector t_i. The tensor product of m of these is taken. A coefficient $c(a_1\cdots a_s,b_1\cdots b_m)$ is constructed to contract with this tensor product. The b's are the indices of the pion isospin. c is symmetric and traceless in the s a's and they are contracted with the indices of y and Y isospin tensor. The behavior of c under a permutation of the indices b is like a specific representation of the permutation group. For $I_y=0$ and 1 these coefficients have been given[8] for $m=1$, 2, 3, 4. Here they are listed for $I_y=2$, 3, 4 and for $m=2$, 3, 4. The generalization for larger numbers is simple but tedious. These coefficients

[8] Francis R. Halpern, Ann. Phys. (N. Y.) **7**, 146 (1959).

are not normalized.

$$c^2(a_1a_2,b_1b_2)=\delta(a_1b_1)\delta(a_2b_2)+\delta(a_1b_2)\delta(a_2b_1)-\tfrac{2}{3}\delta(a_1a_2)\delta(b_1b_2)\,.$$

$$\begin{aligned}c^3(a_1a_2a_3b_1b_2b_3)=&\ \delta(a_1b_1)\delta(a_2b_2)\delta(a_3b_3)+\delta(a_1b_1)\delta(a_2b_3)\delta(a_3b_2)+\delta(a_1b_2)\delta(a_2b_1)\delta(a_3b_3)+\delta(a_1b_2)\delta(a_2b_3)\delta(a_3b_1)\\&+\delta(a_1b_3)\delta(a_2b_1)\delta(a_3b_2)+\delta(a_1b_3)\delta(a_2b_2)\delta(a_3b_1)-\tfrac{2}{5}[\delta(a_1a_2)\delta(a_3b_3)\delta(b_1b_2)\\&+\delta(a_1a_2)\delta(a_3b_2)\delta(b_1b_3)+\delta(a_1a_2)\delta(a_3b_1)\delta(b_2b_3)+\delta(a_1a_3)\delta(a_2b_1)\delta(b_2b_3)\\&+\delta(a_1a_3)\delta(a_2b_2)\delta(b_1b_3)+\delta(a_1a_3)\delta(a_2b_3)\delta(b_1b_2)+\delta(a_2a_3)\delta(a_1b_1)\delta(b_2b_3)\\&+\delta(a_2a_3)\delta(a_1b_2)\delta(b_1b_3)+\delta(a_2a_3)\delta(a_1b_3)\delta(b_1b_2)]\,.\end{aligned}$$

$$c_1^{21}(a_1a_2,b_1b_2b_3)=[\delta(a_1b_1)\epsilon(a_2b_2b_3)+\delta(a_2b_1)\delta(a_1b_2b_3)-\delta(a_1b_2)\epsilon(a_2b_3b_1)-\delta(a_2b_2)\epsilon(a_1b_3b_1)]\,.$$

$$\begin{aligned}c_2^{21}(a_1a_2,b_1b_2b_3)=&\ [2\delta(a_1b_3)\epsilon(a_2b_1b_2)+2\delta(a_2b_3)\epsilon(a_1b_1b_2)-\delta(a_1b_1)\epsilon(a_2b_2b_3)\\&-\delta(a_2b_1)\epsilon(a_1b_2b_3)-\delta(a_1b_2)\epsilon(a_2b_3b_1)-\delta(a_2b_2)\epsilon(a_1b_3b_1)]\,.\end{aligned}$$

$$\begin{aligned}c^4(a_1a_2a_3a_4,b_1b_2b_3b_4)=&\ \textstyle\sum_{24}\delta(a_1b_1)\delta(a_2b_2)\delta(a_3b_3)\delta(a_4b_4)\\&-(2/7)\textstyle\sum_6\delta(a_1a_2)\sum_{12}\delta(b_1b_2)\delta(a_3b_3)\delta(a_4b_4)+(8/7)[\sum_3\delta(a_1a_2)\delta(a_3a_4)][\sum_3\delta(b_1b_2)\delta(b_3b_4)]\,.\end{aligned}$$

The sums are to be taken over all distinct permutations of the indices, the total number of permutations is indicated by the numerical subscript on the summation sign.

$$\begin{aligned}c_1^{31}(a_1a_2a_3,b_1b_2b_3b_4)=&\ \textstyle\sum_6\{\delta(a_1b_1)\delta(a_2b_3)\epsilon(a_3b_2b_4)+\delta(a_1b_1)\delta(a_2b_4)\epsilon(a_3b_2b_4)-\delta(a_1b_2)\delta(a_2b_4)\epsilon(a_3b_1b_3)\\&-\delta(a_1b_3)\delta(a_2b_4)\epsilon(a_3b_1b_2)]-\tfrac{1}{5}\delta(a_1a_2)[\delta(b_1b_2)\epsilon(a_3b_3b_4)+\delta(b_1b_3)\epsilon(a_3b_2b_4)\\&-\delta(b_2b_4)\epsilon(a_3b_1b_3)-\delta(b_3b_4)\epsilon(a_3b_1b_2)]\}\,.\end{aligned}$$

$$\begin{aligned}c_2^{31}(a_1a_2a_3,b_1b_2b_3b_4)=&\ \textstyle\sum_6\{\delta(a_1b_1)\delta(a_2b_3)\epsilon(a_3b_2b_4)+\delta(a_1b_1)\delta(a_2b_4)\epsilon(a_3b_2b_3)-\delta(a_1b_3)\delta(a_2b_4)\epsilon(a_3b_1b_2)\\&-\tfrac{1}{5}\delta(a_1a_2)[\delta(b_1b_3)\epsilon(a_3b_2b_4)+2\delta(a_3b_3)\epsilon(b_1b_2b_4)+\delta(b_1b_4)\epsilon(a_3b_2b_3)-\delta(b_3b_4)\epsilon(a_3b_1b_2)]\}\,.\end{aligned}$$

$$\begin{aligned}c_3^{31}(a_1a_2a_3,b_1b_2b_3b_4)=&\ \textstyle\sum_6\{\delta(a_1b_2)\delta(a_2b_3)\epsilon(a_3b_1b_4)+\delta(a_1b_2)\delta(a_2b_4)\epsilon(a_3b_1b_3)+\delta(a_1b_3)\delta(a_2b_4)\epsilon(a_3b_1b_2)\\&-\tfrac{1}{5}\delta(a_1a_2)[\delta(b_2b_3)\epsilon(a_3b_1b_4)+\delta(a_3b_3)\epsilon(b_2b_1b_4)+\delta(b_2b_4)\epsilon(a_3b_1b_3)\\&+\delta(b_3b_4)\epsilon(a_1b_1b_2)+\delta(a_3b_1)\epsilon(b_4b_3b_2)]\}\,.\end{aligned}$$

$$c^4(a_1a_2,b_1b_2b_3b_4)=\textstyle\sum_{12}\{\delta(a_1b_1)\delta(a_2b_2)\delta(b_3b_4)-\tfrac{1}{3}\delta(a_1a_2)\delta(b_1b_2)\delta(b_3b_4)\}\,.$$

$$\begin{aligned}c_1^{22}(a_1a_2,b_1b_2b_3b_4)=&\ \textstyle\sum_2\{\delta(a_1b_1)\delta(a_2b_2)\delta(b_3b_4)-\delta(a_1b_1)\delta(a_2b_4)\delta(b_2b_3)-\delta(a_1b_2)\delta(a_2b_3)\delta(bb)\\&+\delta(a_1b_2)\delta(a_2b_4)\delta(b_1b_3)-\tfrac{1}{2}\delta(a_1a_2)[\delta(b_1b_2)\delta(b_3b_4)-2\delta(b_1b_4)\delta(b_2b_3)+\delta(b_2b_4)\delta(b_1b_3)]\}\,.\end{aligned}$$

$$\begin{aligned}c_2^{22}(a_1a_2,b_1b_2b_3b_4)=&\ \textstyle\sum_2\{\delta(a_1b_1)\delta(a_2b_3)\delta(b_2b_4)-\delta(a_1b_1)\delta(a_2b_4)\delta(b_2b_3)-\delta(a_1b_2)\delta(a_2b_3)\delta(b_1b_4)\\&+\delta(a_1b_3)\delta(a_2b_4)\delta(b_1b_2)-\tfrac{1}{3}\delta(a_1a_2)[\delta(b_1b_3)\delta(b_2b_4)-2\delta(b_1b_4)\delta(b_2b_3)+\delta(b_1b_2)\delta(b_3b_4)]\}\,.\end{aligned}$$

$$\begin{aligned}c_1^{31}(a_1a_2,b_1b_2b_3b_4)=&\ [\delta(a_1b_1)\delta(a_2b_2)+\delta(a_1b_2)\delta(a_2b_1)]\delta(b_3b_4)+[\delta(a_1b_2)\delta(a_2b_3)\\&+\delta(a_1b_3)\delta(a_2b_2)]\delta(b_1b_4)-\tfrac{2}{3}\delta(a_1a_2)[\delta(b_1b_2)\delta(b_3b_4)+\delta(b_1b_4)\delta(b_2b_3)]\,.\end{aligned}$$

$$\begin{aligned}c_2^{31}(a_1a_2,b_1b_2b_3b_4)=&\ [\delta(a_1b_1)\delta(a_2b_3)+\delta(a_1b_3)\delta(a_2b_1)]\delta(b_2b_4)+[\delta(a_1b_2)\delta(a_2b_4)\\&+\delta(a_1b_4)\delta(a_2b_2)]\delta(b_1b_3)-\tfrac{4}{3}\delta(a_1a_2)\delta(b_1b_3)\delta(b_2b_4)\,.\end{aligned}$$

$$\begin{aligned}c_3^{31}(a_1a_2,b_1b_2b_3b_4)=&\ [\delta(a_1b_1)\delta(a_2b_4)+\delta(a_1b_4)\delta(a_2b_1)]\delta(b_2b_3)+[\delta(a_1b_3)\delta(a_2b_4)\\&+\delta(a_1b_4)\delta(a_2b_3)]\delta(b_1b_2)-\tfrac{2}{3}\delta(a_1a_2)[\delta(b_1b_4)\delta(b_2b_3)+\delta(b_1b_2)\delta(b_3b_4)]\,.\end{aligned}$$

In a similar way it is necessary to make a spin tensor from the momenta of the $m\pi$'s, $k_1,k_2\cdots k_m$. The following combinations effect the transition to the X center of mass.

$$K_i=A_{ij}k_j\,,$$

$$A_{ij}=\begin{cases}0 & i>j\quad i\neq m\\ [(n-i)/(n-I+1)]^{1/2} & i=j\quad i\neq m\\ -[(n-i)(n-i+1)]^{-1/2} & i<j\quad i\neq m\\ n^{-1/2} & i=m\,.\end{cases}$$

In these formulas k_j and K_i are four-vectors. The indices i and j identify the vectors and do not refer to components. The over-all center of mass is given by $K_m=0$. Appropriate spin tensors are made from K_1, $K_2\cdots K_{m-1}$ is the rest system and then transformed into a general coordinate system by the rule

$$\epsilon_{ijk}\rightarrow\epsilon_{kl\mu\nu}(K_m)_\nu\,.$$

The types of combinations to be formed are similar to those listed for isotopic spin although the symmetry properties are more difficult to treat, since the $K_ii=1\cdots m-1$, have fairly intricate transformation properties under permutation. Each K_i may be used an arbitrary number of times. The exact combination to choose is to be fixed by the symmetry requirement that can be treated adequately for any realistic case but is rather difficult to prescribe in general. Some examples are given by Henley and Jacobsohn.[9]

[9] E. M. Henley and B. A. Jacobsohn, Phys. Rev. 128, 1394 (1962).