Shape Independence of Low-Lying States of Nuclei*

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In this paper an attempt is made to show that the calculations in nuclear spectroscopy do not depend much on the radial shape of the two-body potential. Explicit calculations on Ni^{58} and Pb^{206} are made for Gaussian, exponential, and Yukawa potentials to illustrate this point. It is also shown that the insensitivity to the nuclear potential shape, first pointed out by Blatt and Jackson in the case of low-energy scattering data, is useful in systematizing a variety of data.

I. INTRODUCTION

DURING the past fifteen years, independent particle models and collective models have been extensively used to explain a variety of properties initiated in the nuclear structure studies. Though in most of the calculations (e.g., the analysis of the nucleon-nucleon scattering data) the approach has been mainly phenomenological; the Brueckner theory,[1] based on the two-body correlations, provides an excellent understanding regarding the nature of the nucleon-nucleon interaction in nuclei. Even so, this theory poses another problem of complexity; namely, the problem of the hard core. It is, however, shown by Moszkowski[2] that such a singularity can be well replaced by means of a nonsingular velocity-dependent potential. In any case, the purpose of each theory has been to reproduce the binding energies of nuclei (as given by the Bethe-Weiszackar mass formula) on the basis of realistic potentials which explain a large variety of scattering data. Quite a few such empirical and semiempirical potentials which satisfy the above requirements are now available. However, they fail to give the required binding in the He^3 nucleus. In fact, Blatt[3] has pointed out that as the agreement between a "new potential" and the scattering data improves, the deviation from the He^3 binding energy becomes more serious. On the other hand, it is now well known that as far as the low-energy scattering is concerned, the radial dependence of the nucleon-nucleon force is negligible. In other words, the low-energy free nucleon-nucleon scattering data can be well described in terms of the scattering length and the effective range of a two-nucleon potential $V(r)$. This problem is investigated in detail by Blatt and Jackson.[4] It is the purpose of this paper to see explicitly whether their results apply also to the problems of nuclear spectroscopy, where one deals with relatively low energies of the order of 3–4 MeV.

In order to normalize the various potentials for the sake of comparison, these authors[4] introduced two parameters; namely, the strength s and the intrinsic range b of the potential. The parameter b, in fact, is the scattering length of the preadjusted potential $V^R(r)$, which gives resonance at zero energy, and s is then defined by $V(r)=sV^R(r)$. The potential parameters V_0 and r_0 are then related to s and b (in units of $M=\hbar=1$),

$$\begin{aligned}&\text{Gaussian:} && s=0.37261V_0r_0^2, && b=1.4354r_0,\\ &\text{Exponential:} && s=0.69164V_0r_0^2, && b=3.5412r_0,\\ &\text{Yukawa:} && s=0.59531V_0r_0^2, && b=2.1196r_0,\end{aligned}$$

where the potentials themselves have the form

$$\begin{aligned}&\text{Gaussian:} && V(r)=V_0e^{-(r/r_0)^2},\\ &\text{Exponential:} && V(r)=V_0e^{-(r/r_0)},\\ &\text{Yukawa:} && V(r)=V_0e^{-(r/r_0)}/(r/r_0).\end{aligned}$$

In calculations of the low-lying levels of nuclei, the energy of the interacting nucleons would be small. One therefore expects that the splittings of these levels, when the levels arise from the nucleon-nucleon interaction (and not due to collective effects), would not depend upon the radial form of the two-body potential. It is also obvious that such an approach would then be useful to systematize the data obtained by different authors for various configurations. In the following sections we bring out these points explicitly by considering the level spectra of Ni^{58} and Pb^{206}.

II. METHOD OF CALCULATION

Calculations in nuclear spectroscopy can be done in two ways: (a) By Talmi[5] and Thieberger's[6] method, in which the matrix elements of the interaction Hamiltonian are expanded in terms of multipoles, and (b) the method of relative coordinates due to Moshinsky[7]

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[1] See, e.g., K. A. Brueckner, J. L. Gammel, and H. Weitzner, Phys. Rev. **110**, 431 (1958).
[2] S. A. Moszkowski, Phys. Rev. **129**, 1901 (1963).
[3] J. M. Blatt, Lecture given in Physics Seminar, University of California, La Jolla, California, 1964 (unpublished); J. M. Blatt, G. H. Derrick, and J. N. Lyness, Phys. Rev. Letters **8**, 323 (1962).
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and others.[8–10] We sketch briefly the second (b) method here.

We write the matrix elements of the interaction Hamiltonian as

$$\langle j_1 j_2 : JM | H_{12} | j_1' j_2' : JM \rangle . \tag{1}$$

As we are interested in the relative angular-momentum states, it is convenient to transform the jj-coupling wave functions into LS-coupling by means of the LS-jj transformation coefficients.

$$| j_1 j_2 : JM \rangle = \sum_{LS} A \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} | l_1 l_2(L), s_1 s_2(S) ; JM \rangle . \tag{2}$$

The transformation of two-particle wave functions in terms of relative and center-of-mass coordinates is made in the following way:

$$| n_1 l_1, n_2 l_2, L\mu \rangle = \sum_{nlN\lambda} B_{nlN\lambda}{}^{n_1 l_1 n_2 l_2}(L) | nl, N\lambda, L\mu \rangle , \tag{3}$$

where B's are the transformation brackets tabulated by Moshinsky. Collecting (1), (2), and (3) and making some simplifications, one obtains for the matrix elements of a central force two-body operator,

$$\langle j_1 j_2 : JM | H_{12} | j_1' j_2' : JM \rangle = aa' \sum_{\substack{LS \\ N\lambda nln'}} A \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} A \begin{bmatrix} l_1' & s_1' & j_1' \\ l_2' & s_2' & j_2' \\ L' & S' & J' \end{bmatrix} \times B_{N\lambda nl}{}^{n_1 l_1 n_2 l_2}(L) B_{NLn'l}{}^{n_1' l_1' n_2' l_2'}(L') \delta_{LL'} \delta_{SS'} \times \langle nl, S | H_{12} | n'l, S \rangle , \tag{4}$$

where a and a' equal $\frac{1}{2}$ if the particles are equivalent, and equal $1/\sqrt{2}$ if the particles are inequivalent. For singlet forces ($S=0$) the A coefficients give terms of the type δ_{LJ} and $\delta_{L'J}$, and if the particles are equivalent (i.e., both protons or both neutrons), then one obtains only l-even states. In a similar way for triplet forces ($S=1$) one would obtain only odd l states. For simplicity we assume a two-body force of the form

$$H_{12} = (x + y \boldsymbol{\sigma}_1 \cdot \boldsymbol{\sigma}_2) V(r_{12}) , \tag{5}$$

where x and y are constants to be evaluated from experiments. The potential $V(r_{12})$ may be any of the three shapes of the potential discussed in Sec. I. The expression (4) can then be simplified by writing

$$\langle j_1 j_2 : JM | H_{12} | j_1' j_2' : JM \rangle = aa' \sum_{\substack{LS \\ N\lambda nl}} A \begin{bmatrix} l_1 & s_1 & j_1 \\ l_2 & s_2 & j_2 \\ L & S & J \end{bmatrix} A \begin{bmatrix} l_1' & s_1' & j_1' \\ l_2' & s_2' & j_2' \\ L & S & J \end{bmatrix} \times B_{nlN\lambda}{}^{n_1 l_1 n_2 l_2} B_{nlN\lambda}{}^{n_1' l_1' n_2' l_2'} \times [1 + (-)^{S+l}]^2 I_{nl} , \tag{6}$$

where

$$I_{nl} = \langle nl, S \| H_{12} \| nl, S \rangle . \tag{7}$$

The integrals I_{nl} for the Gaussian potential are tabulated by one of us[11] for the quantum numbers $n=1$ to 4 and $l=0$ to 5, for various values of the range parameter $\lambda = r_0/r_l$, where r_0 is the range of the two-body potential $V(r)$ and r_l is the range of the nucleon harmonic oscillator wave function. In order to compare the parameters x and y (and effectively s and V_0) appearing in expression (5) with those of other authors (see Ref. 11), we need to connect the coefficients A_{TS} defined by Barker[12] with x, y, and V_0. The result (for $T=1$) is

$$A_{10} = (1/V_0)(x - 3y) , \qquad A_{11} = (1/V_0)(x + y) . \tag{8}$$

III. RESULTS AND DISCUSSION

The purpose of choosing Ni^{58} and Pb^{206} for our analysis is the following: Recently, there has been some theoretical information available regarding the nature of the effective interaction for these nuclei. For example, while Kearsley[13] has made calculations for Pb^{206} with the Yukawa potential, Band *et al.*[14] have shown that a similar fit can also be obtained for a different range and strength of the Gaussian potential for the Pb^{206} nucleus. The preliminary calculations[15] on Ni^{58} and on p-shell nuclei[11] have been made by one of us (YRW). The result of these calculations is that the effective two-body interaction in the p shell is configuration-dependent. We try to verify this conclusion also in the framework of this paper.

We now calculate the level spectrum of Ni^{58} for the Gaussian, exponential, and Yukawa potentials to give a satisfactory fit with the observed splittings of the 0^+, 2^+, and 4^+ levels for various values of the range parameter λ. The 0^+ and 2^+ levels arise from the ground-state configuration $(p_{3/2})^2$, while the 4^+ arises from the excited $(p_{3/2} f_{5/2})$ configuration. The other low-lying levels that can arise are $J=1^+$, 2^+, 3^+ from $(p_{3/2} f_{5/2})$

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TABLE I. Calculated values of various parameters for different potentials in Ni^{58} and Pb^{206}.

Nucleus	Potential	A_{10}	A_{11}	$-V_0$ in MeV	r_0 in F	s in MeV F^2	b in F
Ni^{58}	Exponential	0.6	0	40	0.85	20	3.0
	Yukawa	0.6	0	28	1.08	19	2.3
	Gaussian	0.6	0.08	22	1.47	19	3.0
Pb^{206}	Yukawa	0.6	−0.34	69	1.37	77	2.9
	Gaussian	0.6	0.26	60	2.00	89	2.9

configuration and $J=0^+$, 2^+, and 4^+ from $(f_{5/2})^2$ configuration. We choose a set of parameters λ, x, and y, which give a satisfactory fit with all the observed levels for each potential shape. Such a set for each potential is given in Table I. The observed levels and levels calculated with the parameters of Table I are shown in Fig. 1. It is interesting to note that the calculated levels, which are in agreement with the observed results for all three potentials, give almost the same values of the parameters s and b. It is thus clear that the position of the levels do not depend much on the radial shape. One point about the general behavior of the matrix elements in all three cases should be mentioned. Though the diagonal matrix elements of the levels vary from potential to potential for the same level, the splittings of the levels do not seem to vary substantially. The off-diagonal or the configuration mixing matrix elements are rather weak and have little affect on the relative level spacings. The maximum shift observed is ~ 0.25 MeV for the ground state 0^+. This has been discussed for the Gaussian potential earlier.[15] It is also shown in that paper[15] that the calculations on Ni^{58} do predict the ground state of Ni^{59} as $\frac{3}{2}^-$ which is experimentally suspected. This shows that the jj coupling is valid in this nucleus.

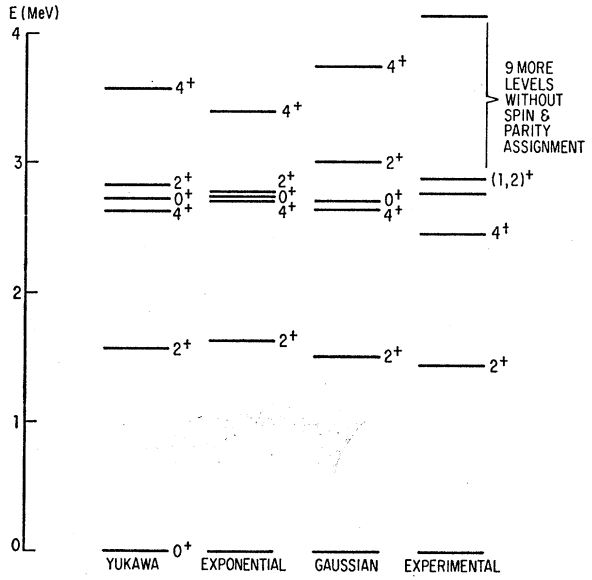

FIG. 1. The observed as well as the calculated levels of Ni^{58} for three potentials discussed in the text.

In Table I we give the values of the parameters for Pb^{206} calculated by Kearsley[13] and Band *et al.*[14] It is clear that the agreement in s and b is satisfactory. It is also clear from the table that the interaction which operates in Ni^{58} does not operate in Pb^{206}. In other words, the two-body effective interaction is configuration-dependent. In Table II, we give the values of the parameters calculated by various authors for different configurations in terms of s and b.

Finally, the variation of parameters s and b with the mass number can probably be derived from Brueckner theory,[1] which gives some idea about the residual interaction in finite nuclei. We feel that the Brueckner theory would give an agreement in heavier nuclei since it is derived primarily from the extension of the theory of nuclear matter.[16]

TABLE II. The parameters s and b evaluated from the analysis of various authors in different mass regions ($M=\hbar=1$).

Authors	Mass number	b in F	s in MeV F^2	Reference
Thankappen, Waghmare, and Pandya	90	3.01	83.80	a
Raz and French	43	3.88	81.49	b
Elliott and Flowers	18	2.97	56.36	c
Barker	16	2.97	90.19	12
Peaslee	16	2.97	70.01	d
Rosenfeld	16	2.97	41.54	e
Waghmare	d,s shells	2.11	54.72	11

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