Theory of Electromagnetic Field Measurement and Photoelectron Counting

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A theory of electromagnetic field measurement by means of photoionization is developed and applied to photoelectron counting. A probability theory involving multitime joint probability functions for a sequence of photoionizations is formulated. A general quantum-theory definition is proposed for the nonexclusive probability function which occurs in the probability theory. Approximations are then introduced to derive expressions for this probability function which involve correlation functions of the photoionization detector and the electromagnetic field-plus-source. The general theory is used to derive quantities of interest in photoionization counting experiments. Expressions are derived for (1) the probability $P_n(t, t+T)$ that n photoionizations are observed in the time interval t to $t+T$, and (2) quantities related to $P_n(t, t+T)$, such as its generating function and various moments. $P_n(t, t+T)$ is found to be a compound Poisson distribution determined by the density operator of the field when the latter is expressed in Glauber's P representation. Using this result, the character of $P_n(t, t+T)$ is examined for several specific density operators. These correspond to a coherent state, various fields with the mode phases distributed independently of the mode amplitudes, and a "spread-out" coherent state.

I. INTRODUCTION

THE recent advent of nearly coherent sources of electromagnetic radiation in the region of optical frequencies has led to the possibility of observing correlations between photoionization events which reflect correlations in the electromagnetic field of a nearly coherent source. Experiments to measure photoelectron correlations using coherent sources have been performed in a number of laboratories. For example, the power spectrum[1–5] and photocounting statistics[6] of a helium-neon laser have been studied. These experiments can be related to the time-correlation measurements of Hanbury Brown and Twiss,[7,8] which were carried out on an incoherent source. The feasibility of such experiments on nearly coherent sources has led to a renewed interest in how the dynamical state of the electromagnetic field is related to photoionization information.

The theory of photomeasurement has until recently been developed almost entirely for the measurement of the field of an incoherent source. This theory, which has been comprehensively reviewed recently,[9] involves a semiclassical treatment of photoionization processes. An expression for $P_n(t, t+T)$, the probability that n photoelectrons are observed in the time interval t to $t+T$, is derived for use in connection with a particular measuring technique—photoelectron counting. The derivation[10] is based on the assumption that the probabilities of photoionization in different, small time intervals are statistically independent. In addition, the latter probabilities are assumed to be proportional to the light intensity which may depend on time. These assumptions lead to a simple Poisson distribution for $P_n(t, t+T)$. Time or ensemble averaging is then performed to obtain a compound Poisson distribution. In the ensemble averaging the real and imaginary parts of the complex field amplitudes are treated as Gaussian random variables.

The desirability of using a thorough quantum treatment as the basis of the theory of measurement of the electromagnetic field and of avoiding the Gaussian random variable assumption for a coherent source was clearly emphasized in recent work of Glauber.[11] Glauber has given an appropriate definition of a coherent state[12] and has developed[13] the formal properties of electromagnetic field correlation functions using quantum electrodynamics. For incoherent fields the second-order correlation function determines all the higher order correlation functions. This is no longer true in general for partly coherent fields, such as can be expected from a laser. Glauber[13] introduces what he calls the P representation, a representation diagonal with respect to coherent states. Use of the P representation allows field-correlation functions to be expressed in a manner formally very similar to the usual classical expressions. This formulation also facilitates making the transition to the classical limit. Sudarshan[14,15] has shown that an arbitrary density operator of the electromagnetic field can be represented formally in Glauber's P represen-

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tation provided one allows for the possibility of a highly singular distribution for the P function.

The present development is concerned with a general formulation of the problem of photomeasurement by photoionization. In most experiments multiple measurements of the electromagnetic field are made by the photomeasuring apparatus. In Sec. II the probability theory of nonindependent photoionization events in various microscopic time intervals is developed to provide for the subsequent treatment of a general starting point involving a minimum of assumptions. Joint probabilities are given for

(1) *exclusive* probabilities, for photoionizations occurring in a certain set of such subintervals and not occurring in others, and for

(2) *nonexclusive* probabilities, for photoionizations occurring at least in a certain set of subintervals.

In Sec. III the joint probabilities of Sec. II are used to obtain quantities related to experiment, in particular, the probability $P_n(t, t+T)$ of n photoionizations in the time interval t to $t+T$. It is shown that $P_n(t, t+T)$ depends on all the nonexclusive joint probabilities of order $\geq n$, without regard to the detailed structure of the field-detector system. On the other hand, the mth factorial moment of $P_n(t, t+T)$ depends only on the nonexclusive joint probabilities of order m.

Section IV has a quantum-mechanical derivation of formulas for the nonexclusive joint probabilities of Sec. II. The resulting formulas involve correlation functions of the detector and the field-plus-source. The joint probabilities are calculated using the repeated random-phase assumption for the detector. It is assumed that at the beginning of each subinterval the detector can be considered to a sufficient approximation to have returned to its ground state or thermal equilibrium state. In addition, the electromagnetic field is assumed to develop independently of the detector between subintervals. The coupling of the detector to the field is considered weak enough so that the time development of the field can be assumed to depend only on its interaction with the source. In this restricted sense, we may think of the measurements made on the field as not disturbing it appreciably. Using this approximation the photoionization correlations in time are determined by the dynamical development of the field plus its sources. In order to make contact with the work of Glauber,[11–13] Sudarshan,[14,15] and others, the full correlation functions are related to the correlation functions of the field alone. This approximation involves effectively treating the field as a closed dynamical system.

In Sec. V the results of Sec. IV are used to express $P_n(t, t+T)$ and related quantities as averages with respect to the density operator of the field and its sources. The quantity averaged is a function of field operators and a detector-correlation function. The result for $P_n(t, t+T)$ shows a clear formal resemblance to a Poisson distribution.

If the time dependence of the field operators is determined solely by the field Hamiltonian, as assumed by others,[11–15] the results can be expressed in terms of Glauber's coherent states. Use of the P representation of Glauber then shows that $P_n(t, t+T)$ is a compound Poisson distribution. Using this result, the character of $P_n(t, t+T)$ is examined for several specific density operators. These correspond to a coherent state, various fields with the mode phases distributed independently of the mode amplitudes, and a "spread-out" coherent state.

Finally, in Sec. VI a brief discussion is given of the implications of the free field motion assumption and of the applicability of the compound Poisson distribution to experiment.

II. PROBABILITY FUNCTIONS

Definitions

To deal with a large class of photomeasurement experiments in a unified way, we introduce hierarchies of distribution functions depending on time coordinates. These functions are related to the rate at which photoionization occurs at various times on various photoemissive surfaces, as a consequence of the presence of an electromagnetic field. The distribution functions are used as a basic ingredient in evaluating systematically various statistical quantities of experimental interest, discussed in Sec. III. The present development has the advantage of allowing one to obtain directly quantities of interest in photomeasurement experiments in terms of joint photoionization probabilities without making any assumptions concerning the nature of the interaction of the electromagnetic field with the detector or of introducing unnecessary statistical assumptions. In this section, the $\mathcal{W}_K$, $\mathcal{P}_K$, w_K, and $\mathcal{P}_K'$ functions are defined and their properties and interrelations are discussed.

The distribution functions are introduced as follows. The time interval t to $t+T$ is divided into N small subintervals. Eventually, one can go to the limit of large N. The basic distribution function is

$$\mathcal{W}_N(y_1,y_2,\cdots,y_N)\,, \tag{2.1}$$

where y_i, a random variable, is a coordinate for the ith time interval. For L photoemissive surfaces in the detector system, $y_i=(y_{i1},y_{i2},\cdots,y_{iL})$ is an L-component vector, each component of which can take on one of the two values: 0 and 1. $\mathcal{W}_N$ incorporates all the information about the system of electromagnetic field in interaction with L photoemissive surfaces that can be obtained by observing when photoemission processes occur on each of the L surfaces during the interval t to $t+T$. For simplicity, the formalism will be developed for a single photoemissive surface. When each member of the set $\{y_i\}$ of y's is assigned a value, $\mathcal{W}_N$ is interpreted as the probability that one photoionization process occurs in each interval with $y=1$ and no photoionization process occurs in each interval with $y=0$. More than one photoionization process in a subinterval of length Δt is precluded by making N sufficiently large, so that only two values of a coordinate y_i are needed. In other words, $\Delta t \ll R^{-1}$, where R is the photoionization rate. Implications of the above condition together with the limitation on the size of Δt required by the quantum-mechanical development are discussed in Sec. IV. In terms of $\mathcal{W}_N$, one can define a set of $[N!/K!(N-K)!]$ $\mathcal{W}_K$'s:

$$\mathcal{W}_K(y_{i_1},y_{i_2},\cdots,y_{i_K}) \equiv \sum \mathcal{W}_N(y_1,y_2,\cdots,y_N)\,, \quad (2.2)$$

where the sum is over the values of the set $\{y_i\}$ except $y_{i_1}, y_{i_2}, \cdots, y_{i_K}$. Clearly, $0\leq K\leq N$, and by conservation of probability, $\mathcal{W}_0=1$. $\mathcal{W}_K(y_{i_1},y_{i_2},\cdots,y_{i_K})$ corresponds for a discrete random series to the $W_K(y_1t_1,y_2t_2,\cdots,y_Kt_K)$ discussed by Wang and Uhlenbeck[16] for a continuous random process; see also Middleton.[17]

Various quantities of physical interest are defined as a mean $\langle\cdots\rangle_{\{y_i\}}$ with respect to $\mathcal{W}_N$ of a function $\mathfrak{F}(y_{i_1},y_{i_2},\cdots,y_{i_K})$ of the y's:

$$\begin{aligned}\langle \mathfrak{F}(y_{i_1},y_{i_2},\cdots,y_{i_K})\rangle_{\{y_i\}} &\equiv \sum_{\{y_i\}=0}^{1} \mathfrak{F}(y_{i_1},y_{i_2},\cdots,y_{i_K})\mathcal{W}_N(y_1,y_2,\cdots,y_N)\\ &= \sum_{y_{i_1},y_{i_2},\cdots,y_{i_K}=0}^{1} \mathfrak{F}(y_{i_1},y_{i_2},\cdots,y_{i_K})\\ &\qquad\times\mathcal{W}_K(y_{i_1},y_{i_2},\cdots y_{i_K})\,. \quad (2.3)\end{aligned}$$

Note that in the particular choices for the $\mathfrak{F}$ function used in this section the y for each subinterval occurs at most linearly. Thus in any of the products of y's used subsequently, the y for a given subinterval appears only once if at all. In this manner, one can introduce the product moment function

$$\begin{aligned}\mathcal{P}_K(i_1,i_2,\cdots,i_K) &\equiv \langle y_{i_1}y_{i_2}\cdots y_{i_K}\rangle_{\{y_i\}} \quad (2.4a)\\ &= \mathcal{W}_K(y_{i_1},y_{i_2},\cdots,y_{i_K})\big|_{y_{i_1}=y_{i_2}=\cdots=y_{i_K}=1} \quad (2.4b)\end{aligned}$$

which is the probability that a photoionization event occurs at least in each of the intervals $i_1, i_2, \cdots, i_K$. Note that in $\mathcal{W}_K$ and $\mathcal{P}_K$, distinct arguments never refer to the same subinterval.

The quantity $w_K(t_1,t_2,\cdots,t_K)$, which is the mean probability per (unit time)^K that photoionizations occur in the intervals t_1 to $t_1+\Delta t_1$, t_2 to $t_2+\Delta t_2$, $\cdots$, t_K to $t_K+\Delta t_K$, is defined by

$$\mathcal{P}_K(i_1,i_2,\cdots,i_K) \equiv w_K(t_1,t_2,\cdots,t_K)\Delta t_1\Delta t_2\cdots\Delta t_K\,. \quad (2.5)$$

Use of this quantity facilitates passage to the continuous limit.

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In Sec. III it is shown that various quantities of experimental interest are expressed naturally in terms of the $\mathcal{P}_K$'s or the w_K's. General quantum-mechanical expressions for $w_K(t_1,t_2,\cdots,t_K)$ are derived in Sec. IV.

One might expect that $w_K(t_1,t_2,\cdots,t_K)$ can be represented in the form

$$w_K = \sum_\alpha \mathcal{K}_K(\alpha)G_K(\alpha)\,, \quad (2.6)$$

of a sum of products of a function $\mathcal{K}_K(\alpha)$ depending only on coordinates of the detector and a function $G_K(\alpha)$ of the coordinates of the electromagnetic field and its source. The quantum-mechanical derivation in Sec. IV does, indeed, yield this form, where the sum over α becomes a $2K$ coordinate integration.

Another probability function $\mathcal{P}_K'$ can be defined conveniently as

$$\begin{aligned}\mathcal{P}_K'(i_1,i_2,\cdots,i_K) &\equiv \Big\langle \prod_{k=1}^{K} y_{i_k} \prod_{l=K+1}^{N}(1-y_{i_l})\Big\rangle_{\{y_i\}} \quad (2.7a)\\ &= \mathcal{W}_N(y_1,y_2,\cdots,y_N)\big|_{\{y_i\}=0 \text{ except } y_{i_1}=y_{i_2}=\cdots=y_{i_K}=1}\,. \quad (2.7b)\end{aligned}$$

$\mathcal{P}_K'(i_1,i_2,\cdots,i_K)$ is the probability that one photoionization process occurs in each of the intervals $i_1, i_2, \cdots, i_K$ and no photoionization processes occur in any of the remaining $N-K$ intervals. The events whose probabilities are given by the $\mathcal{P}_K'$'s are mutually exclusive and altogether comprise all possible 2^N events. The events whose probabilities are given by the $\mathcal{P}_K$'s are, in contrast, not mutually exclusive. This makes the $\mathcal{P}_K'$'s useful in various derivations and calculations, although final results are ordinarily more appropriately expressed in terms of the $\mathcal{P}_K$'s than in terms of the $\mathcal{P}_K'$'s.

Relations Among Probability Functions

Some further relations among the functions $\mathcal{W}_K$, $\mathcal{P}_K$, and $\mathcal{P}_K'$, which have been introduced in this section, will now be discussed. A generating function for the $\mathcal{P}_K'$'s is given by

$$\begin{aligned}\mathcal{G}(u_1,u_2,\cdots,u_N;v_1,v_2,\cdots,v_N) &= \sum_{K=0}^{N}\sum_{\{i_1,i_2,\cdots,i_K\}} \mathcal{P}_K'(i_1,i_2,\cdots,i_K)\\ &\qquad\times\prod_{k=1}^{K}u_{i_k}\prod_{l=K+1}^{N}v_{i_l} \quad (2.8a)\\ &= \Big\langle \prod_{i=1}^{N}[y_iu_i+(1-y_i)v_i]\Big\rangle_{\{y_i\}}\,. \quad (2.8b)\end{aligned}$$

The right-hand summation in (2.8a) is over all distinct sets of K intervals chosen from the N intervals. If u_i is replaced by u_i+v_i, one obtains a generating function for the $\mathcal{P}_K$'s

$$\mathcal{G}(u_1+v_1, u_2+v_2, \cdots, u_N+v_N; v_1,v_2,\cdots,v_N) = \langle \prod_{i=1}^{N} (y_i u_i+v_i)\rangle_{\{y_i\}} \tag{2.9a}$$

$$= \sum_{K=0}^{N} \sum_{\{i_1,i_2,\cdots,i_K\}} \mathcal{P}_K(i_1,i_2,\cdots,i_K) \times \prod_{k=1}^{K} u_{i_K} \prod_{l=K+1}^{N} v_{i_l}. \tag{2.9b}$$

$\mathcal{W}_N$ can be represented in terms of the $\mathcal{P}_K'$'s as

$$\mathcal{W}_N(y_1,y_2,\cdots,y_N) = \sum_{K=0}^{N} \sum_{\{i_1,i_2,\cdots,i_K\}} \mathcal{P}_K'(i_1,i_2,\cdots,i_K) \times \prod_{k=1}^{K} y_{i_k} \prod_{l=K+1}^{N} (1-y_{i_l}) \tag{2.10a}$$

$$= \mathcal{G}(y_1,y_2,\cdots,y_N; 1-y_1, 1-y_2, \cdots, 1-y_N). \tag{2.10b}$$

By using the expansion

$$\prod_{l=K+1}^{N} (1-y_{i_l}) = \sum_{m=K}^{N} (-1)^{m-K} \sum_{\{i_{K+1},\cdots,i_m\}}{}' \prod_{l=K+1}^{m} y_{i_l} \tag{2.11}$$

in (2.7a), one obtains directly the expressions

$$\mathcal{P}_K'(i_1,i_2,\cdots,i_K) = \sum_{m=K}^{N} (-1)^{m-K} \times \sum_{\{i_{K+1},\cdots,i_m\}}{}' \mathcal{P}_m(i_1,i_2,\cdots,i_K,i_{K+1},\cdots,i_m) \tag{2.12a}$$

$$= \sum_{m=K}^{N} \frac{(-1)^{m-K}}{(m-K)!} \times \sum_{i_{K+1},\cdots,i_m=1}^{N}{}' \mathcal{P}_m(i_1,i_2,\cdots,i_K,i_{K+1},\cdots,i_m) \tag{2.12b}$$

for $\mathcal{P}_K'(i_1,i_2,\cdots,i_K)$ in terms of the $\mathcal{P}_m$'s with $K \leq m \leq N$. The prime on the summation symbol in (2.11) and in (2.12a) indicates that the sum is over distinct sets $\{i_{K+1},i_{K+2},\cdots,i_m\}$ of $m-K$ intervals containing none of the intervals $i_1, i_2, \cdots, i_K$. In the corresponding sum in (2.12b), no two of the intervals $i_{K+1}, \cdots, i_m$ in a term are the same, otherwise $i_{K+1}, \cdots, i_m$ each run from 1 to N excluding the intervals $i_1, i_2, \cdots, i_K$. Similarly, by using the expansion

$$1 = \prod_{l=K+1}^{N} [y_{i_l}+(1-y_{i_l})] = \sum_{m=K}^{N} \sum_{\{i_{K+1},\cdots,i_m\}}{}' \prod_{l=K+1}^{m} y_{i_l} \prod_{n=m+1}^{N} (1-y_{i_n}) \tag{2.13}$$

in (2.4a), one obtains the expressions

$$\mathcal{P}_K(i_1,i_2,\cdots,i_K) = \sum_{m=K}^{N} \sum_{\{i_{K+1},\cdots,i_m\}}{}' \mathcal{P}_m'(i_1,i_2,\cdots,i_K,i_{K+1},\cdots,i_m) \tag{2.14a}$$

$$= \sum_{m=K}^{N} \frac{1}{(m-K)!} \times \sum_{i_{K+1},\cdots,i_m=1}^{N}{}' \mathcal{P}_m'(i_1,i_2,\cdots,i_K,i_{K+1},\cdots,i_m), \tag{2.14b}$$

for $\mathcal{P}_K(i_1,i_2,\cdots,i_K)$ in terms of the $\mathcal{P}_m'$'s with $K \leq m \leq N$. The last product in (2.13) includes all $N-m$ intervals not in the set $i_1, i_2, \cdots, i_K, i_{K+1}, \cdots, i_m$; the sums in (2.13) and (2.14a) are as in (2.11) and (2.12a), and the sums in (2.14b) are as in (2.12b).

One can also express $\mathcal{W}_N$ in terms of the $\mathcal{P}_K$'s by using (2.12) to substitute for $\mathcal{P}_K'$ in (2.10a). The resulting relation together with (2.10a), (2.4b), (2.14), (2.7b), and (2.12) constitute six relations among the probability functions of the three types: $\mathcal{W}_N$, $\mathcal{P}_K$, $\mathcal{P}_K'$. A probability function of one type can be represented separately in terms of probability functions of each of the remaining two types.

III. PROBABILITY THEORY OF COUNTING STATISTICS

The output of an instrument for measuring the state of an electromagnetic field by means of photoionization processes is a signal current due to the photoionizations which occur as a function of time. We assume this signal can be represented as

$$S(t) = \sum_{i=-\infty}^{\infty} F_i(t-t_i) y_i, \tag{3.1}$$

where y_i is the random variable defined in Sec. II, and $F_i(t-t_i)$ is a function defining the distribution in time of the signal pulse contributed by a photoionization event at time t_i in interval i. Although $F_i(t-t_i)$ can also be treated as a random variable, for present purposes we will assume it is a fixed function. The sum in (3.1) extends over the whole time axis. [If the instrument has L output signals, associated with different regions of the electromagnetic field, $S(t)$ is an L-dimensional vector.]

Quantities of statistical interest which can be measured with the instrument are in general functionals of $S(t)$. A class of interesting random quantities can be represented in terms of linear functions of products of the signals of the type

$$V_n = \int_{-\infty}^{\infty} \int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f(t_1,t_2,\cdots,t_n) \times S(t_1)S(t_2)\cdots S(t_n) dt_1 dt_2 \cdots dt_n, \tag{3.2}$$

where f is a suitable function of $t_1, t_2, \cdots, t_n$. For example, the integrated signal $\int_t^{t+T} S(t')dt'$ and its products fall in this class, as do various time-correlation functions and Fourier transforms, such as $\int S(t')S(t'+\tau)dt'$, $\int\int e^{i\omega\tau}S(t')S(t'+\tau)dt'd\tau$, and their generalizations. The function f represents an operation applied to the signal to extract from it information such as the number of photoionizations within a time interval, the correlations between photoionizations at different times, or the power spectrum of the signal. Thus, f could represent the effect on the signal of a photocounting circuit, a delay line and coincidence circuit, or a spectrum analyzer. Using the idempotent property $y_i^2 = y_i$ of y_i, we find for $n=1$, 2, and 3

$$V_1 = \sum_i A_i y_i, \tag{3.3a}$$

$$V_2 = \sum_{i,j}{}' A_{ij} y_i y_j + \sum_i A_{ii} y_i, \tag{3.3b}$$

$$V_3 = \sum_{i,j,k}{}' A_{ijk} y_i y_j y_k + \sum_{i,j}{}'(A_{iij}+A_{iji}+A_{jii}) y_i y_j + \sum_i A_{iii} y_i, \tag{3.3c}$$

where the sums are over all intervals, the primes denote that only terms with no summation indices alike are included, and

$$A_{i_1 i_2 \cdots i_n} = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} f(t_1', t_2', \cdots, t_n') \times F_{i_1}(t_1'-t_{i_1})F_{i_2}(t_2'-t_{i_2})\cdots \times F_{i_n}(t_n'-t_{i_n})dt_1'dt_2'\cdots dt_n'. \tag{3.4}$$

It is apparent from (3.3) that the mean of any V_n with respect to the distribution of the y's, given by (2.3), is expressible as a linear combination of the $\mathcal{P}_K$'s of (2.4) with $K=1, 2, \cdots, n$.

In the rest of this section, we confine our attention to a model for an *idealized photoelectron counter* obtained when the values of the A's of (3.4) are given by

$$A_{i_1 i_2 \cdots i_n} = \begin{cases} 1 & t \le t_{i_1}, t_{i_2}, \cdots, t_{i_n} \le t+T, \\ 0 & \text{otherwise}. \end{cases} \tag{3.5}$$

The mth power of $n(t, t+T)$, the number of photoelectron emission events in the interval t to $t+T$, is then given by V_m, so that V_1 yields

$$n(t, t+T) = \sum_{i=1}^{N} y_i. \tag{3.6}$$

Time delays are neglected as a consequence of the choice (3.5). The choice (3.5) for $A_{i_1 i_2 \cdots i_n}$ can be realized, for example, as follows. The mth power of the integrated signal $\int_t^{t+T} S(t')dt'$ is given by V_m when f is a product of m rectangular pulse functions, one for each t_i in f; a single rectangular pulse function is unity inside the interval t to $t+T$ and zero outside. When the signal pulse is then represented by a delta function, $F_i(t-t_i) = \delta(t-t_i)$, it follows that (3.4) yields (3.5). A function of $n(t, t+T)$ which has a relatively simple form when expressed in terms of the y_i's is the mth *factorial moment*[18,19] of $n=(t, t+T)$, the random variable $n^{[m]} \equiv n(n-1)\cdots \times(n-m+1)$. It becomes

$$n^{[m]} = \sum_{i_1, i_2, \cdots, i_m=1}^{N}{}' y_{i_1} y_{i_2} \cdots y_{i_m}, \tag{3.7}$$

where the prime indicates that terms with $i_1, i_2, \cdots, i_m$ not all distinct are omitted. The result (3.7) is evident from (3.3) for $m=1$, 2, and 3 when (3.4) is specialized, and is readily proved for general m by induction. Only products of degree m occur in (3.7). This simplifying feature does not hold in general. For example, the generalization of $n^{[2]} = n(n-1)$ is, from (3.3),

$$V_2 - V_1 = \sum_{i,j}{}' A_{ij} y_i y_j + \sum_i (A_{ii} - A_i) y_i. \tag{3.8}$$

In (3.8), $A_{ii} - A_i$ does not vanish in general.

It may be worthwhile noting that when $F_i(t-t_i) = \delta(t-t_i)$, the signal integrated over the ith interval is simply the random variable y_i. This allows the definition (2.4) of the basic probability $\mathcal{P}_n(i_1, \cdots, i_n)$ to be interpreted as a mean value of a product of n integrated signals, one over each of the intervals $i_1, i_2, \cdots, i_n$.

So far, the discussion in this section has been mainly in terms of random variables. We now consider averages with respect to the y's of functions of the V_n's in order to represent observable quantities. For our purposes, it is sufficient to find the probability function for V_1 of (3.3a) which is given formally by

$$\text{Prob}(V_1) = \langle \delta(V_1 - \sum_i A_i y_i) \rangle_{\{y_i\}}. \tag{3.9}$$

For the idealized photoelectron counter case, where $V_1 = n(t, t+T)$, the probability $\text{Prob}(V_1) \equiv P_n(t, t+T)$ becomes, according to (3.6) and (3.9),

$$P_n(t, t+T) = \langle \delta_n, \sum_{i=1}^{N} y_i \rangle_{\{y_i\}} = \sum_{\{i_1, i_2, \cdots, i_n\}} \mathcal{P}_n'(i_1, i_2, \cdots, i_n) = \frac{1}{n!} \sum_{i_1, i_2, \cdots, i_n=1}^{N}{}' \mathcal{P}_n'(i_1, i_2, \cdots, i_n), \tag{3.10}$$

which is the probability of exactly n photoionizations in the interval t to $t+T$. Note that the natural definition of $P_n(t, t+T)$ is in terms of the exclusive probability function (2.7) rather than in terms of the nonexclusive

[18] M. G. Kendall and A. Stuart, *The Advanced Theory of Statistics* (C. Griffin and Company, Ltd., London, 1958).
[19] S. S. Wilks, *Mathematical Statistics* (John Wiley & Sons, Inc., New York, 1962).

probability function (2.4). It has the generating function

$$G(u)=\sum_{n=0}^{N} P_n(t,\, t+T)u^n$$
$$=\langle\prod_{i=1}^{N}(y_i u+1-y_i)\rangle_{\{y_i\}}$$
$$=\sum_{m=0}^{\infty}\sum_{\{i_1,i_2,\cdots,i_m\}}\mathcal{P}_m(i_1,i_2,\cdots,i_m)(u-1)^m$$
$$\to\sum_{m=0}^{\infty}\int_t^{t+T}\int_t^{t+T}\cdots\int_t^{t+T} w_m(t_1,t_2,\cdots,t_m)\times dt_1dt_2\cdots dt_m\frac{(u-1)^m}{m!},\quad (3.11)$$

derived from (2.8): $G(u)=\mathcal{G}(u,u,\cdots,u;\,1,1,\cdots,1)$. The expression involving $\mathcal{P}_m$ follows directly on expanding

$$\prod_{i=1}^{N}[1+y_i(u-1)]$$

in powers of $u-1$. The arrow indicates passage to the continuous limit. The last expression yields immediately an expression for $P_n(t,\, t+T)$ in terms of the $w_m(t_1,t_2,\cdots,t_m)$'s

$$P_n(t,\, t+T)=\frac{1}{n!}\left(\frac{d}{du}\right)^n G(u)\,|_{u=0}$$
$$\to\frac{1}{n!}\sum_{m=0}^{\infty}\frac{(-1)^{m-n}}{(m-n)!}\int_t^{t+T}\int_t^{t+T}\cdots\times\int_t^{t+T} w_m(t_1,t_2,\cdots,t_m)dt_1dt_2\cdots dt_m.\quad (3.12)$$

This result is also obtained by substituting (2.12b) into (3.10). An expression for the mean with respect to the y's of the mth factorial moment (3.7) can be derived as follows using $G(u)$:

$$\langle n^{[m]}\rangle_{\{y_i\}}=\sum_{n=0}^{N}\frac{n!}{(n-m)!}P_n(t,\, t+T),$$
$$=(d/du)^m G(u)\,|_{u=1}$$
$$=m!\sum_{\{i_1,i_2,\cdots,i_m\}}\langle\prod_{l=1}^{m} y_{i_l}\times\prod_{q=m+1}^{N}(y_{i_q}u+1-y_{i_q})\rangle_{\{y_i\}}\,|_{u=1}$$
$$=m!\sum_{\{i_1,i_2,\cdots,i_m\}}\mathcal{P}_m(i_1,i_2,\cdots,i_m)$$
$$\to\int_t^{t+T}\int_t^{t+T}\cdots\int_t^{t+T} w_m(t_1,t_2,\cdots,t_m)\times dt_1dt_2\cdots dt_m.\quad (3.13)$$

Alternatively, (3.13) follows immediately on averaging (3.7) with respect to the y's. Note that the integral in (3.12) is just $\langle n^{[m]}\rangle_{\{y_i\}}$. The counterpart for $\langle n^{[m]}\rangle_{\{y_i\}}$ of the fact discussed following (3.7), that $n^{[m]}$ contains only products of degree m, is that $\langle n^{[m]}\rangle_{\{y_i\}}$ contains only $w_K(t_1,t_2,\cdots,t_K)$'s with $K=m$. This is to be contrasted with the fact that in (3.12) $P_n(t,\, t+T)$ depends on all $w_K(t_1,t_2,\cdots,t_K)$'s with $K\geq n$ and that in (3.13), $\langle n^{[m]}\rangle_{\{y_i\}}$ depends on all $P_n(t,\, t+T)$'s with $n\geq m$.

Thus, it has been shown how expressions for quantities of experimental interest such as $\langle n^{[m]}\rangle_{\{y_i\}}$ may be given very simply in terms of joint photoionization probabilities. The use of joint photoionization probabilities has avoided any unnecessary statistical assumptions and the introduction of any physical approximations. In Sec. V the approximations used in Sec. IV to relate the joint photoionization probabilities to correlation functions of the electromagnetic field are applied to the results of this section. It is to be noted, however, that the results of this section are general enough so that they can be applied to situations where the approximations of Sec. IV break down.

IV. QUANTUM THEORY EXPRESSION FOR $w_K(t_1,t_2,\cdots,t_K)$

We will now obtain a quantum-mechanical expression for the joint probability $w_K(t_1,t_2,\cdots,t_K)\Delta t_1\Delta t_2\cdots\Delta t_K$ that photoionization events occur in the intervals t_1 to $t_1+\Delta t_1$, t_2 to $t_2+\Delta t_2$, $\cdots$ and t_K to $t_K+\Delta t_K$. Here, all the Δt's are positive and $t_i\geq t_{i-1}+\Delta t_{i-1}$; we order the times for convenience in the derivation. First we introduce a formal expression for this probability.

For the single time interval t_i, $t_i+\Delta t_i$, we shall assume that the probability for a photoionization to the lth (measurable) photoionization state of a detector when the system at time t_i is in state $|t_i\rangle$ is just the absolute value squared of the amplitude

$$|t_i+\Delta t_i^+\rangle=A_{il}|t_i\rangle,\quad (4.1)$$

for the compound event that the lth photoionization state is unoccupied at time t_i and occupied at time $t_i+\Delta t_i$, that is, that the lth photoionization state *becomes* occupied in the interval t_i, $t_i+\Delta t_i$. In (4.1)

$$A_{il}=P_l U^{-1}(t_i+\Delta t_i,\, t_i)(1-P_l).\quad (4.2)$$

Here P_l is the projection operator for the lth photoionization state, and with $\hbar=1$,

$$U^{-1}(t',t)=\exp[-i\mathcal{H}(t'-t)]\quad (4.3)$$

is the time evolution operator for the system between the times t and t', where $\mathcal{H}$ is the Hamiltonian of the total system of detector, electromagnetic field, and sources. An extension can be made to take account of the probability f_l that, if the detector is in the lth photoionization state at time $t_i+\Delta t_i$, the photoionization is actually measured. This could be done simply by inserting the factor $f_l^{\frac{1}{2}}$ in (4.2).

The amplitude analogous to (4.1) for the multitime probability, corresponding to a sequence of such compound events, is then given by

$$|t_K+\Delta t_K^+\rangle = A_{Kl_K}B_K\cdots A_{2l_2}B_2A_{1l_1}B_1|0\rangle\,, \quad (4.4)$$

where B_i is the time evolution operator which relates $|t_{i-1}+\Delta t_{i-1}^+\rangle$ to $|t_i\rangle$ and where for $i=1$, $|t_{i-1}+\Delta t_{i-1}^+\rangle = |0\rangle$, the state at time $t=0$.

The multitime probability is now expressed in terms of a trace of a density operator as follows:

$$w_K(t_1l_1,t_2l_2,\cdots,t_Kl_K)\Delta t_1\Delta t_2\cdots\Delta t_K = \mathrm{Tr}\rho(t_K+\Delta t_K^+)\,, \quad (4.5)$$

where the density operator $\rho(t_K+\Delta t_K^+)$ is given for the amplitude (4.4) by

$$\rho(t_K+\Delta t_K^+) = |t_K+\Delta t_K^+\rangle\langle t_K+\Delta t_K^+|\,. \quad (4.6)$$

We also introduce in place of A_{il} a "superoperator" $\mathcal{A}_i$ given by the following identity:

$$\mathcal{A}_i\theta \equiv \sum_l A_{il}\theta A_{il}^\dagger\,, \quad (4.7)$$

where θ is an arbitrary operator. This allows us to express the relation between $\rho(t_i)$ and $\rho(t_i+\Delta t_i^+)$ as

$$\rho(t_i+\Delta t_i^+) = \mathcal{A}_i\rho(t_i)\,, \quad (4.8)$$

taking into account transitions to all measurable photoionization states. In a similar manner, to replace B_i, we introduce a superoperator $\mathcal{B}_i$. This operator relates $\rho(t_{i-1}+\Delta t_{i-1}^+)$ and $\rho(t_i)$ in the following way:

$$\rho(t_i) = \mathcal{B}_i\rho(t_{i-1}+\Delta t_{i-1}^+)\,. \quad (4.9)$$

Introduction of the superoperator $\mathcal{B}_i$ is done for more than reasons of formal simplicity; we are going to introduce an approximation for $\mathcal{B}_i$ which cannot be simply expressed as an approximation in terms of B_i. A formal expression for the multitime probability for transitions to all measurable photoionization states is then

$$w_K(t_1,t_2,\cdots,t_K)\Delta t_1\Delta t_2\cdots\Delta t_K = \mathrm{Tr}\,\mathcal{A}_K\mathcal{B}_K\cdots\mathcal{B}_2\mathcal{A}_1\mathcal{B}_1\rho(0)\,. \quad (4.10)$$

We now make the following assumptions concerning the nature of $\mathcal{B}_i$. (1) The density operator for the total system of detector and electromagnetic field together with its sources (by sources we mean here all systems other than the detector which interact with the field) is assumed to be given at time t_i by

$$\rho(t_i) = \rho_D(0)\rho_{\mathrm{F+S}}(t_i)\,. \quad (4.11)$$

In other words, just before the interval of interest, we assume that the density operator of the total system is a product of the density operator $\rho_D(0)$ of the detector and the density operator $\rho_{\mathrm{F+S}}(t_i)$ of the electromagnetic field together with its sources. (2) The density operator $\rho_{\mathrm{F+S}}(t_i)$ is assumed to be given by

$$\rho_{\mathrm{F+S}}(t_i) = \exp(-i\mathcal{L}_{\mathrm{F+S}}\{t_i-(t_{i-1}+\Delta t_{i-1})\}) \times \mathrm{Tr}_D\rho(t_{i-1}+\Delta t_{i-1}^+)\,, \quad (4.12)$$

where $\mathcal{L}_{\mathrm{F+S}}$ is the Liouville operator for the electromagnetic field plus sources and Tr_D indicates trace with respect to the detector variables. This amounts to the assumption that the time development of the density operator of the electromagnetic field plus sources between intervals t_i to $t_i+\Delta t_i$ takes place independently of the detector. In other words, the detector has a negligible effect on the temporal character of the field compared with the effect of the sources. The assumptions (1) and (2) are equivalent to assuming that

$$\mathcal{B}_i = \rho_D(0)\exp[-i\mathcal{L}_{\mathrm{F+S}}(t_i-t_{i-1}-\Delta t_{i-1})]\,\mathrm{Tr}_D\,. \quad (4.13)$$

Two other simplifying assumptions are also introduced. (a) The density operator of the detector $\rho_D(0)$ is assumed to be one corresponding to some equilibrium ensemble, and in particular, an ensemble in which the probability of photoionization states is negligible. (b) It is further assumed that the initial density operator of the system can be written as

$$\rho(t_0+\Delta t_0=0) = \rho_D(0)\rho_{\mathrm{F+S}}(0)\,, \quad (4.14)$$

where $\rho_{\mathrm{F+S}}(0)$ is an appropriately chosen initial density operator for the field plus sources.

Before proceeding further, it is worthwhile to mention that assumption (1) and the assumption that $\rho_D(0)$ is an equilibrium ensemble density operator are familiar ones in the statistical theory of the time development of large systems. Pauli[20] was one of the first to use these statistical assumptions when he introduced the repeated-random-phase assumption (rrpa) in deriving the master equation. Wangsness and Bloch[21] modified the rrpa to apply to interacting systems, a problem analogous to the present problem. These assumptions have been found to be justified under certain conditions by approximating the correct dynamical equations. This justification has been given by Van Hove[22] for the master equation and by Argyres[23] and Argyres and Kelley[24] for the system studied by Wangsness and Bloch. It is not the aim of the present paper to justify these assumptions for the problem studied here, but only to appeal to their physical reasonableness.

With regard to assumption (2), our treatment does not explicitly take account of modification of the field distribution engendered by the presence of the detector, particularly in the region of the detector where, for example, one may expect attenuation by the detector. We assume that this modification can be adequately approximated by selecting an appropriate "*effective volume*" of the detector within which the action of the unmodified field is confined. This "effective volume" is related to the damping length of the field inside the detector. It should be noted that the modification of the spatial dependence of the field in the region of the detector does not contradict the assumption that the field

[20] W. Pauli, *Festschrift zum 60 Geburtstage A. Sommerfelds* (S. Hirzel Verlag, Leipzig, 1928), p. 30.
[21] R. K. Wangsness and F. Bloch, Phys. Rev. **89**, 728 (1953).
[22] L. Van Hove, Physica **23**, 441 (1957).
[23] P. N. Argyres, *Proceedings of the Eindhoven Conference on Magnetic and Electric Resonance and Relaxation*, edited by J. Smidt (North-Holland Publishing Company, Amsterdam, 1963), p. 555.
[24] P. N. Argyres and P. L. Kelley, Phys. Rev. **134**, A98 (1964).

just outside the detector is determined to a high degree of approximation by the sources alone.

As a consequence of assumption (a), the second P_l in the definition (4.2) of A_{il} drops out, since it acts on $\rho_D(0)$. Using this fact and (4.11), we rewrite (4.10) in the form

$$w_K(t_1,t_2,\cdots,t_K)\Delta t_1\Delta t_2\cdots\Delta t_K$$
$$=\mathrm{Tr}_{\mathrm{F+S}}T[\prod_{i=1}^{K}\mathrm{Tr}_D\mathcal{R}\mathcal{S}(t_i+\Delta t_i,\,t_i)\rho_D(0)$$
$$\times\exp(-i\mathcal{L}_{\mathrm{F+S}}\{t_i-(t_{i-1}+\Delta t_{i-1})\})]\rho_{\mathrm{F+S}}(0)\,. \quad (4.15)$$

Here we have expressed $\mathcal{A}_i$ in terms of $\mathcal{R}$ and the super time evolution operator $\mathcal{S}(t',t)$ defined by

$$\mathcal{R}\theta=\sum_l P_l\theta P_l\,, \quad (4.16)$$

and

$$\mathcal{S}(t',t)\theta=U^{-1}(t',t)\theta U(t',t)\,, \quad (4.17)$$

θ being an arbitrary operator. In (4.15), $\mathrm{Tr}_{\mathrm{F+S}}$ indicates a trace with respect to the field-plus-source variables, and T indicates time ordering with later times to the left.

The time development of the total system during the intervals t_i to $t_i+\Delta t_i$ is now approximated. It is assumed that the interaction between the field and the detector contains only terms linear in the creation or annihilation of photoionizations. The time development operators are expanded as a power series in the interaction Hamiltonian, and only the linear terms in the interaction Hamiltonian are retained, the higher order terms being neglected and the zeroth-order term vanishing when the projection operator $\mathcal{R}$ is applied, because the time development operator to zeroth order does not produce photoionizations. On expanding $U^{\pm1}(t',t)$ to first order, the following is obtained:

$$U^{\pm1}(t',t)\approx U_0^{\pm1}(t',t)+U_1^{\pm1}(t',t)\,, \quad (4.18)$$

where

$$U_0^{\pm1}(t',t)=\exp[\pm i\mathcal{H}_0(t'-t)]\,; \quad (4.19)$$

$$U_1^{+1}(t',t)=i\,\exp(-i\mathcal{H}_0t)F(t',t)\,\exp(i\mathcal{H}_0t')\,, \quad (4.20a)$$

and

$$U_1^{-1}(t',t)=-i\,\exp(-i\mathcal{H}_0t')F(t',t)\,\exp(i\mathcal{H}_0t)\,. \quad (4.20b)$$

Here $\mathcal{H}=\mathcal{H}_0+\mathcal{H}_I$, where $\mathcal{H}_0$ is the total Hamiltonian except for the interaction term between the field and the detector, and $\mathcal{H}_I$ is the interaction Hamiltonian. The sources are assumed not to interact with the detector. Also, $F(t',t)$ is given by

$$F(t',t)=\int_t^{t'}d\tau\mathcal{H}_I(\tau)\,, \quad (4.21)$$

where

$$\mathcal{H}_I(\tau)=[\exp(i\mathcal{L}_0\tau)\mathcal{H}_I]$$
$$=\exp(i\mathcal{H}_0\tau)\mathcal{H}_I\exp(-i\mathcal{H}_0\tau)\,. \quad (4.22)$$

Here $\mathcal{L}_0=\mathcal{L}_D+\mathcal{L}_{\mathrm{F+S}}$ is the Liouville operator corresponding to $\mathcal{H}_0$. $\mathcal{L}_D$ is the detector part of $\mathcal{L}_0$ and $\mathcal{L}_{\mathrm{F+S}}$ is the field-plus-source part. Furthermore, we write for the part of $\mathcal{H}_I(t)$ which is linear in the fields

$$\mathcal{H}_I(t)=\mathcal{H}_I^{(+)}(t)+\mathcal{H}_I^{(-)}(t)\,, \quad (4.23)$$

where

$$\mathcal{H}_I^{(\pm)}(t)=\sum_\mu\int d\mathbf{r}C_\mu^{(\pm)}(\mathbf{r},t)A_\mu^{(\pm)}(\mathbf{r},t)\,. \quad (4.24)$$

$C^{(+)}$ and $C^{(-)}$ are detector operators which create a particle in one state and annihilate it in another. $A^{(+)}$ and $A^{(-)}$ are, respectively, the positive and negative frequency parts of the vector potential operator.

Substituting (4.20a) and (4.20b) into (4.15), we obtain

$$w_K(t_1,t_2,\cdots,t_K)\Delta t_1\Delta t_2\cdots\Delta t_K$$
$$=\sum_{l_1,l_2,\cdots,l_K}\mathrm{Tr}_{\mathrm{F+S}}\,\mathrm{Tr}_D\{P_{l_K}F(t_K+\Delta t_K,\,t_K)\rho_D(0)$$
$$\times\mathrm{Tr}_D\cdots[P_{l_2}F(t_2+\Delta t_2,\,t_2)\rho_D(0)$$
$$\times\mathrm{Tr}_D\{P_{l_1}F(t_1+\Delta t_1,\,t_1)\rho_D(0)\rho_{\mathrm{F+S}}(0)$$
$$\times F(t_1+\Delta t_1,\,t_1)P_{l_1}\}F(t_2+\Delta t_2,\,t_2)P_{l_2}]\cdots$$
$$\times F(t_K+\Delta t_K,\,t_K)P_{l_K}\}\,. \quad (4.25)$$

The integral denoted by $F(t+\Delta t,\,t)$ becomes on integration

$$F(t+\Delta t,\,t)=\left(\left[\frac{\exp(i(\mathcal{L}_D+\mathcal{L}_{\mathrm{F+S}})\Delta t)-1}{i(\mathcal{L}_D+\mathcal{L}_{\mathrm{F+S}})}\right]\mathcal{H}_I(t)\right)\,. \quad (4.26)$$

The dominant contributions to the multitime probability come from singularities in the F's associated with vanishing energy denominators.

To separate the field-plus-source variables from the detector variables so that the traces can be taken independently, we introduce the following approximations with the eventual result that the square bracket in (4.26) depends only on detector variables. We replace $\mathcal{L}_{\mathrm{F+S}}$ by $\mathcal{L}_{\mathrm{F}}$ in the square bracket term of (4.26), in other words, approximating $\mathcal{L}_{\mathrm{F+S}}$ to zeroth order in the interaction between the field and the sources. The projection operator P_l operating to the left of F, and $\rho_D(0)$ operating to the right give rise to positive matrix elements of the $\mathcal{L}_D$ operator in the square bracket term. In other words, when matrix elements are taken, $\mathcal{L}_D$ becomes the energy of the detector in a photoionization state minus the energy in a nonphotoionized state, a positive quantity. In order for F to give rise to a singularity in energy, the matrix elements of its argument $\mathcal{L}_0$ must vanish. For $\mathcal{L}_0$ to vanish, the matrix elements of $\mathcal{L}_{\mathrm{F}}$ must be negative since $\mathcal{L}_D$ is positive. This will occur only when the term in square brackets operates on the positive frequency part of the vector potential operator in the interaction Hamiltonian. We further replace $\mathcal{L}_F$ by $-\omega_0$, ω_0 being the average frequency of the radiation field. This is the approximation that the radiation field frequencies appearing in the terms involving detector variables can be replaced by some average frequency of the radiation field. Since the interval Δt will be of the order of the electronic correlation time τ_c of the detector, the approximation of replacing $\mathcal{L}_F$ by $-\omega_0$ is valid provided $\Delta\omega\tau_c \ll 1$, where $\Delta\omega$ is the bandwidth of the field. Using the above approximations, we obtain

$$P_l F(t+\Delta t, t) \approx \sum_{\mu'} \int d\mathbf{r}' A_{\mu'}^{(+)}(\mathbf{r}',t) \times P_l\left(\left[\frac{\exp(i(\mathcal{L}_D-\omega_0)\Delta t)-1}{i(\mathcal{L}_D-\omega_0)}\right] C_{\mu'}^{(+)}(\mathbf{r}',t)\right), \quad (4.27)$$

and using similar arguments

$$F(t+\Delta t, t) P_l \approx \sum_{\mu} \int d\mathbf{r} \left(\left[\frac{\exp(i(\mathcal{L}_D+\omega_0)\Delta t)-1}{i(\mathcal{L}_D+\omega_0)}\right] C_{\mu}^{(-)}(\mathbf{r},t)\right) P_l A_{\mu}^{(-)}(\mathbf{r},t). \quad (4.28)$$

Substituting (4.27) and (4.28) into (4.25) and disentangling the field operators from the detector operators, we obtain for the multitime transition rate

$$w_K(t_1,t_2,\cdots,t_K) = \sum_{\substack{\mu_1\cdots\mu_K \\ \mu_1'\cdots\mu_K'}} \int d\mathbf{r}_1\cdots d\mathbf{r}_K d\mathbf{r}_1'\cdots d\mathbf{r}_K' \left(\prod_{i=1}^{K} \mathcal{K}_{\mu_i\mu_i'}(\mathbf{r}_i,\mathbf{r}_i')\right) \times G^{(K)}_{\mu_1\cdots\mu_K,\mu_1'\cdots\mu_K'}(\mathbf{r}_1t_1,\cdots,\mathbf{r}_Kt_K;\mathbf{r}_1't_1,\cdots,\mathbf{r}_K't_K), \quad (4.29)$$

where

$$G^{(K)}_{\mu_1\cdots\mu_K,\mu_1'\cdots\mu_K'}(\mathbf{r}_1t_1,\cdots,\mathbf{r}_Kt_K;\mathbf{r}_1't_1,\cdots,\mathbf{r}_K't_K) \equiv \mathrm{Tr}_{F+S}\{\rho_{F+S}(0) \prod_{i=1}^{K} A_{\mu_i}^{(-)}(\mathbf{r}_i,t_i) \bar{T} \prod_{i=1}^{K} A_{\mu_i'}^{(+)}(\mathbf{r}_i',t_i)\}, \quad (4.30)$$

and

$$\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}') \equiv (\Delta t)^{-1} \sum_l \mathrm{Tr}_D\left\{P_l\left(\left[\frac{\exp(i(\mathcal{L}_D-\omega_0)\Delta t)-1}{i(\mathcal{L}_D-\omega_0)}\right] C_{\mu'}^{(+)}(\mathbf{r}')\right) \times \rho_D(0)\left(\left[\frac{\exp(i(\mathcal{L}_D+\omega_0)\Delta t)-1}{i(\mathcal{L}_D+\omega_0)}\right] C_{\mu}^{(-)}(\mathbf{r})\right) P_l\right\}. \quad (4.31)$$

In (4.30) $\bar{T}$ indicates time ordering with later times to the right in the first product and later times to the left in the second product. Note that with the introduction of time ordering we may drop the restriction on w_K that $t_i < t_{i+1}$. We have used the fact that the detector time development operator $\exp(-i\mathcal{H}_D t)$ commutes with $\rho_D(0)$, $\mathcal{L}_D$, and the P's to eliminate the time dependence from the C operators in obtaining (4.31). The multitime transition rate has here been expressed as integrals of products of detector terms and a term due to the field; it has the form suggested in Sec. II, (2.6). $G^{(K)}$ is the Green's function or correlation function for the field and its sources.

The multitime transition rate can be expressed alternatively as

$$w_K(t_1,t_2,\cdots,t_K) = \mathrm{Tr}_{F+S}\rho_{F+S}(0) N\{\mathfrak{M}(t_1)\mathfrak{M}(t_2)\cdots\mathfrak{M}(t_K)\} \quad (4.32)$$

by rearranging (4.29). Here N indicates the normal product where all the $A^{(-)}$'s lie to the left of all the $A^{(+)}$'s and, in addition, the time ordering occurring in (4.30). Also, $\mathfrak{M}(t)$ is given by

$$\mathfrak{M}(t) = \sum_{\mu\mu'} \int\int d\mathbf{r} d\mathbf{r}' \mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}') \times A_{\mu}^{(-)}(\mathbf{r},t) A_{\mu'}^{(+)}(\mathbf{r}',t). \quad (4.33)$$

The correlation function $\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')$ is Hermitian and $\mathfrak{M}(t)$ is non-negative, facts used in Sec. V. The Hermiticity property, $\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}') = \mathcal{K}_{\mu'\mu}(\mathbf{r}',\mathbf{r})^*$, is seen to follow directly from (4.31) when the trace is written in the form

$$\mathrm{Tr}_D(J_{\mu'}(\mathbf{r}')P_l)^\dagger \rho_D(0)(J_{\mu}(\mathbf{r})P_l), \quad (4.34)$$

where

$$J_{\mu}(\mathbf{r}) = \left(\left[\frac{\exp(i(\mathcal{L}_D+\omega_0)\Delta t)-1}{i(\mathcal{L}_D+\omega_0)}\right] C_{\mu}^{(-)}(\mathbf{r})\right). \quad (4.35)$$

Also, the operator $\mathfrak{M}(t)$ can be written in the form

$$\sum_l \mathrm{Tr}_D(\mathfrak{X}(t)P_l)^\dagger \rho_D(0)(\mathfrak{X}(t)P_l), \quad (4.36)$$

where

$$\mathfrak{X}(t)=\sum_\mu \int d\mathbf{r} J_\mu(\mathbf{r}) A_\mu^{(-)}(\mathbf{r},t). \quad (4.37)$$

Consequently, $\mathfrak{M}(t)$ is non-negative in the sense that it is a non-negative number if $A^{(-)}(\mathbf{r},t)$ is regarded as an arbitrary complex field function; also $\mathrm{Tr_{F+S}}\rho_{\mathrm{F+S}}\mathfrak{M}(t)\geq 0$.

In Sec. V and Appendix B, use is also made of the assumed lattice translational symmetry property, $\mathcal{K}_{\mu\mu'}(\mathbf{r}+\mathbf{R}_m, \mathbf{r}'+\mathbf{R}_m)=\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')$, in case the photoemissive surface is a crystal and $\mathbf{R}_m$ is one of its primitive lattice translation vectors.

Finally, as a consequence of assumption (a), we may rewrite (4.31) as

$$\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')=\sum_l \mathrm{Tr}_D\left\{\rho_D(0)\left(\Delta t\left[\frac{\sin(\{\mathcal{L}_D+\omega_0\}\frac{1}{2}\Delta t)}{\{\mathcal{L}_D+\omega_0\}\frac{1}{2}\Delta t}\right]^2 C_\mu^{(-)}(\mathbf{r})\right)P_l C_{\mu'}^{(+)}(\mathbf{r}')\right\} \quad (4.38)$$

$$=2\pi\sum_l \mathrm{Tr}_D\{\rho_D(0)(\delta(\mathcal{L}_D+\omega_0)C_\mu^{(-)}(\mathbf{r}))P_l C_{\mu'}^{(+)}(\mathbf{r}')\}, \quad (4.39)$$

where we have assumed in the second expression that

$$\Delta t\left[\frac{\sin(\{\mathcal{L}_D+\omega_0\}\frac{1}{2}\Delta t)}{\{\mathcal{L}_D+\omega_0\}\frac{1}{2}\Delta t}\right]^2 \to 2\pi\delta(\mathcal{L}_D+\omega_0), \quad (4.40)$$

making use of the assumption that $\Delta t \gtrsim \tau_c$. We now recall that the condition of Sec. II that, at most, a single count occurs in a subinterval was expressed by the inequality $\Delta t \ll R^{-1}$, where R is the counting rate. Thus, for the derivation of Sec. II and the present quantum-mechanical derivation to be valid, the inequality $R^{-1}\gg\tau_c$ must be satisfied. This inequality is generally not difficult to satisfy in practice.

Thus, we have seen that the multitime transition probabilities can be related to correlation functions of the positive and negative frequency parts of the electromagnetic field. We note that these correlation functions are expectation values taken in the system of field and its sources, since the field operators $A^{(+)}$ and $A^{(-)}$ evolve according to the dynamics of the interacting system.

We now discuss an assumption made implicitly by other authors and which is of far reaching consequence. The time evolution of the field operators under the full Hamiltonian of field-plus-sources is replaced by the evolution under the field Hamiltonian alone. In other words, we make the replacement

$$A_\mu^{(\pm)}(\mathbf{r},t)=\exp(i\mathcal{L}_{\mathrm{F+S}}t)A_\mu^{(\pm)}(\mathbf{r}) \to \exp(i\mathcal{L}_{\mathrm{F}}t)A_\mu^{(\pm)}(\mathbf{r})\equiv A_\mu'^{(\pm)}(\mathbf{r},t). \quad (4.41)$$

The correlation function $G^{(K)}$ of (4.29) is then replaced by one of the same form

$$G^{(K)}{}_{\mu_1\cdots\mu_K,\mu_1'\cdots\mu_K'}(\mathbf{r}_1t_1,\cdots,\mathbf{r}_Kt_K;\mathbf{r}_1't_1,\cdots,\mathbf{r}_K't_K)$$
$$=\mathrm{Tr_F}\rho_{\mathrm{F}}(0)\prod_{i=1}^K A_{\mu_i}'^{(-)}(\mathbf{r}_i,t_i)\prod_{i=1}^K A_{\mu_i'}'^{(+)}(\mathbf{r}_i',t_i), \quad (4.42)$$

in which the $A^{(\pm)}$'s are replaced by $A'^{(\pm)}$'s, $\rho_{\mathrm{F+S}}$ is replaced by ρ_{F} which does not depend on source variables, and $\mathrm{Tr_{F+S}}$ is replaced by $\mathrm{Tr_F}$. The time ordering operator $\bar{T}$ of (4.30) is no longer necessary since $[A'^{(\pm)}(\mathbf{r}_i,t_i),A'^{(\pm)}(\mathbf{r}_j,t_j)]=0$. Similarly, (4.32) is replaced by an expression of the same form in which $A_\mu^{(-)}(\mathbf{r},t)$ and $A_{\mu'}^{(+)}(\mathbf{r}',t)$ in the definition of $\mathfrak{M}(t)$ are replaced by $A_\mu'^{(-)}(\mathbf{r},t)$ and $A_{\mu'}'^{(+)}(\mathbf{r}',t)$, and in which $\rho_{\mathrm{F+S}}$ is replaced by ρ_{F} and $\mathrm{Tr_{F+S}}$ is replaced by $\mathrm{Tr_F}$. There thus results a description of the multitime transition probabilities in terms of field-correlation functions (4.42) involving the free development of the electromagnetic field. The correlation functions used by Glauber,[11–13] Sudarshan,[14,15] and others are of this type. A question which remains unanswered is under what circumstances and how this simplification can be justified.

The discussion up to this point has concerned the calculation of photoionization probabilities for a single interval t to $t+T$. The result can be extended to include the average photoionization probabilities for a number of intervals of length T by appropriately redefining $\rho_{\mathrm{F+S}}(0)$ to be the average of the initial density operators of the intervals. In the case of a nonstationary density operator the average density operator will be, in general, different from the density operator at any one time. This point is discussed further in Sec. V.

V. QUANTUM THEORY OF COUNTING STATISTICS

We proceed now to evaluate (3.12), (3.11), and (3.13) for $P_n(t,t+T)$, $G(u)$, and $\langle n^{[m]}\rangle_{\{y_i\}}$ in terms of an explicit quantum-mechanical expression for $w_m(t_1,t_2,\cdots,t_m)$, namely, (4.32). Then (3.13) becomes

$$\langle n^{[m]}\rangle \to \mathrm{Tr_{F+S}}\rho_{\mathrm{F+S}}N\{\mathfrak{M}^m\}, \quad (5.1)$$

where the subscript $\{y_i\}$ has been suppressed, and the argument of $\rho_{\mathrm{F+S}}$ has been dropped for simplicity; (3.11) becomes

$$G(u)\to \mathrm{Tr_{F+S}}\rho_{\mathrm{F+S}}N\left\{\sum_{m=0}\frac{\mathfrak{M}^m(u-1)^m}{m!}\right\}$$
$$=\mathrm{Tr_{F+S}}\rho_{\mathrm{F+S}}N\{\exp[(u-1)\mathfrak{M}]\}, \quad (5.2)$$

and (3.12) becomes

$$P_n(t,t+T)\to \mathrm{Tr_{F+S}}\rho_{\mathrm{F+S}}N\{(\mathfrak{M}^n/n!)\exp(-\mathfrak{M})\}, \quad (5.3)$$

where

$$\mathfrak{N}=\mathfrak{N}(t,t+T)=\int_t^{t+T}\mathfrak{M}(t')dt', \quad (5.4)$$

and $\mathfrak{M}(t)$ is given by (4.33). The quite general expressions (5.1), (5.2), and (5.3) have an appealing simplicity. The quantity in braces in (5.3) is an operator with the form of the Poisson probability distribution for n counts in the interval $(t, t+T)$, with the operator $\mathfrak{N}$ representing the mean number of counts in the interval. The quantity in braces in (5.2) is the corresponding Poisson distribution generating function operator.

These expressions can be evaluated further most simply by using the "coherent state" representation of the field (see Ref. 13, particularly Sec. IX). To proceed, we make use of the simplifying assumption (4.41) regarding the time evolution of the field operators. Then $\rho_{\mathrm{F+S}}$ is replaced by ρ_{F} and $\mathrm{Tr}_{\mathrm{F+S}}$ is replaced by Tr_{F}, as in (4.42). For simplicity of notation we shall, however, henceforth drop both the prime on $A_\mu'^{(\pm)}(\mathbf{r},t)$ of (4.41) and the subscript F on ρ_{F} and Tr_{F}. Expressed in the coherent state representation, ρ has the form

$$\rho=\int\prod_k(\pi^{-2}d^2\alpha_k d^2\beta_k)\,|\{\alpha\}\rangle\langle\{\alpha\}|\rho|\{\beta\}\rangle\langle\{\beta\}|\,, \quad (5.5)$$

where the coherent state $|\{\alpha\}\rangle$ is an eigenstate of $\mathbf{A}^{(+)}(\mathbf{r},t)$

$$\mathbf{A}^{(+)}(\mathbf{r},t)|\{\alpha\}\rangle=\mathfrak{A}(\{\alpha\},\mathbf{r},t)|\{\alpha\}\rangle\,, \quad (5.6a)$$

$$\langle\{\alpha\}|\mathbf{A}^{(-)}(\mathbf{r},t)=\mathfrak{A}(\{\alpha\},\mathbf{r},t)^*\langle\{\alpha\}|\,. \quad (5.6b)$$

In terms of the mode expansion for $\mathbf{A}^{(+)}(\mathbf{r},t)$

$$\mathbf{A}^{(+)}(\mathbf{r},t)=\sum_k\left(\frac{\hbar c^2}{2\omega_k}\right)^{1/2}a_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t}\,, \quad (5.7)$$

the eigenvalue

$$\mathfrak{A}(\{\alpha\},\mathbf{r},t)=\sum_k\left(\frac{\hbar c^2}{2\omega_k}\right)^{1/2}\alpha_k\mathbf{u}_k(\mathbf{r})e^{-i\omega_k t} \quad (5.8)$$

is given by replacing each mode annihilation operator a_k by its eigenvalue, the mode coordinate α_k. The set $\{\alpha\}$ of all mode coordinates thus characterizes the coherent state $|\{\alpha\}\rangle$ and the eigenvalue (5.8).

In the coherent-state representation, we then have

$$\mathrm{Tr}\rho N\{f(\mathfrak{N})\}=\int\prod_k(\pi^{-2}d^2\alpha_k d^2\beta_k)\times\langle\{\alpha\}|\rho|\{\beta\}\rangle\langle\{\beta\}|\{\alpha\}\rangle f(\mathfrak{N}_{\beta\alpha})\,, \quad (5.9)$$

where $f(x)$ is some function of x and

$$\mathfrak{N}_{\beta\alpha}=\int_t^{t+T}dt'\int\int d\mathbf{r}d\mathbf{r}'\sum_{\mu,\mu'}\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')\times\mathfrak{A}_\mu(\{\beta\},\mathbf{r},t')^*\mathfrak{A}_{\mu'}(\{\alpha\},\mathbf{r}',t')\,, \quad (5.10a)$$

$$=\sum_{k,k'}\mathfrak{R}_{kk'}\beta_k{}^*\alpha_{k'}\,, \quad (5.10b)$$

$$=\boldsymbol{\beta}^\dagger\mathfrak{R}\boldsymbol{\alpha}\,, \quad (5.10c)$$

with

$$\mathfrak{R}_{kk'}=\tfrac{1}{2}\hbar c^2(\omega_k\omega_{k'})^{-1/2}\int_t^{t+T}dt'\int\int d\mathbf{r}d\mathbf{r}'\times\sum_{\mu,\mu'}\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')u_{k\mu}(\mathbf{r})^*u_{k'\mu'}(\mathbf{r}')e^{i(\omega_k-\omega_{k'})t'}. \quad (5.11)$$

On using the mode expansion (5.8) in (5.10a), one obtains (5.10b) which is written in matrix notation in (5.10c). The matrix $\mathfrak{R}$ is Hermitian and non-negative definite, as a consequence of the Hermiticity of $\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')$ and the non-negativity of $\mathfrak{M}(t)$ proved in Sec. IV. One sees that the time integral in (5.11) can be evaluated immediately, and that the diagonal elements of $\mathfrak{R}$ are proportional to T.

Assuming $\Delta\omega/\omega_0\ll 1$, where ω_0 is a central frequency and $\Delta\omega$ is the bandwidth of the field [see the discussion after (4.26)], we can approximate $(\omega_k\omega_{k'})^{-1/2}$ in (5.11) by $(\omega_k\omega_{k'})^{1/2}/\omega_0{}^2$. In this way, $\mathfrak{N}_{\beta\alpha}$ can be expressed in terms of eigenvalues $\boldsymbol{\mathcal{E}}(\{\alpha\},\mathbf{r},t)$ of the positive frequency part of the electric field operator $\mathbf{E}^{(+)}(\mathbf{r},t)=c^{-1}(\partial/\partial t)\mathbf{A}^{(+)}(\mathbf{r},t)$

$$\mathfrak{N}_{\beta\alpha}=(c^2/\omega_0{}^2)\int_t^{t+T}dt'\int\int d\mathbf{r}d\mathbf{r}'\sum_{\mu,\mu'}\mathcal{K}_{\mu\mu'}(\mathbf{r},\mathbf{r}')\times\mathcal{E}_\mu(\{\beta\},\mathbf{r},t')^*\mathcal{E}_{\mu'}(\{\alpha\},\mathbf{r}',t') \quad (5.12)$$

rather than in terms of the $\mathfrak{A}(\{\alpha\},\mathbf{r},t)$'s.

Equations (5.1), (5.2), and (5.3) yield special cases of (5.9), with $f(x)=x^m$, $e^{(u-1)x}$, and $x^ne^{-x}/n!$, respectively.

If, in the coherent-state representation, ρ is represented entirely by diagonal matrix elements

$$\langle\{\alpha\}|\rho|\{\beta\}\rangle=\langle\{\alpha\}|\rho|\{\alpha\}\rangle\prod_k\delta^{(2)}(\alpha_k-\beta_k) \quad (5.13a)$$

$$=P(\{\alpha\})\prod_k\pi^2\delta^{(2)}(\alpha_k-\beta_k)\,, \quad (5.13b)$$

then

$$\rho=\int P(\{\alpha\})|\{\alpha\}\rangle\langle\{\alpha\}|\prod_k d^2\alpha_k\,. \quad (5.14)$$

The representation (5.14), in which ρ is expressed as an incoherent superposition of coherent states, is called the *P representation* by Glauber.[13] Sudarshan has shown[14,15] that any ρ can be represented formally in the P representation. However, this requires in some cases that the $P(\{\alpha\})$ in (5.14) be highly singular. In particular, this is clearly the case when the number of photons represented by ρ is bounded, for then the sums in Eq. (6) of Sudarshan[14] are finite, and $P(\{\alpha\})$ involves derivatives of δ functions. Such highly singular $P(\{\alpha\})$'s have no classical analogs,[25] and ρ may then be more usefully described in the more general coherent state representation of (5.5).

On using (5.14), (5.9) reduces to

$$\mathrm{Tr}\rho N\{f(\mathfrak{N})\}=\int P(\{\alpha\})f(\mathfrak{N}_{\alpha\alpha})\prod_k d^2\alpha_k \quad (5.15a)$$

$$=\int_0^\infty f(\nu)W(\nu)d\nu\,, \quad (5.15b)$$

where

$$W(\nu)=\int \delta(\nu-\mathfrak{N}_{\alpha\alpha})P(\{\alpha\})\prod_k d^2\alpha_k \quad (5.16)$$

and (5.3), (5.2), and (5.1) simplify correspondingly to

$$P_n(t,t+T)=\int_0^\infty \frac{\nu^n}{n!}e^{-\nu}W(\nu)d\nu\,, \quad (5.17)$$

$$G(u)=\int_0^\infty e^{(u-1)\nu}W(\nu)d\nu\,, \quad (5.18)$$

$$\langle n^{[m]}\rangle=\int_0^\infty \nu^m W(\nu)d\nu\,. \quad (5.19)$$

The probability $P_n(t, t+T)$ is thus a *compound Poisson distribution*,[26,27] some general properties of which are summarized in Appendix A.

The case of a field generated by a prescribed source is treated by Glauber.[13] Equations (5.15)–(5.19) apply also in this case provided $\mathfrak{N}_{\alpha\alpha}$ is replaced by $\overline{\mathfrak{N}}_{\alpha\alpha}$. The latter is obtained from $\mathfrak{N}_{\alpha\alpha}$, as given by (5.10a), by replacing α_k by $\alpha_k+\alpha_k(t')$, where $\alpha_k(t)$ is a function of time determined by the electric current distribution source, and is given by Eq. (9.21) of Ref. 13. Thus, $P_n(t, t+T)$ is a compound Poisson distribution when the field is generated by a prescribed source, as well as when the field moves freely, the latter having been assumed in deriving (5.17)–(5.19).

The compound Poisson distribution has been obtained by Mandel[9,10] for the photocounting problem using other methods described in the Introduction. Note that it is not simply ensemble averaging which gives rise to the compound Poisson distribution. Even in the absence of ensemble averaging, $P_n(t, t+T)$ is not a simple Poisson distribution unless the nonstatistical state is also a coherent state. An example of such a noncoherent state is an eigenstate of the number operator of the field.

The variable $\mathfrak{N}_{\alpha\alpha}$, with respect to which averaging occurs in the compound Poisson distribution (5.17), contains the correlation function of the photoemissive surface. This correlation function deals with two properties of the detector not treated in the earlier derivation[9] of the compound Poisson distribution. It takes account of spatial correlations in the detector, which may be characterized conveniently by a detector correlation length. Also, it takes account of the region in the detector in which the field effectively acts. These properties are examined in some detail in Appendix B. In the form given by (5.12), $\mathfrak{N}_{\alpha\alpha}$ reduces to $\alpha\int_t^{t+T} I(t')dt'$ for small enough detector correlation length and small enough effective detector volume so that $\mathfrak{S}$, given by (B2), is expressed by (B12) and is effectively a constant. Here $\alpha\int_t^{t+T} I(t')dt'$, in the notation of Ref. 9, denotes the intensity at the detector surface integrated over the interval t to $t+T$ multiplied by α, the quantum sensitivity of the photodetector.

We now examine properties of $P_n(t, t+T)$ for some particular $P(\{\alpha\})$'s.

Coherent field. If ρ represents a pure coherent state $|\{\alpha\}\rangle$, generated perhaps by a single coherent source, then

$$P(\{\alpha'\})=\prod_k \delta^{(2)}(\alpha_k'-\alpha_k)\,, \quad (5.20)$$

so that

$$W(\nu)=\delta(\nu-\mathfrak{N}_{\alpha\alpha})\,, \quad (5.21)$$

and $P_n(t, t+T)$ becomes simply the Poisson distribution

$$P_n(t,t+T)=\nu^n e^{-\nu}/n! \quad (5.22)$$

with $\nu=\mathfrak{N}_{\alpha\alpha}$.

A coherent state is a special case of a pure (nonstatistical) quantum state of the field $|\rangle$. It is noteworthy that the distribution $P_n(t, t+T)$ does not have any simple form, such as the form (5.22), when ρ represents a general pure state of the field $\rho=|\rangle\langle|$. This can be seen by using $\rho=|\rangle\langle|$ in (5.9).

Equation (5.20) is an appropriate choice for the density operator of a coherent field when the initial state is known. It is unnecessary to include explicitly the distribution of over-all phase, since the distribution of over-all phase does not affect the field-correlation functions.[28] For many types of experiments in particular, for a typical photoelectron counting experiment, the measurements are not performed by repeatedly returning the system to a known initial state (apart from the arbitrary over-all phase factor). A coherent field evolves with time from one coherent state to another. Which of these coherent states the field happens to be in at the initial time of any counting interval is unknown in the ordinary counting experiment. Consequently, an appropriate density operator for such an experiment on a coherent field is not a coherent-state density operator, but rather some average of the coherent-state density operators through which the field evolves with time.

Field with amplitudes independent of phases (the product

[25] See Ref. 13, footnotes 11 and 15.

[26] W. Feller, *An Introduction to Probability Theory and Its Applications* (John Wiley & Sons, Inc., New York, 1957), 2nd ed.

[27] Emanuel Parzen, *Stochastic Processes* (Holden-Day, Inc., San Francisco, 1962).

[28] See Ref. 13, Sec. X.

P case). A density operator for the field with mode amplitudes distributed independently of the phases may be useful in describing experiments involving randomization of the phases. For such a field $P(\{\alpha\})$ has the product form

$$P(\{\alpha\})=Q(\{|\alpha|\})R(\{\phi\}), \tag{5.23}$$

with

$$\int Q(\{|\alpha|\})\prod_k |\alpha_k| d|\alpha_k|=1, \tag{5.24a}$$

and

$$\int R(\{\phi\})\prod_k d\phi_k=1. \tag{5.24b}$$

If, in addition, the mode phases are mutually independent, then

$$R(\{\phi\})=\prod_k f_k(\phi_k), \tag{5.25}$$

where the function f_k is normalized according to $\int_0^{2\pi} f_k(\phi)d\phi=1$. The function f_k can be defined as

$$f_k(\phi)=\sum_{n=-\infty}^{\infty} g_k(\phi-2\pi n)\Big/\int_{-\infty}^{\infty} g_k(\phi)d\phi \tag{5.26}$$

in terms of a function $g(\phi)$ defined for $-\infty<\phi<\infty$.

The properties (5.23) and (5.25) yield a simplification in the expression for $\langle n^{[m]}\rangle$, but do not appear to lead to any significant simplification of the expressions for $G(u)$ and $P_n(t,t+T)$.

To evaluate the moments (5.19) let us first calculate the phase average of $\mathfrak{N}_{\alpha\alpha}{}^m$

$$\begin{aligned}\langle n^{[m]}\rangle_\phi=\langle\mathfrak{N}_{\alpha\alpha}{}^m\rangle_\phi&\equiv\int\mathfrak{N}_{\alpha\alpha}{}^m R(\{\phi\})\prod_k d\phi_k\\&=\sum_{k_1,k_2,\cdots,k_{2m}} C_{k_1k_{m+1}}C_{k_2k_{m+2}}\cdots C_{k_mk_{2m}}\\&\quad\times\langle e^{-i(\phi_{k_1}+\phi_{k_2}+\cdots+\phi_{k_m}-\phi_{k_{m+1}}-\phi_{k_{m+2}}-\cdots-\phi_{k_{2m}})}\rangle_\phi.\end{aligned} \tag{5.27}$$

Here we have used

$$\mathfrak{N}_{\alpha\alpha}=\sum_{k,k'} C_{kk'}e^{-i(\phi_k-\phi_{k'})} \tag{5.28}$$

obtained from (5.10) by inserting $\alpha_k=|\alpha_k|e^{i\phi_k}$. In (5.27) and (5.28),

$$C_{kk'}=\mathfrak{N}_{kk'}|\alpha_k\alpha_{k'}|=C_{k'k}{}^*. \tag{5.29}$$

The remaining amplitude average, determined by $Q(\{|\alpha|\})$, affects only the products of C's.

On assuming the product form (5.25) for $R(\{\phi\})$, the phase average becomes a product of phase averages of the type

$$\langle e^{il\phi}\rangle_\phi=\int_0^{2\pi} e^{il\phi}f(\phi)d\phi=B(l), \tag{5.30}$$

where l is an integer. If f_k is real, then $B_k(l)^*=B_k(-l)$. In case f_k is given by (5.26), B becomes the Fourier transform of g, assuming now that $\int_{-\infty}^{\infty} g(\phi)d\phi=1$:

$$B(l)=\int_{-\infty}^{\infty} e^{il\phi}g(\phi)d\phi. \tag{5.31}$$

When $m=1$, (5.27) becomes

$$\begin{aligned}\langle\mathfrak{N}_{\alpha\alpha}\rangle_\phi&=\sum_{12} C_{12}\langle e^{-i(\phi_1-\phi_2)}\rangle_\phi\\&=\sum_{12}{}' C_{12}\langle e^{-i(\phi_1-\phi_2)}\rangle_\phi+\sum_1 C_{11}\\&=\sum_{12}{}' C_{12}B_1(-1)B_2(1)+\sum_1 C_{11},\end{aligned} \tag{5.32a}$$

where, for clarity and simplicity of notation, a subscript k_i has been abbreviated to i. The first summation of (5.32a) has been decomposed into summations in which no two indices are alike, indicated by a prime on the summation symbol. The same procedure is used for $m>1$. We find for the phase averaged part of the second and third moments

$$\begin{aligned}\langle\mathfrak{N}_{\alpha\alpha}{}^2\rangle_\phi=&\sum_{1234}{}' C_{13}C_{24}B_1(-1)B_2(-1)B_3(1)B_4(1)\\&+\sum_{123}{}'\{(2C_{11}C_{23}+2C_{13}C_{21})B_2(-1)B_3(1)+[C_{12}C_{13}B_1(-2)B_2(1)B_3(1)+\text{H.c.}]\}\\&+\sum_{12}{}'\{C_{11}C_{22}+C_{12}C_{21}+C_{12}{}^2B_1(-2)B_2(2)+[2C_{11}C_{12}B_1(-1)B_2(1)+\text{H.c.}]\}+\sum_1 C_{11}{}^2,\end{aligned} \tag{5.32b}$$

$$\langle\mathfrak{N}_{\alpha\alpha}{}^3\rangle_\phi=\sum_{123456}{}' F_6+\sum_{12345}{}' F_5+\sum_{1234}{}' F_4+\sum_{123}{}' F_3+\sum_{12}{}' F_2+\sum_1 F_1, \tag{5.32c}$$

$$F_6=C_{14}C_{25}C_{36}B_1(-1)B_2(-1)B_3(-1)B_4(1)B_5(1)B_6(1), \tag{5.33a}$$

$$\begin{aligned}F_5=&(3C_{11}C_{24}C_{35}+6C_{14}C_{21}C_{35})B_2(-1)B_3(-1)B_4(1)B_5(1)\\&+(3C_{13}C_{14}C_{25}B_1(-2)B_2(-1)B_3(1)B_4(1)B_5(1)+\text{H.c.}),\end{aligned} \tag{5.33b}$$

$$F_4=[(3C_{13}C_{14}C_{21}+6C_{11}C_{13}C_{24})B_1(-1)B_2(-1)B_3(1)B_4(1)+C_{12}C_{13}C_{14}B_1(-3)B_2(1)B_3(1)B_4(1)$$
$$+(3C_{13}C_{14}C_{22}+6C_{12}C_{13}C_{24})B_1(-2)B_3(1)B_4(1)+\text{H.c.}]+(3C_{13}{}^2C_{24}+6C_{13}C_{14}C_{23})B_1(-2)B_2(-1)B_3(2)B_4(1)$$
$$+(3C_{11}C_{22}C_{34}+3C_{12}C_{21}C_{34}+6C_{11}C_{24}C_{32}+6C_{14}C_{21}C_{32})B_3(-1)B_4(1)\,, \quad (5.33c)$$

$$F_3=(3C_{11}{}^2C_{23}+6C_{11}C_{13}C_{21})B_2(-1)B_3(1)+C_{11}C_{22}C_{33}+2C_{12}C_{23}C_{31}+3C_{11}C_{23}C_{32}$$
$$+(3C_{13}{}^2C_{22}+6C_{12}C_{13}C_{23})B_1(-2)B_3(2)+[3C_{11}C_{12}C_{13}B_1(-2)B_2(1)B_3(1)$$
$$+(6C_{11}C_{12}C_{23}+6C_{12}C_{21}C_{13}+6C_{11}C_{13}C_{22})B_1(-1)B_3(1)+(3C_{13}{}^2C_{21}+6C_{11}C_{13}C_{23})B_1(-1)B_2(-1)B_3(2)$$
$$+3C_{12}{}^2C_{13}B_1(-3)B_2(2)B_3(1)+\text{H.c.}]\,, \quad (5.33d)$$

$$F_2=[3C_{11}{}^2C_{12}B_1(-1)B_2(1)+3C_{11}C_{12}{}^2B_1(-2)B_2(2)+\text{H.c.}]$$
$$+3C_{11}{}^2C_{22}+6C_{11}C_{12}C_{21}+C_{12}{}^3B_1(-3)B_2(3)+(3C_{12}{}^2C_{21}+6C_{11}C_{12}C_{22})B_1(-1)B_2(1)\,, \quad (5.33e)$$

$$F_1=C_{11}{}^3\,, \quad (5.33f)$$

where H.c. denotes the Hermitian conjugate.

Consider the case in which the $f_k(\phi')$ are specified by using the Gaussian functions

$$g_k(\phi')=(2\pi\sigma_k{}^2)^{-1/2}e^{-\frac{1}{2}[(\phi'-\phi_k)/\sigma_k]^2} \quad (5.34)$$

in (5.26). Then (5.31) yields

$$B_k(l)=e^{il\phi_k-\frac{1}{2}l^2\sigma_k{}^2}=B_k(-l)^*. \quad (5.35)$$

Two limiting cases are of particular interest. If all the $\sigma_k\to 0$, then

$$R(\{\phi'\})\to\prod_k \delta(\phi_k'-\phi_k).$$

A coherent state results as a special case if

$$Q(\{|\alpha'|\})=\prod_k \delta(|\alpha_k'|-|\alpha_k|)|\alpha_k'|^{-1}.$$

Correspondingly, $\langle\mathfrak{N}_{\alpha'\alpha'}{}^m\rangle=\mathfrak{N}_{\alpha\alpha}{}^m$. The other limiting case is the following.

Stationary field. The $P(\{\alpha\})$ representing a stationary density operator diagonal in the number representation depends only on the mode amplitudes,[13] the mode phases being distributed at random. This case results when all the $f_k=1/2\pi$. It is obtained in particular when all the $\sigma_k\to\infty$ in (5.34). In this case (5.25) becomes

$$R(\{\phi\})=\prod_k(2\pi)^{-1}$$

and (5.23) becomes

$$P(\{\alpha\})=Q(\{|\alpha|\})\prod_k(1/2\pi). \quad (5.36)$$

On using (5.15a) with $f(\mathfrak{N}_{\alpha\alpha})=\mathfrak{N}_{\alpha\alpha}{}^m$ and using the multinomial expansion for $\mathfrak{N}_{\alpha\alpha}{}^m$ with $\mathfrak{N}_{\alpha\alpha}$ given by (5.10b), the expression (5.19) for the mth factorial moment of $P_n(t,t+T)$, which is also the mth power moment of $W(\nu)$, can be reduced to

$$\langle n^{[m]}\rangle=m!\sum_{\{m_{12}\}}\Big(\prod_{1,2}(m_{12}!)^{-1}\mathfrak{R}_{12}{}^{m_{12}}\Big)$$
$$\times\Big\langle\prod_1\delta\big(\sum_2 m_{21},\sum_2 m_{12}\big)|\alpha_1|^{2\sum_2 m_{12}}\Big\rangle, \quad (5.37)$$

where the sum is over all sets of (non-negative) $m_{k_1k_2}=m_{12}$ such that $\sum m_{12}=m$, where the δ is a Kronecker δ, and where $\langle\cdots\rangle$ denotes the average

$$\int\cdots Q(\{|\alpha|\})\prod_k d^2\alpha_k/2\pi.$$

It is convenient to express these results in terms of the cumulants κ_q of $W(\nu)$ [see (A1)]. The cumulant κ_1 is simply the mean of the distribution $W(\nu)$, while κ_2 is the variance and κ_3 is the third moment about the mean. The cumulants κ_q' of $P_n(t,t+T)$ are given in terms of the cumulants κ_q by (A5). In particular, $\kappa_1'=\kappa_1$, $\kappa_2'=\kappa_1+\kappa_2$, and $\kappa_3'=\kappa_1+3\kappa_2+\kappa_3$. We note that $\kappa_1'=\langle n\rangle$, the mean of the distribution $P_n(t,t+T)$, $\kappa_2'=\langle(n-\langle n\rangle)^2\rangle$, the variance, and $\kappa_3'=\langle(n-\langle n\rangle)^3\rangle$, the third moment. We have for the first three cumulants, introducing $a_i=|\alpha_i|^2$, $\bar{a}_i=\langle|\alpha_i|^2\rangle$,

$$\kappa_1=\sum_1\mathfrak{R}_{11}\bar{a}_1\,, \quad (5.38a)$$

$$\kappa_2=\sum_{12}{}'\{\mathfrak{R}_{11}\mathfrak{R}_{22}\langle(a_1-\bar{a}_1)(a_2-\bar{a}_2)\rangle+\mathfrak{R}_{12}\mathfrak{R}_{21}\langle a_1a_2\rangle\}+\sum_1\mathfrak{R}_{11}{}^2\langle(a_1-\bar{a}_1)^2\rangle\,, \quad (5.38b)$$

$$\kappa_3=\sum_{123}{}'\{\mathfrak{R}_{11}\mathfrak{R}_{22}\mathfrak{R}_{33}\langle(a_1-\bar{a}_1)(a_2-\bar{a}_2)(a_3-\bar{a}_3)\rangle+2\mathfrak{R}_{12}\mathfrak{R}_{23}\mathfrak{R}_{31}\langle a_1a_2a_3\rangle+3\mathfrak{R}_{11}\mathfrak{R}_{23}\mathfrak{R}_{32}\langle(a_1-\bar{a}_1)(a_2a_3-\langle a_2a_3\rangle)\rangle\}$$
$$+3\sum_{12}{}'\{\mathfrak{R}_{11}{}^2\mathfrak{R}_{22}\langle(a_1-\bar{a}_1)^2(a_2-\bar{a}_2)\rangle+2\mathfrak{R}_{11}\mathfrak{R}_{12}\mathfrak{R}_{21}\langle(a_1-\bar{a}_1)(a_1a_2-\langle a_1a_2\rangle)\rangle\}+\sum_1\mathfrak{R}_{11}{}^3\langle(a_1-\bar{a}_1)^3\rangle\,, \quad (5.38c)$$

where a prime on a summation symbol indicates as before that no two indices are equal. Equation (5.38) can be obtained alternatively using (5.32) and (5.35) with all $\sigma_k\to\infty$ in (5.35) so that all $B(l)=0$, $l\neq 0$. One sees from (5.32) that additional types of amplitude averages appear in general if $P=QR$ is not stationary.

When the mode amplitudes are independently distributed the amplitude averages break up into factors, and the cumulants (5.38) for a stationary field simplify further to

$$\kappa_1=\sum_1 \mathfrak{R}_{11}\bar{a}_1\,, \tag{5.39a}$$

$$\kappa_2=\sum_{12}{}' \mathfrak{R}_{12}\mathfrak{R}_{21}\bar{a}_1\bar{a}_2+\sum_1 \mathfrak{R}_{11}{}^2\langle(a_1-\bar{a}_1)^2\rangle\,, \tag{5.39b}$$

$$\kappa_3=2\sum_{123}{}' \mathfrak{R}_{12}\mathfrak{R}_{23}\mathfrak{R}_{31}\bar{a}_1\bar{a}_2\bar{a}_3 +6\sum_{12}{}' \mathfrak{R}_{11}\mathfrak{R}_{12}\mathfrak{R}_{21}\langle(a_1-\bar{a}_1)^2\rangle\bar{a}_2 +\sum_1 \mathfrak{R}_{11}{}^3\langle(a_1-\bar{a}_1)^3\rangle\,. \tag{5.39c}$$

Ideally incoherent field. An important special case of a stationary field with independently distributed amplitudes is an ideally incoherent state, generated say by a "completely chaotic" source.[13] In this case, $P(\{\alpha\})$ is given by (5.36) in which Q is given by the Gaussian expression

$$Q(\{|\alpha|\})=\prod_k 2\langle n_k\rangle^{-1}e^{-|\alpha_k|^2/\langle n_k\rangle} \tag{5.40a}$$

$$=\exp(-\tfrac{1}{2}\boldsymbol{\alpha}^\dagger\mathfrak{B}\boldsymbol{\alpha})\prod_k 2\langle n_k\rangle^{-1}\,, \tag{5.40b}$$

where $\langle n_k\rangle$ is the mean number of photons in the kth mode, and $\mathfrak{B}$ is a diagonal matrix, $\mathfrak{B}_{kk'}=2\delta_{kk'}(\langle n_k\rangle)^{-1}$. In the photon-number representation, ρ is given by its diagonal elements

$$\langle\{n\}|\rho|\{n\}\rangle=\prod_k \frac{\langle n_k\rangle^{n_k}}{(\langle n_k\rangle+1)^{n_k+1}}\,. \tag{5.41}$$

In this distribution the mode photon numbers are independently distributed, each with a *geometric distribution* (see Appendix A). In thermal equilibrium, $\langle n_k\rangle$ is given by $\langle n_k\rangle=[\exp(\hbar\omega_k/k_BT)-1]^{-1}$, corresponding to blackbody radiation.

The averages of (5.39) can be evaluated in this case using

$$\langle|\alpha_k|^{2r}\rangle=(\pi\langle n_k\rangle)^{-1}\times\int|\alpha_k|^{2r}e^{-|\alpha_k|^2/\langle n_k\rangle}d^2\alpha_k=r!\langle n_k\rangle^r\,, \tag{5.42}$$

since the modes are independent for the $P(\{\alpha\})$ given by (5.40). The first three cumulants are then given by (5.39), with $\bar{a}=\langle n\rangle$, $\langle(a-\bar{a})^2\rangle=\langle n\rangle^2$, and $\langle(a-\bar{a})^3\rangle=2\langle n\rangle^3$. However, as shown in Appendix C, where a more detailed discussion of this case is given, the general expression

$$\kappa_q=(q-1)!\sum_{12\cdots q}\langle n_1\rangle\langle n_2\rangle\cdots\langle n_q\rangle\mathfrak{R}_{12}\mathfrak{R}_{23}\cdots\mathfrak{R}_{q1}\,, \tag{5.43}$$

for the cumulants of $W(\nu)$ can be derived by evaluating the characteristic function for $W(\nu)$.

Gaussian squared-amplitude field. We consider another special case of a stationary field with independently distributed mode amplitudes. A Gaussian squared-amplitude field is specified by

$$Q(\{|\alpha'|\})=\prod_k\left(\frac{\pi\sigma_k{}^2}{2}\right)^{-1/2}\times\exp[-\tfrac{1}{2}((|\alpha_k'|^2-|\alpha_k|^2)/\sigma_k)^2]\,, \tag{5.44}$$

with moments in (5.39) given by $\bar{a}=|\alpha|^2$, $\langle(a-\bar{a})^2\rangle=\sigma^2$, and $\langle(a-\bar{a})^3\rangle=0$. The approximation of extending the range of integration for $|\alpha_k'|$ from $0\leq|\alpha_k'|<\infty$ to $-\infty<|\alpha_k'|<\infty$ has been made to obtain the normalization in (5.44) and the moment expressions above. The error is small if $\sigma_k\ll|\alpha_k|^2$. Equation (5.44) reduces to the case of fixed amplitudes when the $\sigma_k\to 0$. Note that in this limit the cumulants (5.39) have the same form as the corresponding ones given by (5.43) for the ideally incoherent field except that no diagonal elements of $\mathfrak{R}$ occur in (5.39).

As a simple illustration, for a field with only a single mode excited ($\alpha_k=0$ when $k\neq 1$) the mean number of photoionizations and the mean-square deviation in the number of photoionizations for the case (5.44) are, according to (5.39), given by

$$\langle n\rangle=\mathfrak{R}_{11}|\alpha_1|^2\,, \tag{5.45}$$

and

$$\langle n^2\rangle-\langle n\rangle^2=\langle n\rangle\{1+\langle n\rangle(\sigma_1/|\alpha_1|^2)^2\}\,. \tag{5.46}$$

Note that $\langle n\rangle$ may be larger than $(|\alpha_1|^2\sigma_1{}^{-1})^2$ despite the condition $|\alpha_1|^2\sigma_1{}^{-1}>1$. In this case a significant departure from Poisson statistics may occur with only a small uncertainty in the amplitude of a single mode.

Spread-out coherent state. As a final example we consider a simple nonstationary distribution of the mode coordinates which can be used to represent a state of the field in which noise is superposed on a field in the coherent state $|\{\alpha\}\rangle$. It is given by

$$P(\{\alpha'\})=\prod_k(\pi\sigma_k{}^2)^{-1}\exp[-|(\alpha_k'-\alpha_k)/\sigma_k|^2]\,. \tag{5.47}$$

It reduces to the coherent state field of (5.20) when the $\sigma_k\to 0$ and to the ideally incoherent field of (5.40) when $\sigma_k{}^2=\langle n_k\rangle$ and the $\alpha_k\to 0$. Its cumulants can be found by an extension of the derivation in Appendix C. They are given by

$$\kappa_q=\kappa_{q,\mathrm{inc}}+\kappa_q(\alpha)\,, \tag{5.48}$$

where $\kappa_{q,\mathrm{inc}}$, the entire contribution when all $\alpha_k=0$, is just the cumulant (5.43) for the ideally incoherent field with $\langle n_k\rangle$ replaced by σ_k^2. The additional contribution arising when not all $\alpha_k=0$ is given by

$$\kappa_q(\alpha)=q!\sum_{12\cdots q+1}(\sigma_2\sigma_3\cdots\sigma_q)^2 \times\alpha_1{}^*\mathfrak{R}_{12}\mathfrak{R}_{23}\cdots\mathfrak{R}_{q,q+1}\alpha_{q+1}. \quad (5.49)$$

We note finally that the examples of field density operators discussed in this section in connection with photoelectron counting experiments may be useful in the theory of other types of photodetection experiments.

VI. DISCUSSION

The assumption was made in Sec. IV that free field motion given by (4.41) together with a statistical model of the initial field density operator can adequately simulate the dynamical effects of a field interacting with its sources. Concerning the statistical description, not only do we use the choice of a nonpure case density operator ($\rho\neq|\rangle\langle|$) to represent in the customary way our ignorance about the initial state of the system, but we also try to take into account in our choice of density operator dynamical effects which would otherwise be neglected as a consequence of the free field approximation. The statistical description also might be used in this way if the approximation of free field motion were replaced by the less restrictive approximation, discussed in Sec. V, that the field evolves in time according to a prescribed current source which may be a random variable.

The justification of the above assumption has not been undertaken here. Justification might be given in terms of a derivation of an appropriate set of equations of motion of the field coupled to its sources. For example, for the case of a maser source this approach has been used by Grivet and Blaquiere,[29] Haus,[30] and Lamb[31]; however, there does not appear to be an adequate fully quantum-mechanical treatment of the equation of motion problem including nonlinear effects, in particular, one which can, in general, be used to justify the assumption used in the present paper. The assumption of a free field together with a statistical model can, of course, be alternatively justified by its usefulness in describing the results of experiment.

A classic application of the statistical method mentioned above employs the relatively simple, ideally incoherent field description [(5.40) in our formalism] to simulate the very complicated dynamical description of the field generated by a large number of uncorrelated sources. The ideally incoherent field has been used almost exclusively to describe fields from available sources until recent years when laser sources became available.

A laser field, because of its supposed high degree of coherence, might well be represented by a coherent state (5.20) or, more generally, by a coherent state with "noise" (5.47). Use of a coherent-state density operator implies that we know what pure state the field is in initially (aside from the unknown over-all phase) and that the effect of the sources is to leave the field moving as though unperturbed by sources. To the extent that use of (5.47) for a laser is to represent effects of coupling of the field to sources rather than ignorance of the initial state, it might be assumed for a counting experiment that the parameters of this distribution depend on the duration T of the counting interval. Thus, (5.47) with the σ_k's depending on T could represent "diffusion" away from a coherent state. If we use the Gaussian function (5.34) in (5.26), then the distribution (5.23) together with (5.25) and (5.26) could similarly represent diffusion of the mode phases. This approach could be used also if the field evolves according to prescribed current sources rather than as if in the absence of sources.

For an ordinary counting experiment, the discussion of the preceding paragraph unfortunately does not apply. This is because in an ordinary counting experiment the initial state of the field is not known to the extent that (5.20) and (5.47) would represent suitable density operators [see comments after (5.22)]. More appropriate density operators for an ordinary counting experiment on a laser would be obtained by phase averaging (5.20) and (5.47). Alternatively, the Gaussian-squared amplitude random-phase choice (5.44) could be useful.

In view of the foregoing discussion it is clear that more detailed information could be obtained about the field from more refined photocounting experiments in which the state of the field at the beginning of the counting interval is better known than it is in an ordinary counting experiment.

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APPENDIX A: PROPERTIES OF THE COMPOUND POISSON DISTRIBUTION

The generating function $G(u)$ for $P_n(t,t+T)$ is given in terms of the characteristic function

$$\phi(s)=\int_0^\infty \exp(is\nu)W(\nu)d\nu\,, \quad \text{(A1a)}$$

$$=\int \exp(is\mathfrak{N}_{\alpha\alpha})P(\{\alpha\})\prod_k d^2\alpha_k\,, \quad \text{(A1b)}$$

$$=\exp\left[\sum_{q=1}^\infty \kappa_q\frac{(is)^q}{q!}\right], \quad \text{(A1c)}$$

[29] P. Grivet and A. Blaquiere, Ref. 15, p. 69.
[30] H. A. Haus, Quarterly Progress Report No. 72, Research Laboratory of Electronics, MIT, p. 53, 1964 (unpublished).
[31] W. E. Lamb (to be published).

and the cumulants κ_q of $W(\nu)$ by

$$G(u)=\phi(i(1-u)), \tag{A2a}$$

$$=\exp\left[\sum_{q=1}^{\infty}\kappa_q\frac{(u-1)^q}{q!}\right]. \tag{A2b}$$

It is apparent from $P_n(t,t+T)=(d/du)^nG(u)|_{u=0}/n!$ that each $P_n(t,t+T)$ is a nonlinear function of all the κ_q's. According to (5.19), the mth factorial moment of $P_n(t,t+T)$ is just the mth power moment of $W(\nu)$, given by

$$\int_0^\infty \nu^m W(\nu)d\nu=\left(\frac{d}{d(is)}\right)^m\phi(s)|_{s=0}. \tag{A3}$$

Expressions for the mth power moment of a distribution as a linear combination of the mth and lower cumulants of the distribution are tabulated by Kendall and Stuart.[18]

On the other hand, the characteristic function of $P_n(t,t+T)$ is given by

$$\phi'(s)=G(e^{is})=\exp\left[\sum_{q=1}^{\infty}\kappa_q'\frac{(is)^q}{q!}\right] \tag{A4}$$

in terms of G and the cumulants κ_q' of $P_n(t,t+T)$. Consequently, using (A2b) for $G(u)$, we obtain an expression for κ_q' as a linear combination of the first q κ_j's

$$\kappa_q'=\left(\frac{d}{dv}\right)^q\ln\phi'(-iv)\bigg|_{v=0}=\sum_{j=1}^{q}C_{qj}\kappa_j, \tag{A5}$$

where, in general,

$$C_{qj}=\frac{1}{j!}\left(\frac{d}{dv}\right)^q(e^v-1)^j\bigg|_{v=0}=\sum_{m=1}^{j}\frac{(-1)^{j-m}}{m!(j-m)!}m^q, \tag{A6}$$

and, in particular, $C_{q1}=C_{qq}=1$. This is in contrast to the dependence of $P_n(t,t+T)$ on all the κ_q's. This is also in contrast to the nonlinear dependence of $P_n(t,t+T)$ and $\langle n^{[m]}\rangle$ on the κ_q's. The C_{qj}'s for $q\leq 4$ can be found in Ref. 9.

We give for reference the first three factorial moments $\langle n^{[m]}\rangle$ in terms of the cumulants κ_q.[18]

$$\langle n^{[1]}\rangle=\langle n\rangle=\kappa_1, \tag{A7}$$

$$\langle n^{[2]}\rangle=\langle n^2\rangle-\langle n\rangle=\kappa_2+\kappa_1^2, \tag{A8}$$

$$\langle n^{[3]}\rangle=\langle n^3\rangle-3\langle n^2\rangle+2\langle n\rangle=\kappa_3+3\kappa_2\kappa_1+\kappa_1^3. \tag{A9}$$

As a formal example, suppose ν is distributed according to a *gamma distribution*

$$W(\nu)=(\mu\Gamma(r))^{-1}(\nu/\mu)^{r-1}e^{-\nu/\mu}. \tag{A10}$$

Then

$$\phi(s)=(1-is\mu)^{-r}, \tag{A11}$$

and the generating function

$$G(u)=\left[\left(\frac{1}{1+\mu}\right)\bigg/\left(1-\frac{\mu}{1+\mu}u\right)\right]^r \tag{A12}$$

generates a *negative binomial distribution* for $P_n(t,t+T)$

$$P_n(t,t+T)=\binom{r+n-1}{n}\left(\frac{1}{1+\mu}\right)^r\left(\frac{\mu}{1+\mu}\right)^n. \tag{A13}$$

When $r=1$, $W(\nu)$ is an *exponential distribution*

$$W(\nu)=\mu^{-1}e^{-\nu/\mu}, \tag{A14}$$

and $P_n(t,t+T)$ is a *geometric distribution*

$$P_n(t,t+T)=\frac{\mu^n}{(\mu+1)^{n+1}}. \tag{A15}$$

APPENDIX B: DETECTOR CORRELATIONS

To get some insight into the significance of the matrix $\mathfrak{R}$ of (5.11), let us consider the spatial integral part of $\mathfrak{R}_{kk'}$ in the case of plane-wave modes

$$u_{k,\lambda}(\mathbf{r})=V^{-1/2}e_\lambda\exp(i\mathbf{k}\cdot\mathbf{r}) \tag{B1}$$

with $\omega_k=ck$. Then,

$$\begin{aligned}\mathfrak{S}_{kk'}&\equiv V^{-1}\sum_{\mu\mu'}e_\mu{}^*e_{\mu'}\int\int\mathcal{K}_{\mu\mu'}(\mathbf{r}_1,\mathbf{r}_2)\\&\qquad\times\exp[i(\mathbf{k}'\cdot\mathbf{r}_2-\mathbf{k}\cdot\mathbf{r}_1)]d\mathbf{r}_1d\mathbf{r}_2\\&=\int\int L(\mathbf{R},\mathbf{r})\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{R}]\\&\qquad\times\exp\left[i\left(\frac{\mathbf{k}'+\mathbf{k}}{2}\right)\cdot\mathbf{r}\right]d\mathbf{R}d\mathbf{r},\end{aligned} \tag{B2}$$

where we have introduced the center of mass and relative coordinates

$$\mathbf{R}=\tfrac{1}{2}(\mathbf{r}_1+\mathbf{r}_2),\quad \mathbf{r}=\mathbf{r}_2-\mathbf{r}_1 \tag{B3}$$

and the function $L(\mathbf{R},\mathbf{r})$ defined by

$$L(\mathbf{R},\mathbf{r})=V^{-1}\sum_{\mu\mu'}\mathcal{K}_{\mu\mu'}(\mathbf{r}_1,\mathbf{r}_2)e_\mu{}^*e_{\mu'}. \tag{B4}$$

Also,

$$\mathfrak{R}_{kk'}=\tfrac{1}{2}\hbar c^2(\omega_k\omega_{k'})^{-1/2}\mathfrak{S}_{kk'}\int_t^{t+T}dt'e^{i(\omega_k-\omega_{k'})t'}. \tag{B5}$$

Assuming now that the region where photoemissions occur is crystalline with primitive lattice translations $\mathbf{R}_n$ and reciprocal lattice translations $\boldsymbol{\kappa}_j/2\pi$, we observe that

$$\mathcal{K}_{\mu\mu'}(\mathbf{r}_1+\mathbf{R}_n,\ \mathbf{r}_2+\mathbf{R}_n)=\mathcal{K}_{\mu\mu'}(\mathbf{r}_1,\mathbf{r}_2), \tag{B6}$$

and consequently that L is periodic in $\mathbf{R}$

$$L(\mathbf{R}+\mathbf{R}_n,\,\mathbf{r})=L(\mathbf{R},\mathbf{r})=\sum_{\kappa_j} M(\kappa_j,\mathbf{r})\exp(i\kappa_j\cdot\mathbf{R}). \quad \text{(B7)}$$

The last expression is a Fourier series representation of L with respect to $\mathbf{R}$. The crystalline nature of the detector is manifested by the $\mathbf{R}$ dependence of $L(\mathbf{R},\mathbf{r})$. We also introduce the Fourier transforms

$$N(\kappa_j,\mathbf{q})=\int M(\kappa_j,\mathbf{r})\exp(i\mathbf{q}\cdot\mathbf{r})d\mathbf{r}. \quad \text{(B8)}$$

Assuming that the correlation length for $M(\kappa_j,\mathbf{r})$, the dimension of the region about $\mathbf{r}=0$ where $M(\kappa_j,\mathbf{r})$ is appreciable, is small compared to the dimensions of the effective photoemissive volume, we may, without significant error extend to infinity the limits of integration in the definition of $N(\kappa_j,\mathbf{q})$. Equation (B2) can then be written in the alternative forms

$$\mathfrak{S}_{kk'}=\sum_{\kappa_j}\int M(\kappa_j,\mathbf{r})\times\exp\left[i\left(\frac{\mathbf{k}'+\mathbf{k}}{2}\right)\cdot\mathbf{r}\right]d\mathbf{r}\Delta(\kappa_j-\mathbf{k}+\mathbf{k}'), \quad \text{(B9a)}$$

$$=\sum_{\kappa_j} N\left(\kappa_j,\frac{\mathbf{k}'+\mathbf{k}}{2}\right)\Delta(\kappa_j-\mathbf{k}+\mathbf{k}'), \quad \text{(B9b)}$$

where

$$\Delta(\mathbf{k})=\int \exp(i\mathbf{k}\cdot\mathbf{R})d\mathbf{R} \quad \text{(B10)}$$

is a function peaked about $\mathbf{k}=0$ with range of the order of the reciprocal dimensions of the effective photoemissive volume; it approaches $(2\pi)^3\delta(\mathbf{k})$ in the limit of large effective photoemissive volume.

For the photodetection problem of interest here k, $k'\gtrsim 10^4\ \text{cm}^{-1}$, while $\kappa_j\gtrsim 10^8\ \text{cm}^{-1}$ unless $\kappa_j=0$. Therefore, only the $\kappa_j=0$ term contributes to the series, and (B9) reduces to

$$\mathfrak{S}_{kk'}=\int M(0,\mathbf{r})\exp\left[i\left(\frac{\mathbf{k}'+\mathbf{k}}{2}\right)\cdot\mathbf{r}\right]d\mathbf{r}\Delta(\mathbf{k}'-\mathbf{k}) \quad \text{(B11a)}$$

$$=N[0,(\mathbf{k}'+\mathbf{k})/2]\Delta(\mathbf{k}'-\mathbf{k}). \quad \text{(B11b)}$$

In (B11) the $\mathbf{R}$ dependence of L does not enter, so that the detector behaves as a continuum.

When the correlation length of $M(0,\mathbf{r})$ is small compared to the wavelengths of the field, as when a photoemission involves only an atomic volume, then $N(0,(\mathbf{k}+\mathbf{k}')/2)\cong N(0,0)$ and (B11b) reduces to

$$\mathfrak{S}_{kk'}=N(0,0)\Delta(\mathbf{k}'-\mathbf{k}). \quad \text{(B12)}$$

This is practically independent of $\mathbf{k}'-\mathbf{k}$ if the appropriate dimension of the effective photoemissive volume is much less than the differences in wavelengths involved.

The finite spread in Δ is necessary for the observation of beats in photodetection. If Δ were actually a Dirac delta function, $\mathfrak{N}$ would be diagonal, and the interference terms between modes would vanish. The interference terms arise because of the presence of spatial as well as temporal beats. These spatial beats would average to zero if the effective photodetector length were much larger than $|\mathbf{k}-\mathbf{k}'|^{-1}$.

APPENDIX C: IDEALLY INCOHERENT FIELD

In this case the characteristic function for $W(\nu)$ can be evaluated further. From (A1b), (5.10c), and (5.40b), we have

$$\phi(s)=\frac{\int \exp[-\frac{1}{2}\alpha^\dagger(\mathfrak{B}-2is\mathfrak{N})\alpha]\prod_k d^2\alpha_k}{\int \exp[-\frac{1}{2}\alpha^\dagger\mathfrak{B}\alpha]\prod_k d^2\alpha_k}. \quad \text{(C1)}$$

We next make two changes of variables: First, a change of scale, $\alpha'=\mathfrak{B}^{1/2}\alpha$, so that the exponent of the numerator becomes $-(1/2)\alpha'^\dagger(1-2is\mathfrak{B}^{-1/2}\mathfrak{N}\mathfrak{B}^{-1/2})\alpha'$ and that of the denominator becomes $-(1/2)\alpha'^\dagger\alpha'$; the second, $\alpha'=\mathfrak{U}\alpha''$, is a unitary transformation, $\mathfrak{U}^{-1}=\mathfrak{U}^\dagger$, which diagonalizes the Hermitian and non-negative matrix

$$\mathfrak{T}=\mathfrak{B}^{-1/2}\mathfrak{N}\mathfrak{B}^{-1/2}, \quad \text{(C2a)}$$

with

$$\mathfrak{T}_{kk'}=\tfrac{1}{2}(\langle n_k\rangle\langle n_{k'}\rangle)^{1/2}\mathfrak{N}_{kk'}, \quad \text{(C2b)}$$

so that

$$\phi(s)=\frac{\int \exp[-\frac{1}{2}\alpha''^\dagger(1-2is\mathfrak{D})\alpha'']\prod_l d^2\alpha_l''}{\int \exp(-\frac{1}{2}\alpha''^\dagger\alpha'')\prod_l d^2\alpha_l''}, \quad \text{(C3)}$$

where

$$\mathfrak{D}=\mathfrak{U}^{-1}\mathfrak{T}\mathfrak{U}=\mathfrak{U}^{-1}\mathfrak{B}^{-1/2}\mathfrak{N}\mathfrak{B}^{-1/2}\mathfrak{U} \quad \text{(C4)}$$

is diagonal, real, and non-negative. For both changes of variables, the Jacobian for the change in volume element is independent of the variables, and therefore cancels between numerator and denominator. The method used here is an extension of a method[18,32] often used in other contexts corresponding to the case when $\mathfrak{N}$ is real. Evaluating the integrals then yields

$$\phi(s)=\prod_l(1-2is\mathfrak{D}_l)^{-1}, \quad \text{(C5a)}$$

$$=[\det(1-2is\mathfrak{D})]^{-1}, \quad \text{(C5b)}$$

$$=[\det(1-2is\mathfrak{T})]^{-1}, \quad \text{(C5c)}$$

[32] U. Grenander, H. O. Pollak, and D. Slepian, J. Soc. Ind. Appl. Math. 7, 374 (1959).

the $\mathrm{Re}\alpha_l''$ integral and the $\mathrm{Im}\alpha_l''$ integral each contributing a factor $(1-2is\mathfrak{D}_l)^{-1/2}$. The cumulants of $W(\nu)$ are then found from (A1c) and (C5) to be

$$\kappa_q=2^q(q-1)!\sum_l \mathfrak{D}_l{}^q \tag{C6a}$$

$$=2^q(q-1)!\,\mathrm{Tr}\mathfrak{T}^q \tag{C6b}$$

$$=(q-1)!\sum_{k_1,k_2,\cdots,k_q}\langle n_{k_1}\rangle\langle n_{k_2}\rangle\cdots \times\langle n_{k_q}\rangle\mathfrak{R}_{k_1k_2}\mathfrak{R}_{k_2k_3}\cdots\mathfrak{R}_{k_qk_1}. \tag{C6c}$$

It is apparent that $W(\nu)$ and $P_n(t,t+T)$ do not depend on the matrix $\mathfrak{T}$ in its full generality, but rather only on its eigenvalues.

In case $\mathfrak{R}$ is a diagonal, so is $\mathfrak{T}$, with $\mathfrak{T}_{kk}=\frac{1}{2}\langle n_k\rangle\mathfrak{R}_{kk}$. Also, $\mathfrak{D}=\mathfrak{T}$ if we choose $\mathfrak{U}=1$. The expression (C6c) then simplifies to $\kappa_q=(q-1)!\sum_k(\langle n_k\rangle\mathfrak{R}_{kk})^q$.

The characteristic function $\phi(s)$ of (C5a) is a product of characteristic functions (A11) for exponential distributions ($r=1$), so that $W(\nu)$ is the convolution of these exponential distributions with mean κ_1 and variance κ_2. Similarly, $G(u)$ is a product of generating functions (A12) for geometric distributions, so that $P_n(t,t+T)$ is the convolution of these distributions with, according to (A5), mean $\kappa_1'=\kappa_1$ and variance $\kappa_2'=\kappa_1+\kappa_2$. These expressions for $W(\nu)$ and $P_n(t,t+T)$ could become asymptotically normal with increasing number of modes excited, according to the central limit theorem[33] but only if $\kappa_2\to\infty$.

On the other hand, when few modes are excited, one can more readily exhibit the results explicitly for $W(\nu)$ and $P_n(t,t+T)$. Thus, the case of a single-mode excited ($\langle n_k\rangle=0$, $k\neq1$) yields simply the exponential distribution (A14) for $W(\nu)$ and the geometric distribution (A15) for $P_n(t,t+T)$; μ in these equations is given by $\mu=\kappa_1=2\mathfrak{D}_1=\langle n_1\rangle\mathfrak{R}_{11}$.

If just two modes 1 and 2 are excited, carrying out the convolution integral yields

$$W(\nu)=(e^{-\nu/\mu_1}-e^{-\nu/\mu_2})/(\mu_1-\mu_2)\,, \tag{C7}$$

where the $\mu_l=2\mathfrak{D}_l$ with $l=1,2$ determine the component exponential distributions. The two roots $\mathfrak{D}_1$ and $\mathfrak{D}_2$ of the determinantal equation $\det(\mathfrak{T}-\lambda)=0$, are, of course, easily expressed in terms of the invariant quantities $\mathrm{Tr}\mathfrak{T}$ and $\det\mathfrak{T}$, or in terms of the invariant quantities from (C6)

$$\kappa_1=2(\mathfrak{D}_1+\mathfrak{D}_2)=\langle n_1\rangle\mathfrak{R}_{11}+\langle n_2\rangle\mathfrak{R}_{22}\,, \tag{C8a}$$

$$\kappa_2=4(\mathfrak{D}_1{}^2+\mathfrak{D}_2{}^2) =\langle n_1\rangle^2\mathfrak{R}_{11}{}^2+\langle n_2\rangle^2\mathfrak{R}_{22}{}^2+2\langle n_1\rangle\langle n_2\rangle\mathfrak{R}_{12}\mathfrak{R}_{21}. \tag{C8b}$$

Corresponding to $W(\nu)$ of (C7) is

$$P_n(t,t+T)=\left[\left(\frac{1}{1+(1/\mu_1)}\right)^{n+1}-\left(\frac{1}{1+(1/\mu_2)}\right)^{n+1}\right]\Big/(\mu_1-\mu_2). \tag{C9}$$

A detailed examination[34,35] of the special case in which the two modes represent orthogonal modes of linear polarization of a plane-wave beam gives the results (C7) and (C9) with $\mu_1=\frac{1}{2}(1+P)\bar{n}$ and $\mu_2=\frac{1}{2}(1-P)\bar{n}$, where $\bar{n}$ is the mean number of counts, and P is the degree of polarization in the notation of Ref. 34.

[33] Reference 19, p. 257.

[34] L. Mandel, Proc. Phys. Soc. (London) **81**, 1104 (1963).

[35] L. Mandel and E. Wolf, J. Opt. Soc. Am. **53**, 1315 (1963).