Evaluation of High-Frequency Ultrasonic Attenuation in Superconductors in the Bardeen-Cooper-Schrieffer Theory of Superconductivity*

V. M. Bobetic†
Department of Physics, University of Illinois, Urbana, Illinois
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Calculations are given of the ratio of the attenuations of longitudinal sound waves in the superconducting and normal states according to the BCS theory of superconductivity for an isotropic model at arbitrary frequencies and temperatures. It is predicted that there is a discontinuity in attenuation at a frequency corresponding to the energy gap.

I. INTRODUCTION

Measurement of ultrasonic attenuation has become an important method for getting information about the energy gap in superconductors. For a simple isotropic model and $\hbar\omega \ll \Delta(T)$, Bardeen, Cooper, and Schrieffer[1] derived the following expression for the ratio r of attenuations of longitudinal sound waves in superconducting and normal states:

$$r=\frac{\alpha_s}{\alpha_n}=\frac{2}{1+\exp[\beta\Delta(T)]}=2f(\Delta)\,, \qquad (1)$$

where $\beta=1/k_BT$ and $2\Delta(T)$ is the temperature-dependent energy gap. This expression has been much used in the interpretation of experimental data. With the possibility of making measurements in the microwave region, it becomes desirable to extend the calculations to cover the range $\hbar\omega\sim\Delta$. Even at ordinary ultrasonic frequencies of $\sim 10^7$ cps, $\hbar\omega$ may be of the order of Δ close to T_c where Δ becomes small. Calculations in this temperature range by Privorotskii[2] are in good agreement with experiment.[3] He also gave expressions valid for other limiting cases. We give here more complete calculations which apply over a considerable range of temperature and frequency. The main restrictions are the use of the isotropic model and a mean free path large compared with the wavelength.

When $\hbar\omega > 2\Delta$, it is possible to create a pair of quasiparticles, and this adds to the attenuation. As we shall see, the theory predicts that with increasing frequency, the attenuation should be discontinuous at $\hbar\omega=2\Delta$, with a large jump in attenuation, even at finite temperatures. A discontinuity should also appear for fixed frequency as the temperature is varied when $\hbar\omega=2\Delta(T)$. If such a discontinuity could be observed, it would give a precise measure of the gap. In actual materials, effects of anisotropy may give a smooth transition rather than an abrupt discontinuity.

The expression to be evaluated as given in Ref. 1 for case I (destructive interference):

$$r=\frac{1}{\hbar\omega}\int\left(1-\frac{\Delta^2}{EE'}\right)(f(E)-f(E'))\rho(E)\rho(E')dE\,, \qquad (2)$$

where $E'=E+\hbar\omega$, $f(E)=1/(1+\exp(\beta E))$, and $\rho(E)$ is the relative density of states:

$$\begin{aligned}\rho(E)&=0\,, & |E|&<\Delta\\ &=|E|/(E^2-\Delta^2)^{1/2}\,, & |E|&>\Delta\,.\end{aligned}$$

Integration is over positive and negative values of E. It is here assumed that the integration is symmetric in the normal quasiparticle energies ϵ and ϵ' so that a term proportional to $\epsilon\epsilon'$ in the coherence factors cancels out. This requires the additional condition that Δ and kT be much smaller than $\hbar qv_F$, where q is the magnitude of the wave vector of the sound wave and v_F is the Fermi velocity. This condition is normally satisfied, since $\hbar qv_F=\hbar\omega(v_F/c)$, and v_F is large compared with the sound velocity c. It is further required that $ql\gg 1$, where l is the mean free path.

Equation (1) may be written in the following form,

$$\begin{aligned}r=&\frac{2}{\hbar\omega}\int_\Delta^\infty (f(E)-f(E+\hbar\omega))\\ &\qquad\times\frac{[E(E+\hbar\omega)-\Delta^2]dE}{\{(E^2-\Delta^2)[(E+\hbar\omega)^2-\Delta^2]\}^{1/2}}\\ &-\frac{1}{\hbar\omega}\int_{\Delta-\hbar\omega}^{-\Delta}(1-2f(E+\hbar\omega))\\ &\qquad\times\frac{[E(E+\hbar\omega)-\Delta^2]dE}{\{(E^2-\Delta^2)[(E+\hbar\omega)^2-\Delta^2]\}^{1/2}}\,, \qquad (3)\end{aligned}$$

where the first line corresponds to E and E' having the same sign in (1) and the second to E and E' having opposite signs. Some changes in the variables of integration have been made to reduce the integral to this form. The second integral, corresponding to creation of a pair of quasiparticles, appears only when $\hbar\omega > 2\Delta$. The form of the integral is such as to give a finite contribution even when $\hbar\omega$ is only infinitesimally

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† Present address: University of Windsor, Windsor, Ontario, Canada.

[1] J. Bardeen, L. N. Cooper, and J. R. Schrieffer, Phys. Rev. **108**, 1175 (1957).
[2] I. A. Privorotskii, Zh. Eksperim. i Teor. Fiz. **43**, 1331 (1962) [English transl.: Soviet Phys.—JETP **16**, 945 (1963)].
[3] P. A. Begriglyi, A. A. Galkin, and A. P. Korolyuk, Zh. Eksperim. i Teor. Fiz. **39**, 7 (1960) [English transl.: Soviet Phys.—JETP **12**, 4 (1961)].

greater than 2Δ, and this leads to the discontinuity in the attenuation.

We introduce the reduced variables:

$$x=E/\hbar\omega\,,\quad a=\hbar\omega/kT\,,\quad c=\Delta/\hbar\omega\,. \tag{4}$$

Equation (2) then becomes

$$r(a,c)=2I_1(a,c)+I_2(a,c)\,, \tag{5}$$

where

$$I_1(a,c)=\int_c^\infty\left[\frac{1}{e^{ax}+1}-\frac{1}{e^{a(x+1)}+1}\right]\times\frac{(x^2+x-c^2)dx}{\{(x^2-c^2)[(x+1)^2-c^2]\}^{1/2}} \tag{6a}$$

$$I_2(a,c)=-\int_{c-1}^{-c}\left[1-\frac{2}{e^{a(x+1)}+1}\right]\times\frac{(x^2+x-c^2)dx}{\{(x^2-c^2)[(x+1)^2-c^2]\}^{1/2}}. \tag{6b}$$

Since $I_2(a,c)$ vanishes for $c\geq\frac{1}{2}$, we may distinguish the region for which $c\geq\frac{1}{2}$ from that for which $c<\frac{1}{2}$. This corresponds to the separation of frequencies which are less than the energy gap from those which are greater.

$$\begin{aligned} r&=2I_1(a,c)+I_2(a,c)\,, &&\text{for}\quad 0\leq c<\tfrac{1}{2}\quad\text{or}\quad \hbar\omega>2\Delta\\ r&=2I_1(a,c)\,, &&\text{for}\quad c\geq\tfrac{1}{2}\quad\text{or}\quad \hbar\omega\leq2\Delta. \end{aligned} \tag{7}$$

II. EVALUATION OF THE INTEGRAL $I_1(a,c)$

The integral $I_1(a,c)$ has a singular point at $x=c$. The difficulties in the evaluation of this integral come from this singularity as well as from the form of the integrand. We give the cases where simple and useful analytic expressions can be derived. For other cases we give a suitable expression for numerical integration.

1. $c=0$, $a\neq0$

For $c=0$ the second factor in the integrand becomes 1 and the first factor can be analytically integrated. The result is

$$I_1=(1/a)\ln[2e^a/(1+e^a)]. \tag{8}$$

2. $a\to\infty$, $c\geq0.25$

For large a, the first factor in the integrand rapidly decreases when x increases and the expansion in power series of the exponential functions is useful. Performing this expansion we find

$$I_1(a,c)=\int_c^\infty\left[\sum_0(-1)^n(1-e^{-(n+1)a})e^{-(n+1)ax}\right]\times\left\{\frac{1}{(x-c)^{1/2}}\left[\psi(c)+\psi'(c)(x-c)+\frac{1}{2!}\psi''(c)(x-c)^2+\cdots\right]\right\}dx, \tag{9}$$

where

$$\psi(x)=\frac{x^2+x-c^2}{\{(x+c)[(x+1)^2-c^2]\}^{1/2}}.$$

It is convenient now to introduce the new variable: $x-c=u^2$. The expression (9) then becomes

$$I_1(a,c)=\left(\frac{\pi}{a}\right)^{1/2}\sum_1(-1)^{k-1}\frac{e^{-kac}}{k^{1/2}}(1-e^{-ka})A_k(a,c)\,, \tag{10}$$

where

$$A_k(a,c)=\psi(c)+\frac{1}{2ka}\psi'(c)+\frac{3}{8k^2a^2}\psi''(c)+\frac{5}{16k^3a^3}\psi'''(c)+\cdots.$$

For a large a and $c\geq0.25$ this series converges rapidly and numerical evaluation for given a and c is convenient.

3. $a\to0$, $c\to\infty$ ($\omega\to0$, $T\neq0$)

This is the case which we have mentioned in the introduction that was evaluated with the result:

$$r=2f(\Delta).$$

4. Other Cases

For other cases we can evaluate I_1 by numerical integration. In order to eliminate the singular point we introduce the following substitution: $x=u^2+c$ and find

$$I_1(a,c)=2\int_0^\infty\left[\frac{1}{1+e^{a(u^2+c)}}-\frac{1}{1+e^{a(u^2+c+1)}}\right]\times\frac{[u^2(u^2+2c)+u^2+c]du}{[(u^2+1)(u^2+2c)(u^2+2c+1)]^{1/2}}. \tag{11}$$

When u^2/c is somewhat larger than 1, the second factor of the integrand is approximately equal to u. So, from a certain point u_0, the integral may be analytically performed in the following form:

$$I_1\Big|_{u_0}^{\infty}=1+\frac{1}{a}\ln\frac{1+e^{a(u_0^2+c)}}{1+e^{a(u_0^2+c+1)}}. \tag{12}$$

III. EVALUATION OF INTEGRAL $I_2(a,c)$

The integral $I_2(a,c)$ has to be considered only for $c<\frac{1}{2}$. Its integrand contains two singular points: $x=-c$ and $x=-1+c$. We again list the cases where analytical integration can be performed and the appropriate expression for numerical integration in other cases.

1. Small $a\neq0$, Arbitrary c

Since the integral I_2 has finite limits, it is useful to expand the first factor of the integrand in power series

TABLE I. The ratio $r=\alpha_s/\alpha_n$ as a function of the parameters $a=\hbar\omega/k_BT$, $c=\Delta(T)/\hbar\omega$.

$a \backslash c$	0.00	0.10	0.20	0.25	0.40	0.4999	0.50	1.00	2.00	3.00	5.00
0.10	1.0000	1.0000	1.0000	1.0000	1.0001	1.0001	0.9608	0.9351	0.8920	0.8450	0.7489
0.50	1.0000	1.0000	1.0000	1.0003	1.0010	1.0069	0.8116	0.7016	0.5024	0.3402	0.1413
1.00	1.0000	1.0009	1.0028	1.0036	1.0074	1.0113	0.6266	0.4483	0.1989	0.0790	0.0148
2.00	1.0000	1.0066	1.0144	1.0214	1.0398	1.0704	0.3446	0.1552	0.0238	0.0033	0.0001
3.00	1.0000	1.0111	1.0368	1.0585	1.1356	1.1819	0.1843	0.0494	0.0027	0.0001	0.0000
5.00	1.0000	1.0315	1.0898	1.1333	1.2858	1.3864	0.0540	0.0050	0.0004	0.0000	0.0000
8.00	1.0000	1.0355	1.1223	1.1808	1.3905	1.5234	0.0093	0.0002	0.0000	0.0000	0.0000
∞	1.0000	1.0507	1.1508	1.2111	1.4180	1.5707	0.0000	0.0000	0.0000	0.0000	0.0000

of x and then to reduce the integration to elliptic integrals. For sufficiently small a one can keep only the zero- and first-order terms in this expansion. We can perform the power-series expansion at $x=-0.50$ and find

$$I_2(a,c)=E(k)[1-2/(1+e^{0.50a})]+\cdots, \quad (13)$$

where $E(k)$ is the complete elliptic integral with $k^2=1-4c^2$. The integrals which lead to the elliptic integrals were performed by the following substitutions:

$$x=(1/t)+c \quad (14a)$$

which transforms the polynomial of the denominator into $t^{-4}Q(t)$, where $Q(t)$ is a third degree polynomial with the null points

$$\alpha=-1/2c, \quad \beta=-1, \quad \gamma=-1/(1+2c).$$

Introducing the new substitution

$$t=\alpha+(\beta-\alpha)\sin^2\Phi, \quad (14b)$$

one finds

$$\int_{c-1}^{-c}\frac{(x^2+x-c^2)dx}{\{(x^2-c^2)[(x+1)^2-c^2]\}^{1/2}}=-E(k) \quad (15a)$$

and

$$\int_{c-1}^{-c}\frac{x(x^2+x-c^2)dx}{\{(x^2-c^2)[(x+1)^2-c^2]\}^{1/2}}=-(1/2)E(k). \quad (15b)$$

2. $c\to 0.50_{(-)}$, Arbitrary a

When c approaches 0.50 from the left side, the limits of I_2 approach -0.50. It makes possible the expansion of the first factor of the integrand in a power series again and one finds the same formula (13) as in the previous case.

3. $c=0$, Arbitrary a

For $c=0$, the second factor in the integrand is 1 and the first factor can be analytically integrated. One finds

$$I_2=1-(2/a)\ln(2e^a/(1+e^a)). \quad (16)$$

4. $a\to\infty$, Arbitrary c

For a large a, the factor $2/(1+e^{a(x+1)})$ can be expanded in geometrical series as follows:

$$\frac{2}{1+e^{a(x+1)}}=2\sum_0(-1)^k e^{-(k+1)a(x+1)}. \quad (17)$$

Now, for a sufficiently large a this sum is negligibly small so that we find for this case

$$I_2=E(k)+\cdots, \quad (18)$$

where $E(k)$ is a complete elliptic function with $k^2=1-4c^2$. This case corresponds to $T=0$.

5. Other Cases

For other cases the numerical integration seems to be again the simplest way. The singular points are eliminated by the substitutions

$$u^2=-x-c \quad\text{and}\quad u^2=x+1-c,$$

for $x=-c$ and $x=-1+c$, respectively, and we find

$$I_2(a,c)=4\int_0^{(\frac{1}{2}-c)^{1/2}}\left[1-\left(\frac{1}{1+e^{a(u^2+c)}}-\frac{1}{1+e^{a(1-c-u^2)}}\right)\right]\times\frac{[u^2(u^2+2c)-(u^2+c)]du}{[(u^2-1)(u^2+2c)(u^2+2c-1)]^{1/2}}. \quad (19)$$

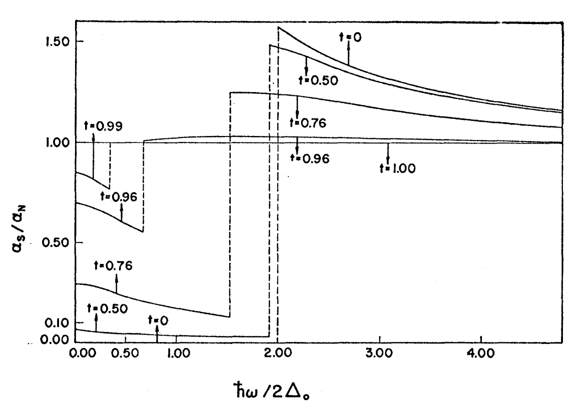

FIG. 1. Ratio of attenuations in superconducting and normal states as a function of frequency for different temperatures.

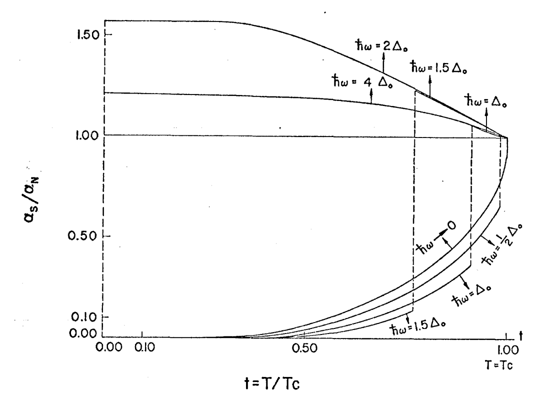

FIG. 2. Ratio of attenuations in superconducting and normal states as a function of temperature for different frequencies.

IV. RESULTS OF THE CALCULATION

1. r as a Function of a and c

Making use of the numerical values of the integrals I_1 and I_2, we find r as a function of a for given c; the numerical values are listed in Table I.

Figure 1 is a plot of r as a function of the frequency for fixed temperatures. Note the large discontinuities that occur when the frequency corresponds to the gap, $\hbar\omega=2\Delta(T)$; when $\hbar\omega>2\Delta$ it is possible to create a pair of excitations. At $T=0°\mathrm{K}$, there is no absorption for $\hbar\omega<2\Delta$. In Fig. 2, plots are given of the absorption ratio as a function of temperature for fixed frequencies. The discontinuities are again evident. It is not possible to compare fully these results with experiment since as yet there are not available experimental data in the frequency range $\hbar\omega>2\Delta$, except for temperatures very near T_c. It is hoped that recent advances in technology for generating ultrasonic waves at microwave frequencies will make such measurements possible.

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Momentum-Transfer Cross Sections for Slow Electrons in He, Ar, Kr, and Xe from Transport Coefficients*

L. S. FROST AND A. V. PHELPS
Westinghouse Research Laboratories, Pittsburgh, Pennsylvania

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Momentum-transfer cross sections for electrons in He, Ar, Kr, and Xe are obtained from a comparison of theoretical and experimental values of the drift velocities and of the ratio of the diffusion coefficient to the mobility coefficient for electrons in these gases. The theoretical transport coefficients are obtained by calculating accurate electron-energy distribution functions for energies below excitation using an assumed energy-dependent momentum-transfer cross section. The resulting theoretical values are compared with the available experimental data and adjustments made in the assumed cross sections until good agreement is obtained. The final momentum cross section for helium is $5.0\pm0.1\times10^{-16}$ cm^2 for an electron energy of 5×10^{-3} eV and rises to $6.6\pm0.3\times10^{-16}$ cm^2 for energies near 1 eV. The cross sections obtained for Ar, Kr, and Xe decrease from 6×10^{-16}, 2.6×10^{-15}, and 10^{-14} cm^2, respectively, at 0.01 eV to minimum values of 1.5×10^{-17} cm^2 at 0.3 eV for Ar, 5×10^{-17} cm^2 at 0.65 eV for Kr, and 1.2×10^{-16} cm^2 at 0.6 eV for Xe. The agreement of the very-low-energy results with the effective-range theory of electron scattering is good.

I. INTRODUCTION

THE total scattering cross sections for electrons in the rare gases have been studied extensively at energies above about one electron volt using electron-beam techniques.[1] However, these methods are difficult to apply at lower energies. In this low-energy range one generally obtains the elastic-scattering cross sections, actually the cross sections for momentum transfer, from analyses of electron-transport coefficient data.[2] Until recently, such analyses avoided the complexities of solving the Boltzmann transport equation for each gas

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[2] For reviews of the earlier analyses see R. H. Healey and J. W. Reed, *The Behavior of Slow Electrons in Gases* (Amalgamated Wireless Ltd., Sydney, Australia, 1941); H. S. W. Massey and E. H. S. Burhop, *Electronic and Ionic Impact Phenomena* (Clarendon Press, Oxford, 1952); L. B. Loeb, *Basic Processes in Gaseous Electronics* (University of California Press, Berkeley, California, 1955); and L. G. H. Huxley and R. W. Crumpton, in *Atomic and Molecula, Processes*, edited by D. R. Bates (Academic Press Inc., New York, 1962), Chap. 10.