Application of N/D and Determinantal Methods to Yukawa Potential Scattering*

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The S-, P-, and D-wave amplitudes for the Yukawa potential are found by the N/D method. The solution with the first and second Born cuts gives a reasonable approximation to the exact amplitude in the low-energy region for potential strengths up to values strong enough to give an S-wave bound state. The solution yields a fair prediction of the S-wave binding energy. The first- and second-order determinantal solutions are also obtained. Because of the distortion of the left-hand cut, the determinantal method gives less reliable results than the N/D method.

I. INTRODUCTION

SINCE it was first introduced by Chew and Mandelstam,[1] the N/D method has been used extensively to calculate relativistic partial-wave amplitudes for processes involving strong interactions. In applying the method, it is necessary to start with some known information about the discontinuities of the amplitude across the left-hand cuts, i.e., across the branch cuts located below the threshold. In principle, if the discontinuities across all the left-hand cuts are known, one could find the exact amplitude by the N/D method. However, in practice, one is only able to specify the branch cuts close to the threshold which arise from the exchange of one or two particles, while the discontinuities of the other cuts arising from many-particle exchanges are difficult to obtain. As an illustration, consider nucleon-nucleon scattering. The exchange of a single pion of mass μ gives rise to a branch point in the partial-wave amplitude at $k^2=-\mu^2/4$, where k is the momentum in the center-of-mass system. The exchange of two, three, etc., pions gives rise to branch points at $k^2=-\mu^2$, $-9\mu^2/4$, etc., which lie progressively farther away from the threshold. The assumption is generally made that the behavior of the amplitude in the physical region $k^2>0$ is influenced largely by the characteristics of the "nearby" cuts, while the effects of the more "distant" cuts involving exchange of many particles may be neglected. This assumption is based on the intuitive reasoning that an analytic function is determined through the Cauchy integral formula by a sort of Coulomb effect, with the branch cuts being analogous to line charges; so nearby cuts have more influence than the distant cuts.[2]

One of the purposes of this paper is to examine this assumption by applying the N/D method to a case of potential scattering whose amplitude has a similar analytic structure, and see to what extent is the approximation valid. In potential scattering, the partial-wave amplitude can be found exactly by solving the Schrödinger equation; thus, the N/D solution with the approximation of neglecting "faraway" cuts may be compared with the exact solution. It has been shown by several authors[3-5] that the scattering amplitude for the Yukawa potential satisfies a simple Mandelstam representation similar to the representation for relativistic amplitudes; that is, the scattering amplitude $f(k^2,t)$ is an analytic function of k^2 and t except for singularities along the real axes of these variables. $t=-2k^2\times(1-\cos\theta)$ is the negative momentum transfer squared. For real k^2, $f(k^2,t)$ has a pole at $t=\mu^2$ coming from the first Born term, μ being the inverse Yukawa potential range. In addition, it has branch points at $t=(n\mu)^2$, $n=2, 3,\cdots$, and each of these comes from the nth Born term. These singularities give rise to branch points in the partial wave amplitude at $k^2=-\mu^2/4$, $-\mu^2$, $-9\mu^2/4$, etc., where the branch point at $k^2=-\frac{1}{4}(n\mu)^2$ comes from the nth Born term. We note that the analytic structure of the partial-wave amplitude is similar to the relativistic case; there is a sequence of branch cuts in the region $k^2<0$ and a branch cut due to unitarity in the region $k^2>0$. In analogy to the relativistic problem, we keep only the nearby left-hand cuts and use the N/D analysis to find an approximation to the amplitude. First, we take only the nearest branch cut from the first Born approximation, which is analogous to considering only single-particle exchange in the relativistic case. Next, we consider both the first and second Born cuts, which is analogous to considering single- and two-particle exchanges.

We note that a similar work using potential scattering to check the N/D approximation was done by Bjorken and Goldberg[6]; however, they studied only the case of S-wave scattering by an exponential potential. In that case, the amplitude has a sequence of poles along the negative k^2 axis instead of a sequence of branch points as we have in the Yukawa potential and relativistic problems.

Another purpose of this paper is to investigate the determinantal method by applying it to the Yukawa potential. This method of approximation has frequently

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[1] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).
[2] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin, Inc., New York, 1961), p. 6.
[3] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).
[4] A. Klein, J. Math. Phys. **1**, 41 (1960); J. Math. Phys. **1**, 274 (1960).
[5] J. Bowcock and A. Martin, Nuovo Cimento **14**, 516 (1959).
[6] J. D. Bjorken and A. Goldberg, Nuovo Cimento **16**, 539 (1960).

been used in relativistic calculations in place of solving the integral equation required by the exact N/D method. First, we consider only the first Born cut, in which case we are able to find an explicit determinantal solution for arbitrary angular momentum. Next, we construct the higher order determinantal solution by using information from both the first and second Born cuts. The results are compared to the N/D and exact solutions.

A well-known difficulty with the N/D method when applied to partial waves with $l>0$ is the problem of imposing the threshold condition. The basic reason is that the vanishing of the amplitude at the threshold for $l>0$ is a result of cancelations coming from effects due to *all* the left-hand cuts. If we put into the N/D formalism all the left-hand cuts, we automatically get a solution satisfying the threshold behavior. But as long as we are forced to neglect faraway cuts, we must impose the threshold condition by some artifice. A common method is to divide the amplitude by the threshold factor k^{2l} and do N/D on the new amplitude. The result is that the dispersion integrals for the new amplitude now diverges and a cutoff is required. This amounts to approximating the right-hand cut by a cut of finite length. Here, we apply the same type of cutoff procedure to treat the higher partial-wave amplitudes for the Yukawa potential and compare the results with the exact solution. The use of cutoffs in dispersion integrals is a common practice in relativistic N/D calculations. It is required not only because of the threshold problem but also because of the spin of exchanged particles. It necessarily introduces an artificial branch point at the cutoff energy. The assumption is tacitly made that if the cutoff is sufficiently far away, it would not influence the low-energy behavior of the amplitude. In this paper, we study this assumption by comparing the cutoff solution with the exact solution.

Another object of this work is to find a suitable N/D approximation that will work well for different l values, primarily the first few low values of l, which can be extended to noninteger values of l, and can be used to trace Regge trajectories.

In Sec. II, we review the known analytic properties of the partial-wave amplitude for a Yukawa potential and derive the N/D equations. The case of including only the first Born cut is discussed. In Sec. III, we study the addition of the second Born cut to the N/D method. In Sec. IV, we derive an exact solution to the first-order determinantal method for the Yukawa potential for arbitrary l. We also work out the second-order determinantal solution which uses information from the first and second Born terms. In Sec. V, we give the results of our analysis and the conclusions of this work.

II. N/D EQUATIONS

We consider the attractive Yukawa potential

$$V(r)=-g^2e^{-\mu r}/r\,, \tag{2.1}$$

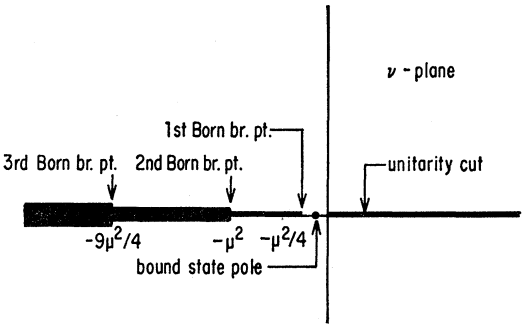


FIG. 1. Nearby singularities of the Yukawa partial-wave amplitude.

where we have chosen units such that $\hbar=2M=1$, M being the reduced mass of the scattering system. The radial Schrödinger equation with this potential has the form

$$\frac{d^2u(r)}{dr}+[k^2-l(l+1)/r^2+g^2e^{-\mu r}/r]u(r)=0\,, \tag{2.2}$$

where $u(r)/r$ is the radial wave function and $u(r)$ satisfies the boundary condition

$$u(r)\sim r^{l+1};\quad r\to 0\,, \tag{2.3}$$

k is the momentum, and we denote k^2 by the symbol ν. The partial-wave amplitude is defined as

$$f_l(\nu)=\frac{e^{i\delta_l(\nu)}\sin\delta_l(\nu)}{\nu^{1/2}}=\frac{1}{(\nu^{1/2})\cot\delta_l(\nu)-i\nu^{1/2}} \tag{2.4}$$

which is related to the scattering amplitude by

$$f(\nu,\cos\theta)=\sum_l(2l+1)f_l(\nu)P_l(\cos\theta)\,. \tag{2.5}$$

The exact values for the phase shifts $\delta_l(\nu)$ can be obtained in the usual manner by solving Eq. (2.2) numerically, subject to the boundary condition (2.3).

It has been proved by several authors[3–5] that the analytic continuation of the function $f_l(\nu)$ into the complex ν plane has the following properties:

(1) It is an analytic function of ν in the cut plane shown on Fig. 1.

(2) $f_l(\nu)$ is real when ν is real and $-\mu^2/4<\nu<0$. Or equivalently, $f_l(\nu)$ satisfies Schwartz's reflection principle across the real axis

$$f_l(\nu)^*=f_l(\nu^*) \tag{2.6}$$

so the discontinuity across the cuts is given by

$$f_l(\nu+i\epsilon)-f_l(\nu-i\epsilon)=2i\,\mathrm{Im}f_l(\nu+i\epsilon)\,. \tag{2.7}$$

(3) $f_l(\nu)$ has branch points at $\nu=-\frac{1}{4}\mu^2, -\mu^2, -9\mu^2/4, \cdots, -\frac{1}{4}(n\mu)^2, \cdots$. If we denote the Born series for the partial-wave amplitude by

$$f_l(\nu)=b_l^{(1)}(\nu)+b_l^{(2)}(\nu)+\cdots+b_l^{(n)}(\nu)+\cdots\,, \tag{2.8}$$

where $b_l^{(n)}(\nu)$ is the nth Born term of $O(g^{2n})$, then the discontinuities across the left-hand branch cuts are given by

$$\begin{aligned}\mathrm{Im}f_l(\nu+i\epsilon)&=\mathrm{Im}b_l^{(1)}(\nu+i\epsilon), \quad -\mu^2<\nu<-\tfrac{1}{4}\mu^2\\ &=\mathrm{Im}b_l^{(1)}(\nu+i\epsilon)+\mathrm{Im}b_l^{(2)}(\nu+i\epsilon)\\ &\qquad -9\mu^2/4<\nu<-\mu^2\\ &=\mathrm{Im}b_l^{(1)}(\nu+i\epsilon)+\mathrm{Im}b_l^{(2)}(\nu+i\epsilon)\\ &\quad +\mathrm{Im}b_l^{(3)}(\nu+i\epsilon), \; -4\mu^2<\nu<-9\mu^2/4 \quad (2.9)\end{aligned}$$

and so forth. (Refer to Fig. 1.)

(4) For values of g^2, μ, and l such that a bound state exists, $f_l(\nu)$ has a pole on the first Riemann sheet (physical sheet).

The above analytic properties form the basis of our analysis. In addition, we also know that $f_l(\nu)$ must satisfy the unitarity condition. Unitarity requires simply that the phase shift be real in the physical region, which we define as $\nu+i\epsilon$, $\nu>0$. For the phase shift to be real, $f_l(\nu)$ must have an imaginary part which is given by

$$\mathrm{Im}f_l(\nu+i\epsilon)=\nu^{1/2}|f_l(\nu+i\epsilon)|^2, \quad \nu\geqslant 0. \tag{2.10}$$

It follows from (2.7) that there is a branch cut in the region $\nu>0$, the so-called unitarity cut. The purpose of the N/D analysis is to evaluate this cut, given information about the left-hand cuts.

Let

$$f_l(\nu)\equiv N_l(\nu)/D_l(\nu), \tag{2.11}$$

where $N_l(\nu)$ contains only left-hand cuts and is analytic everywhere else; $D_l(\nu)$ contains only the unitarity cut and is analytic everywhere else. It can be proved that $f_l(\nu)$ can always be written in this way.[7] From the identity

$$\mathrm{Im}f_l(\nu)^{-1}=-\mathrm{Im}f_l(\nu)/|f_l(\nu)|^2, \tag{2.12}$$

and (2.10), we have

$$\mathrm{Im}f_l(\nu+i\epsilon)^{-1}=-\nu^{1/2} \quad \nu\geqslant 0. \tag{2.13}$$

Therefore,

$$\mathrm{Im}D_l(\nu+i\epsilon)=-(\nu)^{1/2}N_l(\nu), \quad \nu\geqslant 0, \tag{2.14}$$

or

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^\infty d\nu'\frac{\nu'^{1/2}N_l(\nu')}{\nu'-\nu}, \tag{2.15}$$

where we have normalized the D function such that $D_l(\infty)=1$. The D function is real for negative real values of ν. If it vanishes, the amplitude has a pole at that energy, which corresponds to a bound state.

Let us define the following function which is analytic everywhere except for cuts along the negative real axis, and whose discontinuities across these cuts are equal to that of the amplitude.

$$B_l(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4} d\nu'\frac{\mathrm{Im}f_l(\nu')}{\nu'-\nu}. \tag{2.16}$$

[7] Reference 2, p. 48.

We will call this the "potential function," since it is the input to the N/D equation which is analogous to the fact that the ordinary potential is the input to the Schrödinger equation. The function $B_l(\nu)D_l(\nu)$ contains the correct discontinuities required by the N function along the left-hand cuts, but it also contains a right-hand cut coming from $D_l(\nu)$ which we must remove; therefore,

$$N_l(\nu)=B_l(\nu)D_l(\nu)-\frac{1}{\pi}\int_0^\infty d\nu'\frac{B_l(\nu')\,\mathrm{Im}D_l(\nu')}{\nu'-\nu}. \tag{2.17}$$

In this equation, we have set $N_l(\infty)=0$, which is required by the fact that $f_l(\infty)=0$ as seen from Eq. (2.4). Substituting (2.14) and (2.15) into the above equation, we have the following integral equation for the N function:

$$N_l(\nu)=B_l(\nu)+\frac{1}{\pi}\int_0^\infty d\nu'\frac{B_l(\nu')-B_l(\nu)}{\nu'-\nu}(\nu')^{1/2}N_l(\nu'). \tag{2.18}$$

If we were given the exact $B_l(\nu)$ describing completely all the left-hand cuts of the amplitude, then the N function we obtain from solving this integral equation, when substituted into (2.15) and (2.11), would give us the exact amplitude.

As in the relativistic case, it is not possible to specify all the left-hand cuts. As a first approximation, we consider the whole left-hand cut to be the same as the first Born cut; that is, instead of (2.9), we assume

$$\mathrm{Im}f_l(\nu+i\epsilon)=\mathrm{Im}b_l^{(1)}(\nu) \quad \nu<-\tfrac{1}{4}\mu^2. \tag{2.19}$$

This means that between $-\frac{1}{4}\mu^2$ and $-\mu^2$, the discontinuity of $f_l(\nu)$ is exact, while for $\nu<-\mu^2$, the discontinuity is only approximate. The first Born approximation for the scattering amplitude is

$$f^{(1)}(\nu,\cos\theta)=\frac{g^2}{\mu^2+2\nu(1-\cos\theta)}. \tag{2.20}$$

Its partial-wave projection is

$$\begin{aligned}b_l^{(1)}(\nu)&=\tfrac{1}{2}\int_{-1}^{1} d(\cos\theta)f^{(1)}(\nu,\cos\theta)P_l(\cos\theta)\\ &=\frac{g^2}{2\nu}Q_l\left(1+\frac{\mu^2}{2\nu}\right),\end{aligned} \tag{2.21}$$

where $Q_l(z)$ is the Legendre function of the second kind. For integer l, $Q_l(z)$ is analytic everywhere in z except for a cut between $z=\pm1$; hence $b_l^{(1)}(\nu)$ is an analytic function of ν with a cut from $\nu=-\frac{1}{4}\mu^2$ to $-\infty$. In accordance with assumption (2.19), we let

$$B_l(\nu)=b_l^{(1)}(\nu). \tag{2.22}$$

Equation (2.18) becomes

$$N_l(\nu)=b_l^{(1)}(\nu)+\frac{1}{\pi}\int_0^\infty d\nu'\frac{b_l^{(1)}(\nu')-b_l^{(1)}(\nu)}{\nu'-\nu}(\nu')^{1/2}N_l(\nu'). \quad (2.23)$$

For a short-range potential, due to the centrifugal potential effects, the amplitude must satisfy the threshold behavior

$$f_l(\nu)\propto\nu^l, \quad \nu\to 0. \quad (2.24)$$

Since $D_l(\nu)\to$ constant as $\nu\to 0$, this requires that the N function satisfies the threshold behavior

$$N_l(\nu)\propto\nu^l, \quad \nu\to 0. \quad (2.25)$$

From the relation[8]

$$Q_l\left(1+\frac{\mu^2}{2\nu}\right)=\frac{(\pi)^{1/2}\Gamma(l+1)}{\Gamma(l+\frac{3}{2})}\left(\frac{\nu}{2\nu+\mu^2}\right)^{l+1}\times F\left[l/2+1,\,l/2+\tfrac{1}{2};\,l+\tfrac{3}{2};\left(\frac{2\nu}{2\nu+\mu^2}\right)^2\right], \quad (2.26)$$

we have

$$b_l^{(1)}(\nu)\to\frac{g^2\pi^{1/2}\Gamma(l+1)}{2\Gamma(l+\frac{3}{2})}\nu^l, \quad \text{as} \quad \nu\to 0. \quad (2.27)$$

Referring to (2.23), we see that the inhomogeneous term and the second part of the integral expression vanishes like $O(\nu^l)$ at $\nu=0$, but the first part of the integral expression does not; namely,

$$\frac{1}{\pi}\int_0^\infty d\nu'\nu'^{-1/2}b_l^{(1)}(\nu')N_l(\nu')=\text{constant}\neq 0. \quad (2.28)$$

This is because $b_l^{(1)}(\nu)$ is positive in the region of integration and $N_l(\nu)$ is a monotonic function. Therefore, $N_l(\nu)$ and hence $f_l(\nu)$ does not satisfy the threshold behavior for $l\geqslant 1$.

To remedy this, we define a new amplitude

$$\bar{f}_l(\nu)=f_l(\nu)/\nu^l=\bar{N}_l(\nu)/\bar{D}_l(\nu). \quad (2.29)$$

The unitarity condition (2.13) now becomes

$$\text{Im}[\bar{f}_l(\nu)]^{-1}=-\nu^{l+1/2} \quad \nu\geqslant 0. \quad (2.30)$$

Or

$$\text{Im}\bar{D}_l(\nu)=-\nu^{l+1/2}\bar{N}_l(\nu) \quad \nu\geqslant 0, \quad (2.31)$$

$$\bar{D}_l(\nu)=1-\frac{1}{\pi}\int_0^{\nu_c}d\nu'\frac{\nu'^{l+1/2}\bar{N}_l(\nu')}{\nu'-\nu} \quad \nu\geqslant 0, \quad (2.32)$$

where we have introduced a cutoff at ν_c to keep the integral finite. This is required by the fact that $N_l(\nu)$ does not vanish faster than $O(\nu^{-l-1/2})$ as $\nu\to\infty$. Usually $N_l(\nu)$ contains terms of order $1/\nu$ multiplied by factors of $(\ln\nu)$, as one may see from Eq. (2.34). We define the new potential function containing the left-hand cuts of $\bar{f}_l(\nu)$ by

$$\bar{B}_l(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4}d\nu'\frac{\text{Im}\bar{f}_l(\nu')}{\nu'-\nu}. \quad (2.33)$$

Following a similar derivation as before, one finds the integral equation for $\bar{N}_l(\nu)$

$$\bar{N}_l(\nu)=\bar{B}_l(\nu)+\frac{1}{\pi}\int_0^{\nu_c}d\nu'\frac{\bar{B}_l(\nu')-\bar{B}_l(\nu)}{\nu'-\nu}\nu'^{l+1/2}\bar{N}_l(\nu'). \quad (2.34)$$

If we considered only the first Born cut, we have

$$\bar{B}_l(\nu)=b_l^{(1)}(\nu)/\nu^l=(g^2/2\nu^{l+1})Q_l(1+\mu^2/2\nu). \quad (2.35)$$

The threshold condition now requires that as $\nu\to 0$, $\bar{f}_l(\nu)\to$ constant, and $\bar{N}_l(\nu)\to$ constant. The solution of (2.34) satisfies this condition. Equation (2.34) with the input (2.35) may be solved numerically by the method of matrix inversion. Knowing the N function in the physical region, the D function is calculated by (2.32). For energies above threshold, the principal value of the integral is related to the phase shift. To find the binding energy, one searches for the zero of the D function for negative values of ν.

III. ADDITION OF SECOND BORN CUT

An improvement of the N/D approximation can be made by including the next branch cut, which starts at $\nu=-\mu^2$ and runs to $\nu=-\infty$. This cut comes from the second Born term. By including both the first and second Born cuts, we have

$$\begin{aligned}\text{Im}f_l(\nu+i\epsilon)&=\text{Im}b_l^{(1)}(\nu+i\epsilon) \quad -\mu^2<\nu<-\tfrac{1}{4}\mu^2\\&=\text{Im}b_l^{(1)}(\nu+i\epsilon)+\text{Im}b_l^{(2)}(\nu+i\epsilon) \quad \nu<-\mu^2. \quad (3.1)\end{aligned}$$

Therefore, the discontinuity of $f_l(\nu)$ between $-9\mu^2/4$ and $-\frac{1}{4}\mu^2$ is exact, while the discontinuity for $\nu<-9\mu^2/4$ is approximate.

The second-order term in the Born series for the scattering amplitude is[9]

$$f^{(2)}(\nu,\cos\theta)=\frac{g^4}{2\nu^{1/2}\sin\frac{1}{2}\theta A^{1/2}}\times\left\{\tan^{-1}\frac{\mu\nu^{1/2}\sin\frac{1}{2}\theta}{A^{1/2}}+\frac{i}{2}\ln\frac{A^{1/2}+2\nu\sin\frac{1}{2}\theta}{A^{1/2}-2\nu\sin\frac{1}{2}\theta}\right\}, \quad (3.2)$$

[8] Bateman Manuscript Project, *Higher Transcendental Functions* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 122.

[9] P. M. Morse and H. Feshbach, *Methods of Theoretical Physics* (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 2, p. 1082.

where

$$A=\mu^4+4\nu(\mu^2+\nu\sin^2\tfrac{1}{2}\theta).\qquad(3.3)$$

The partial-wave projection is

$$b_l^{(2)}(\nu)=\frac{g^4}{4\nu^{1/2}}\int_{-1}^{1}d(\cos\theta)\frac{P_l(\cos\theta)}{\sin\frac{1}{2}\theta(A)^{1/2}}\times\left\{\tan^{-1}\frac{\mu\nu^{1/2}\sin\frac{1}{2}\theta}{A^{1/2}}+\frac{i}{2}\ln\frac{A^{1/2}+2\nu\sin\frac{1}{2}\theta}{A^{1/2}-2\nu\sin\frac{1}{2}\theta}\right\}.\qquad(3.4)$$

For $l=0$, the imaginary part of (3.4) can be integrated by parts, and we find explicitly

$$\mathrm{Im}b_0^{(2)}(\nu)=(g^4/4\nu^{3/2})[Q_0(1+\mu^2/2\nu)]^2,\quad\nu\geqslant0.\qquad(3.5)$$

From the unitarity relation (2.10)

$$\mathrm{Im}\{b_l^{(1)}(\nu)+b_l^{(2)}(\nu)+\cdots\}=\nu^{1/2}|b_l^{(1)}(\nu)+b_l^{(2)}(\nu)+\cdots|^2\quad\nu\geqslant0.\qquad(3.6)$$

Upon expanding the right-hand side and equating terms of the same order, we have

$$\mathrm{Im}b_l^{(1)}(\nu)=0,\qquad\nu\geqslant0,\qquad(3.7\text{a})$$

$$\mathrm{Im}b_l^{(2)}(\nu)=\nu^{1/2}[b_l^{(1)}(\nu)]^2,\quad\nu\geqslant0,\qquad(3.7\text{b})$$

and so forth. Therefore, relation (3.5) should hold for $l>0$ also.

$$\mathrm{Im}b_l^{(2)}(\nu)=\frac{g^4}{4\nu^{3/2}}\left[Q_l\left(1+\frac{\mu^2}{2\nu}\right)\right]^2\quad\nu\geqslant0.\qquad(3.8)$$

The real part of (3.4) cannot be integrated explicitly. We will denote it by $T_l(\nu)$. Equation (3.4) then becomes

$$b_l^{(2)}(\nu)=T_l(\nu)+i\frac{g^4}{4\nu^{3/2}}\left[Q_l\left(1+\frac{\mu^2}{2\nu}\right)\right]^2\quad\nu\geqslant0.\qquad(3.9)$$

In contrast to the first Born term, the second Born term contains a unitarity cut; therefore, to find its contribution to the left-hand integral $B_l(\nu)$, we must subtract off the unitarity cut.

To guarantee that our solution for $f_l(\nu)$ satisfies the proper threshold behavior, we have to do the N/D analysis on $\bar{f}_l(\nu)$. The Born series for $\bar{f}_l(\nu)$ is given by the expression

$$\bar{f}_l(\nu)=b_l^{(1)}(\nu)/\nu^l+b_l^{(2)}(\nu)/\nu^l+O(g^6).\qquad(3.10)$$

On the basis of its analytic property, we express the second term as a Cauchy integral over the left- and right-hand cut as follows:

$$\frac{b_l^{(2)}(\nu)}{\nu^l}=\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4}\frac{\mathrm{Im}b_l^{(2)}(\nu')}{\nu'^l(\nu'-\nu)}+\frac{1}{\pi}\int_0^\infty d\nu'\frac{g^4[Q_l(1+\mu^2/2\nu)]^2}{4\nu'^{3/2}\nu'^l(\nu'-\nu)}.\qquad(3.11)$$

In the approximation indicated by (3.1), we take the potential function for $\bar{f}_l(\nu)$ to be

$$\bar{B}_l(\nu)=\frac{b_l^{(1)}(\nu)}{\nu^l}+\frac{b_l^{(2)}(\nu)}{\nu^l}-\frac{1}{\pi}\int_0^\infty d\nu'\frac{g^4[Q_l(1+\mu^2/2\nu)]^2}{4\nu'^{3/2}\nu'^l(\nu'-\nu)},\qquad(3.12)$$

where we have subtracted off the unitarity cut in the second Born term. Substituting (3.9) into the above equation, one obtains

$$\bar{B}_l(\nu)=\frac{b_l^{(1)}(\nu)}{\nu^l}+\frac{T_l(\nu)}{\nu^l}-\mathcal{P}\frac{1}{\pi}\int_0^\infty d\nu'\frac{g^4[Q_l(1+\mu^2/2\nu')]^2}{4\nu'^{3/2}\nu'^l(\nu'-\nu)}.\qquad(3.13)$$

In working with $\bar{f}_l(\nu)$, we are forced to introduce a cutoff in the unitarity cut for partial waves with $l>0$. On the other hand, our second Born approximation already contains some information about the unitarity cut. In the region of high energy, where the phase shift is small, the discontinuity given by the second Born term may be taken as an approximation to the correct discontinuity. This leads one to the modification of the input function $\bar{B}_l(\nu)$ by allowing it to contain, in addition to left-hand cuts, a portion of the unitarity cut extending from the cutoff to infinity. This would then "patch up" our solution so that it now has a unitarity cut extending from zero to infinity. We denote the modified B function by $\tilde{B}_l(\nu)$. The discontinuity of this function along the real ν axis has the following values:

$$\begin{aligned}\mathrm{Im}\tilde{B}_l(\nu+i\epsilon)&=\mathrm{Im}b_l^{(1)}(\nu+i\epsilon)/\nu^l+\mathrm{Im}b_l^{(2)}(\nu+i\epsilon)/\nu^l\quad\nu<-\mu^2/4\\&=\mathrm{Im}b_l^{(2)}(\nu+i\epsilon)/\nu^l\qquad\nu>\nu_c.\end{aligned}\qquad(3.14)$$

Therefore, the Cauchy integral representation of $\tilde{B}_l(\nu)$ is

$$\tilde{B}_l(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4}d\nu'\frac{\mathrm{Im}[b_l^{(1)}(\nu')+b_l^{(2)}(\nu')]}{\nu'^l(\nu'-\nu)}+\frac{1}{\pi}\int_{\nu_c}^\infty d\nu'\frac{\nu'^{1/2}/b_l^{(1)}(\nu')/^2}{\nu'^l(\nu'-\nu)}.\qquad(3.15)$$

This is equivalent to subtracting off from $b_l^{(2)}(\nu)/\nu^l$ only a portion of the second Born unitarity cut from zero to ν_c, rather than the entire cut from zero to infinity as indicated in (3.12); that is,

$$\tilde{B}_l(\nu)=\frac{b_l^{(1)}(\nu)}{\nu^l}+\frac{b_l^{(2)}(\nu)}{\nu^l}-\frac{1}{\pi}\int_0^{\nu_c}d\nu'\frac{g^4[Q_l(1+\mu^2/2\nu')]^2}{4\nu'^{3/2}\nu'^l(\nu'-\nu)}.\qquad(3.16)$$

We define the modified D function with a cutoff at ν_c as

$$\tilde{D}_l(\nu)=1-\frac{1}{\pi}\int_0^{\nu_c} d\nu' \frac{(\nu')^{1/2}\nu'^l \tilde{N}_l(\nu')}{\nu'-\nu}, \quad (3.17)$$

which carries the unitarity cut from zero to ν_c. The function $\tilde{N}_l(\nu)$ is defined as the function that carries the two left-hand cuts and the right-hand cut (ν_c to ∞) as contained in $\tilde{B}_l(\nu)$, or

$$\tilde{N}_l(\nu)=\frac{1}{\pi}\int_{-\infty}^{-\mu^2/4} d\nu' \frac{\mathrm{Im}\tilde{B}_l(\nu')\tilde{D}_l(\nu')}{\nu'-\nu} +\frac{1}{\pi}\int_{\nu_c}^{\infty} d\nu' \frac{\mathrm{Im}\tilde{B}_l(\nu')\tilde{D}_l(\nu')}{\nu'-\nu}. \quad (3.18)$$

To construct an integral equation for determining $\tilde{N}_l(\nu)$, we start by considering the function $\tilde{B}_l(\nu)\tilde{D}_l(\nu)$. This function contains the two left hand cuts corresponding to the branch points at $-\frac{1}{4}\mu^2$ and $-\mu^2$, and two right-hand cuts: one from zero to ν_c coming from $\tilde{D}_l(\nu)$ and another from ν_c to ∞ coming from $\tilde{B}_l(\nu)$. To obtain $\tilde{N}_l(\nu)$, we simply subtract off the right-hand cut coming from $\tilde{D}_l(\nu)$; that is,

$$\tilde{N}_l(\nu)=\tilde{B}_l(\nu)\tilde{D}_l(\nu)-\frac{1}{\pi}\int_0^{\nu_c} d\nu' \frac{\tilde{B}_l(\nu')\,\mathrm{Im}\tilde{D}_l(\nu')}{\nu'-\nu}. \quad (3.19)$$

Substituting (3.17) into the above equation one finds the integral equation

$$\tilde{N}_l(\nu)=\tilde{B}_l(\nu) +\frac{1}{\pi}\int_0^{\nu_c} d\nu' \frac{\tilde{B}_l(\nu')-\tilde{B}_l(\nu)}{\nu'-\nu}\nu'^{l+1/2}\tilde{N}_l(\nu'), \quad (3.20)$$

which is similar to Eq. (2.34), except the input function is $\tilde{B}_l(\nu)$, which contains the left-hand cuts and a portion of the right-hand cut from ν_c to ∞. The above equation is solved numerically with $\tilde{B}_l(\nu)$ given by (3.16), and the results are discussed in Sec. V. The amplitude obtained by $\tilde{N}_l(\nu)/\tilde{D}_l(\nu)$ has a right-hand cut extending from zero to infinity. The discontinuity across the cut from zero to ν_c satisfies unitarity exactly, whereas from ν_c to ∞ it satisfies unitarity to the extent given by the second Born approximation; namely, as given in (3.7b).

IV. DETERMINANTAL METHOD

An approximation that is often used is to assume that the N function is equal to the first Born approximation: Then the D function is given directly by

$$D_l(\nu)=1-\frac{1}{\pi}\int_0^{\infty} d\nu' \frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}. \quad (4.1)$$

This method is called the determinantal method.[10] While it has the advantage of simplicity, it has the disadvantage of destroying what little exact information we do know about the left-hand cut between the first and second branch points. In this method, one assumes that

$$f_l(\nu)=b_l^{(1)}(\nu)/D_l(\nu), \quad (4.2)$$

which has a discontinuity on the left given by

$$\mathrm{Im} f_l(\nu)=\mathrm{Im} b_l^{(1)}(\nu)/D_l(\nu) \quad \nu<-\tfrac{1}{4}\mu^2, \quad (4.3)$$

while we know that the discontinuity of the exact amplitude across the cut between the first and second branch points should be

$$\mathrm{Im} f_l(\nu)=\mathrm{Im} b_l^{(1)}(\nu) \quad -\mu^2<\nu<-\tfrac{1}{4}\mu^2 \quad (4.4)$$

even though we do not know what it should be for $\nu<-\mu^2$.

It is of interest to apply the determinantal method to Yukawa potential scattering and check its reliability. It turns out that for the Yukawa potential, we can get an explicit expression for the amplitude for arbitrary l when we use the determinantal method. Substituting the first Born term into (4.1), we have[11]

$$D_l(\nu)=1-\frac{g^2}{2\pi}\int_0^{\infty} d\nu' \frac{Q_l(1+\mu^2/2\nu')}{\nu'^{1/2}(\nu'-\nu)}\equiv 1-\frac{g^2}{2}K_l(\nu). \quad (4.5)$$

For $l=0$,

$$Q_0(1+1/2\nu)=\tfrac{1}{2}\ln(4\nu+1) \quad (4.6)$$

and

$$K_0(\nu)=\frac{1}{2\pi}\int_0^{\infty} d\nu' \frac{\ln(4\nu'+1)}{\nu'^{1/2}(\nu'-\nu)}. \quad (4.7)$$

Let us introduce the parameter β and define

$$J(\beta,\nu)\equiv\int_0^{\infty} d\nu' \frac{\ln(\beta^2\nu'+1)}{\nu'^{1/2}(\nu'-\nu)}. \quad (4.8)$$

Differentiating with respect to β,

$$\frac{\partial J(\beta,\nu)}{\partial\beta}=\frac{2}{\beta}\int_0^{\infty} d\nu' \frac{\nu'^{1/2}}{(\nu'-\nu)(\nu'+\beta^{-2})}. \quad (4.9)$$

The integrand

$$\frac{\nu'^{1/2}}{(\nu'-\nu-i\epsilon)(\nu'+\beta^{-2})}$$

is an analytic function of ν' which has poles at $\nu+i\epsilon$ and $-\beta^{-2}$, and a branch cut from 0 to $+\infty$ coming from $\nu'^{1/2}$, whose value along the cut we define as $(\nu'\pm i\epsilon)^{1/2}=\pm\nu'^{1/2}$. Applying Cauchy's theorem around the con-

[10] M. Baker, Ann. Phys. (N. Y.) 4, 271 (1958).

[11] In this section we set $\mu=1$ for convenience.

tour formed by the cut and an infinite circle, we find

$$-\frac{1}{2\pi i}\int_{-\infty}^{0} d\nu'\frac{\nu'^{1/2}}{(\nu'-\nu)(\nu'+\beta^{-2})}+\frac{1}{2\pi i}$$

$$\times\int_{0}^{\infty} d\nu'\frac{\nu'^{1/2}}{(\nu'-\nu)(\nu'+\beta^{-2})}=-\frac{i}{(\nu+\beta^{-2})}+\frac{\nu^{1/2}}{(\nu+i\epsilon+\beta^{-2})},$$

which simplifies to

$$\int_{0}^{\infty} d\nu'\frac{\nu'^{1/2}}{(\nu'-\nu)(\nu'+\beta^{-2})}=\frac{\pi\beta}{1-i\beta(\nu)^{1/2}}. \quad (4.10)$$

Integrating $\partial J/\partial\beta$ with respect to β

$$J(\beta,\nu)=2\pi\int^{\beta}\frac{d\beta}{1-i\beta(\nu)^{1/2}}+C(\nu)$$

$$=2\pi i\ln[1-i\beta(\nu)^{1/2}]/\nu^{1/2}+C(\nu). \quad (4.11)$$

From (4.8), we observe that when $\beta=0$, $J(0,\nu)=0$; therefore, $C(\nu)$ is zero, and we have

$$K_0(\nu)=i\frac{\ln[1-2i(\nu)^{1/2}]}{\nu^{1/2}}. \quad (4.12)$$

Therefore, the S-wave D function is given explicitly by

$$D_0(\nu)=1-\frac{g^2 i\ln[1-2i(\nu)^{1/2}]}{2(\nu)^{1/2}}. \quad (4.13)$$

For real positive ν,

$$D_0(\nu+i\epsilon)=1-[g^2\tan^{-1}2(\nu)^{1/2}]/2(\nu)^{1/2}$$

$$-ig^2\frac{\ln(4\nu+1)}{4(\nu)^{1/2}}. \quad (4.13a)$$

For real negative ν,

$$D_0(\nu)=1-\frac{g^2\ln[1+2(-\nu)^{1/2}]}{2(-\nu)^{1/2}}. \quad (4.13b)$$

The S-wave binding energy ν_0 is given by the roots of the equation

$$2(\nu_0)^{1/2}-g^2\ln[1+2(\nu_0)^{1/2}]=0. \quad (4.13c)$$

The coupling constant corresponding to zero-energy bound state is $g_0^2=1$.

To derive the D function for higher l values, we start by recalling that for integer l

$$Q_l(z)=P_l(z)Q_0(z)-W_{l-1}(z), \quad (4.14)$$

where $W_{l-1}(z)$ is a polynomial of z of order $(l-1)$. The functions $P_l(1+\frac{1}{2}\nu)$ and $W_{l-1}(1+\frac{1}{2}\nu)$ are analytic everywhere in the ν plane except at the origin, where they have lth and $(l-1)$th-order poles, respectively; therefore, the function

$$g_l(\nu)=iP_l\left(1+\frac{1}{2\nu}\right)$$

$$\times\frac{\ln[1-2i\nu^{1/2}]}{\nu^{1/2}}-i\frac{W_{l-1}(1+1/2\nu)}{\nu^{1/2}} \quad (4.15)$$

is an analytic function containing an lth-order pole at the origin and a branch cut from 0 to $+\infty$ with the discontinuity

$$g_l(\nu+i\epsilon)-g_l(\nu-i\epsilon)=2iQ_l(1+1/2\nu)/\nu^{1/2} \quad \nu>0, \quad (4.16)$$

and as $|\nu|\to\infty$, $g_l(\nu)\to 0$. Therefore, if we remove the pole at the origin we would have the function $K_l(\nu)$ defined in (4.5); that is

$$K_l(\nu)=iP_l\left(1+\frac{1}{2\nu}\right)\frac{\ln[1-2i(\nu)^{1/2}]}{\nu^{1/2}}$$

$$-i\frac{W_{l-1}(1+1/2\nu)}{\nu^{1/2}}-A_l(\nu), \quad (4.17)$$

where $A_l(\nu)$ is defined as a function analytic everywhere except for an lth-order pole at the origin and its singularity at that point cancels exactly the singularity of $g_l(\nu)$. For any given l, we can find $A_l(\nu)$ by examining the behavior of $g_l(\nu)$ near the origin. For instance, take $l=1$, we have

$$Q_1(z)=zQ_0(z)-1. \quad (4.18)$$

The function

$$g_1(\nu)=i\left(1+\frac{1}{2\nu}\right)\frac{\ln[1-2i(\nu)^{1/2}]}{\nu^{1/2}}-\frac{i}{\nu^{1/2}} \quad (4.19)$$

has the following behavior near the origin:

$$g_1(\nu)\approx 1/\nu+2+2i(\nu)^{1/2}+O(\nu). \quad (4.19a)$$

Therefore,

$$A_1(\nu)=1/\nu \quad (4.20)$$

and the P-wave D function is given explicitly by

$$D_1(\nu)=1-\frac{g^2}{2}i\left(1+\frac{1}{2\nu}\right)\frac{\ln[1-2i(\nu)^{1/2}]}{\nu^{1/2}}$$

$$+\frac{ig^2}{2(\nu)^{1/2}}+\frac{g^2}{2\nu}. \quad (4.21)$$

For $l=2$, we find

$$A_2(\nu)=2/\nu+3/4\nu^2. \quad (4.22)$$

In a similar manner we can derive the function $A_l(\nu)$ for other values of l and find for the D function in the physical region

$$D_l(\nu+i\epsilon)=1-\frac{g^2}{2}P_l(1+1/2\nu)[\tan^{-1}2(\nu)^{1/2}]/\nu^{1/2}+\frac{g^2}{2}A_l(\nu)-\frac{ig^2Q_l(1+1/2\nu)}{2(\nu)^{1/2}},\quad \nu>0. \quad (4.23)$$

The phase shift is given by

$$(\nu)^{1/2}\cot\delta_l=\left[1-\frac{g^2}{2}P_l\left(1+\frac{1}{2\nu}\right)\frac{\tan^{-1}2(\nu)^{1/2}}{\nu^{1/2}}+\frac{g^2}{2}A_l(\nu)\right][(g^2/2\nu)Q_l(1+1/2\nu)]^{-1}. \quad (4.24)$$

The phase shifts are calculated by the above relation for $g^2=3$ and 1, and for $l=0$, 1, and 2. The results are discussed in Sec. V.

We might consider improving the determinantal solution by including some information from the second Born term. The complete solution expresses the amplitude as the ratio of two integral functions of the coupling constant g^2. Let us include the next term in the numerator which is of order g^4 and call it $N_l^{(2)}(\nu)$; then,

$$f_l(\nu)=\frac{b_l^{(1)}(\nu)+N_l^{(2)}(\nu)}{1-\frac{1}{\pi}\int_0^\infty d\nu'(\nu')^{1/2}[b_l^{(1)}(\nu')+N_l^{(2)}(\nu')](\nu'-\nu)^{-1}} \quad (4.25)$$

satisfies the unitarity condition (2.13). In the limit of small g^2, we expand the above equation

$$f_l(\nu)=b_l^{(1)}(\nu)+N_l^{(2)}(\nu)+\frac{b_l^{(1)}(\nu)}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}+O(g^6) \quad (4.26)$$

and compare with the Born series (2.8); hence

$$N_l^{(2)}(\nu)=b_l^{(2)}(\nu)-\frac{b_l^{(1)}(\nu)}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}. \quad (4.27)$$

This function has no branch cut along the positive real axis as the right-hand cut in $b_l^{(2)}(\nu)$ is canceled by the right-hand cut in the second expression. The modified N function is

$$N_l(\nu)=b_l^{(1)}(\nu)\times\left\{1-\frac{1}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}\right\},\quad -1<\nu<-\tfrac{1}{4}. \quad (4.28)$$

Between the first and second branch points,

$$\mathrm{Im}N_l(\nu)=\mathrm{Im}b_l^{(1)}(\nu)\times\left\{1-\frac{1}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}\right\},\quad -1<\nu<-\tfrac{1}{4}. \quad (4.29)$$

Therefore, with the modification, the discontinuity of the amplitude in that section of the cut now satisfies (4.4) to order g^4. Referring to (4.23)

$$\mathcal{P}\frac{1}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}b_l^{(1)}(\nu')}{\nu'-\nu}=\frac{g^2}{2}P_l\left(1+\frac{1}{2\nu}\right)\times\frac{\tan^{-1}2\nu^{1/2}}{\nu^{1/2}}-\frac{g^2}{2}A_l(\nu). \quad (4.29a)$$

From (4.27) the function $N_l^{(2)}(\nu)$ along the positive real axis is

$$N_l^{(2)}(\nu)=\mathrm{Re}b_l^{(2)}(\nu)-\frac{g^4}{4\nu}Q_l\left(1+\frac{1}{2\nu}\right)\times\left[P_l\left(1+\frac{1}{2\nu}\right)\frac{\tan^{-1}2\nu^{1/2}}{\nu^{1/2}}-A_l(\nu)\right]. \quad (4.30)$$

From (4.25), we have for $\nu>0$

$$\mathrm{Re}D_l(\nu)=1-\frac{g^2}{2}P_l\left(1+\frac{1}{2\nu}\right)\frac{\tan^{-1}2(\nu)^{1/2}}{\nu^{1/2}}-\frac{g^2}{2}A_l(\nu)-\mathcal{P}\frac{1}{\pi}\int_0^\infty d\nu'\frac{(\nu')^{1/2}N_l^{(2)}(\nu')}{\nu'-\nu}. \quad (4.31)$$

Using (4.30), we evaluate the principal value integral over $N_l^{(2)}(\nu)$ numerically, find $\mathrm{Re}D_l(\nu)$, and calculate the phase shifts for $g^2=3$ and 1, and for $l=0$, 1, and 2. The results are discussed in the following section. The D function is also calculated for negative energies to look for bound-state poles of the amplitude.

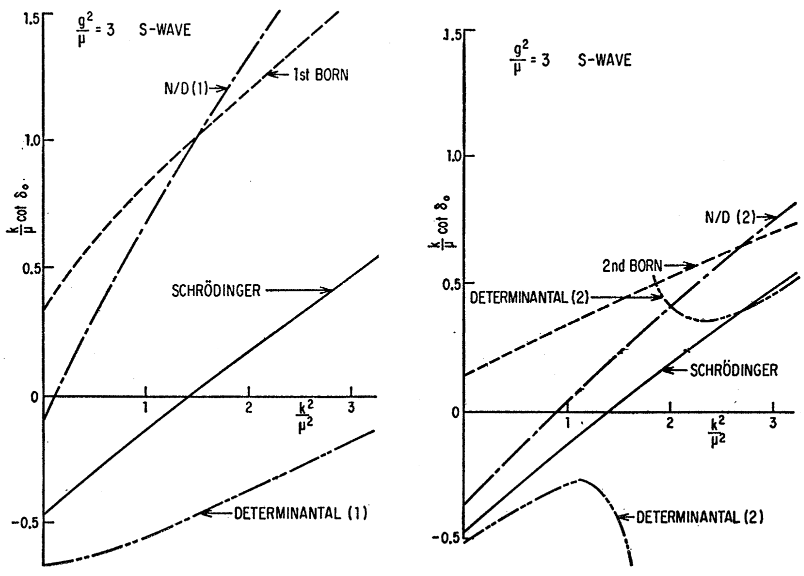
V. RESULTS AND CONCLUSIONS

For different potential strengths, the S-, P-, and D-wave phase shifts are computed by the following methods:

(*1*) *Schrödinger equation:* The exact values of phase shifts are computed by integrating Eq. (2.2) numerically, subject to the boundary condition (2.3).

(*2*) *N/D method:* (A) With only the first Born cut. This is designated as N/D (1). The integral equation (2.34) is solved numerically with the input $\bar{B}_l(\nu)$ given by (2.35). From the N and D functions we find the phase shifts. (B) With the first and second Born cuts. This is designated as N/D (2). The integral equation (3.20) is solved numerically with the input $\tilde{B}_l(\nu)$ given by (3.16). The phase shifts are found from the N and D functions.

(*3*) *Determinantal method:* (A) With first Born term.



(a) (b)

FIG. 2. S-wave effective range plot for $g^2/\mu=3$. Meaning of labels: N/D (1): N/D with first Born cut; N/D (2): N/D with first and second Born cuts; Determinantal (1): first-order determinantal; Determinantal (2): second-order determinantal (N/D solutions have a cutoff at $\nu_c/\mu^2=100$).

This is designated as DETERMINANTAL (1). The phase shifts are calculated from (4.24). (B) With first and second Born terms. This is designated as DETERMINANTAL (2). The phase shifts are calculated by the use of (4.31) and (4.30).

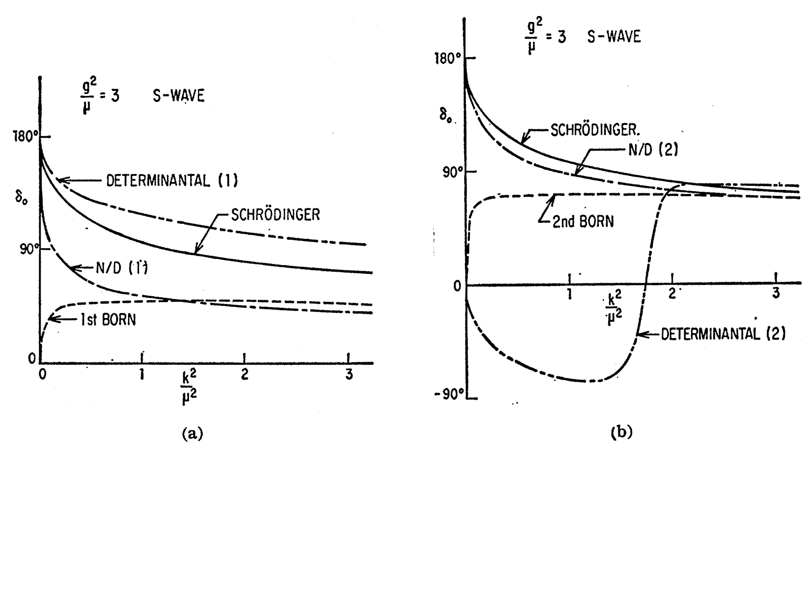
(*4*) *Born approximation:* (A) First Born approximation. We define the first Born phase shift by $(\nu)^{1/2}\cot\delta_l = 1/b_l^{(1)}(\nu)$ or $\tan\delta_l=[g^2/2(\nu)^{1/2}]Q_l(1+\mu^2/2\nu)$. (B) Second Born approximation. We define the phase shift as $(\nu)^{1/2}\cot\delta_l=\mathrm{Re}(1/f_l)$, where $f_l=b_l^{(1)}(\nu)+b_l^{(2)}(\nu)$.

Some of the results of these calculations are shown from Figs. 2 to 9. For each value of g^2/μ and l, we plot the data in two separate figures: The figure denoted by (a) contains the results of the various approximations using information from the first Born term only, whereas the figure denoted by (b) contains the results of the various approximations using information from both the first and second Born terms.

For the S wave, no cutoff is necessary in the N/D equation; whereas, for the P and D waves, a cutoff is required to take care of the threshold behavior. As seen in Table I, if we also apply a cutoff at $\nu_c/\mu^2=100$ to the S-wave solution, the phase shifts differ only slightly

TABLE I. S-wave N/D phase shifts for $g^2/\mu=3$ with and without cutoff (N/D with first and second Born cuts).

ν/μ^2	($\nu_c/\mu^2=10$) δ_0(rad)	($\nu_c/\mu^2=100$) δ_0(rad)	($\nu_c/\mu^2=1000$) δ_0(rad)	(no cutoff) δ_0(rad)
0.05	2.565	2.571	2.569	2.568
0.25	2.042	2.052	2.050	2.049
0.5	1.779	1.790	1.788	1.786
1.0	1.520	1.533	1.532	1.530
3.0	1.152	1.161	1.160	1.159
5.0	0.974	1.011	1.009	1.009
7.0	0.849	0.919	0.918	0.918
10.00		0.830	0.828	0.828
20.0		0.675	0.673	0.673
50.0		0.504	0.504	0.504
80.0		0.411	0.432	0.432



(a) (b)

FIG. 3. S-wave phase shifts for $g^2/\mu=3$. For meaning of labels see Fig. 2 caption.

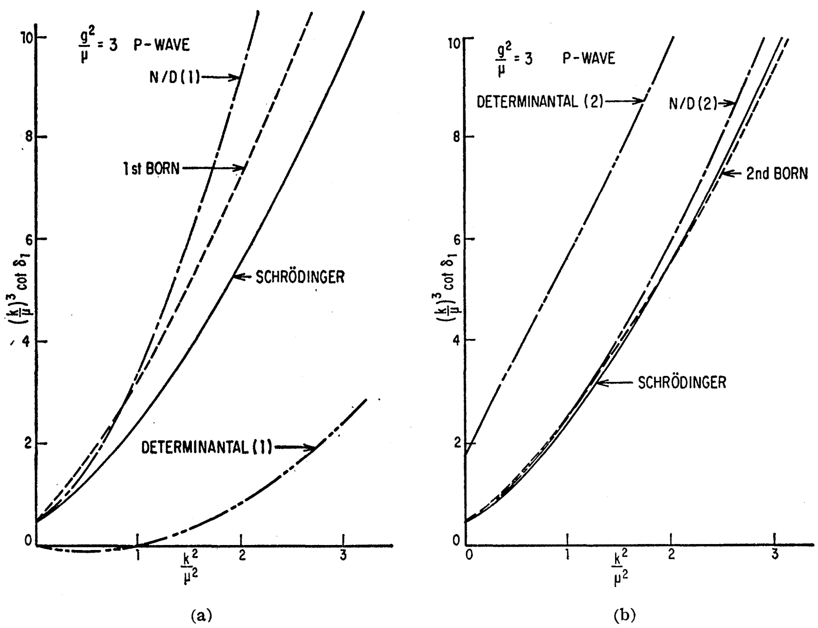


FIG. 4. P-wave effective range plot for $g^2/\mu=3$. For meaning of labels see Fig. 2 caption.

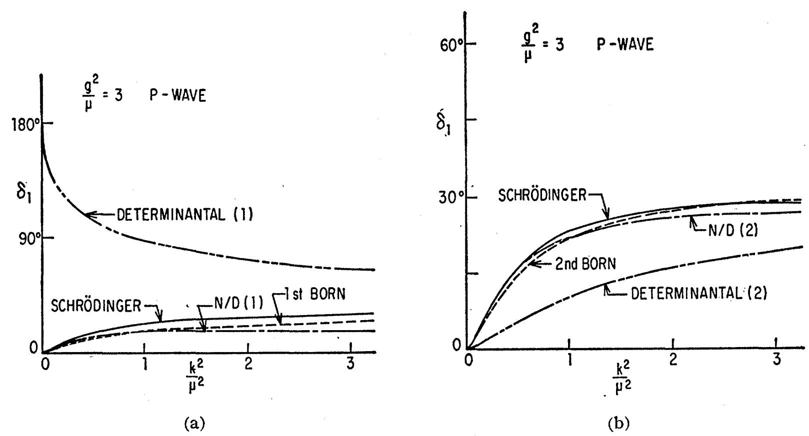


FIG. 5. P-wave phase shifts for $g^2/\mu=3$. For meaning of labels see Fig. 2 caption.

from the values for no cutoff. To obtain a uniform N/D equation for the S, P, and D states, we apply a single cutoff at $\nu_c/\mu^2=100$ for the three cases. The sensitivity of the P- and D-wave phase shifts to a change in the cutoff is shown in Tables II and III. It is seen that in the low-energy region far away from the cutoff, a change in the cutoff does not change the solution very much. On the other hand, at moderately high energies ($\nu\gtrsim 10$),

TABLE II. P-wave N/D solution for $g^2/\mu=3$ with different cutoffs. (N/D with first and second Born cuts.)

ν/μ^2	($\nu_c/\mu^2=10$) δ_1(rad)	($\nu_c/\mu^2=100$) δ_1(rad)	($\nu_c/\mu^2=1000$) δ_1(rad)
0.05	0.0231	0.0248	0.0231
0.25	0.155	0.155	0.156
0.5	0.273	0.271	0.271
1.0	0.385	0.384	0.383
3.0	0.466	0.464	0.464
5.0	0.459	0.457	0.456
7.0	0.441	0.440	0.439
10.0		0.414	0.414
20.0		0.348	0.347
50.0		0.243	0.237
80.0		0.197	0.173

TABLE III. D-wave N/D phase shifts for $g^2/\mu=3$ with different cutoffs (N/D with first and second Born cuts).

ν/μ^2	($\nu_c/\mu^2=10$) δ_2(rad)	($\nu_c/\mu^2=100$) δ_2(rad)	($\nu_c/\mu^2=1000$) δ_2(rad)
0.05	0.00076	0.00081	0.00077
0.25	0.0174	0.0176	0.0179
0.50	0.0484	0.0488	0.0490
1.0	0.104	0.104	0.104
3.0	0.209	0.209	0.209
5.0	0.241	0.240	0.242
7.0	0.252	0.250	0.254
10.0		0.249	0.256
20.0		0.206	0.232
50.0		0.017	0.147
80.0		0.007	0.082

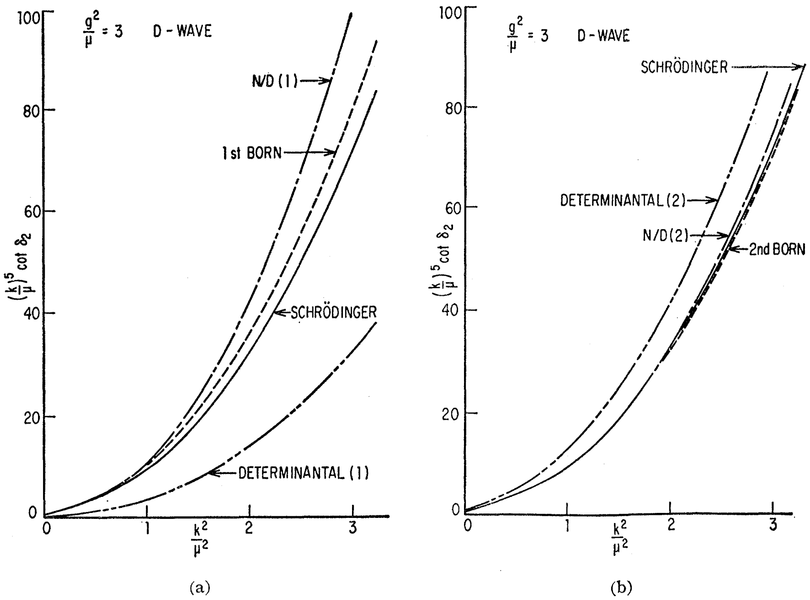


FIG. 6. *D*-wave effective range plot for $g^2/\mu=3$. For meaning of labels see Fig. 2 caption.

the solution is influenced by the position of the cutoff.

In the case of *S*-wave scattering, Bjorken and Goldberg[6] have studied the exponential potential for which the amplitude has a sequence of poles at $\nu=-\frac{1}{4}(n\mu)^2$. They showed that for that case the N/D approximation with only the first pole did not do too well, but when the first two poles are included, reasonable results in the low-energy region were obtained. As they pointed out, the exponential potential is "smoother" (or less singular) than potentials such as the Yukawa potential and the scattering at high energy is small, so it is a more favorable case for the neglect of faraway singularities. From our present results with the Yukawa potential, we find that although this potential is more singular, similar conclusions are still obtained regarding the N/D approximation. When the first two cuts are included, the approximation is a reasonable one in the low-energy region, provided the coupling is not too strong; when only the first cut is included, the approximation is good only in a limited region near the threshold. The Yukawa amplitude has a sequence of branch cuts instead of poles, which is more similar to the relativistic situation, and we consider it to be a better analog of the relativistic amplitude.

In the low-energy region, the "Coulomb" effect of the nearby cuts is relatively more important than the more distant cuts. At higher energies, the faraway cuts become relatively as important as the nearby cuts, and

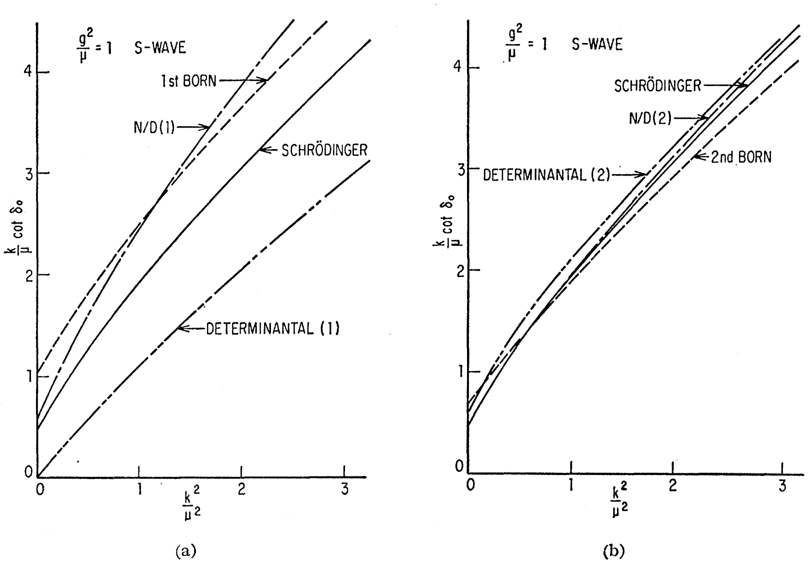


FIG. 7. *S*-wave effective range plot for $g^2/\mu=1$. For meaning of labels see Fig. 2 caption.

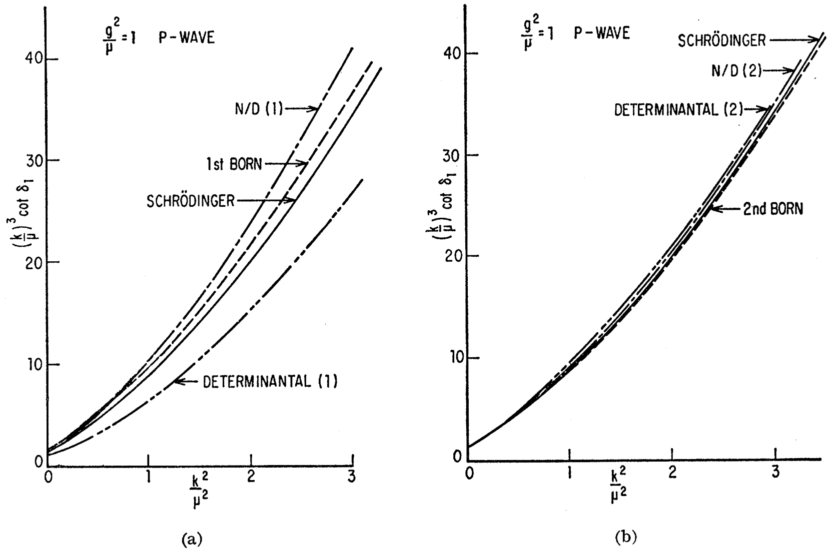


FIG. 8. P-wave effective range plot for $g^2/\mu=1$. For meaning of labels see Fig. 2 caption.

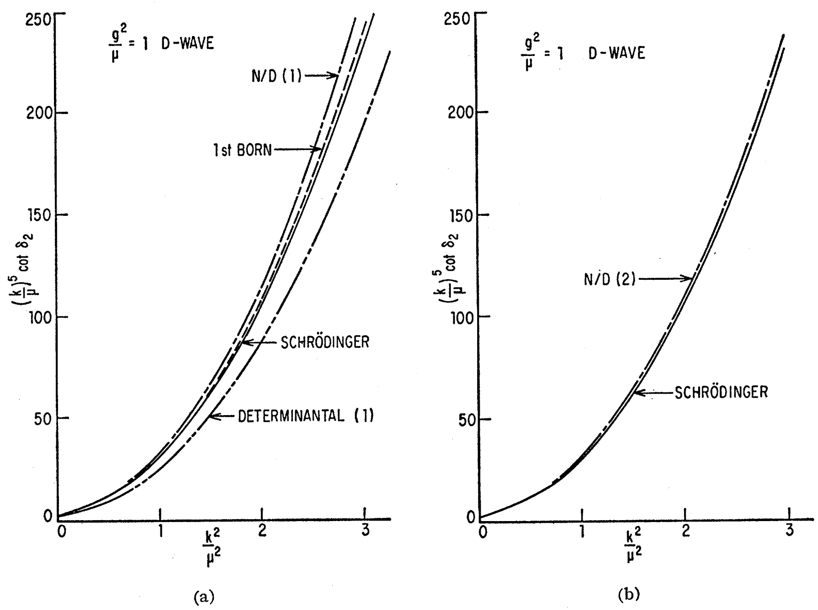


FIG. 9. D-wave effective range plot for $g^2/\mu=1$. For meaning of labels see Fig. 2 caption. The determinantal (2) curve is nearly the same as the N/D (2) curve, and the second Born curve is nearly the same as the Schrödinger curve.

the N/D approximation begins to fail. It actually becomes worse than the Born approximation when the energy exceeds the region of validity.

We should keep in mind that for the curves with $g^2/\mu=1$, the Born series converges for all energies. For the curves with $g^2/\mu=3$, the Born series would still converge in the region where $k/\mu \ln(k/\mu)\gg\frac{3}{2}$. (The exact value of energy beyond which it converges can be known only if we know the exact radius of convergence in the g^2 plane as a function of energy.[12]) For the S wave, the N/D solution tends to the first Born approximation as $\nu\rightarrow\infty$. This is due to the fact that the first Born approximation goes like $(\ln\nu)/\nu$, which eventually dominates the dispersion integral, which only goes like $1/\nu$. [Refer to Eq. (2.17).] This effect is not shown in the figures, but the N/D curve for the S wave eventually tends to the Born curve at extremely high energies. For the P and D waves, the N/D solution does not tend to the Born approximation as $\nu\rightarrow\infty$. This is due to the fact that, in taking care of the threshold, we have divided the Born term by the factor ν^l which makes it tend to a lower order than the dispersion integral as $\nu\rightarrow\infty$. This discrepancy in asymptotic behavior would be of concern to us if we were trying to use N/D to solve a potential problem, which we are not. The situa-

[12] The first pole in the g^2 plane is real for $k^2\leqslant 0$ but complex for $k^2>0$. [R. Jost and A. Pais, Phys. Rev. 82, 840 (1951).] The position of this pole as a function of energy for $k^2>0$ can only be found numerically by finding the complex eigenvalues in g^2 for a given value of k^2 in the Schrödinger equation.

tion has no relativistic analog, since, as we go up in energy in the relativistic problem, we must consider the inelastic channels and our potential analog would break down.

In the N/D method, we have constructed an approximation to the amplitude which satisfies unitarity along the right-hand cut and has the same discontinuity as the exact amplitude over a finite portion of the left-hand cut. The $N/D(1)$ solution contains the exact discontinuity from $k^2=-\frac{1}{4}\mu^2$ to $-\mu^2$, and it appears to be a reasonable approximation up to $k^2\approx\mu^2$. On the other hand, the $N/D(2)$ solution, which contains the exact discontinuity from $k^2=-\frac{1}{4}\mu^2$ to $-9\mu^2/4$, appears to be good up to $k^2\approx 3\mu^2$. One might argue that an N/D approximation which contains the exact left-hand discontinuity out to an energy $-\nu_n$ would be a valid approximation from threshold up to energies of the order of ν_n. The goodness of the approximation depends on the coupling strength. The stronger is the coupling strength, the more nearby cuts one must consider to obtain a good approximation, since the discontinuities of the cuts are of increasing order in g^2/μ as we go towards the left. The exact radius of convergence in g^2/μ for our B function has not been investigated in this paper. However, taking g^2/μ up to ~ 3 already represents a sufficiently large coupling constant typical of strong interaction. As an example, the single pion cut in the singlet S-wave amplitude for nucleon-nucleon scattering has a discontinuity equivalent to the first cut of a Yukawa potential with $g^2/\mu=0.53$.

Our N/D results for the Yukawa potential tend to lend support to relativistic N/D calculations that have been made for low energies; for example, the analyses of low-energy nucleon-nucleon scattering.[13] One might consider applying the N/D method to other similar situations where the dominant forces are of a long-range nature and one is only interested in the energy region where two particle states are dominant, for example, low-energy lambda-nucleon scattering. On the other hand, for problems such as the $I=1$ $\pi\pi$ scattering, where the ρ meson is found, it is known that phenomenological analyses of the scattering amplitude show that either the force is of a very short range nature or that the inelastic effects are important. Our present analysis of the N/D method has little relevance to these cases.

In addition to giving scattering phase shifts at energies above threshold, both the N/D and determinantal methods can give bound-state poles when the D function vanishes below the threshold energy. We have used the N/D and determinantal methods to calculate S-wave binding energies of the Yukawa potential for different potential strengths. In each method, the first- and second-order approximations were used. The results are shown in Fig. 10 as plots of potential strengths versus

[13] For example, see A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963); D. Amati, E. Leader, and B. Vitale, Phys. Rev. **130**, 750 (1963).

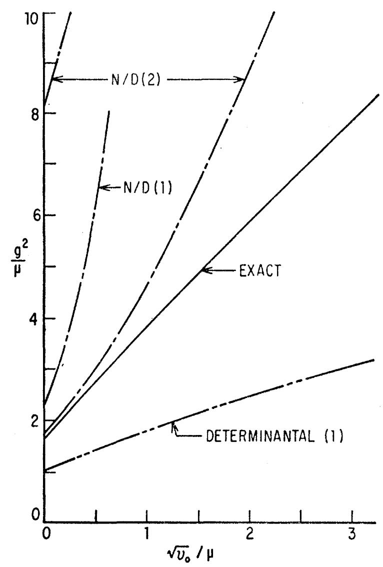


FIG. 10. Potential strengths versus S-state binding energy.

$\nu_0^{1/2}/\mu$, where ν_0 is the binding energy in units such that $\hbar=2m=1$. The curve for the exact binding energies is based on data taken from Lovelace and Masson.[14] Figure 10 has the same general features as a similar figure (Fig. 9) shown in the work of Bjorken and Goldberg for the exponential potential. Our results and their results have the following points in common: (1) The second-order N/D method appears to work well in predicting S-wave bound-state energies provided the potential strength is not too large. (2) The determinantal method in second order predicts no bound state whatsoever. (3) The first-order determinantal method gives an excessively strong binding energy. (4) The first-order N/D method gives too weak a binding energy. We note that for g^2/μ greater than approximately 6.5, the exact solution for the Yukawa potential gives two S-wave bound states. On the other hand, the second-order N/D approximation does not begin to show two bound states until $g^2/\mu=8.2$. From our results and the results of Bjorken and Goldberg, it appears that a useful criterion for the validity of the second-order N/D method for calculating S-wave binding energies is that the coupling strength be limited to strengths where only one S-wave bound state occurs. It also appears that the first-order N/D method cannot be relied upon to give good estimates of the S-wave binding energy.

In contrast to the N/D method, the determinantal method does not necessarily give an improved solution when the second Born term is added. Consider the S-wave determinantal solution. For $g^2/\mu=1$, the addition of the second Born cut does improve the answer. (See Fig. 7.) However, for $g^2/\mu=3$ (see Fig. 2), the addition of the second Born cut causes the N function to have a "spurious" zero in the physical region, which

[14] C. Lovelace and D. Masson, Nuovo Cimento **26**, 472 (1962), Table II.

appears as a pole in $k\cot\delta_0$. As we go from large k^2 to small k^2, a "legitimate" zero in the N function is one that appears after a zero appears in the real part of D, which corresponds to the phase shift going through 90° first, then increasing towards 180°. By a spurious zero, we mean a zero in N which appears before any zero in ReD develops, which corresponds to the phase shift going through zero and becoming negative, even though the potential is purely attractive. The reason why a spurious zero may appear in the N function is the fact that the analytic property of the amplitude in the ν plane is given incorrectly. Although the determinantal solution gives the correct location of the two branch points, it does not give the correct discontinuity across the branch cuts. For $g^2/\mu=3$, we know from the N/D solution that the effect from the second Born cut is important. This cut with a branch point at $k^2=-\mu^2$ appears not only in the $O(g^4)$ term in the N function of the determinantal solution but also in the $O(g^6)$ and higher order terms. Apparently, when g^2 is large, the inclusion of only the $O(g^4)$ term gives an erroneous effect in the second cut that resembles a form of repulsion. For this reason, the solution does not give any S-wave bound state at all. When the determinantal method is carried to higher orders than the first order, a difficulty of this kind might appear when the coupling is strong. The determinantal solution in the exact form is a ratio of two integral functions of g^2. However, in practice, one cannot find these two functions exactly but must truncate the series in g^2 up to some order. The truncated numerator function does not describe the analytic property in the ν plane adequately. However, in the first-order determinantal solution, where the numerator is just the first Born term which has no zeros in the physical region, this kind of difficulty does not appear.

In the N/D method, the N function is constrained by the boundary condition along the first and second Born cuts so that it has the same discontinuity as the exact N function from $k^2=-\frac{1}{4}\mu^2$ to $-9\mu^2/4$. This constraint prevents the N function from developing a spurious zero in the physical region.

In reviewing the results in Figs. 2 to 9, we find that in general the first-order determinantal solution gives unreliable results. The distortion of the discontinuity along the nearest left hand-cut causes a large error in the behavior of the amplitude in the physical region, even though unitarity is satisfied exactly along the right hand cut. For instance, in the S wave with $g^2/\mu=1$ [see Fig. 7(a)], the first-order determinantal solution shows a zero-energy bound state, whereas the exact solution is far from having a zero-energy bound state. In fact, one requires a value of $g^2/\mu\approx 1.67$ to produce a zero-energy bound state in the exact solution. We see in Fig. 7(a) that the $N/D(1)$ solution with only the discontinuity given correctly in the segment from $k^2=-\mu^2$ to $-\frac{1}{4}\mu^2$ is able to give a good approximation to the scattering length. Also, in the P wave with $g^2/\mu=3$, the first-order determinantal solution gives the apparent effect of a very strong attraction which produces a P-wave bound state, even though the exact solution is far from having such a bound state. [See Fig. 5(a)]. On the other hand, the $N/D(1)$ solution does not produce such erratic behavior. Our results emphasize the importance of preserving what little exact information we do know about the discontinuity along the nearest portion of the left-hand cut, which is what we do in the N/D method. In the determinantal method, this information is disregarded and the discontinuity of the entire left-hand cut up to $k^2=-\frac{1}{4}\mu^2$ is left arbitrary. The bad effect due to this distortion in the cut becomes more severe as the coupling strength increases. It appears that the determinantal approach is not as reliable a method for calculating partial-wave amplitudes in strong interactions as the N/D method where the known exact discontinuity of the amplitude is preserved in the nearby region.

Since the determinantal method is not able to give uniformly good results for arbitrary physical values of l, we should not attempt to use it for noninteger values of l for tracing Regge trajectories. The N/D method with the same cutoff for all l values gives uniformly good approximations for the S, P, and D waves, and it may be extended to noninteger l to trace Regge trajectories.

Finally, we should note that for $l>0$ the second Born approximation is superior to both the N/D and determinantal methods. Apparently, for $l>0$ and for the range of coupling considered, the phase shifts are small, and the right-hand cut determined from perturbation expansion is actually better than that obtained from N/D or the determinantal method.

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