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Nonlinear Interaction of Light in a Plasma

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The elastic scattering of light off light in the presence of a plasma is investigated. The cross section for this process is expressed in terms of matrix elements of the charge-density operator between exact eigenstates of the many-body system (plasma). The cross section is evaluated to the lowest order in the plasma parameter $r_s \equiv (n/q_d^3)^{-1}$. A study of the dependence of the differential cross section on momentum and energy transferred in the scattering will allow one to measure the plasma dielectric functions. Although the cross sections are small, the development of intense light sources (lasers) makes a measurement feasible.

I. INTRODUCTION

TWO electromagnetic waves incident on an electron-ion plasma will interact with one another and scatter. The plasma may be regarded as a polarizable continuum. Electrons may be virtually excited from the equilibrium distribution by the absorption of radiation to form excited pairs. The pairs in turn annihilate themselves giving rise to the scattered radiation.

It is clear that the nonlinear interaction between electromagnetic fields in a plasma depends in an essential way on the local screening properties of the plasma. We will show that, to a good approximation, a measurement of the differential elastic-scattering cross section of two light beams as a function of the energy and momentum transferred in the scattering is a direct measure of the plasma dielectric function. The elastic-scattering experiments discussed here, in principle, allow one to measure the behavior of the screening function over a region of the energy-momentum plane.

Recently there has been considerable theoretical and experimental interest in the incoherent scattering of optical or microwave beams from plasmas.[1,2] The experiments are difficult because the cross sections (related to the Thomson cross section $\cong 5\times 10^{-25}$ cm^2) are small. Since the elastic scattering of light off light is a higher order process (of order e^4), the cross section is expected to be even smaller than that for incoherent scattering. In spite of this fact, we will show that under suitable conditions the light-off-light scattering experiment offers certain advantages over single-beam scattering, so much so that the counting rate in the light-off-light scattering experiment can exceed the rate in the single-beam scattering. The salient features of these results were reported earlier.[3]

We will compute the cross section for light-off-light scattering in an electron-gas plasma. This cross section will be expressed in terms of matrix elements of the charge-density operator between exact eigenstates of the many-body system. The cross section will be evaluated to lowest order in the plasma parameter, $r_s \equiv (n/q_d^3)^{-1}$ where n is the electron density and q_d the Debye wave number.[4]

II. COMPUTATION

Consider a system of electrons and ions interacting with one another through an instantaneous Coulomb potential and simultaneously with an external applied transverse electromagnetic field. The Hamiltonian describing such a system is of the form

$$H = H_p + H_I + H_{\mathrm{e.m.}}, \tag{1a}$$

$$H_p = \sum_i \int d\mathbf{x}\psi_i^\dagger(\mathbf{x})(-\nabla^2/2m_i)\psi_i(\mathbf{x}) + \tfrac{1}{2}\int d\mathbf{x}d\mathbf{x}'\rho(\mathbf{x})\frac{1}{|x-x'|}\rho(\mathbf{x}'), \tag{1b}$$

$$H_{\mathrm{e.m.}} = \frac{1}{8\pi}\int(\mathbf{E}^2+\mathbf{H}^2)d\mathbf{x}, \tag{1c}$$

$$H_I = -\int d\mathbf{x}\mathbf{j}(\mathbf{x})\cdot\mathbf{A}(\mathbf{x}) + \sum_i \frac{e^2}{2m_i}\int d\mathbf{x}\rho_i(\mathbf{x})\mathbf{A}^2(\mathbf{x}), \tag{1d}$$

where

$$\rho_i(\mathbf{x}) = \psi_i^\dagger(\mathbf{x})\psi_i(\mathbf{x});\ \rho(\mathbf{x}) = \sum_i e_i\rho_i(\mathbf{x}), \tag{1e}$$

$$\mathbf{j}(\mathbf{x}) = \sum_i (e_i/m_i)[\psi_i^\dagger(\mathbf{x})\{\nabla\psi_i(\mathbf{x})\} - \{\nabla\psi_i^\dagger(\mathbf{x})\}\psi_i(\mathbf{x})] \equiv \sum_i j_i(\mathbf{x}). \tag{1f}$$

[1] W. E. Gordon, Proc. IRE **46**, 1824 (1955); E. E. Salpeter, Phys. Rev. **120**, 1528 (1960); M. N. Rostoker and N. Rosenbluth, Phys. Fluids **5**, 776 (1962); A. Ron, J. Dawson, and C. Oberman, Phys. Rev. **132**, 497 (1963); D. F. Dubois and V. Gilinsky, *ibid.* **133**, A1308, A1317 (1964) (the first of these two papers will heretofore be referred to as DG).

[2] K. W. Bowles, Phys. Rev. Letters **1**, 454 (1958); V. C. Pineo, L. G. Craft, and H. W. Briscoe, J. Geophys. Res. **65**, 2629 (1960); G. Fiocco and E. Thompson, Phys. Rev. Letters **10**, 89 (1963).

[3] P. M. Platzman, S. J. Buchsbaum, and N. Tzoar, Phys. Rev. Letters **12**, 531 (1964).

[4] For a nondegenerate plasma $q_d^2 = 4\pi ne^2\beta$, where $\beta = 1/kT$. For

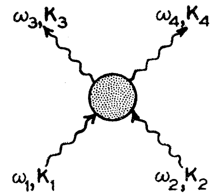

FIG. 1. The general Feynman diagram for the light-off-light scattering amplitude.

We have used the Coulomb gauge div$\mathbf{A}=0$ and chosen our units such, the $\hbar=c=1$. The operators ψ_i are the usual second quantized field operators for the particles, and $\mathbf{A}$ is the corresponding operator for the electromagnetic field. $\mathbf{E}$ and $\mathbf{H}$ are the electric and magnetic fields, respectively. The sum on i corresponds to a sum over species with species 1 being the electrons and species 2 the ions.

In order to calculate the transition rate we make use of the familiar golden rule,

$$\Gamma=2\pi \sum_i{}^{\text{av}} \sum_f |\langle \varphi_f | H_I | \psi_i^+ \rangle|^2 \delta(E_i - E_f)\,, \qquad (2)$$

where

$$\psi_i^+ = \varphi_i + (E_i - H + i\epsilon)^{-1} H_I \psi_i^+\,. \qquad (3)$$

The $\sum_i{}^{\text{av}}$ is an appropriate average over initial states. The states φ_i and φ_f are, respectively, the initial and final states of the uncoupled system: transverse electromagnetic field and the complete interacting (via the Coulomb interaction) many-body system. The small positive imaginary $i\epsilon$ in the denominator of Eq. (3) produces the outgoing wave boundary condition on the scattered radiation.

Since we will be interested in the problem of two photons going to two photons (Fig. 1) we should consider Eq. (2) to fourth order in the interaction Hamiltonian, Eq. (1d). Usually one makes the argument that the $\mathbf{j}\cdot\mathbf{A}$ terms are relativistic terms which may be neglected in computing the low-frequency scattering of light off a nonrelativistic system such as a plasma. This argument is in general false since, for example, the $\mathbf{j}\cdot\mathbf{A}$ terms are the dominant terms in the incoherent scattering of light from a simple atomic system which is nonrelativistic. They also contribute terms of order one (in the ratio v/c) to the dissipative part of the conductivity of an electron-ion plasma.[5] For the Compton scattering of light-off *free* electrons the $\mathbf{j}\cdot\mathbf{A}$ terms are negligible and the $\mathbf{A}^2$ term gives the correct Thompson cross section. In order to determine when the $\mathbf{j}\cdot\mathbf{A}$ terms can be neglected relative to the $\mathbf{A}^2$ term we must compare the matrix element of $\mathbf{j}\cdot\mathbf{A}$ taken twice with the matrix element of a single $\mathbf{A}^2$ term. We do this comparison in the Appendix. Our conclusions are summarized here.

For a given process the scattering amplitude may be expanded as a power series in the plasma parameter r_s. The lowest order terms may be computed neglecting the $\mathbf{j}\cdot\mathbf{A}$ term as long as $\hbar\omega_e \ll mc^2$ (where ω_e is the frequency of the external field). If one is interested in corrections to this amplitude (i.e., the higher order terms in r_s), then there are a number of different cases which must be distinguished. Suppose one considers a quantum plasma at $T=0$. If the $\mathbf{A}^2$ term gives a nonzero contribution to the amplitude to this order, then as long as $E_F \ll \hbar\omega_e$, the $\mathbf{j}\cdot\mathbf{A}$ terms may be neglected. In practice, $E_F \gg \hbar\omega_e$ so that the $\mathbf{j}\cdot\mathbf{A}$ term must be included.

For a classical plasma the conclusions are modified. If the $\mathbf{A}^2$ term gives a nonzero contribution to the amplitude to order r_s, then as long as $(\Delta\omega/\omega_e) \ll 1$, the $\mathbf{j}\cdot\mathbf{A}$ terms may be neglected. The quantity $\Delta\omega$ is the frequency transferred to the medium by a two-photon event (one in and one out). Typically, $\Delta\omega \sim \omega_p$. For the conductivity the $\mathbf{A}^2$ term produces *no* terms of order r_s so that the entire part proportional to r_s comes from the $\mathbf{j}\cdot\mathbf{A}$ term.[5]

We will be interested in computing the differential scattering cross section of light by light (see Fig. 1) *to lowest order in the plasma parameter*. Consequently, we will neglect the $(\mathbf{j}\cdot\mathbf{A})$ term in H_I. Photons 1 and 2 are the incoming photons with wave vectors $\mathbf{k}_1$ and $\mathbf{k}_2$, and photons 3 and 4 are the outgoing photons (scattered) with wave vectors $\mathbf{k}_3$ and $\mathbf{k}_4$.

We normalize $\mathbf{A}$ so that there is a unit probability per cubic centimeter of finding a photon. That is to say

$$\mathbf{A}(\mathbf{x}) = \sum_{\mathbf{k}} (2\pi/\omega)^{1/2} [a_{\mathbf{k}} e^{i\mathbf{k}\cdot\mathbf{x}} + a_{\mathbf{k}}^\dagger e^{-i\mathbf{k}\cdot\mathbf{x}}] \mathbf{e}_{\mathbf{k}}\,, \qquad (4)$$

where $a_{\mathbf{k}}$, $a_{\mathbf{k}}^\dagger$ are the photon annihilation and creation operators and $\mathbf{e}_{\mathbf{k}}$ is the polarization vector. With this normalization the density of final states for the photons is $d\mathbf{k}/(2\pi)^3$ per unit volume. Then,

$$\begin{aligned} M_{fi} &\equiv \langle \varphi_f | H_I | \psi_i^+ \rangle \\ &= \sum_{n,m,o,p} \frac{4(2\pi)^2 e^4}{(\omega_n \omega_m \omega_o \omega_p)^{1/2}} \frac{(\mathbf{e}_p\cdot\mathbf{e}_o)(\mathbf{e}_n\cdot\mathbf{e}_m)}{m_1^2} \\ &\quad \times \sum_l \langle f | \rho_{\mathbf{k}_p+\mathbf{k}_o} | l \rangle \\ &\quad \times \langle l | \rho_{\mathbf{k}_p+\mathbf{k}_m} | i \rangle / (\mathcal{E}_i + \omega_n + \omega_m - \mathcal{E}_l + i\epsilon)\,. \end{aligned} \qquad (5)$$

The quantities $\rho_{\mathbf{k}}$ are the Fourier transforms of the electron density,

$$\rho_{\mathbf{k}} = \int \psi_1^\dagger(\mathbf{x}) \psi_1(\mathbf{x}) e^{i\mathbf{k}\cdot\mathbf{x}} d\mathbf{x}\,, \qquad (6)$$

and $\mathcal{E}$ is the energy and $|j\rangle$ the wave function of the interacting many-body system. The integers n, m, o, p range from 1 to 4. Terms in the sum include those combinations of n, m, o, and p which are all different.

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a degenerate plasma at zero degrees, we define q_d as the inverse Fermi Thomas screening length, i.e., $q_d^2 = 4\pi n e^2/E_F$, where E_F is the Fermi energy.

[5] A. Ron and N. Tzoar, Phys. Rev. **131**, 12 (1963).

Two terms differing in the interchange of o and p, or n and m are not counted twice. The factor four in Eq. (5) takes these symmetric terms into account. The sign of $\mathbf{k}_n$ and ω_n is plus or minus depending on whether the photon is incoming (1,2) or outgoing (3,4).

We will only be interested in elastic scattering, i.e., there is no energy transferred to the system. Thus the over-all conservation of energy, the delta function in Eq. (2), becomes a conservation requirement on the photons. Since the system is translationally invariant transitions occur only for $\mathbf{k}_1+\mathbf{k}_2=\mathbf{k}_3+\mathbf{k}_4$. The six terms in the sum for M_{fi}, Eq. (5), are conveniently broken up into three sets of two, corresponding to a crossed and an uncrossed diagram (see Fig. 2). A typical term in M_{fi} is

$$\tilde{M}_{1,2}(k)\sim(\mathbf{e}_1\cdot\mathbf{e}_2)(\mathbf{e}_3\cdot\mathbf{e}_4)\times\sum_l\left\{\frac{\langle f|\rho_{-\mathbf{k}_{12}}|l\rangle\langle l|\rho_{\mathbf{k}_{12}}|i\rangle}{\mathcal{E}_i-\mathcal{E}_l+\omega_{12}+i\epsilon}+\frac{\langle f|\rho_{\mathbf{k}_{12}}|l\rangle\langle l|\rho_{-\mathbf{k}_{12}}|i\rangle}{\mathcal{E}_i-\mathcal{E}_l-\omega_{12}+i\epsilon}\right\},\quad(7)$$

where

$$\mathbf{k}_{12}=\mathbf{k}_1+\mathbf{k}_2\quad\text{and}\quad\omega_{12}=\omega_1+\omega_2.\quad(8)$$

$\tilde{M}_{1,2}$ may be written as the Fourier transform of a time-ordered product.

$$\tilde{M}_{1,2}(k)\sim(\mathbf{e}_1\cdot\mathbf{e}_2)(\mathbf{e}_3\cdot\mathbf{e}_4)(-i)\times\int^{+\infty}e^{-i\omega_{12}t}\langle f|T[\rho_{\mathbf{k}}(t),\rho_{-\mathbf{k}}(0)]|i\rangle,\quad(9)$$

where

$$\rho_{\mathbf{k}}(t)=e^{iH_pt}\rho_{\mathbf{k}}e^{-iH_pt}\quad(10)$$

and T is the usual time ordering operator.

In deriving Eq. (9) from Eq. (7) we have used the fact that $\mathcal{E}_i=\mathcal{E}_f$. The total matrix element which still has to be averaged over all initial states and summed over all final states is

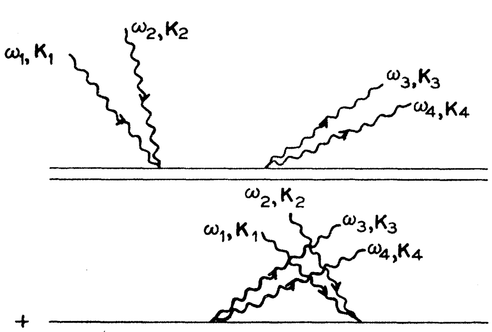

FIG. 2. The crossed and uncrossed Feynman diagrams corresponding to the term $M_{1,2}$ in Eq. (7). The wiggly lines are photons, the heavy solid line is the many-body system (the plasma).

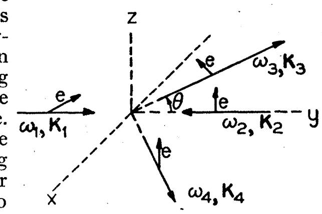

FIG. 3. Diagram showing the kinematics and polarizations for the light-off-light scattering. All the light beams are in the X-Y plane (the scattering plane). The scattering angle is measured in this plane. Beams one and three are polarized in the scattering plane. Beams two and four are polarized perpendicular to the scattering plane.

$$M_{fi}=\frac{4(2\pi)^2e^4}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}[(\mathbf{e}_1\cdot\mathbf{e}_2)(\mathbf{e}_3\cdot\mathbf{e}_4)\langle f|R_{1,2}|i\rangle+(\mathbf{e}_1\cdot\mathbf{e}_3)(\mathbf{e}_2\cdot\mathbf{e}_4)\langle f|R_{1,3}|i\rangle+(\mathbf{e}_1\cdot\mathbf{e}_4)(\mathbf{e}_2\cdot\mathbf{e}_3)\langle f|R_{1,4}|i\rangle];\quad(11)$$

where

$$\langle f|R_{i,j}|i\rangle=(-i)\times\int_{-\infty}^{+\infty}e^{-i\omega_{ij}t}\langle f|T(\rho_{\mathbf{k}_{ij}}(t),\rho_{-\mathbf{k}_{ij}}(0)]|i\rangle.\quad(12)$$

In general, all three terms in the polarization sum will contribute to the scattering matrix element M_{fi}.

Consider for the sake of definiteness, the scattering when the two beams are incident head on (see Fig. 3). Each beam has two independent polarization directions, in the plane of scattering +, and perpendicular to the plane of scattering −. It is easy to see that there are, as in the scattering of light in a vacuum,[6] only eight independent amplitudes, (M_{++++}, M_{----}, M_{++--}, M_{+-+-}, M_{+--+}, M_{--++}, M_{-+-+}, M_{-++-}). If we polarize our initial beams with beam 1 in the + direction, 2 in the − direction and measure 3 in the + direction, we get only one term from the eight possible amplitudes M_{+-+-}. Our matrix element M then reduces to

$$M_{fi}=\frac{4(2\pi)^2e^4}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}(\mathbf{e}_1\cdot\mathbf{e}_3)\frac{1}{m_1^2}\langle f|R_{1,3}|i\rangle.\quad(13)$$

If we now substitute M_{fi} into the expression for the transition rate and divide by the flux per cc, which is 2 in these units, we find the differential cross section for scattering photon number 3 into a solid angle $d\Omega_3$ with its polarization vector perpendicular to the scattering plane.

$$d\sigma/d\Omega_3=\sum_i\sum_f\int d\omega_3\delta(\omega_1+\omega_2-\omega_3-\omega_4)\times\eta|\langle f|R_{1,3}|i\rangle|^2\exp[\beta(\Omega-\mathcal{E}_i+\mu N_i)],\quad(14)$$

where

$$\eta=(8\pi^2e^8\omega_3/\omega_1\omega_2\omega_4)(\cos^2\theta).\quad(15)$$

[6] J. McKenna and P. M. Platzman, Phys. Rev. **129**, 2354 (1963).

We have specified an average over an initial canonical ensemble for the many-body system at temperature $\beta=1/kT$. The sum is only over final states of the many-body system which conserve energy. By setting $\omega_1+\omega_2=\omega_3+\omega_4$ and putting in the δ function as an additional time integration the sum over final states can be formally performed. However, we still must evaluate an expression of the form

$$S=\int\int\int dtdt'dt''e^{i\omega t}e^{-i\omega' t}\times\langle T\{\rho_{-\mathbf{q}}(t'+t'')\rho_{\mathbf{q}}(t')\}T\{\rho_{\mathbf{q}}(t),\rho_{-\mathbf{q}}(0)\}\rangle\,, \quad (16)$$

where $\omega\equiv\omega_{13}$ and $q=k_{13}$. The bracket symbol is the usual thermal ensemble average

$$\langle O\rangle=\mathrm{Tr}\{e^{+\beta(\Omega+\mu N-H)}O\}\,. \quad (17)$$

We are only interested in that part of the average which is proportional to a delta function, i.e., is infinite like $\mathbf{T}$ (the time interval in the integration). This infinity just cancels the $d\omega_3$ to give terms of order unity in the differential scattering cross section. It is possible to show that only products of the form

$$\langle T[\rho_{-\mathbf{q}}(t'),\rho_{\mathbf{q}}(0)]\rangle\langle T[\rho_{\mathbf{q}}(t),\rho_{-\mathbf{q}}(0)]\rangle \quad (18)$$

give terms proportional to $\mathbf{T}$. The net result is that we only have to consider the expectation value of two Heisenberg density operators (the two-point density correlation function). If we wanted to go to higher order in the plasma parameter, the $\mathbf{j}\cdot\mathbf{A}$ terms in H_I would contribute to the scattering cross section and the final answer could not simply be expressed in terms of the two-point density correlation function.

We can then write the cross section as[7]:

$$\left(\frac{d\sigma}{d\Omega_3}\right)=r_0^2\left(\frac{\omega_3}{\omega_4}\right)\frac{1}{(k_1k_2)}\frac{8\pi^2}{m^2}\left|\frac{e^2Q(q,\omega)q^2}{q^2-4\pi e^2Q(q,\omega)}\right|^2, \quad (19)$$

with

$$Q(q,\omega)=\int_{-\infty}^{+\infty}dte^{i\omega t}\langle T\{\rho_q(t)\rho_{-q}\}\rangle_0, \quad (20)$$

$$\rho_q(t)=e^{iH_0t}\rho_qe^{-iH_0t}, \quad (21)$$

where H_0 is the free electron Hamiltonian and $|\rangle_0$ its eigenfunctions, which obey

$$H_0|\rangle_0=E|\rangle_0. \quad (22)$$

The quantity r_0 is the classical radius of the electron, $\cos\theta$ is the scattering angle (see Fig. 3), and

$$1-(4\pi e^2/q^2)Q(q,\omega)\equiv\epsilon(q,\omega)\,, \quad (23)$$

where $\epsilon(q,\omega)$ is the so-called "causal" dielectric function. The causal dielectric function differs from the usual retarded dielectric function because it has a different imaginary part

$$\mathrm{Im}[\epsilon(q,\omega)]^{-1}=\coth(\beta\omega/2)\,\mathrm{Im}[\epsilon_+(q,\omega)]^{-1}\,, \quad (24)$$

where $\epsilon_+(q,\omega)$ is the usual Lindhard dielectric function.[8] Equation (19) may then be rewritten as (replacing all $\hbar$'s and c's)

$$\frac{d\sigma}{d\Omega_3}=\frac{r_0^2}{2}\left(\frac{\omega_3}{\omega_4}\right)\frac{q^2}{k_1k_2}\left(\frac{q\hbar}{mc}\right)^2\left|\frac{1-\epsilon(q,\omega)}{\epsilon(q,\omega)}\right|^2\cos^2\theta. \quad (25)$$

III. EVALUATION OF CROSS SECTION AND DISCUSSION OF RESULTS

Typically, $\omega_3/\omega_4\approx1$, $q^2/k_1k_2\approx1$ (if we are away from the forward direction). For light sources operating in the visible at a frequency $\hbar\omega_e\approx2$ eV, the quantity $q\hbar/mc$ is of the order of 10^{-5}. For these values of the parameters Eq. (25) may be written as

$$\frac{d\sigma}{d\Omega_3}\approx10^{-36}\left|\frac{\epsilon(q,\omega)-1}{\epsilon(q,\omega)}\right|^2\ \mathrm{cm^2/sr}. \quad (26)$$

When $\epsilon(q,\omega)\neq0$, the ratio inside the absolute value sign in Eq. (26) is less than or equal to one. Near a zero of $\epsilon(q,\omega)$, there is a resonance in the scattering cross section. This simply means that the intermediate state to which the many-body system was excited in our fourth-order process could have been a real state. Of course, these states have a finite lifetime, i.e., $\epsilon(q,\omega)$ has an imaginary part. The total cross sections are not infinite but just large.

Away from a resonance the ratio involving the dielectric function in Eq. (26) is of the order of unity. To see if an experiment is feasible, consider two cw lasers each capable of emitting 10^{20} photons/sec or approximately 20 W. Now suppose that the output of each laser is focused cown to an approximately parallel beam which is 10^{-5} cm^2 in cross-sectional area. If we allow these two beams to scatter along the length of a plasma column 30 cm long, then there would be a total of ten photons/sec scattered into 4π solid angle. This would be an extremely difficult experiment to perform.

Suppose instead we used two pulsed lasers each capable of emitting twenty joules per pulse in the same experimental set up. If we assume that the pulse lasers have a pulse width of 10^{-8} sec, then the total number of scattered photons per pulse would be increased to 10^9 photons in 4π solid angle. This is an appreciable number of photons. The main problem in such an experiment would be the problem of eliminating unwanted background. This would come mainly from incoherent scattering of the single beams and from light emitted from the plasma itself. If the two lasers used in the experiment were at different frequencies[9] photon 3 would be at a

[7] This result was independently obtained by D. F. Dubois and V. Gilinsky, Phys. Rev. **135**, A985 (1964).

[8] A. J. Glick and R. A. Ferrell, Ann. Phys. (N. Y.) **11**, 359 (1960).

[9] D. F. Dubois, V. Gilinsky, and M. Kivelson, Phys. Rev. **129**, 2376 (1963) [see Eq. (26) and Fig. 11].

frequency ω_3 determined by the scattering angle θ.

$$\omega_3=\frac{2\omega_2\omega_1}{\omega_1(1-\cos\theta)+\omega_2(1+\cos\theta)}. \qquad (27)$$

To lowest order in the difference frequency $\Delta\omega'\equiv\omega_2-\omega_1$, $\omega_3-\omega_1\approx\Delta\omega'\cos\theta$. The difference frequency for the scattered photon, by going to angles other than $\frac{1}{2}\pi$, will be approximately equal to the difference frequency between the two incident beams. A ruby laser and a neodinium laser, the first operating in the visible and the second operating in the infrared, could be used to produce frequency shifts in the infrared. A spectrometer could be used as a means for discriminating against unwanted background. In addition, coincidence counters could be used to further discriminate against background.

Although the cross section is small away from resonance and would not be easy to measure, an experiment using pulsed lasers is not out of the question. However, near resonance, i.e., a zero of $\epsilon(q,\omega)$, the cross section is increased by many orders of magnitude and an experiment is a definite possibility.

To determine how large the cross section can be at a resonance, let us assume that a fully ionized nondegenerate high-temperature plasma acts as the scattering medium. At resonance, $\omega^2\simeq\omega_p{}^2+q^2\langle v^2\rangle$, $\mathrm{Re}\epsilon_+(q,\omega)=0$, and $\mathrm{Im}[\epsilon_+(\omega,q)]\simeq\omega_p\tau$ where τ represents the lifetime of the intermediate state of the many-body system. The lifetime τ is determined either by Landau damping or by collisions, whichever leads to shorter τ,

$$(\omega_p\tau)^{-1}=(\omega_p\tau_l)^{-1}+(\omega_p\tau_c)^{-1}, \qquad (28)$$

where

$$(\omega_p\tau_c)^{-1}=\frac{r_s}{12\sqrt{2}\pi^{3/2}}\ln[C_a(\omega_p)/\beta\hbar\omega_p] \qquad (29)$$

is the part due to electron-ion collisions and

$$(\omega_p\tau_l)^{-1}=\tfrac{1}{8}\pi^{1/2}(q_d{}^3/q^3)e^{-q_d{}^2/q^2} \qquad (30)$$

is the part due to Landau damping. The quantity $C_a(\omega_p)$ is a definite integral which has been evaluated numerically in Ref. 8. Both parts have been calculated in the long-wavelength limit, i.e., $q\ll q_d$. The collisional part, Eq. (29), will vary as $q^2/q_d{}^2$ as one moves away from the origin in q space. At resonance then Eq. (25) may be written as

$$\frac{d\sigma}{d\Omega_3}=\frac{r_0{}^2}{2}\left(\frac{\omega_3}{\omega_4}\right)\left(\frac{q^2}{k_1k_2}\right)\left(\frac{q\hbar}{mc}\right)^2(\omega_p\tau)^2\cos^2\theta. \qquad (31)$$

In an actual experiment near the plasma resonance one would like to maximize the total cross section at resonance [Eq. (31)]. In a fully ionized plasma at high temperatures ($kT\approx10^2$ eV, $n\approx5\times10^{12}$) the collisional part in the $\mathrm{Im}\epsilon_+$ is extremely small of the order of 10^{-6}. In these types of plasmas, light in the visible part of the spectrum has a k of the order of q_d. Therefore, unless q is made appreciably smaller than k by measuring the scattered light in the nearly forward direction, Landau damping will dominate the collisional damping and $|\mathrm{Im}(1/\epsilon_+)|^2$ will be severely decreased. However, by going to the forward direction the factor q^4 in Eq. (25) tends to reduce cross section. In an actual case one should maximize the quantity $q^4|\mathrm{Im}[1/\epsilon_+(q,\omega)]|^2$ to find the optimum conditions for doing an experiment.

At resonance the elastic light-off-light scattering experiment gives the same information about the plasma as the incoherent scattering. The light-off-light scattering may be thought of as incoherent scattering squared. We have previously compared the total counting rates in the light-off-light scattering experiment with that in the single-beam scattering,[3] and have shown that it is possible to make the counting rate in the light-off-light scattering larger than in the single-beam experiment. The physical reason for this is as follows. One of the beams in the light-off-light scattering excites a large density of plasma oscillations in a very small energy interval. The other beam then scatters from this "nonequilibrium plasma." The result is an enhancement of the counting rate.

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Note added in proof. Recently there has appeared a paper by N. Kroll, A. Ron, and N. Rostoker (KRR) [Phys. Rev. Letters **13**, 83 (1964)] in which the nonlinear interaction of light in a plasma is considered. KRR calculated a counting rate for a particular nonlinear process using a classical approach but neglecting thermal fluctuations.

We would like to point out that the cross section as calculated here and in Ref. 3 may be used to determine the counting rates in a variety of different experimental arrangements, in particular, the arrangement considered by KRR. The KRR arrangement corresponds to putting in energy in beam (1) and in beam (3), and looking at the scattering of beam (2) into beam (4). The total scattering rate into beam (4) would then be proportional to $n_1n_2n_3$ (i.e., $E_1{}^2E_2{}^2E_3{}^2$) in the standard way. [See for example, J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), Chap. 8].

If one normalizes all electromagnetic fields to a single photon per unit volume, then the KRR results, Eqs. (6) and (1) in their paper may be compared with our result, Eq. (1), Ref. 3, or Eq. (19) of this paper. At resonance, i.e., near zero of $\epsilon(q,\omega)$, this comparison shows that the *cross sections* are identical except that the two dielectric functions which appear in the denominator are different. The two functions, at resonance, and in the high temperature limit, differ by a factor $(kT/\hbar\omega_p)^2$. This factor is the square of the number of plasmons in thermal equilibrium. It arises from the macroscopic occupation of the plasmon states. We believe that had KRR included thermal fluctuations in their analysis, they too would have found a factor corresponding to our $(kT/\hbar\omega_p)^2$ multiplying their cross section.

APPENDIX A

In this appendix we make a qualitative comparison between the matrix element of $\mathbf{j}\cdot\mathbf{A}$ taken twice with the matrix element of $\mathbf{A}^2$. Schematically then;

$$M_{\mathbf{j}\cdot\mathbf{A}}\approx\langle\mathbf{j}_1\cdot\mathbf{A}\rangle\langle\mathbf{j}_1\cdot\mathbf{A}\rangle/\Delta E\,, \tag{A1}$$

$$M_{\mathbf{A}^2}\approx e^2\langle\mathbf{A}^2\rho_1\rangle/m\,, \tag{A2}$$

$$R\equiv\frac{M_{\mathbf{j}\cdot\mathbf{A}}}{M_{\mathbf{A}^2}}=\frac{1}{m\Delta E}\frac{\langle\mathbf{p}_1\cdot\mathbf{A}\rangle\langle\mathbf{p}_1\cdot\mathbf{A}\rangle}{\langle\mathbf{A}^2\rho_1\rangle}\,. \tag{A3}$$

In Eqs. (A1)–(A3) the matrix elements are to be taken between eigenstates of H_p, Eq. (1b). The energy denominator ΔE is a sum or difference of an excitation energy of the many-body system and a typical photon energy and $m_1=m$ mass of the electron. For a plasma the light couples to the electrons alone. The electromagnetic coupling to the ions is reduced by a factor $(m_1/m_2=m/M\approx 1/2000)$. Since the total momentum of the electron-ion system is a conserved quantity for the Hamiltonian H_p, we assume that it is zero, i.e., that the system carries no total current. If we neglect collisions between electrons and ions (these collisions are of higher order in the plasma parameter, and will be considered shortly), the total momenta of both electrons and ions are separately constants of the motion. They are both independently zero. With the neglect of collisions then, it is clear that to all orders in the electron-electron interaction the $\langle\mathbf{p}_1\cdot\mathbf{A}\rangle\langle\mathbf{p}_1\cdot\mathbf{A}\rangle$ term is of the order $k_e^2\langle\mathbf{A}^2\rho\rangle$. The only momentum available for the $\mathbf{p}_1\cdot\mathbf{A}$ term to couple to is the recoil momentum picked up from the light itself. In this case k_e is the wave vector of the incident or scattered light beams. Thus,

$$R\approx k_e^2/m_1\Delta E\,. \tag{A4}$$

It remains to estimate ΔE. It is easy to show that as long as $v/c\ll 1$ (where v is a typical particle velocity) $\Delta E\sim\omega_e$, the frequency of the light. The energy difference of the recoiling many-body system is negligible compared with ω_e. Putting $\Delta E\approx\omega_e$ into Eq. (A3), and replacing c and $\hbar$ we find

$$R\simeq(\hbar\omega_e/mc^2)\,. \tag{A5}$$

The ratio is completely negligible for the usual experiment so we conclude that the $\mathbf{A}^2$ term in this case gives the entire amplitude.

Suppose we go to the next order in the electron-ion interaction (i.e., we include electron ion collisions), then R is not small and cannot be neglected. Since the electrons collide with the ions, only the total momentum $\mathbf{P}=\mathbf{P}_1+\mathbf{P}_2$ commutes with H_p. The quantities $\mathbf{P}_1$ and $\mathbf{P}_2$ are not zero separately. In fact, both P_1 and P_2 in general will be of order P_F where P_F is typical momentum, the Fermi momentum for a degenerate plasma and a thermal momentum for a classical plasma. In this case then

$$R\approx P_F^2/2m_1\hbar\omega_e\,, \tag{A6}$$

which can and is usually larger than one so that the $\mathbf{P}_1\cdot\mathbf{A}$ term is dominant not the $\mathbf{A}^2$ term. Another way of saying this is to remark that collisions with ions are equivalent to allowing the electrons in the plasma to interact with external Coulomb fields which have relatively high Fourier components. The Coulomb fields bring in wave numbers of the order of k_F producing a recoiling electron system with momentum of the order of P_F.

For quantum plasmas the preceding discussion is a valid one. For classical plasmas the situation is not so clear, since the ratio R should not depend on $\hbar$. The extra factor of $\hbar$ arises from the identification of the momentum and energy of the electromagnetic wave with its wave number k_e and frequency ω_e. Detailed calculations of a few diagrams show that an additional statistical weighting factor $A\coth(\beta\hbar\Delta\omega/2)$ (where A is a number of the order of unity) appears as a multiplicative factor in the ratio R. In the quantum zero temperature limit $\beta\rightarrow\infty$, $\coth(\beta\hbar\Delta\omega/2)\rightarrow 1$, and R is unchanged. In the classical limit $\beta\rightarrow 0$, $\coth(\hbar\beta\Delta\omega/2)\rightarrow 1/\beta(\Delta\omega)\hbar$ and

$$R\approx(\Delta\omega/\omega_e)\,. \tag{A7}$$

Typically, $\Delta\omega$ is of the order of ω_p so that $R\approx\omega_p/\omega_e$ which is small compared to one.

In the classical case our conclusion then is that the term $\mathbf{j}\cdot\mathbf{A}$ can always be neglected as long as $\omega_p/\omega_e\ll 1$.