PHYSICAL REVIEW VOLUME 135, NUMBER 3B 10 AUGUST 1964

Inelastic Effects in Nucleon-Nucleon Scattering*†

YAN-CHOK LEUNG
Department of Physics, Massachusetts Institute of Technology, Cambridge, Massachusetts
and
Department of Physics, University of Colorado, Boulder, Colorado
(Received 16 March 1964)

In this study we examine theoretically the possibility that the elastic scattering amplitude may be substantially enhanced due to the opening of a strong inelastic channel as the elastic and inelastic channels are coupled through the unitarity condition. Some recent experimental data on proton-proton scattering have suggested rather convincingly such a phenomenon. The theoretical investigation of this feature is carried out by means of the current method of dispersion relations. The dynamical inputs of the scattering amplitudes are derived from the perturbation diagrams of the Feynman theory using only the one-pion-exchange diagrams. A multichannel unitarity condition is used by retaining the three-particle intermediate states. This provides a coupling between the elastic and the production amplitudes. Finally, the problem is solved in terms of the N/D method. The results obtained demonstrate a maximum in the elastic scattering amplitude, which seems to explain the observed phenomenon.

I. INTRODUCTION

EXPERIMENTAL work on nucleon-nucleon elastic scattering by Martelli *et al.*[1] has shown that certain partial wave amplitudes in the total isotopic spin $T=1$ state is enhanced over the $T=0$ state at energies somewhat above the production threshold. This feature is usually not apparent from comparing the total cross sections of scattering in these two states, since at such energies many partial waves would have contributed to the cross sections, and the detail features would have been obscured. In the experimental conducted by Martelli *et al.*, the differential cross sections for p-p and n-p scattering at 90° in the center-of-mass (c.m.) system are measured. Since many angular momentum states do not contribute to right angle scattering, this experiment is able to eliminate some of the nonessential interferences. Unfortunately, not all odd orbital angular momentum states are removed by this procedure, contrary to what is being claimed.[2] Therefore, in the $T=1$ channel, for example, part of the ^3P states mix in with the ^1S and ^1D states, all of which are believed to be contributing substantially to the cross sections.

Measurements are made at 595, 775, and 1010 MeV. These together with the differential cross sections at lower energies given by Hess[3] and by Amaglobeli and Kazarinov[4] indicate clearly the trend of the cross sections for the $T=0$ and $T=1$ states (see Fig. 1). Around 600 MeV we see that the $T=1$ state is definitely enhanced relative to the $T=0$ state, whereas around 1100 MeV the $T=0$ state is enhanced. The presentation of the first enhancement can be made more appealing by taking the difference of the cross sections in these two states, assuming that the $T=1$ state would behave similar to the $T=0$ state if not for the enhancement, as shown dotted in Fig. 1, where we see a resonance-like behavior at around 600 MeV.

Since the position of the enchancement occurs at an energy which is very near to that required for the production of the $\pi-N$ (3,3) isobar, and since this isobar can only be produced in the $T=1$ state and not in $T=0$ state, it has already been pointed out by Martelli *et al.* that the enhancement is a result of the coupling of the $T=1$ elastic amplitude to this production amplitude. Similarly, the enhancement at higher energies in the $T=0$ elastic amplitude may be the result of its coupling with the production of another $\pi-N$ isobar which occurs at the higher energy region of the $\pi-N$ spectrum. We shall not, however, be involved with the $T=0$ channel, but concentrate with the $T=1$ channel for the time being.

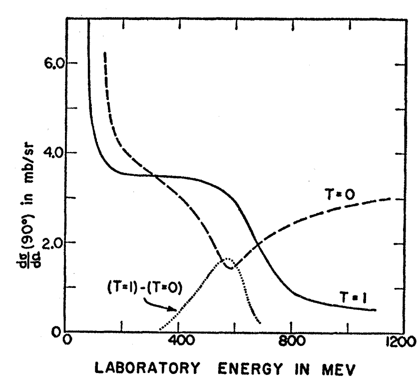

FIG. 1. Elastic differential cross sections for nucleon-nucleon scattering in $T=0$ and $T=1$ states at 90° c.m. system.

* This work is supported in part through funds provided by the U. S. Atomic Energy Commission under Contract At(30-1)-2098 and by the National Science Foundation.
† Based in part on a thesis submitted to the Physics Department of the Massachusetts Institute of Technology by the author in partial fulfillment of the requirements for the degree of Doctor of Philosophy.
[1] G. Martelli, H. B. Van der Raay, R. Rubinstein, K. R. Chapman, J. D. Dowell, W. R. Frisken, B. Musgrave, and D. H. Reading, Nuovo Cimento **21**, 581 (1961).
[2] I wish to thank Dr. D. Sprung for pointing this out to me.
[3] W. H. Hess, Rev. Mod. Phys. **30**, 368 (1958).
[4] N. S. Amaglobeli, and Yu. M. Kazarinov, Zh. Eksperim. i Teor. Fiz. **37**, 1587 (1959) [English transl.: Soviet Phys.—JETP **37**, 1125 (1960)].

Taking the suggested mechanism for granted, we see that if the (3,3) isobar is produced in a $l=0$ state relative to the nucleon, then because of the conservation of total angular momentum, isotopic spin, and parity, the only possible two-nucleon initial state is ${}^1D_2(T=1)$. Hence, the partial wave amplitude that is of interest will be in $T=1$, $J=2$ state. In fact, this is why the right angle scattering experiment is performed—to bring out the full effect of this possibility.

We might remark that the maximum occurring at 600 MeV cannot be identified with a resonance in the $J=2$ state. For if it were a resonance, the inelastic cross section at this energy must be at least as large as 23 mb in order to be consistent with unitarity. This is known not to be the case.[3] Therefore, if we insist that this maximum occurs in the $J=2$ state, then it is probably more adequate to call it a "woolly cusp" after Pais and Nauenberg.[5]

Recently, many authors[6–9] have discussed scattering models which include isobar production. It seems interesting to put the above conjecture that the enhancement is due entirely to the production of the (3,3) isobar to a test. Formalism related to the treatment of multichannel problem within the framework of dispersion relations has also been well developed, and we shall follow closely that of Cook and Lee.[8]

Briefly, the method is as follows. A set of equations for the scattering amplitudes T_{22}, T_{23}, and T_{33}, which correspond to two-body elastic, production, and three-body elastic processes, respectively, are developed based on a truncated unitarity condition that involves only these three processes. Then these equations are solved according to an input information on the production amplitude, which is calculated from a single-pion-exchange diagram of an isobar model. All connections are discussed in terms of the analytic properties of the scattering amplitudes. This amounts to assigning to each scattering amplitude in the complex plane of its energy variable a "unitary cut" with the discontinuity over this cut given by the imaginary part of the amplitude as prescribed by the unitarity condition, and in addition, certain "dynamical singularities" to the production amplitude, which are evaluated from the single-pion-exchange diagram.

Inelastic effects on the elastic amplitude can be seen very easily in a formulation based on the usual dispersion relations. The inclusion of all inelastic processes in the unitarity condition modifies the approximate "elastic unitarity" by just the factor $(\sigma_T{}^l/\sigma_{el}{}^l)$, where $\sigma_T{}^l$ and $\sigma_{el}{}^l$ denote, respectively, the total and elastic cross sections in the lth partial wave state. In analogy to the formulations by Chew and Low[10] or Frautschi and Walecka,[11] the partial wave elastic amplitude can be written as

$$f_l^{-1}(W)=\frac{D(W_0)}{N(W)}-\frac{(W-W_0)}{\pi N(W)}\oint\frac{dW'p(W')N(W')}{(W'-W_0)(W'-W)}\times\frac{\sigma_T{}^l(W')}{\sigma_{el}{}^l(W')}-ip(W)\frac{\sigma_T{}^l(W)}{\sigma_{el}{}^l(W)},\quad(1)$$

where $p(W)$ is the center of mass momentum, $D(W_0)$ is a constant, and $N(W)$ is some function of dynamical origin. This is a general solution of the unitarity equations. Whatever the real part of f_l^{-1} may be, f_l satisfies the unitarity condition. The imaginary part of f_l^{-1} provides the "unitary limit" of the scattering amplitude, and this limit is reached only when the real part of the denominator vanishes. We see that this limit for f_l is reduced whenever $\sigma_T{}^l>\sigma_{el}{}^l$, and in fact if $\sigma_T{}^l$ takes on a sudden increase, as in S-wave production processes, this sudden depletion of the elastic channel manifests as a phenomenon identified as cusp in the elastic cross section. While the meaning of the imaginary part of f_l^{-1} is strictly kinematical, the real part bears the dynamics of the interaction. We note that the factor $(\sigma_T{}^l/\sigma_{el}{}^l)$ now occurs within the integral, and hence the correction it introduces is felt long before the production threshold is reached. Hence, a strong production channel may cause substantial enhancement in the elastic channel at a lower energy through this mechanism. Assuming $N(W)$ to be nonvarying, a factor $(\sigma_T{}^l/\sigma_{el}{}^l)$ different from one in the production region will cause the real part of f_l^{-1} to decrease, and hence an increase in the cross section until the production threshold is reached; then the increase in the imaginary part will cause the cross section to decrease. The combined effects give the cross section an appearance which looks deceptively like a resonance. This seems to be the case in nucleon-nucleon scattering, which we are about to investigate.

In Sec. II, the kinematics of the multiparticle states is discussed and the helicity amplitudes are defined. In Sec. III, the dynamical problem is formulated and the discontinuity equations over the unitary cuts are derived. In Secs. IV and V, the one-pion-exchanged amplitudes for the production process are evaluated and singularities of the amplitudes are discussed. Finally, in Sec. VI, the equations are solved with the input information coming from the production amplitudes, and the results are exhibited and discussed. The contributions from the complex singularities which have been neglected in the calculations are estimated in Sec. VII.

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[6] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) **18**, 198 (1962).

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[8] L. F. Cook and B. W. Lee, Phys. Rev. **127**, 283 (1962); **127**, 297 (1962).

[9] J. S. Ball, W. R. Frazer, and M. Nauenberg, Phys. Rev. **128**, 478 (1962).

[10] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

[11] S. C. Frautschi, and J. D. Walecka, Phys. Rev. **120**, 1486 (1960).

II. KINEMATICS AND HELICITY AMPLITUDES

The processes that we are going to investigate are the following:

$$\text{(I)}\qquad N(p_1)+N(p_2)\leftrightarrow N(p_1')+N(p_2')\,,$$
$$\text{(II)}\qquad N(p_1)+N(p_2)\leftrightarrow N(p_1')+N(p_2')+\pi(k')\,,$$
$$\text{(III)}\qquad N(p_1)+N(p_2)+\pi(k)\leftrightarrow N(p_1')+N(p_2')+\pi(k')\,, \tag{2}$$

where we use N to denote the nucleons and π pions, and have indicated the momenta that these particles carry.

The two-particle elastic process is described by the independent variables s and t:

$$s=-(p_1+p_2)^2,\quad t=-(p_1-p_1')^2, \tag{3}$$

where the metric is $p^\mu p_\mu=-p_0{}^2+\mathbf{p}^2=-m^2$.

The description of the production process is given after the following decomposition of the three-particle state.[8,12] The three-particle system is first converted into a two-particle system by combining one of the nucleons with the pion to form a subsystem. This nucleon is denoted by the subscript 4. The momentum of the $\pi-N$ subsystem is $\mathbf{q}$ and in the center-of-mass frame of the total system $\mathbf{q}=-\mathbf{p}_3$. This subsystem will, of course, have internal degrees of freedom, and these will be described by the variables σ, α, and β, where

$$\sigma=-(p_4+k)^2, \tag{4}$$

is the square of the "mass" of the subsystem, α and β are the spherical angles measured from $\mathbf{q}$ to $\mathbf{P}_4'$ (where $\mathbf{P}_4'$ is the transformed momentum of nucleon 4 in the rest frame of the subsystem). The production amplitudes describing reaction II are therefore given by five variables, s, t, σ, α, and β, with

$$s=-(p_3+p_4+k)^2,$$
$$t=-(p_1-p_3')^2=(p-p'\cos\theta)^2 -[(p^2+m^2)^{1/2}-(p'^2+m^2)^{1/2}]^2, \tag{5}$$

where θ is the scattering angle between $\mathbf{p}_1$ and $\mathbf{p}_3'$ in the center of mass of the total system, and

$$p^2=(s/4)-m^2,$$
$$p'^2=[s-(\sigma^{1/2}+m)^2][s-(\sigma^{1/2}-m)^2](4s)^{-1}. \tag{6}$$

The three-particle scattering amplitudes describing reaction (III) are decomposed similarly, and are given by variables s, t, σ, α, β, σ', α', β', where the unprimed and primed variables of the pion-nucleon subsystems refer to the initial and final states, respectively.

The scattering amplitudes will be helicity amplitudes, which we shall designate by the momenta of the particles and the helicities of the nucleons, as follows:

$$\text{(I)}\quad \langle p_1'\lambda_1',\, p_2'\lambda_2'|T_{22}(s,t)|p_1\lambda_1,\, p_2\lambda_2\rangle,$$
$$\text{(II)}\quad \langle p_3'\lambda_3',\, p_4'\lambda_4',\, k'\alpha'\beta'|T_{23}(s,t,\sigma')|p_1\lambda_1,\, p_2\lambda_2\rangle,$$
$$\text{(II}')\quad \langle p_1'\lambda_1',\, p_2'\lambda_2'|T_{32}(s,t,\sigma)|p_3\lambda_3,\, p_4\lambda_4,\, k\alpha\beta\rangle, \tag{7}$$
$$\text{(III)}\quad \langle p_3'\lambda_3',\, p_4'\lambda_4',\, k'\alpha'\beta'|T_{33}(s,t,\sigma,\sigma')|p_3\lambda_3,\, p_4\lambda_4,\, k\alpha\beta\rangle.$$

These states are normalized according to

$$\langle p'\lambda'|p\lambda\rangle=(2\pi)^3 2(p^2+m^2)^{1/2}\delta(\mathbf{p}-\mathbf{p}')\delta_{\lambda\lambda'}. \tag{8}$$

Although most of the formalism will be discussed in terms of the invariant variable s, the final calculations are carried in the variable $W=s^{1/2}$ which is slightly easier to work with.

Following Jacob and Wick,[13] the partial-wave amplitudes in the center-of-mass system are defined in the following manner:

$$\text{(I)}\quad \langle p_1'\lambda_1',\, p_2'\lambda_2'|T_{22}(s,t)|p_1\lambda_1,\, p_2\lambda_2\rangle =(2\pi)^3(4p_0/p)\textstyle\sum_J(2J+1) \times\langle\lambda_1'\lambda_2'|T_{22}{}^J(s)|\lambda_1\lambda_2\rangle d_{\mu,\nu}{}^J(\theta), \tag{9}$$

where θ is the scattering angle, and μ, ν are, respectively, the algebraic sums of the helicities of the initial state along $\mathbf{p}_1$ and the final state along $\mathbf{p}_1'$,

$$\mu=\lambda_1-\lambda_2,\quad \nu=\lambda_1'-\lambda_2'.$$

Decomposition of the three-particle state in terms of the variables introduced has been given by Wick.[12] The $\pi-N$ subsystem is treated as a single particle and will be characterized by its rest frame quantum numbers such as mass, spin, and helicity. To accomplish this the $\pi-N$ subsystem is transformed from the center of mass of the total system to its own rest frame; however, a Lorentz transformation of this kind will in general change the axis of quantization of the nucleon, and the helicity is decomposed into a new set of helicities along the new direction of the nucleon momentum vector. Since this complication is unnecessary for our purpose, we shall choose to characterize the spin states of the nucleon 4, which forms the pion-nucleon subsystem, not by helicities but by the spin projection along the direction opposite to $\mathbf{p}_3$ (i.e., along $\mathbf{q}$). Let us again denote this spin projection by λ_4. Then λ_4 is unaltered by the Lorentz transformation along $\mathbf{q}$. Although the nucleons are identical particles, this unsymmetrical treatment of the nucleons, however, does not generate other difficulties as long as we neglect certain "mixed terms," which we shall define later, in the unitarity condition or the cross-section formula. By a procedure similar to that discussed by Wick,[12] we can write

$$|p_3\lambda_3,p_4\lambda_4,k\alpha\beta\rangle=\sum_{j=\frac12}\sum_{\lambda=-j}^{j}|p_3\lambda_3,q\sigma\lambda\rangle \times D_{\lambda,\lambda_4}{}^{j*}(-\beta,\alpha,\beta)N_j(\sigma), \tag{10}$$

where j is the spin of the $\pi-N$ subsystem, λ is its helicity, $q=p'$ in the center of mass of the total system, and

$$N_j(\sigma)=(2\pi)[(2j+1)2\sigma^{1/2}/q]^{1/2}. \tag{11}$$

We see that in the present choice of variables and decomposition scheme, the three-particle system can be

[12] G. C. Wick, Ann. Phys. (N. Y.) **18**, 65 (1962).

[13] M. Jacob, and G. C. Wick, Ann. Phys. (N. Y.) **7**, 404 (1959).

treated as a set of two-particle systems, each particle having definite spin and helicity. The usefulness of this decomposition is obvious when we want to make the approximation that only one of the many j states of the subsystem is important. Since we have in mind an isobar model, the fact that the spin of the nucleon 4 is not in the usual helicity description makes no difference.

The amplitudes for the production process are

$$\text{(II)}\quad \langle p_3'\lambda_3',p_4'\lambda_4',k'\alpha'\beta'|T_{23}(s,t,\sigma')|p_1\lambda_1,p_2\lambda_2\rangle = (4\pi)(2q_0W/pq)^{1/2}\sum_{j,\lambda'}\sum_{J=0}(2J+1) \times\langle\lambda_3'\lambda'|T_{23}{}^{J,j}(s,\sigma')|\lambda_1\lambda_2\rangle d_{\mu,\nu}{}^J(\theta) \times D_{\lambda',\lambda_4'}{}^j(-\beta',\alpha',\beta')N_j(\sigma'),\quad (12a)$$

$$\text{(II}'\text{)}\quad \langle p_1'\lambda_1',p_2'\lambda_2'|T_{32}(s,t,\sigma)|p_3\lambda_3,p_4\lambda_4,k\alpha\beta\rangle = (4\pi)(2q_0W/pq)^{1/2}\sum_{j,\lambda}\sum_{J=0}(2J+1) \times\langle\lambda_1'\lambda_2'|T_{32}{}^{J,j}(s,\sigma)|\lambda_3\lambda\rangle d_{\mu,\nu}{}^J(\theta) \times D_{\lambda,\lambda_4}{}^{j*}(-\beta,\alpha,\beta)N_j(\sigma),\quad (12b)$$

where $q_0=(q^2+\sigma)^{1/2}$, $\mu=\lambda_1-\lambda_2$, $\nu=\lambda_3'-\lambda'$ in II; $\mu=\lambda_1'-\lambda_2'$, $\nu=\lambda_3-\lambda$ in II′.

The amplitudes for the three-body process are

$$\text{(III)}\quad \langle p_3'\lambda_3',p_4'\lambda_4',k'\alpha'\beta'|T_{33}(s,t,\sigma_1,\sigma_2)|p_3\lambda_3,p_4\lambda_4,k\alpha\beta\rangle = (8\pi)(q_0q_0'/qq')^{1/2}\sum_{j,\lambda}\sum_{j',\lambda'}\sum_{J}(2J+1) \times\langle\lambda_3'\lambda'|T_{33}{}^{J,j,j'}(s,\sigma_1,\sigma_2)|\lambda_3\lambda\rangle d_{\mu,\nu}{}^J(\theta) \times D_{\lambda,\lambda_4}{}^{j*}(-\beta,\alpha,\beta)D_{\lambda',\lambda_4'}{}^{j}(-\beta',\alpha',\beta') \times N_j(\sigma_1)N_{j'}(\sigma_2),\quad (13)$$

where $\mu=\lambda_3-\lambda$, $\nu=\lambda_3'-\lambda'$.

In terms of the differential cross sections, the helicity amplitudes are defined as

$$\frac{d\sigma}{d\Omega_{(\text{elastic})}}=(4p^2)^{-1}\sum_{J,J'}(2J+1)(2J'+1) \times\langle\lambda_1'\lambda_2'|T_{22}{}^J(s)|\lambda_1\lambda_2\rangle\langle\lambda_1'\lambda_2'|T_{22}{}^{J'}(s)|\lambda_1\lambda_2\rangle^* \times d_{\mu,\nu}{}^J(\theta)d_{\mu,\nu}{}^{J'}(\theta),$$

$$\frac{d\sigma}{d\Omega_{(\text{inelastic})}}=(4p^2)^{-1}\sum_{J,J'}(2J+1)(2J'+1)d_{\mu,\nu}{}^J(\theta)d_{\mu,\nu}{}^{J'}(\theta) \times\int_{\sigma_0}^{\sigma_1}d\sigma'\langle\lambda_3'\lambda'|T_{23}{}^J(s,\sigma')|\lambda_1\lambda_2\rangle \times\langle\lambda_3'\lambda'|T_{23}{}^{J'}(s,\sigma')|\lambda_1\lambda_2\rangle^*,\quad (14)$$

where $\sigma_0=(2m+\mu)^2$, $\sigma_1=(s^{1/2}-\mu)^2$, with m the nucleon mass and μ the pion mass.

We shall have occasion in the future to make connections with the perturbation amplitudes in the Feynman theory. For this purpose let us define the Feynman amplitude F, for the scattering of two fermions, by

$$S=1-i(2\pi)^4\delta^4(\Delta p)(2\pi)^{-6} \times\prod_{i=1}^{4}(m_i/E_i)^{1/2}F(\lambda_3\lambda_4;\lambda_1\lambda_2),\quad (15)$$

where E_i are the center-of-mass energies of the particles.

The relation between the partial wave helicity amplitude and the Feynman amplitude is

$$\langle\lambda_3\lambda_4|T^J|\lambda_1\lambda_2\rangle=\int d\cos\theta d_{\mu,\nu}{}^J(\theta) \times(m_1m_2m_3m_4pp'/4\pi^2s)^{1/2}F(\lambda_3\lambda_4;\lambda_1\lambda_2),\quad (16)$$

where p' is the final state center-of-mass momentum.

III. FORMULATION OF THE DYNAMICAL PROBLEM

The application of the technique of dispersion relations to the determination of the scattering process consists in treating the scattering amplitude as the boundary value of an analytic scattering function. The scattering function will have certain preassigned analytic properties. Due to the incompleteness of the theory, the exact analytic properties of the function is yet unknown; however, one realizes that it should possess a branch cut extending over the entire physical region of the process. The cut arises purely out of kinematical considerations, and is important because the discontinuities across this cut are known and are dictate entirely by unitarity. This cut is referred to as the unitary cut. Amplitudes evaluated from scattering functions containing this cut and satisfying the discontinuity equations will automatically satisfy the unitarity condition. The proper treatment of the unitarity condition has led to many fruitful investigations of the strongly interacting systems, even though they base on dynamical theories which are incomplete.

Other singularities of the amplitudes are not definite, and all we can say now is that we should consider whatever singularities suggested by field theory. In particular, we shall incorporate into the scattering functions the analytic properties of the one-pion-exchange term for the elastic and production amplitudes. Therefore, the entire approach falls within the framework of the Feynman perturbative method where the lowest perturbation diagram is taken, with the exception that the amplitude is required to satisfy the unitarity condition by the analytic requirement. It is interesting to note the marked improvement of the result due to this modification.

Let us now determine the set of discontinuities over the unitary cuts of the different scattering functions. The following discussion is generally valid irrespective of isobar approximation. Since we shall be working entirely with partial wave amplitudes, we shall consider scattering functions which are functions of the variables s and σ only. The physical scattering amplitudes will be defined to be the boundary values obtained by approaching the unitary cuts from the upper half of the plane, which we shall indicate by the subscript $+$.

The requirement of *CPT* invariance provides the connection that

$$\begin{aligned} T_{22}(s_+) &= T_{22}^*(s_-)\,, \\ T_{32}(s_+,\sigma_+) &= T_{23}^*(s_-,\sigma_-)\,, \\ T_{33}(s_+,\sigma_+,\sigma_+') &= T_{33}^*(s_-,\sigma_-,\sigma_-')\,. \end{aligned} \tag{17}$$

These relations have been shown to be generally valid by Olive.[14] Since these amplitudes contain spinor parts, what we mean by Eq. (17) is that if the amplitudes are decomposed into a sum of terms consisting of the products of the scalar functions of the invariants s and σ, and the spinor parts are chosen such that the complex conjugation is equal to the reversed process, then the scalar functions satisfy the above relations in the s and σ variables. These relations imply that T_{22} and T_{33} are real functions of the variables s, σ, σ', although we cannot conclude the same for T_{32}. The assumption of time reversal invariance separately is however sufficient to make T_{32} a real function. Since T_{23} and T_{32} are now the same function (in the sense of the scalar function discussed above), we shall carry only one of them.

Unitarity of the S matrix gives in the s channel, when all variables are having physical values ($s\geqslant 4m^2$, $t<0$, σ, $\sigma'\geqslant(m+\mu)^2$), the following set of equations:

$$\begin{aligned} &T_{22}(s_+,t)-T_{22}(s_-,t) \\ &\quad =2i\sum_2 T_{22}(s_+,t')T_{22}(s_-,t'') \\ &\qquad +2i\sum_3 T_{32}(s_+,t',\sigma_+')T_{32}(s_-,t'',\sigma_-'')\,, \\ &T_{32}(s_+,t,\sigma_+)-T_{32}(s_-,t,\sigma_-) \\ &\quad =2i\sum_2 T_{32}(s_+,t',\sigma_+)T_{22}(s_-,t'') \\ &\qquad +2i\sum_3 T_{33}(s_+,\sigma_+,\sigma_+',t')T_{32}(s_-,\sigma_-'',t'')\,, \\ &T_{33}(s_+,t,\sigma_{1+},\sigma_{2+})-T_{33}(s_-,t,\sigma_{1-},\sigma_{2-}) \\ &\quad =2i\sum_2 T_{32}(s_+,t',\sigma_{1+})T_{32}(s_-,t'',\sigma_{2-}) \\ &\qquad +2i\sum_3 T_{33}(s_+,t',\sigma_{1+},\sigma_+')T_{33}(s_-,t'',\sigma_{2-},\sigma_-'')\,, \end{aligned} \tag{18}$$

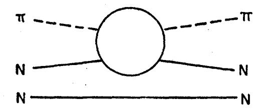

FIG. 2. Disconnected part of the three-body scattering amplitudes.

where we have neglected intermediate states with two or more pions; the primes designate the intermediate variables which are to be integrated over, and the symbols $\sum_2$ and $\sum_3$ represent two- and three-particle phase space integrals over the momenta of the nucleons and pion in the intermediate states. Letting the total 4-momenta of the intermediate states be P, then

$$\begin{aligned} \sum_2 &= \frac{1}{2}\int d^4p'(2\pi)^{-3}\delta_+(p'^2+m^2)d^4p''(2\pi)^{-3} \\ &\qquad \times\delta_+(p''^2+m^2)(2\pi)^4\delta^4(p'+p''-P)\,, \\ \sum_3 &= \frac{1}{2}\int d^4p'(2\pi)^{-3}\delta_+(p'^2+m^2)d^4p''(2\pi)^{-3}\delta_+(p''^2+m^2) \\ &\qquad \times d^4k(2\pi)^{-3}\delta_+(k^2+\mu^2)(2\pi)^4\delta^4(p'+p''+k-P)\,, \end{aligned} \tag{19}$$

where δ_+ denotes the mass shell lying in the future cone. We note that Eqs. (18) do not yet give us the discontinuities across the unitary cuts in the s variable. However, Blankenbecler has shown that discontinuities in the s variable are obtained if we replace T_{33} by its connected part, $T_{33}{}^3=T_{33}-T_{33}{}^d$, where $T_{33}{}^d$ is the disconnected part, as shown schematically in Fig. 2.[15] Furthermore, following Ball, Frazer, and Nauenberg,[9] we shall introduce isobar amplitudes which are defined as

$$\begin{aligned} M_{22}{}^J(s) &= T_{22}{}^J(s)\,, \\ M_{32}{}^{J,j}(s,\sigma) &= T_{32}{}^{J,j}(s,\sigma)/f^j(\sigma)\,, \\ M_{33}{}^{J,j,j'}(s,\sigma,\sigma') &= T_{33}{}^{J,j,j'}(s,\sigma,\sigma')/f^j(\sigma)f^{j'}(\sigma')\,, \end{aligned} \tag{20}$$

where $f^j(\sigma)$ is the jth partial wave in the center of mass of the $\pi-N$ scattering amplitude. The M amplitudes are called isobar amplitudes where the interactions within the isobar are factored out explicitly. These amplitudes have no discontinuities across the unitary cut in the σ variable.[9] In terms of the isobar amplitudes, the discontinuity equations for the partial wave

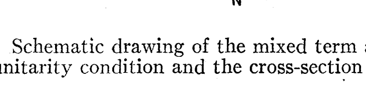

FIG. 3. Schematic drawing of the mixed term appearing in the unitarity condition and the cross-section formula.

[14] D. I. Olive, Nuovo Cimento **26**, 73 (1962).

[15] R. Blankenbecler, Phys. Rev. **122**, 983 (1961); see also Ref. 9.

amplitudes are

$$
\begin{aligned}
&(2i)^{-1}\langle\lambda_1'\lambda_2'|M_{22}{}^J(s_+)-M_{22}{}^J(s_-)|\lambda_1\lambda_2\rangle \\
&\quad=\tfrac{1}{4}\Theta(s-4m^2)\sum_{\lambda_m,\lambda_n=-\frac{1}{2}}^{\frac{1}{2}}\langle\lambda_1'\lambda_2'|M_{22}{}^J(s_+)|\lambda_m\lambda_n\rangle\langle\lambda_m\lambda_n|M_{22}{}^J(s_-)|\lambda_1\lambda_2\rangle+\tfrac{1}{4}\Theta[s-(2m+\mu)^2] \\
&\qquad\times\int_{\sigma_0}^{\sigma_1}d\sigma'\sum_{j=\frac{1}{2}}^{\infty}\sum_{\lambda_m=-j}^{j}\sum_{\lambda_n=-\frac{1}{2}}^{\frac{1}{2}}|f^j(\sigma')|^2\langle\lambda_1'\lambda_2'|M_{32}{}^{J,j}(s_+,\sigma')|\lambda_M\lambda_n\rangle\langle\lambda_M\lambda_n|M_{32}{}^{J,j}(s_-,\sigma')|\lambda_1\lambda_2\rangle, \\
&(2i)^{-1}\langle\lambda_3'\lambda_4'|M_{32}{}^{J,j}(s_+,\sigma)-M_{32}{}^{J,j}(s_-,\sigma)|\lambda_1\lambda_2\rangle \\
&\quad=\tfrac{1}{4}\Theta(s-4m^2)\sum_{\lambda_m,\lambda_n}\langle\lambda_3'\lambda_4'|M_{32}{}^{J,j}(s_+,\sigma)|\lambda_m\lambda_n\rangle\langle\lambda_m\lambda_n|M_{22}{}^J(s_-)|\lambda_1\lambda_2\rangle+\tfrac{1}{4}\Theta[s-(2m+\mu)^2] \\
&\qquad\times\int_{\sigma_0}^{\sigma_1}d\sigma'\sum_{j',\lambda_M,\lambda_n}|f^{j'}(\sigma')|^2\langle\lambda_3'\lambda_4'|M_{33}{}^{J,j,j'}(s_+,\sigma,\sigma')|\lambda_M\lambda_n\rangle\langle\lambda_M\lambda_n|M_{32}{}^{J,j'}(s_-,\sigma')|\lambda_1\lambda_2\rangle, \\
&(2i)^{-1}\langle\lambda_3'\lambda_4'|M_{33}{}^{J,j,j'}(s_+,\sigma_1,\sigma_2)-M_{33}{}^{J,j,j'}(s_-,\sigma_1,\sigma_2)|\lambda_3\lambda_4\rangle \\
&\quad=\tfrac{1}{4}\Theta(s-4m^2)\sum_{\lambda_m,\lambda_n}\langle\lambda_3'\lambda_4'|M_{32}{}^{J,j}(s_+,\sigma_1)|\lambda_m\lambda_n\rangle\langle\lambda_m\lambda_n|M_{32}{}^{J,j'}(s_-,\sigma_2)|\lambda_3\lambda_4\rangle+\tfrac{1}{4}\Theta[s-(2m+\mu)^2] \\
&\qquad\times\int_{\sigma_0}^{\sigma_1}d\sigma'\sum_{j'',\lambda_M,\lambda_n}|f^{j''}(\sigma')|^2\langle\lambda_3'\lambda_4'|M_{33}{}^{J,j,j''}(s_+,\sigma_1,\sigma')|\lambda_M\lambda_n\rangle\langle\lambda_M\lambda_n|M_{33}{}^{J,j',j''}(s_-,\sigma_2,\sigma')|\lambda_3\lambda_4\rangle, \qquad (21)
\end{aligned}
$$

where the Θ function is the usual step function, which is equal to one when the argument is positive and zero otherwise.

The treatment of the three-particle intermediate states in the above equations is not entirely correct, where we have consistently assumed that the isobar in both the "bra" and "ket" vectors are formed by the same pair of particles. This, of course, need not be the case, as the isobar may be formed by the first nucleon in the bra vector while the isobar isobar is formed by the second nucleon in the ket vector.[7] Especially, when the nucleons are identical, an antisymmetrization of the nucleon wave function will naturally bring in the mixed situation. In other words we have neglected terms like

$$\sum_3\langle\lambda_1'\lambda_2'|T_{32}|(\lambda_3\pi)\lambda_4\rangle\langle\lambda_3(\pi\lambda_4)|T_{32}|\lambda_1\lambda_2\rangle, \qquad (22)$$

where we have denoted the nucleons by their helicities and pion by π, and put parentheses around the pairs that form the isobars. Let us call these the "mixed terms." Schematically, the mixed terms are illustrated in Fig. 3; however, there are reasons to believe that the contributions from these mixed terms are small, and because they add considerable difficulties to the problem we shall exclude them from our calculations. These contributions can always be estimated.

The rest of the information about the scattering amplitudes comes from perturbation theory. The one-pion-exchange diagrams for the production amplitudes will be evaluated in the following sections. Since we shall be interested mainly in the $J=2$ state, this state for the elastic amplitudes as evaluated from the one-pion-exchange diagram is very small, and therefore can be neglected.

IV. PRODUCTION AMPLITUDES IN THE ISOBAR MODEL

In view of the strong resonance in the $J=\frac{3}{2}$, $l=1$, $T=\frac{3}{2}$ state in the low-energy region of the $\pi-N$ interaction, commonly called the (3,3) resonance, we shall assume that this state dominates all others. This means, referring to Eq. (10), that the sum over the angular momentum states in the center of mass of the $\pi-N$ subsystem will be restricted to $j=\frac{3}{2}$ only. In the isobar approximation, the $\pi-N$ resonance is replaced by a stable particle, denoted by N^*, for which we shall introduce a spin-$\frac{3}{2}$ field explicitly. The $\frac{3}{2}$ spinor will satisfy the Rarita-Schwinger equation[16]

$$(\gamma_\mu\partial_\mu+M)\psi_{3/2}{}^\nu=0, \qquad (23)$$

with the subsidiary conditions

$$\gamma_\mu\psi_{3/2}{}^\mu=0, \quad \partial_\mu\psi_{3/2}{}^\mu=0, \qquad (24)$$

where M is the mass of N^*, which shall be taken as 1237 MeV.

In short, we are saying that the nucleon can exist in, besides in its normal state, one further isobaric state (at least for low-energy processes) which has the above-described properties. This approach is well known, and in fact many kinematical aspects of pion production in nucleon-nucleon scattering can be accounted for by the isobar model. Lindenbaum and Sternheimer[17] have shown that using N^* alone they can get very reasonable fit for such experimental features as momentum spectra of the pion and the recoil nucleon, angular correlations

[16] W. Rarita and J. Schwinger, Phys. Rev. Letters **60**, 61 (1941).
[17] R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. **123**, 333 (1961).

between pions and nucleons, etc., for energies below several BeV. Furthermore, from Lindenbaum and Sternheimer's analysis of the energy spectra of the produced π meson in nucleon-nucleon scattering, one can deduce that at laboratory incident energies below 1 BeV or so the N^* is produced essentially isotropically. This is to justify our emphasis on processes involving S-wave production of N^*.

Dynamically we shall assume N^* to be coupled to the nucleon and the pion by a derivative coupling

$$(G/m)\bar{\psi}_{3/2}{}^{\mu}(\boldsymbol{\psi}\overleftrightarrow{\partial}_{\mu}\boldsymbol{\phi})+\text{H.c.}, \tag{25}$$

where G is the dimensionless coupling constant, and the bold-face symbols denote vectors in isotopic spin space. Computing the decay rate τ of such a particle in a zeroth order theory, we get

$$\frac{1}{\tau}=\frac{2}{3}\frac{G^2}{m^2}\frac{1}{2\pi}\frac{k^3(E+m)}{(E+w)}, \tag{26}$$

where k is the momentum of the pion in the rest frame of N^*, and $E=(k^2+m^2)^{1/2}$, $w=(k^2+\mu^2)^{1/2}$. Since τ is related to the half-width Δ of the resonance by $\tau^{-1}=\Delta/\hbar$, and taking Δ to be 120 MeV, we get $G^2=53$.

Let us denote the Fourier transform of $\psi_{3/2}{}^{\mu}$, or the momentum space spinor, by $u^{\mu}(p')$, which will be constructed out of the direct product of a $\frac{1}{2}$ spinor, $u(p')$, and a polarization vector

$$u^{\mu}(p')=\epsilon^{\mu}u(p'). \tag{27}$$

The subsidiary condition $\partial_{\mu}\psi_{3/2}{}^{\mu}=0$ means $p_{\mu}'u^{\mu}(p')=0$, and the polarization vector satisfying this condition is

$$\epsilon_{\mu}=e_{\mu}-p_{\mu}'(p_{\nu}'e^{\nu})(p')^{-2}[1-(p_{\nu}'e^{\nu})^2(p')^{-2}]^{-1/2}, \tag{28}$$

where $e=(e_x,e_y,e_z,0)$ is the direction vector, with $e^2=1$. Due to the presence of the subsidiary condition there are just three directions of polarization, from which we can form a new orthonormal basis by the linear combinations

$$\begin{aligned}\epsilon_1&=(2)^{-1/2}(-e_x-ie_y),\\ \epsilon_0&=e_z,\\ \epsilon_{-1}&=(2)^{-1/2}(e_x-ie_y).\end{aligned} \tag{29}$$

Polarization vectors along these directions will be characterized by their helicities Λ, and their components

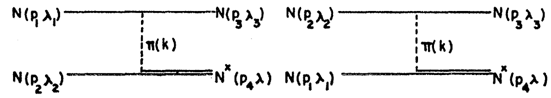

FIG. 4. Feynman diagrams of the isobar amplitudes.

will be

$$\begin{aligned}\epsilon_{\Lambda=1}&=-(2)^{-1/2}(e_x,ie_y,0,0),\\ \epsilon_{\Lambda=0}&=[0,0,(p'^2+M^2)^{1/2}/M,p'/M],\\ \epsilon_{\Lambda=-1}&=(2)^{-1/2}(e_x,-ie_y,0,0).\end{aligned} \tag{30}$$

The second subsidiary condition that $\gamma_{\mu}\psi_{3/2}{}^{\mu}=0$ guarantees that spin $\frac{3}{2}$ is obtained from the direct product representation. Denoting the helicity of the isobar by λ and that of the nucleon by λ_4, the combinations are as follows, where we write $(\lambda)=\sum C_{\lambda_4\Lambda}(\lambda_4,\Lambda)$ with $C_{\lambda_4,\Lambda}$ being the Clebsch-Gordon coefficients.

$$\begin{aligned}(\tfrac{3}{2})&=(\tfrac{1}{2},1),\\ (-\tfrac{1}{2})&=(\tfrac{2}{3})^{1/2}(\tfrac{1}{2},0)+(\tfrac{1}{3})^{1/2}(-\tfrac{1}{2},1),\\ (-\tfrac{1}{2})&=(\tfrac{1}{3})^{1/2}(\tfrac{1}{2},-1)+(\tfrac{2}{3})^{1/2}(-\tfrac{1}{2},0),\\ (-\tfrac{3}{2})&=(-\tfrac{1}{2},-1).\end{aligned} \tag{31}$$

The input information that we shall incorporate into the production channel will be limited to the one-pion-exchange amplitude calculated using the isobar model. The Feynman diagrams that we shall concern ourselves with are shown in Fig. 4. We shall take $p_1=(-\mathbf{p},E)$, $p_2=(\mathbf{p},E)$, $p_3=(-\mathbf{p}',E_3)$, and $p_4=(\mathbf{p}',E_4)$, with $E=(\mathbf{p}^2+m^2)^{1/2}$, $E_3=(\mathbf{p}'^2+m^2)^{1/2}$ and $E_4=(\mathbf{p}'^2+M^2)^{1/2}$. The amplitude corresponding to the second diagram will be denoted by the superscript c.

The Feynman amplitudes are given by

$$\begin{aligned}&F_0(\lambda_3\lambda;\lambda_1\lambda_2)\\ &\quad=-(gG/m)\Gamma[(p_1-p_3)^2+\mu^2]^{-1}(\bar{u}_{\lambda_3}(-p'),\gamma_5u_{\lambda_1}(-p))\\ &\qquad\times(\bar{u}_{\lambda}{}^{\nu}(p'),(k-p_2)_{\nu}u_{\lambda_2}(p)),\\ &F_0{}^c(\lambda_3\lambda;\lambda_1\lambda_2)\\ &\quad=-(gG/m)\Gamma[(p_2-p_3)^2+\mu^2]^{-1}(\bar{u}_{\lambda_3}(-p'),\gamma_5u_{\lambda_2}(p_2))\\ &\qquad\times(\bar{u}_{\lambda}{}^{\nu}(p'),(k-p_1)_{\nu}u_{\lambda_1}(-p)),\end{aligned} \tag{32}$$

where Γ denoted the isotopic spin matrix element, and for the $T=1$ state $\Gamma=-8/3$. The matrix elements evaluated are (we use the same representation of the γ matrices as that used in Ref. 18)

$$(\bar{u}_{\lambda_3}(-p'),\gamma_5u_{\lambda_1}(-p))=(m\xi_0)^{-1}[\xi_0{}^2p'\lambda_3-p\lambda_1]\{|\lambda_3+\lambda_1|\cos\tfrac{1}{2}\theta-(\lambda_3-\lambda_1)\sin\tfrac{1}{2}\theta\}, \tag{33a}$$

$$(\bar{u}_{\lambda_3}(-p'),\gamma_5u_{\lambda_2}(p))=(m\xi_0)^{-1}[\xi_0{}^2p'\lambda_3-p\lambda_2]\{|\lambda_2-\lambda_3|\cos\tfrac{1}{2}\theta-(\lambda_3+\lambda_2)\sin\tfrac{1}{2}\theta\}, \tag{33b}$$

$$(\bar{u}_{\lambda}{}^{\nu}(p'),(k-p_2)_{\nu}u_{\lambda_2}(p))=\sum_{\lambda_4,\Lambda}C_{\lambda_4,\Lambda}(\bar{u}_{\lambda_4}(p'),u_{\lambda_2}(p))[\epsilon_{\nu}{}^*(k-p_2)^{\nu}]_{\Lambda}, \tag{34}$$

with

$$(\bar{u}_{\lambda_4}(p'),u_{\lambda_2}(p))=\xi_1(1-4\lambda_2\lambda_4\xi_2)\{|\lambda_4+\lambda_2|\cos\tfrac{1}{2}\theta+(\lambda_4-\lambda_2)\sin\tfrac{1}{2}\theta\}, \tag{35a}$$

$$[\epsilon_{\nu}{}^*(k-p_2)^{\nu}]_{\Lambda=0}=-\xi_3+(2E_4p/M)\cos\theta, \tag{35b}$$

[18] M. L. Goldberger, M. T. Grisaru, S. W. MacDowell, and D. Y. Wong, Phys. Rev. **120**, 2250 (1960).

$$[\epsilon_\nu^*(k-p_2)^\nu]_{\Lambda=\pm1}=\mp\sqrt{2}p\sin\theta\exp(\pm i\phi),\qquad(35c)$$

and

$$(\bar{u}_{\lambda}{}^\nu(p'),(k-p_1)_\nu u_{\lambda_1}(-p))=\sum_{\lambda_4,\Lambda}C_{\lambda_4,\Lambda}(\bar{u}_{\lambda_4}(p'),u_{\lambda_1}(-p))[\epsilon_\nu^*(k-p_1)^\nu]_\Lambda,\qquad(36)$$

with

$$(\bar{u}_{\lambda_4}(p'),u_{\lambda_1}(-p))=\xi_1(1-4\lambda_1\lambda_4\xi_2)\{|\lambda_4-\lambda_1|\cos\tfrac{1}{2}\theta+(\lambda_4+\lambda_1)\sin\tfrac{1}{2}\theta\},\qquad(37a)$$

$$[\epsilon_\nu^*(k-p_1)^\nu]_{\Lambda=0}=-\xi_3-(2E_4p/M)\cos\theta,\qquad(37b)$$

$$[\epsilon_\nu^*(k-p_1)^\nu]_{\Lambda=\pm1}=\pm\sqrt{2}p\sin\theta\exp(\pm i\phi),\qquad(37c)$$

where

$$\begin{aligned}\xi_0&=(E+m)^{1/2}(E_3+m)^{-1/2},\\\xi_1&=(E+m)^{1/2}(E_4+M)^{1/2}(mM)^{-1/2},\\\xi_2&=pp'(E+m)^{-1}(E_4+M)^{-1},\\\xi_3&=p'W/M.\end{aligned}\qquad(38)$$

Now, let us list a number of things that we want to do. First, we want the Feynman amplitudes to be properly antisymmetrized, which means for the isotopic triplet state ($T=1$) we should take the difference of the amplitudes calculated from the graphs, and let us define

$$F_{32}(\lambda_3\lambda_4;\lambda_1\lambda_2)=F_0(\lambda_3\lambda_4;\lambda_1\lambda_2)-F_0^c(\lambda_3\lambda_4;\lambda_1\lambda_2).\qquad(39)$$

F_0 and F_0^c are in general not related to each other by changing θ to $\pi-\theta$, although further linear combinations of the helicity states will have this property. Secondly, we want to project out the partial wave amplitudes as defined by Eq. (16). Finally, we want to combine the partial wave amplitudes into amplitudes of definite parity. It will be apparent later that it is easier to work with amplitudes of definite parities, since the amplitudes will be characterized by definite relative angular momenta. To achieve this we shall make linear combinations of the helicity states according to the property of the state under inversion. Assuming all particles to have positive intrinsic parities, the helicity states under parity operator P behave as [Jacob and Wick, Eq. (41)]

$$P|JM\lambda_1\lambda_2\rangle=(-)^{J-s_1-s_2}|JM-\lambda_1-\lambda_2\rangle,\qquad(40)$$

where s_1 and s_2 are the spins of the particles. Hence, for the NN system, states of definite parity are

$$\begin{aligned}&(2)^{-1/2}|\tfrac{1}{2}\tfrac{1}{2}\rangle\pm|-\tfrac{1}{2}-\tfrac{1}{2}\rangle,\\&(2)^{-1/2}|\tfrac{1}{2}-\tfrac{1}{2}\rangle\pm|-\tfrac{1}{2}\tfrac{1}{2}\rangle,\end{aligned}\qquad(41)$$

where the states with the minus sign have orbital angular momentum $l=J$, while the states with the plus sign have $l=J\pm1$, and the first one taken with the minus belongs to the spin singlet state, while the others belong to the spin triplet states.

For the NN^* system, states of definite parity are

$$\begin{aligned}&(2)^{-1/2}|\tfrac{1}{2}\tfrac{3}{2}\rangle\pm|-\tfrac{1}{2}-\tfrac{3}{2}\rangle,\\&(2)^{-1/2}|\tfrac{1}{2}\tfrac{1}{2}\rangle\pm|-\tfrac{1}{2}-\tfrac{1}{2}\rangle,\\&(2)^{-1/2}|\tfrac{1}{2}-\tfrac{1}{2}\rangle\pm|-\tfrac{1}{2}\tfrac{1}{2}\rangle,\\&(2)^{-1/2}|\tfrac{1}{2}-\tfrac{3}{2}\rangle\pm|-\tfrac{1}{2}\tfrac{3}{2}\rangle,\end{aligned}\qquad(42)$$

where the states with minus sign have orbital angular momentum $l=J\pm1$, while the states with plus sign have $l=J$ or $J\pm2$.

The 16 amplitudes with definite parity will be grouped according to their initial spin state (singlet or triplet of the two-nucleon system) and their initial and final orbital momenta, l_i and l_f. Although not all of them will be useful to us in the present calculation, we shall include them here for completeness. They are

(A) initial spin singlet, $J=l_i=l_f$, $l_f\pm2$:

$$\begin{aligned}P_1{}^J&=\langle\tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^J|\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^J|-\tfrac{1}{2}-\tfrac{1}{2}\rangle\\&=R_J\sqrt{3}\{\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1-\kappa)\}\\&\quad\times[Q_{J+1}(\kappa)-Q_{J-1}(\kappa)][J(J+1)]^{1/2}(2J+1)^{-1},\quad J\text{ even}\qquad(43a)\\&=0,\quad J\text{ odd},\end{aligned}$$

$$\begin{aligned}P_2{}^J&=\langle\tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^J|\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^J|-\tfrac{1}{2}-\tfrac{1}{2}\rangle\\&=R_J\{\zeta_0[\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1+\kappa)]\\&\quad+(1-\kappa^2)[-\zeta_1(1+\xi_2)+\zeta_2(1-\xi_2)]\}Q_J(\kappa)\\&\quad+R_J\{\zeta_0[-\zeta_1(1-\xi_2)+\zeta_2(1+\xi_2)]+(4E_4/M)\zeta_4\\&\quad-2\kappa\zeta_5\}_{J=0\text{ only}},\quad J\text{ even}\\&=0,\quad J\text{ odd},\end{aligned}\qquad(43b)$$

$$\begin{aligned}P_3{}^J&=\langle\tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^J|\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^J|-\tfrac{1}{2}-\tfrac{1}{2}\rangle\\&=R_J\{\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1-\kappa)+\zeta_0\zeta_5\}\\&\quad\times[Q_{J+1}(\kappa)-Q_{J-1}(\kappa)][J(J+1)]^{1/2}(2J+1)^{-1},\quad J\text{ even}\\&=0,\quad J\text{ odd},\end{aligned}\qquad(43c)$$

$$P_4{}^J=\langle\tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^J|\tfrac{1}{2}\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^J|-\tfrac{1}{2}-\tfrac{1}{2}\rangle=0.\qquad(43d)$$

(B) Initial spin triplet, $J=l_i=l_f$, $l_f\pm2$:

$$\begin{aligned}P_{11}{}^J&=\langle\tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^J|\tfrac{1}{2}-\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^J|-\tfrac{1}{2}\tfrac{1}{2}\rangle\\&=0\quad J\text{ even}\\&=\sqrt{3}R_J\{[(1-\kappa^2)\zeta_5Q_J(\kappa)-\zeta_3Q_{J-1}(\kappa)]\\&\quad+[\tfrac{1}{3}\zeta_5-\zeta_3\kappa]_{J=1\text{ only}}\}\quad J\text{ odd},\end{aligned}\qquad(44a)$$

$$\begin{aligned}P_{12}{}^J&=\langle\tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^J|\tfrac{1}{2}-\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^J|-\tfrac{1}{2}\tfrac{1}{2}\rangle\\&=0\quad J\text{ even}\\&=2R_J\{[\zeta_0\zeta_5+\kappa\zeta_6-\zeta_4][Q_{J+1}(\kappa)-Q_{J-1}(\kappa)]\\&\quad\times[J(J+1)]^{1/2}(2J+1)^{-1}-(\sqrt{2}/3)\\&\quad\times[(2E_4/M)\zeta_5+\zeta_6]_{J=1\text{ only}}\}\quad J\text{ odd},\end{aligned}\qquad(44b)$$

$$\begin{aligned}P_{13}{}^J&=\langle\tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^J|\tfrac{1}{2}-\tfrac{1}{2}\rangle-\langle\tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^J|-\tfrac{1}{2}\tfrac{1}{2}\rangle\\&=0\quad J\text{ even}\\&=-R_J\{[\zeta_0(\zeta_4-\kappa\zeta_6)+\zeta_5(1-\kappa^2)]Q_J(\kappa)\\&\quad+[\zeta_0(\kappa\zeta_4-\zeta_6)+\zeta_3(1-\kappa^2)]Q_{J-1}(\kappa)\\&\quad+[-\zeta_0\zeta_4+(2E_4/3M)\zeta_6+\kappa\zeta_3+\tfrac{1}{3}\zeta_5]_{J=1\text{ only}}\}\quad J\text{ odd},\end{aligned}\qquad(44c)$$

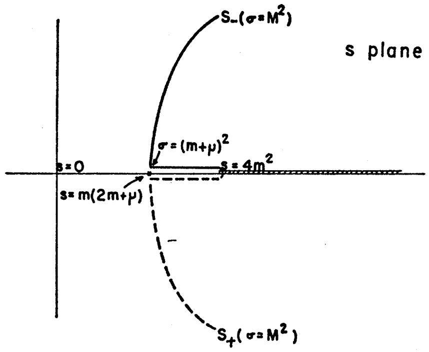

FIG. 5. Singularities of the production amplitudes.

$$
\begin{aligned}
P_{14}{}^{J} &= \langle \tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^{J}|\tfrac{1}{2}-\tfrac{1}{2}\rangle - \langle \tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^{J}|-\tfrac{1}{2}\tfrac{1}{2}\rangle \\
&= 0 \quad J \text{ even} \\
&= \sqrt{3}R_J(J-1)^{1/2}(J+2)^{1/2}\{(\zeta_4-\zeta_6\kappa) \\
&\quad\times[Q_{J+1}(\kappa)-Q_{J-1}(\kappa)](2J+1)^{-1}+(\kappa\zeta_4-\zeta_6) \\
&\quad\times[Q_J(\kappa)-Q_{J-2}(\kappa)](2J-1)^{-1}\} \quad J \text{ odd}. \qquad (44d)
\end{aligned}
$$

(C) Initial spin triplet, $J=l_i\pm 1=l_f\pm 1$;

$$
\begin{aligned}
P_{21}{}^{J} &= \langle \tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^{J}|\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^{J}|-\tfrac{1}{2}\tfrac{1}{2}\rangle \\
&= 0 \quad J \text{ odd} \\
&= \sqrt{3}R_J\{(1-\kappa^2)[\zeta_3 Q_J(\kappa)-\zeta_5 Q_{J-1}(\kappa)] \\
&\quad +(\kappa\zeta_3)_{J=0\text{ only}}-\tfrac{1}{3}(\zeta_5)_{J=2\text{ only}}\} \quad J \text{ even}, \qquad (45a)
\end{aligned}
$$

$$
\begin{aligned}
P_{22}{}^{J} &= \langle \tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^{J}|\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^{J}|-\tfrac{1}{2}\tfrac{1}{2}\rangle \\
&= 0 \quad J \text{ odd} \\
&= 2R_J\{(\zeta_0\zeta_3+\zeta_6-\kappa\zeta_4)[Q_{J+1}(\kappa)-Q_{J-1}(\kappa)] \\
&\quad \times[J(J+1)]^{1/2}(2J+1)^{-1}\} \quad J \text{ even}, \qquad (45b)
\end{aligned}
$$

$$
\begin{aligned}
P_{23}{}^{J} &= \langle \tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^{J}|\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^{J}|-\tfrac{1}{2}\tfrac{1}{2}\rangle \\
&= 0 \quad J \text{ odd} \\
&= -R_J\{[\zeta_0(\kappa\zeta_4-\zeta_6)+\zeta_3(1-\kappa^2)]Q_J(\kappa) \\
&\quad +[\zeta_0(\zeta_4-\kappa\zeta_6)-\zeta_5]Q_{J-1}(\kappa)+(-\zeta_0\zeta_4+\kappa\zeta_3)_{J=0\text{ only}} \\
&\quad +\tfrac{1}{3}[(2E_4/M)\zeta_6+\zeta_5]_{J=2\text{ only}}\} \quad J \text{ even}, \qquad (45c)
\end{aligned}
$$

$$
\begin{aligned}
P_{24}{}^{J} &= \langle \tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^{J}|\tfrac{1}{2}-\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^{J}|-\tfrac{1}{2}\tfrac{1}{2}\rangle \\
&= 0 \quad J \text{ odd} \\
&= \sqrt{3}R_J\{(J-1)^{1/2}(J+2)^{1/2}[(\kappa\zeta_4-\zeta_6) \\
&\quad \times(Q_{J+1}(\kappa)-Q_{J-1}(\kappa))(2J+1)^{-1}+(\zeta_4-\kappa\zeta_6) \\
&\quad \times(Q_J(\kappa)-Q_{J-2}(\kappa))(2J-1)^{-1}]+\tfrac{2}{3}(\zeta_6)_{J=2\text{ only}}\} \\
&\qquad J \text{ even}, \qquad (45d)
\end{aligned}
$$

$$
\begin{aligned}
P_{31}{}^{J} &= \langle \tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^{J}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}\tfrac{3}{2}|T_{32}{}^{J}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle \\
&= -\sqrt{3}R_J\{[-\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1-\kappa)] \\
&\quad \times[Q_{J+1}(\kappa)-Q_{J-1}(\kappa)][J(J+1)]^{1/2}(2J+1)^{-1}\} \\
&\qquad J \text{ even} \\
&= 0 \quad J \text{ odd}, \qquad (45e)
\end{aligned}
$$

$$
\begin{aligned}
P_{32}{}^{J} &= \langle \tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^{J}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}\tfrac{1}{2}|T_{32}{}^{J}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle \\
&= R_J\{\zeta_0[-\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1-\kappa)] \\
&\quad +(1-\kappa^2)[\zeta_1(1+\xi_2)+\zeta_2(1-\xi_2)]\}Q_J(\kappa) \\
&\quad +R_J\{\zeta_0[-\zeta_1(1-\xi_2)+\zeta_2(1+\xi_2)]+(4E_4/M)\zeta_6 \\
&\quad +2\kappa\zeta_3\}_{J=0\text{ only}} \quad J \text{ even} \qquad (45f) \\
&= 0 \quad J \text{ odd},
\end{aligned}
$$

$$
\begin{aligned}
P_{33}{}^{J} &= \langle \tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^{J}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}-\tfrac{1}{2}|T_{32}{}^{J}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle \\
&= -R_J\{[-\zeta_1(1-\xi_2)(1+\kappa)+\zeta_2(1+\xi_2)(1-\kappa) \\
&\quad +\zeta_0\zeta_3][Q_{J+1}(\kappa)-Q_{J-1}(\kappa)][J(J+1)]^{1/2} \\
&\quad \times(2J+1)^{-1}\} \quad J \text{ even} \\
&= 0 \quad J \text{ odd}, \qquad (45g)
\end{aligned}
$$

$$
\begin{aligned}
P_{34}{}^{J} &= \langle \tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^{J}|\tfrac{1}{2}\tfrac{1}{2}\rangle + \langle \tfrac{1}{2}-\tfrac{3}{2}|T_{32}{}^{J}|-\tfrac{1}{2}-\tfrac{1}{2}\rangle \\
&= 0, \qquad (45h)
\end{aligned}
$$

where

$$
\begin{aligned}
&\zeta_0=(2E_4\kappa/M)-(\xi_3/p), \quad \zeta_4=(p'\xi_0)+(p\xi_2/\xi_0), \\
&\zeta_1=(\xi_0 p')-(p/\xi_0), \quad \zeta_5=(p'\xi_0\xi_2)-(p/\xi_0), \\
&\zeta_2=(\xi_0 p')+(p/\xi_0), \quad \zeta_6=(p'\xi_0\xi_2)+(p/\xi_0), \\
&\zeta_3=(p'\xi_0)-(p\xi_2/\xi_0), \\
&\kappa=(s-\sigma-3m^2+2\mu^2)(4pp')^{-1}, \\
&R_J=(gG/m)\tfrac{2}{3}(p/p')^{1/2}(mM)^{1/2}\xi_1(2\pi W)^{-1}. \qquad (46)
\end{aligned}
$$

In deriving these formulas we have made use of the fact that

$$
\begin{aligned}
\cos\theta \textstyle\sum_J (2J+1)P_J(\cos\theta)Q_J(\kappa) \\
=\kappa \textstyle\sum_J(2J+1)P_J(\cos\theta)Q_J(\kappa)-P_0(\cos\theta)
\end{aligned}
$$

for the successive reduction of the $\cos\theta$ terms.

V. SINGULARITIES OF THE PRODUCTION AMPLITUDES

The production amplitudes in the isobar model and one-pion-exchange approximation, as given by Eqs. (43)–(45), contain many singularities which are of kinematical origin, such as the normalization factors $(E_3+m)^{1/2}$, etc., and these shall be removed. We are interested only in the singularities which are of dynamical origin, by which we mean the singularities arising from the propagator. In the partial wave amplitudes these singularities appear as branch lines of the Q_J functions, and the branch points occur at $\kappa=\pm 1$. Denoting the branch points by $s_\pm$, the equations for the branch points are

$$
\begin{aligned}
s_\pm=\tfrac{1}{2}(3m^2+\sigma-\mu^2)\pm\tfrac{1}{2}[(3m^2+\sigma-\mu^2)^2 \\
-(\sigma-m^2)^2(4m^2/\mu^2)]^{1/2}. \qquad (47)
\end{aligned}
$$

These branch points become complex as soon as $\sigma>(m+\mu)^2$. This is a well-known fact associated with the instability of the vertex. As we have mentioned before that we shall assume the production amplitude to possess, in addition to this branch line, a unitary cut running from $s=4m^2$ to ∞. We then encounter the difficult situation that the two cuts run into each other, and it becomes difficult to determine which of the several Reimann sheets should be taken as the physical sheet. To circumvent this the physical sheet is often taken to be the one obtained by continuing the s-plane branch points in the σ variable from $\sigma<(m+\mu)^2$ to the desired σ value, with σ taking on a positive imaginary part. The loci of $s_\pm$ of this continuation is illustrated in Fig. 5.

Part of the s_+ locus is shown dotted as it enters onto the second Reimann sheet of the unitary cut. If σ is continued analytically with a negative imaginary part, the loci are changed and s_- will move onto the second sheet on the upper half-plane while s_+ remains on the first sheet in the lower half-plane.

As the dynamical branch line moves pass the elastic threshold and enters the second sheet of the unitary cut, any integration along the unitary cut is no longer defined. This situation is often treated by deforming the path of integration into the second sheet in front of the advancing branch points s_+ or s_-. This modification at the elastic threshold is referred to as an anomalous threshold. However, a straight forward analytic continuation of the amplitudes alters the properties of the amplitudes. Ball, Frazer, and Nauenberg[9] have looked into this problem in detail. We give here a slightly different version of the treatment of the analytic continuation in a later section.

To work with amplitudes which have all kinematical singularities removed, we shall introduce new amplitudes $\bar{M}$, defined as

$$\begin{aligned}\bar{M}_{22}{}^J&=g_2{}^I M_{22}{}^J,\\ \bar{M}_{32}{}^J&=(g_3{}^{I'})^{1/2}M_{32}{}^J(g_2{}^I)^{1/2},\\ \bar{M}_{33}{}^J&=g_3{}^{I'}M_{33}{}^J,\end{aligned}\tag{48}$$

where

$$\begin{aligned}g_2{}^I&=(E+m)/p^{2I+1},\\ g_3{}^{I'}&=(E_4+\sigma^{1/2})(E_3+m)p'^{-2}(W+2m)(W+\sigma^{1/2}+m)^{-1}\\ &\quad\times[(W+\sigma^{1/2}+m)/2p']^{2I'-1},\end{aligned}\tag{49}$$

and I, I' are the smallest orbital angular momenta of the initial and final states, respectively. The bar amplitudes also have the correct threshold behavior. Using these amplitudes, the unitarity equations for M are modified by replacing the factor $\frac{1}{4}$ by $\pi\rho_2{}^I$ and $\pi\rho_3{}^{I'}$ for the two-particle and three-particle intermediate states, respectively, with

$$\begin{aligned}\rho_2{}^I&=(4\pi)^{-1}[(W-2m)/(W+2m)]^{I+1/2},\\ \rho_3{}^{I'}&=(4\pi)^{-1}\left[\frac{W-(\sigma^{1/2}+m)}{W+(\sigma^{1/2}+m)}\right]^{I'+1/2}\\ &\quad\times\left[\frac{W^2-(\sigma^{1/2}-m)^2}{W^2}\right]^{I'-1/2}\left[\frac{W+\sigma^{1/2}+m}{W+2m}\right].\end{aligned}\tag{50}$$

These quantities are expressed in the total energy W, which is the variable we shall use for our calculations.

The production process that we are mainly concerned with are the ones given by $J=2$, initial spin singlet, $l_i=2$, and $l_f=0$. These in the one-pion-exchange approximation are P_1, P_2, P_3, and P_4 (where J is understood to be 2). The modified amplitudes, $\bar{P}_i=(g_2)^{1/2}P_i(g_3)^{1/2}$, as we specialize to $\sigma^{1/2}=M$, can be written as

$$\begin{aligned}\bar{P}_1&=\sqrt{2}\eta(W+2m)(W+M-m)[Q_3(\kappa)-Q_1(\kappa)]\\ &\quad\times[\mu^2+(M+m)^2m(M-m)(W)^{-2}],\\ \bar{P}_2&=\tfrac{5}{6}\eta(W+M+m)[W^2-(M-m)^2](W)^{-2}Q_2(\kappa)\\ &\quad\times\{[(W^2+M^2-m^2)\mu^2+(W^2-M^2-m^2)(M^2-m^2)]\\ &\quad\times(2M)^{-1}[(W^2-M^2-3m^2+2\mu^2)\\ &\quad-(W^2+2m(M-m))(W^2-(M+m)^2)](W)^{-2}\\ &\quad-[m^2(M^2-m^2)^2+\mu^2W^2(W^2-M^2-3m^3+\mu^2)]\\ &\quad\times(M+m)(W)^{-2}\},\\ \bar{P}_3&=\tfrac{1}{3}{}^{1/2}\bar{P}_1-\tfrac{2}{3}{}^{1/2}[Q_3(\kappa)-Q_1(\kappa)][(M+m)/2m]\\ &\quad\times[W^2-(M-m)^2](W+M+m)[(M^2-m^2)\\ &\quad\times(W^2-M^2-m^2)+\mu^2(W^2+M^2-m^2)](W)^{-2},\\ \bar{P}_4&=0,\end{aligned}\tag{51}$$

where

$$\eta=(gG/m)(W+2m)^2(80Wp^2p'^2)^{-1}.\tag{52}$$

We see that these amplitudes contain the dynamical cuts due to the branch lines in the Q functions, but no other singularities in the W plane except for simple pole at $W=0$.

VI. APPROXIMATE SOLUTIONS OF THE PROBLEM

As we have stated before, the dynamical problem is formulated by assuming the amplitudes as analytic functions which possess the unitary cuts and some dynamical cuts. The discontinuity across the unitary cut is given by a modified form of the unitarity condition and the discontinuities across the dynamical cuts serve as the input information. It has been shown by Blankenbecler[15] (see also Bjorken and Nauenberg[19]) that amplitudes satisfying these constraints can be developed into the following set of multichannel N/D equations:

$$\begin{aligned}N_{22}(W)&=M_{22}(W)D_{22}(W)+\sum_j\int d\sigma M_{32}{}^j(W,\sigma_+)D_{32}{}^j(W,\sigma_-),\\ N_{23}{}^i(W,\sigma)&=M_{22}(W)D_{23}{}^i(W,\sigma)+\sum_j\int d\sigma' M_{32}{}^j(W,\sigma_+{}')D_{33}{}^{ij}(W,\sigma,\sigma_-{}'),\\ N_{32}{}^i(W,\sigma)&=M_{32}{}^i(W,\sigma)D_{22}(W)+\sum_j\int d\sigma' M_{33}{}^{ij}(W,\sigma,\sigma_+{}')D_{32}{}^j(W,\sigma_-{}'),\\ N_{33}{}^{ii'}(W,\sigma_1,\sigma_2)&=M_{32}{}^i(W,\sigma_1)D_{23}{}^{i'}(W,\sigma_2)+\sum_j\int d\sigma' M_{33}{}^{ij}(W,\sigma_1,\sigma_+{}')D_{33}{}^{i'j}(W,\sigma_2,\sigma_-{}'),\end{aligned}\tag{53}$$

[19] J. D. Bjorken and M. Nauenberg, Phys. Rev. 121, 1250 (1961).

where

$$
\begin{aligned}
D_{22}(W) &= 1-\int dW'(W'-W)^{-1}\rho_2(W')N_{22}(W')\,,\\
D_{23}{}^{i}(W,\sigma) &= -\int dW'(W'-W)^{-1}\rho_2(W')N_{23}{}^{i}(W',\sigma)\,,\\
D_{32}{}^{i}(W,\sigma) &= -\int dW'(W'-W)^{-1}\rho_3(W',\sigma)N_{32}{}^{i}(W',\sigma)\,,\\
D_{33}{}^{ij}(W,\sigma_1,\sigma_2) &= 1-\int dW'(W'-W)^{-1}\rho_3(W',\sigma_1)N_{33}{}^{ij}(W',\sigma_1,\sigma_2)\,,
\end{aligned}
\tag{54}
$$

and

$$
\begin{aligned}
N_{22}(W) &= \int dW'(W'-W)^{-1}\Big\{A_{22}(W')D_{22}(W')+\sum_j \int d\sigma' A_{23}{}^{j}(W',\sigma_+')D_{32}{}^{j}(W',\sigma_-')\Big\}\,,\\
N_{23}{}^{i}(W,\sigma) &= \int dW'(W'-W)^{-1}\Big\{A_{22}(W')D_{23}{}^{i}(W',\sigma)+\sum_j \int d\sigma' A_{23}{}^{j}(W',\sigma_+')D_{33}{}^{ij}(W',\sigma,\sigma_-')\Big\}\,,\\
N_{32}{}^{i}(W,\sigma) &= \int dW'(W'-W)^{-1}\Big\{A_{32}{}^{i}(W',\sigma)D_{22}(W')+\sum_j \int d\sigma' A_{33}{}^{ij}(W',\sigma,\sigma_+')D_{32}{}^{j}(W',\sigma_-')\Big\}\,,\\
N_{33}{}^{ii'}(W,\sigma_1,\sigma_2) &= \int dW'(W'-W)^{-1}\Big\{A_{32}{}^{i}(W',\sigma_1)D_{23}{}^{i'}(W',\sigma_2)+\sum_j \int d\sigma' A_{33}{}^{ij}(W',\sigma_1,\sigma_+')D_{33}{}^{i'j}(W',\sigma_2,\sigma_-')\Big\}\,,
\end{aligned}
\tag{55}
$$

where the superscripts denote the different spin states, the σ integration extends between σ_0 and σ_1, the W' integration in the D functions extends between $2m$ and ∞, and the W' integration in N runs over A_{ij}, which are the discontinuities of $\bar{M}_{ij}$ across the dynamical cuts.

$$
\begin{aligned}
\bar{M}_{22}(W_+)-\bar{M}_{22}(W_-) &= 2\pi i A_{22}(W)\,,\\
\bar{M}_{32}{}^{i}(W_+,\sigma)-\bar{M}_{32}{}^{i}(W_-,\sigma) &= 2\pi i A_{32}{}^{i}(W,\sigma)=2\pi i A_{23}{}^{i}(W,\sigma)\,,\\
\bar{M}_{33}{}^{ij}(W_+,\sigma_1,\sigma_2)-\bar{M}_{33}{}^{ij}(W_-,\sigma_1,\sigma_2) &= 2\pi i A_{33}{}^{ij}(W,\sigma_1,\sigma_2)\,.
\end{aligned}
\tag{56}
$$

Once the A_{ij} are given the coupled integral equations for N_{ij} and D_{ij} can be solved. After substituting N_{ij} and D_{ij} into Eqs. (53), the $\bar{M}_{ij}$ are determined.

Since the discontinuities across the dynamical cuts of $\bar{M}_{22}$ in the $J=2$ state are known to be small, it is best to ignore it completely in order to illustrate the dynamical effects of the production channel $\bar{M}_{32}$ on the elastic channel. Hence, we shall take A_{22} and A_{33} to be zero. From an inspection of Eq. (51), we see the discontinuity A_{32} is most singular at $W=2m$, where it diverges as $P_3(x)$ as $x\to\infty$. This motivates Cook and Lee[8] in doing their problem to approximate the entire dynamical cut by a pole at $W=2m$. The pole approximation has great merit that it reduces the entire set of coupled integral equations to a set of coupled algebraic equations as the kernels of the integral equations are then separable. The simplicity of the solution serves to illustrate clearly the mechanism of excitation of the elastic channel by the production channel.

The amplitudes obtained from the pole approximation, however, do not usually fit the calculated amplitudes too well in the physical region ($2m\leqslant W<\infty$), and this may be serious. To improve on the approximation, we shall take the amplitudes in the physical region to be exactly those given by the one-pion-exchange calculation, but simplify the calculation by introducing an approximation which is suggested by the pole approximation, that the contribution of D_{ij} to N_{ij} in

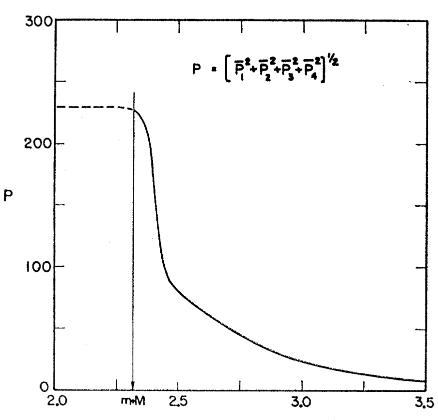

FIG. 6. Plot of the value of P.

the integration over the dynamical cut is such that only its value at $W=2m$ is meaningful (as the integrand is most singular there). Since we are looking at the singlet state of the elastic process, different spin amplitudes of the production process contribute in such a manner that we can introduce conveniently a single amplitude, $P=(\bar{P}_1^2+\bar{P}_2^2+\bar{P}_3^2+\bar{P}_4^2)^{1/2}$, which will account for the summation over the spin states. P is plotted on Fig. 6, and is calculated in units $mc^2=1$. This means we shall introduce P throughout and drop the spin summation, and we get

$$
\begin{aligned}
N_{22}(W)&=\int d\sigma f^*(\sigma)P(W)D_{32}(2m,\sigma)\,,\\
N_{23}(W,\sigma)&=\int d\sigma' f^*(\sigma')P(W)D_{33}(2m,\sigma,\sigma')\,,\\
N_{32}(W,\sigma)&=f(\sigma)P(W)D_{22}(2m)\,,\\
N_{33}(W,\sigma,\sigma')&=f(\sigma)P(W)D_{23}(2m,\sigma')\,.
\end{aligned}
\tag{56a}
$$

By successive substitution of D_{32}, N_{32}, and D_{22} into the equation for N_{22} we get an integral equation for N_{22} which is a Fredholm equation with a separable kernel. N_{22} can be solved by the standard technique, and similarly all N_{ij} and D_{ij} can thus be solved.

Finally, the solution for the scattering amplitudes are

$$
\begin{aligned}
\bar{M}_{22}(W)&=P(W)G(W,2m)[1-K(W,2m)\\
&\qquad\times G(W,2m)]^{-1}\,,\\
\bar{M}_{23}{}^i(W,\sigma)&=\bar{M}_{32}{}^{i*}(W,\sigma)=f^*(\sigma)\bar{P}_i(W)\\
&\qquad\times\bar{M}_{22}(W)[P(W)G(W,2m)]^{-1}\,,\\
\bar{M}_{33}(W,\sigma,\sigma')&=f(\sigma)f(\sigma')K(W,2m)\bar{M}_{22}(W)[G(W,2m)]^{-1}\,,
\end{aligned}
\tag{57}
$$

where

$$
\begin{aligned}
G(W,2m)&=(W-2m)\int d\sigma|f(\sigma)|^2\int dW'(W'-W-i\epsilon)^{-1}\\
&\qquad\times P(W')\rho_3(W',\sigma)(W'-2m)^{-1}\,,\\
K(W,2m)&=(W-2m)\int dW'(W'-W-i\epsilon)^{-1}P(W')\\
&\qquad\times\rho_2(W')(W'-2m)^{-1}\,,
\end{aligned}
\tag{58}
$$

and the superscript i denotes the four different spin states of $\bar{M}_{23}$ or $\bar{M}_{32}$ corresponding to those of $\bar{P}_i$. Here $|f(\sigma)|^2$ is assumed to be given by a Breit-Wigner distribution with a half-width Δ of the (3,3) resonance in the $\pi-N$ scattering, i.e.,

$$|f(\sigma)|^2=(2\sigma^{1/2})^{-1}(\Delta/2\pi)[(\sigma^{1/2}-M)^2+(\Delta/2)^2]^{-1}\,,\tag{59}$$

where we take $(\Delta/2)=0.06$ (in unit of mc^2). $G(W,2m)$ and $K(W,2m)$ are then evaluated numerically, and their real and imaginary parts are plotted on Fig. 7. In passing let us mention that although in this particular problem the G and K curves computed in this manner look quite different from those computed using the pole approximation, the final differential cross sections computed are about the same.

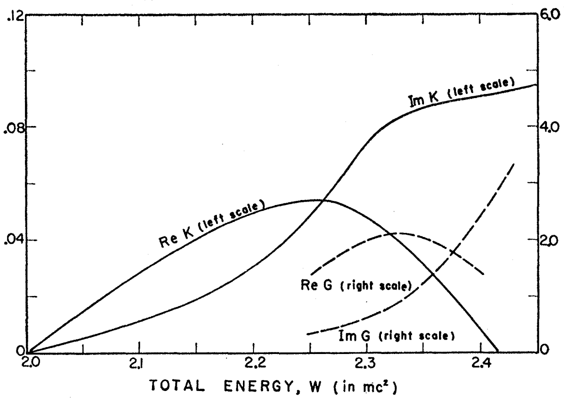

FIG. 7. Numerical values of the real and imaginary parts of $K(W,2m)$ and $G(W,2m)$.

The differential cross section at 90° in the $J=2$ state for the elastic amplitude is roughly

$$
\begin{aligned}
d\sigma/d\Omega&\cong(25/16p^2)|T_{22}|^2\\
&=(25/16p^2)(W-2m)^3(W+2m)^{-5}\\
&\qquad\times[(\mathrm{Re}G^{-1}-\mathrm{Re}K)^2+(\mathrm{Im}G^{-1}+\mathrm{Im}K)^2]^{-1}.
\end{aligned}
\tag{60}
$$

The computed cross section, as plotted on Fig. 8, shows a bump at around 600 MeV. Due to the lack of partial wave data at this energy, this bump is matched with the difference of the differential cross-section curves in the $T=1$ and $T=0$ states. This provides at least a comparison of the order of magnitude. The agreement is surprisingly good, but this should not be taken seriously, as on Fig. 8 another curve is shown which is calculated from a coupling constant G which is 20% larger than that estimated, and the computed curve is moved up quite a bit. This sensitivity is of course not unexpected, since unitarity connects the imaginary part of M_{22} with the square of M_{32}; a change in M_{32} is magnified at least two times in M_{22} which is then squared to give the cross section. This also explains why small inelasticity usually have no effect on the elastic amplitude, since once it is squared it becomes negligible. Therefore, for a production amplitude to contribute substantially to an elastic amplitude it must be as large as the elastic amplitude, and to achieve this it is almost always necessary to have some resonance situation in the production channel so as to obtain maximum coherence.

VII. COMPLEX SINGULARITIES AND ANOMALOUS THRESHOLD

In the previous calculation, the contribution from the anomalous threshold is completely neglected. It is well to have in mind the order of magnitude of this feature, and therefore we shall give an estimate of its contribution to the present problem.

The problem of analytic continuation is most care-

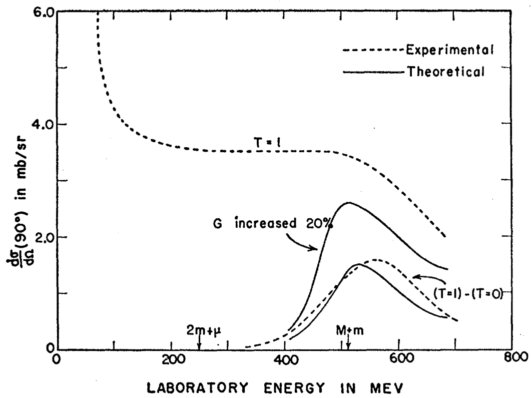

FIG. 8. Computed elastic differential cross sections at 90° c.m.s.

fully discussed by Ball, Frazer, and Nauenberg,[9] and since it is very lengthy, it will not be repeated here. However, let us illustrate with a few lines what is meant by the anomalous threshold. For example, the production amplitude with the assigned analytic properties can be written as

$$\bar{M}_{32}(s,\sigma)=-\frac{1}{\pi}\int_{\sigma^{1/2}} ds'(s'-s)^{-1}A_{32}(s',\sigma) + \int_{4m^2}^{\infty} ds'(s'-s)^{-1}\bar{M}_{32}(s_+',\sigma)\rho_2(s_+') \times\bar{M}_{22}(s_-',\sigma)+\cdots, \quad (61)$$

where the first term on the right is the Cauchy integral over the dynamical singularities and the following terms are those over the unitary cut. The representation is true as long as $\sigma<m^2-(\mu^2/2)$, when the dynamical cut and the unitary cut are separated from each other and the s plane thus defined is the physical sheet. However, as σ is continued with a positive imaginary part to $\sigma=M^2$, the dynamical branch line enters the unitary cut in the region $4m^2<s<(2m+\mu)^2$ onto the second sheet. To avoid this protruding cut, the integral of second term is deformed, and we get:

$$\bar{M}_{32}(s, M^2+i\epsilon) =-\frac{1}{\pi}\int_{C(M^2)} ds'(s'-s)^{-1}A_{32}(s',M^2) +\int_{s_+(M^2)}^{4m^2} ds'(s'-s)^{-1}\,\mathrm{disc}[\bar{M}_{32}{}^{II}(s',M^2)]\rho_2(s') \times\bar{M}_{22}(s')+\int_{4m^2}^{\infty} ds'(s'-s)^{-1}\bar{M}_{32}(s_+', M^2) \times\rho_2(s_+')\bar{M}_{22}(s_-')+\cdots, \quad (62)$$

where $\bar{M}_{32}{}^{II}$ is the continuation of $\bar{M}_{32}$ through the unitary cut in the interval $4m^2<s<(2m+\mu)^2$, and we have written "disc" to denote the discontinuities across the anomalous cut, which is what has been referred to as the anomalous threshold (see Fig. 9).

Since the anomalous threshold is over the complex region of the s plane, $\bar{M}_{32}(s, M^2+i\epsilon)$ is, in general, complex. This is also an indication of the fact that $\bar{M}_{32}$ contains a cut in the real axis of the σ plane and $\bar{M}_{32}$ can never reach the real axis, and becomes a real-valued function at $\sigma=M^2$. This cut, however, can be shown to be absent. This problem has been solved in a very elaborate manner in Ref. 9. In our case, in the interest of simplicity, we shall only impose the condition that the continued amplitudes be real valued, and to accomplish this we shall use what we call the "principal value continuation," which means continuation in σ is carried out on the function $\frac{1}{2}[\bar{M}_{32}(s,\sigma+i\epsilon)+\bar{M}_{32}(s,\sigma-i\epsilon)]$, which is the principal value part of $\bar{M}_{32}$. It can be shown that the continued amplitudes as well as the N_{ij} and D_{ij} functions agree with those obtained by Ball, Frazer, and Nauenberg.

To obtain any result by solving this new set of integral equations for the N_{ij} and D_{ij} functions would certainly mean a formidable task. Let us instead get an estimate of the contributions from such modifications by assuming their effects are small and the resulting amplitudes are not serious affected. The modification on $\bar{M}_{32}$ is given as

$$\bar{M}_{32}(W)=i\int_{M_-}^{W_+} dW'(W'-W)^{-1}\rho_2(W') \times A_{32}(W')\bar{M}_{22}(W'), \quad (63)$$

where $W_\pm=s^{1/2}{}_\pm$. Since the discontinuity across this cut is of logarithmic type, it is most singular at the end points $W_\pm$, hence let us approximate A_{32} by

$$A_{32}(W)=\pi\Gamma_+\delta(W-W_+)+\pi\Gamma_-\delta(W-W_-), \quad (64)$$

with $\Gamma_+=\Gamma_-{}^*$. Assuming $\bar{M}_{22}$ is unmodified and is given by Eq. (57),

$$\Delta\bar{M}_{32}=\frac{\pi i\Gamma_+\rho_2(W_+)}{(W_+-W)(W_+-2m)}\frac{G(W_+,2m)}{1-K(W_+,2m)G(W_+,2m)}+\mathrm{c.c.} =0.03[\Gamma_+(W_+-W)^{-1}+\Gamma_-(W_--W)^{-1}], \quad (65)$$

which is evaluated by taking $W_\pm=(2m\pm im)$, and G

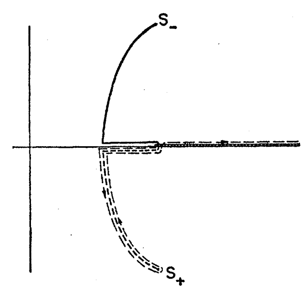

FIG. 9. Deformation of the unitary integral.

and K approximated by a pole at $W=2m$ with residue equal to 50, which fits the one-pion-exchange amplitude approximately in the physical region. This result shows that even if the entire input amplitude were fitted by two poles at W_+ and W_-, with residues Γ_+ and Γ_-, respectively, the contribution from the anomalous threshold can at most amount to 3%. Hence, the initial assumption that its contribution is small is justified. The reason that $\bar{M}_{32}$ is small in this case is because the real parts of $W_\pm$ are exactly at threshold, and consequently $\rho_2(W_\pm)$ are very small, and so are G and K evaluated there. However, if the real parts of $W_\pm$ are large then the contribution from such complex singularities would also be large. Hence, the calculation carried out by neglecting the complex singularities are actually realistic within the framework of our program.

ACKNOWLEDGMENTS

I wish to express my gratitude to Professor P. Federbush for suggesting this problem and for his guidance throughout the course of investigation. I am also very grateful to many of my colleagues for valuable discussions and assistance; in particular, H. C. Lam, M. Forrest, M. Hoenig, D. Swanson, F. Tabakin, M. Weiss, and C. T. Wu. I am indebted to Dr. D. Sprung for a discussion of the experimental information relating to nucleon-nucleon scattering.

PHYSICAL REVIEW VOLUME 135, NUMBER 3B 10 AUGUST 1964

Exploration of S-Matrix Theory*

DAVID I. OLIVE†
Department of Physics, Carnegie Institute of Technology, Pittsburgh, Pennsylvania
(Received 5 August 1963; revised manuscript received 20 April 1964)

The possibility of constructing an S-matrix theory from postulates concerning unitarity, analyticity, connectedness, the $i\epsilon$ prescription and the spin-statistics connection is explored. The existence and residues of the physical region poles are shown to follow from the connected unitarity equations. The validity of certain fundamental theorems known from field theory, Hermitian analyticity, extended unitarity, the existence of antiparticles, the substitution law for crossed processes and the TCP theorem is reduced, in simple cases, to the question of whether the S-matrix singularity structure permits specific distortions of certain paths. These distortions are shown to be possible in a "model" singularity structure consisting of the normal thresholds, and depend only upon simple properties of these singularities. It is explained that it is logically impossible to deduce the complete singularity structure without the results we are trying to prove. A suggested resolution of this difficulty is to set up a scheme of successive iterations in singularity structure to be justified by self-consistency. Then our work is the first step in such a scheme.

1. INTRODUCTION

RECENTLY some degree of understanding of the working of unitarity in S-matrix theory[1] has been developed, e.g., the way it evaluates discontinuities,[2–6] generates singularities,[6,7] and enables analytic continuations to be made onto unphysical sheets[3–5,8,9] In this sort of work a large number of properties or ingredients have been used. Apart from the quantum and Lorentz assumptions these are: (1) unitarity, (2) connectedness structure,[2,10] (3) maximal analyticity,[1] (4) the $i\epsilon$ prescription (see Sec. 3), (5) Hermitian analyticity,[2] (6) extended unitarity,[3] (7) existence of unphysical region stable poles on physical sheets, (8) the existence of antiparticles, (9) the substitution law for crossed processes, (10) the TCP theorem, (11) special physical sheet properties, (12) properties of physical region poles,[4,5] (13) connection between spin and statistics.

Several of these ideas can be grouped together. (5), (6), and (7) can be thought of as unphysical versions of the unitarity equations for T-matrix elements, valid at energies below the physical threshold of the amplitude concerned. The number of intermediate states included decreases with the energy so that (5) derives from the equation with no intermediate states and (7) from that with a single-particle intermediate state.

* This paper is a revised version of an unpublished Cambridge preprint circulated in July 1963 under the title "Towards an Axiomatisation of S-Matrix Theory." Compared with this the conclusions are restated more precisely. The work has been rearranged and more explanation given but no new results are included.

† Permanent address: Churchill College, Cambridge, England.

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