7%, respectively. Bartholomew and Gunye,[10] who have reported values of the spins for the same levels we have studied in Ni^{59}, agree with our assignments for the levels at 0.470 and 0.870 MeV, but disagree in assigning the spin $J=\frac{3}{2}$ to the level at 1.31 MeV. However, the disagreement is not as large as might be inferred from these different assignments, since the measured correlations[11] from which these spins are deduced are the same within the statistical errors.

DISCUSSION

The levels in Fe^{55}, Ni^{59}, and Ni^{61} studied in this experiment all fall within the gross structure in the (d,p) spectrum which was interpreted earlier as the $2p_{3/2}$ single-particle neutron state. The results of the present experiment indicate strongly that the first excited p state in all of these nuclides does, in fact have $J=\frac{1}{2}$, inconsistent with the earlier interpretation. In Ni^{63} our results indicate $J=\frac{3}{2}$ and $J=\frac{1}{2}$ states so close to each other in excitation that they would earlier have been interpreted as part of the same gross-structure group. In fact, a low-lying $p_{3/2}$, $p_{1/2}$ doublet at very low excitation seems to be characteristic of all the odd-neutron nuclei in this mass range. This has been discussed in more detail in an earlier note.[12]

It is then apparent that there is considerable mixing of spins within the (d,p) gross structure, at least for the $l=1$ groups. Each p-wave gross-structure peak contains states with both $p_{3/2}$ and $p_{1/2}$ strength and the simple interpretation advanced earlier[1] is not correct. It is disappointing that the simple and naive interpretation is not the correct one. However, it is now possible, with the aid of fast computers, to calculate the details of the fine structure for particularly favorable cases. A recent calculation by Ramavataram,[13] for instance, predicts our value of $J=\frac{1}{2}$ for the 413-keV first excited state of Fe^{55}. The calculation does not, however, produce the correct spins for some of the higher excited states which are populated strongly in the (d,p) reaction.[14] Better success was achieved for Cr^{53}, where the calculation did remarkably well in fitting the known spins for a number of levels.

There remains, however, the question of what interaction is responsible for the experimentally observed splitting of the p-wave gross-structure peaks. It is evidently not the spin-orbit force, which had been suggested earlier. Nor is it apparently an isotopic-spin splitting, which should not affect the results of (d,p) reaction.[15] It will be interesting to see if more sophisticated nuclear-structure calculations can reproduce these unexplained effects.

[10] G. A. Bartholomew and M. R. Gunye, Bull. Am. Phys. Soc. **8**, 367 (1963).
[11] G. A. Bartholomew (private communication).
[12] L. L. Lee, Jr., J. P. Schiffer, and D. S. Gemmell, Phys. Rev. Letters **10**, 496 (1963).
[13] K. Ramavataram, Phys. Rev. **132**, 2255 (1963).
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[15] J. B. French and M. H. Macfarlane, Nucl. Phys. **26**, 168 (1961).

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Exact Calculation of Bremsstrahlung from Polarized Electrons*

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The production of bremsstrahlung by the interaction of polarized electrons with the Coulomb field of a nucleus is considered. An exact calculation of the angular distribution of the outgoing photons, and the azimuthal asymmetry in this distribution is presented. Numerical calculations were done for an incident electron energy $W_1=1.25m$, a photon energy $k=0.75$ (W_1-m), and a nuclear charge $Ze=79e$.

I. INTRODUCTION

APPROXIMATE expressions for the asymmetry in the distribution of photons in bremsstrahlung production from polarized electrons have been developed by the authors and others.[1,2] The numerical results of these calculations indicate the asymmetry to be a maximum for an incident electron energy $W_1=1.25m$ and for a photon energy $k=0.75(W_1-m)$. At these energies the validity of the Born approximation is doubtful since $\alpha ZW_1/p_1$ for the incident electron is of order one for gold. For this reason a more detailed analysis of the asymmetry seems desirable. Using a method similar to that used by Jaeger and Hulme, one can compute this asymmetry exactly.[3]

An exact calculation of the differential cross section

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[1] W. R. Johnson and J. D. Rozics, Phys. Rev. **128**, 192 (1962).
[2] E. S. Sobolak and P. Stehle, Phys. Rev. **129**, 403 (1963).
[3] J. C. Jaeger and H. R. Hulme, Proc. Roy. Soc. (London) **A138**, 708 (1935).

for the production of unpolarized bremsstrahlung from polarized electrons, summed over the spins and integrated over the angles of the outgoing electron; and of the azimuthal asymmetry in the photon distribution, is presented here. The wave function used was the Darwin series solution of the Dirac equation for an electron in the Coulomb field of a nucleus, given by

$$\Psi^{(i,0)}(\mathbf{p},\mathbf{r},\hat{\zeta})=4\pi\sum_{\kappa,m}P_{\kappa m}(\hat{p},\hat{\zeta})\begin{pmatrix} if_\kappa^{(i,0)}(pr)\Omega_{\kappa m}(\hat{r}) \\ g_\kappa^{(i,0)}(pr)\Omega_{-\kappa m}(\hat{r})\end{pmatrix},$$

where $\mathbf{p}$ is the linear momentum of the electron and $\hat{\zeta}$ is a unit vector in the direction of the electron's spin.

$$P_{\kappa m}(\hat{p},\hat{\zeta})=\Omega_{\kappa m}^\dagger(\hat{p})v,$$

where v is the "large component" of the Dirac plane wave spinor.

$$\Omega_{\kappa\mu}=\sum_m C(l\tfrac{1}{2}j;\mu-m,m)Y_{l,\mu-m}(\hat{r})\mathrm{X}^m,$$

where $C(abc;m_am_b)$ is the Clebsch-Gordan coefficient, $Y_{l,m}$ are the spherical harmonics, and X^m are the two component Pauli spinors.[4]

j, l, and l' are determined from κ in the following way:

$$k=|\kappa|,\quad j=k-\tfrac{1}{2},$$

$$l=\begin{Bmatrix} k\ , & \kappa<0 \\ k-1, & \kappa>0\end{Bmatrix},\quad l'=\begin{Bmatrix} k-1, & \kappa<0 \\ k\ , & \kappa>0\end{Bmatrix}.$$

$f_\kappa^{(i,0)}(pr)$ and $g_\kappa^{(i,0)}(pr)$ are the radial parts of the wave function normalized so that asymptotically the wave function looks like a plane wave plus an incoming or outgoing spherical wave.[5]

The numerical results of this calculation indicate the absolute value of the asymmetry to be a maximum at about 70° and 150°.

II. CALCULATION OF THE CROSS SECTION

The cross section for this process is given by

$$d^3\sigma=\frac{\alpha}{(2\pi)^4}\frac{W_1}{p_1}p_2W_2kdkd\Omega_k\sum_\epsilon\int d\Omega_{p_2}\sum_{\zeta_2}|M|^2,$$

where

$$M=\int d\mathbf{r}\,\Psi^{(i)\dagger}(\mathbf{p}_2,\mathbf{r},\zeta_2)\boldsymbol{\alpha}\cdot\boldsymbol{\varepsilon}e^{-i\mathbf{k}\cdot\mathbf{r}}\Psi^{(0)}(\mathbf{p}_1,\mathbf{r},\zeta_1),$$

where the subscript 1 refers to the incident electron, 2 to the final electron; $\mathbf{k}$ is the momentum of the outgoing photon, and $\boldsymbol{\varepsilon}$ is its polarization vector. Substituting for the Ψ's and writing

$$e^{-i\mathbf{k}\cdot\mathbf{r}}=4\pi\sum_{l,m}i^{-l}j_l(kr)Y_{l,m}(\hat{k})Y_{l,m}^*(\hat{r}),$$

where $j_l(kr)$ is the spherical Bessel function of order l, $|M|^2$ is then given by

$$|M|^2=(4\pi)^6\sum_{\kappa_1,m_1,\kappa_2,m_2,l,m}\ \sum_{\bar\kappa_1,\bar m_1,\bar\kappa_2,\bar m_2,\bar l,\bar m}P_{\kappa_1m_1}(\hat{p}_1,\hat{\zeta}_1)P_{\bar\kappa_1\bar m_1}^\dagger(\hat{p}_1,\hat{\zeta}_1)P_{\kappa_2m_2}^\dagger(\hat{p}_2,\hat{\zeta}_2)P_{\bar\kappa_2\bar m_2}(\hat{p}_2,\hat{\zeta}_2)Y_{l,m}(\hat{k})Y_{\bar l,\bar m}^*(\hat{k})$$

$$\times\{A_{\kappa_1m_1\kappa_2m_2lm}I_{\kappa_1\kappa_2l}-A_{-\kappa_1,m_1,-\kappa_2,m_2,l,m}J_{\kappa_1\kappa_2l}\}\times\{A^*_{\bar\kappa_1\bar m_1\bar\kappa_2\bar m_2\bar l\bar m}I_{\bar\kappa_1\bar\kappa_2\bar l}^*-A^*_{-\bar\kappa_1,\bar m_1,-\bar\kappa_2,\bar m_2,\bar l,\bar m}J^*_{\bar\kappa_1\bar\kappa_2\bar l}\},$$

where

$$I_{\kappa_1\kappa_2l}=\int r^2dri^{1-l}j_l(kr)g_{\kappa_2}^{(i)*}(p_2r)f_{\kappa_1}^{(0)}(p_1r)$$

and

$$J_{\kappa_1\kappa_2l}=\int r^2dri^{1-l}j_l(kr)f_{\kappa_2}^{(i)*}(p_2r)g_{\kappa_1}^{(0)}(p_1r)$$

and

$$A_{\kappa_1m_1\kappa_2m_2lm}=\int d\Omega_rY_{l,m}^*(\hat{r})\Omega_{-\kappa_2m_2}^\dagger\boldsymbol{\sigma}\cdot\boldsymbol{\varepsilon}\Omega_{\kappa_1m_1}.$$

Writing $\boldsymbol{\sigma}$ and $\boldsymbol{\varepsilon}$ in a spherical basis and making use of the Wigner-Eckart theorem, $A_{\kappa_1m_1\kappa_2m_2lm}$ can be written,[6]

$$A_{\kappa_1m_1\kappa_2m_2lm}=\left(\frac{3[l][l_2']}{4\pi[l_1]}\right)^{1/2}C(ll_2'l_1;00)\epsilon^*_{m+m_2-m_1}$$

$$\times\sum_{\mu_1}C(l_1\tfrac{1}{2}j_1;m_1-\mu_1,\mu_1)C(l_2'\tfrac{1}{2}j_2;m_2-\mu_1-\lambda,\mu_1+\lambda)C(\tfrac{1}{2}1\tfrac{1}{2};\mu_1,\lambda)C(ll_2'l_1;m,m_2-\mu_1-\lambda)\delta_{\lambda,m+m_2-m_1},$$

with $[a]=2a+1$. After summing over spins $\hat{\zeta}_2$, integrating over the angles of $\mathbf{p}_2$, and summing over $\boldsymbol{\varepsilon}$, one is then able to carry out the sums over $\bar\kappa_2$, $\bar m_2$, $\bar m$, and m. Introducing the spin projection operator for the incident electron,

[4] The angular momentum coupling coefficients and the spherical harmonics used in this paper are those defined in M. E. Rose, *Elementary Theory of Angular Momentum* (John Wiley & Sons, Inc., New York, 1957).
[5] See, for example, W. R. Johnson and R. T. Deck, J. Math. Phys. **3**, 319 (1962).
[6] See Ref. 4, p. 85.

the sums over $\bar{m}_1$ and m_1 can also be done. The resulting expression is given by

$$\sum_{\epsilon}\int d\Omega_{p2}\sum_{\zeta_2}|M|^2=\sum_{L=0}^{\infty}A_LP_L(\cos\theta)+\hat{n}\cdot\hat{\zeta}_1\sum_{L=1}^{\infty}B_LP_{L,1}(\cos\theta)\,,$$

where

$$A_L=192(2\pi)^3\sum_{\kappa_1\kappa_2\bar{\kappa}_1}\sum_{l,\bar{l}}k_1k_2\bar{k}_1[l][\bar{l}][l_1]^{1/2}[\bar{l}_1]^{1/2}(-1)^{\bar{l}+\bar{l}_1}C(l_1\bar{l}_1L;00)W(l_1\tfrac{1}{2}L\bar{j}_1;j_1\bar{l}_1)\sum_{f\bar{f}}T_{k_1k_2\bar{k}_1l\bar{l}}{}^{f\bar{f}L}\{\ \}\,,$$

and

$$B_L=192(2\pi)^3i\left(\frac{6[L]}{L(L+1)}\right)^{1/2}\sum_{\kappa_1\kappa_2\bar{\kappa}_1}\sum_{l,\bar{l}}k_1k_2\bar{k}_1[l][\bar{l}][l_1]^{1/2}[\bar{l}_1]^{1/2}(-1)^{\bar{l}+\bar{l}_1}$$
$$\times C(l_1\bar{l}_1L;00)X(\bar{j}_1Lj_1;\bar{l}_1Ll_1;\tfrac{1}{2}1\tfrac{1}{2})\sum_{f\bar{f}}T_{k_1k_2\bar{k}_1l\bar{l}}{}^{f\bar{f}L}\{\ \}\,,$$

with

$$T_{k_1k_2\bar{k}_1l\bar{l}}{}^{f\bar{f}L}=\tfrac{2}{3}\delta_{f\bar{f}}[f]C(l\bar{l}L;00)W(L l\bar{j}_1\bar{f};\bar{l}j_1)+\sqrt{5}C(112;00)[f][\bar{f}]W(fj_221;1\bar{f})\sum_{L'}[L']^{1/2}C(l\bar{l}L';00)$$
$$\times C(L'2L;00)X(2LL';fj_1l;\bar{f}\bar{j}_1\bar{l})$$

and

$$\{\ \}=D_{\kappa_1\kappa_2\bar{\kappa}_1\kappa_2l\bar{l}}{}^{f\bar{f}}I_{\kappa_1\kappa_2l}I_{\bar{\kappa}_1\kappa_2\bar{l}}{}^*+D_{-\kappa_1,-\kappa_2,-\bar{\kappa}_1-\bar{\kappa}_2l\bar{l}}{}^{f\bar{f}}J_{\kappa_1\kappa_2l}J_{\bar{\kappa}_1\kappa_2\bar{l}}{}^*-D_{\kappa_1\kappa_2-\bar{\kappa}_1-\bar{\kappa}_2l\bar{l}}{}^{f\bar{f}}I_{\kappa_1\kappa_2l}J_{\bar{\kappa}_1\kappa_2\bar{l}}{}^*-D_{-\kappa_1-\kappa_2\bar{\kappa}_1\kappa_2l\bar{l}}{}^{f\bar{f}}J_{\kappa_1\kappa_2l}I_{\bar{\kappa}_1\kappa_2\bar{l}}{}^*$$

with

$$D_{\kappa_1\kappa_2\bar{\kappa}_1\bar{\kappa}_2l\bar{l}}{}^{f\bar{f}}=[l_2']^{1/2}[\bar{l}_2']^{1/2}W(ll_2'j_1\tfrac{1}{2};l_1f)W(\bar{l}\bar{l}_2'\bar{j}_1\tfrac{1}{2};\bar{l}_1\bar{f})W(1\tfrac{1}{2}fl_2';\tfrac{1}{2}j_2)W(1\tfrac{1}{2}\bar{f}\bar{l}_2';\tfrac{1}{2}j_2)C(ll_2'l_1;00)C(\bar{l}\bar{l}_2\bar{l}_1;00)$$

θ is the angle between $\mathbf{p}_1$ and $\mathbf{k}$, and $\hat{n}=\mathbf{k}\times\mathbf{p}_1/|\mathbf{k}\times\mathbf{p}_1|$. $P_{l,m}(\cos\theta)$ are the associated Legendre polynomials. $W(abcd;ef)$ is the Racah coefficient and $X(abc;def;ghi)$ is the 9-j symbol.

Substituting this expression into the differential cross section one obtains

$$d^3\sigma=\frac{\alpha}{(2\pi)^4}\frac{W_1}{p_1}p_2W_2kdkd\Omega_k\{\sum_{L=0}^{\infty}A_LP_L(\cos\theta)+\hat{n}\cdot\hat{\zeta}_1\sum_{L=1}^{\infty}B_LP_{L,1}(\cos\theta)\}\,.$$

The asymmetry function $P(\theta)$ is given by

$$P(\theta)=-\hat{n}\cdot\hat{\zeta}_1\sum_{L=1}^{\infty}B_LP_{L,1}(\cos\theta)/\sum_{L=0}^{\infty}A_LP_L(\cos\theta)\,.$$

If one integrates over the angles of $\mathbf{k}$, then the cross section differential in photon energy only is given by

$$d\sigma=\frac{4\pi\alpha}{(2\pi)^4}\frac{W_1}{p_1}p_2W_2kdkA_0\,.$$

III. THE RADIAL INTEGRALS

Following a procedure similar to that used by Jaeger and Hulme, one can express the radial integrals $I_{\kappa_1\kappa_2l}$ and $J_{\kappa_1\kappa_2l}$ in terms of the F_A functions which are generalized hypergeometric functions of three complex variables.[3,7] Since these integrals are not absolutely convergent, it was necessary to introduce a factor of $e^{-\lambda r}$ in the integrand and then take the limit as λ went to zero. The integrals are given by:

$$I_{\kappa_1\kappa_2l}=\left(\frac{W_1+m}{2W_1}\right)^{1/2}\left(\frac{W_2-m}{2W_2}\right)^{1/2}\frac{e^{i\pi l_1+i\pi/2+(\nu_1+\nu_2)\pi/2}}{8p_1p_2k}\frac{\Gamma(a_1-1)\Gamma(a_2-1)\Gamma(a_3)\Gamma(d)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}$$
$$\times x^{\gamma_1}y^{\gamma_2}z^{l+1}\{(\gamma_1-i\nu_1)[(\gamma_2-i\nu_2)A-(\kappa_2+i\nu_2')B]-(\kappa_1+i\nu_1')[(\kappa_2+i\nu_2')D-(\gamma_2-i\nu_2)C]\}\,,$$

$$J_{\kappa_1\kappa_2l}=\left(\frac{W_1-m}{2W_1}\right)^{1/2}\left(\frac{W_2+m}{2W_2}\right)^{1/2}\frac{e^{i\pi l_1+i\pi/2+(\nu_1+\nu_2)\pi/2}}{8p_1p_2k}\frac{\Gamma(a_1-1)\Gamma(a_2-1)\Gamma(a_3)\Gamma(d)}{\Gamma(b_1)\Gamma(b_2)\Gamma(b_3)}$$
$$\times x^{\gamma_1}y^{\gamma_2}z^{l+1}\{(\gamma_1-i\nu_1)[(\gamma_2-i\nu_2)A+(\kappa_2+i\nu_2')B]-(\kappa_1+i\nu_1')[(\kappa_2+i\nu_2')D+(\gamma_2-i\nu_2)C]\}\,,$$

[7] P. Appell, *Fonctions hypergeometriques et hyperspheriques-polynomes d'hermite* (Gauthier-Villars, Paris, 1926), p. 114.

with

$$A=F_A(d;a_1,a_2,a_3;b_1,b_2,b_3;x,y,z)\,,$$
$$B=F_A(d;a_1,a_2-1,a_3;b_1,b_2,b_3;x,y,z)\,,$$
$$C=F_A(d;a_1-1,a_2,a_3;b_1,b_2,b_3;x,y,z)\,,$$
$$D=F_A(d;a_1-1,a_2-1,a_3;b_1,b_2,b_3;x,y,z)\,,$$

where

$$b_1=2\gamma_1+1\,,\quad b_2=2\gamma_2+1\,,\quad b_3=2l+2\,,\quad \gamma_i=(k_i^2-\alpha^2Z^2)^{1/2}\,,$$
$$a_1=\gamma_1+1-i\nu_1\,,\quad a_2=\gamma_2+1-i\nu_2\,,\quad a_3=l+1\,,$$
$$x=\frac{2p_1}{p_1+p_2+k+i\lambda}\,,\quad y=\frac{2p_2}{p_1+p_2+k+i\lambda}\,,\quad z=\frac{2k}{p_1+p_2+k+i\lambda}\,,$$
$$\nu_i=\alpha ZW_i/p_i\,,\quad \nu_i'=\alpha ZM/p_i\,.$$

Since $|x|>1$, it was necessary to analytically continue F_A. The series used to evaluate F_A is as follows:

$$F_A(d;a_1,a_2,a_3;b_1,b_2,b_3;x,y,z)$$
$$=\zeta^{-d}(1-y/\zeta)^{b_2-a_2}\sum_{\mu=0}^{\infty}\frac{(d,2u)(\bar{z}/2)^{2u}}{(a_3+\frac{1}{2},u)(1,u)}\left\{\frac{\Gamma(b_1)\Gamma(a_1-d-2u)}{\Gamma(a_1)\Gamma(b_1-d-2u)}\frac{e^{-i\pi(d+2u)}}{\bar{x}^{d+2u}}\sum_{s=0}^{\infty}\frac{(d+2u,s)(1-b_1+d+2u,s)}{(1-a_1+d+2u,s)(1,s)\bar{x}^s}\right.$$
$$\times\sum_{t=0}^{\infty}\frac{(b_2+s,t)(b_2-a_2,t)}{(b_2,t)(1,t)}\bar{y}^t+\frac{\Gamma(b_1)\Gamma(d+2u-a_1)}{\Gamma(d+2u)\Gamma(b_1-a_1)}\frac{e^{-i\pi a_1}}{\bar{x}^{a_1}}\sum_{s=0}^{\infty}\frac{(a_1,s)(1-b_1+a_1,s)}{(1-d-2u+a_1,s)(1,s)}\frac{1}{\bar{x}^s}$$
$$\left.\times\sum_{t=0}^{\infty}\frac{(b_2+s-d-2u,t)(b_2-a_2,t)}{(b_2,t)(1,t)}\bar{y}^t\right\}\,,$$

with

$$\bar{x}=\frac{2p_1}{p_1-p_2+i\lambda}\,,\quad \bar{y}=\frac{2p_2}{p_1+p_2+i\lambda}\,,\quad \bar{z}=\frac{k}{p_1-p_2+i\lambda}\,,\quad \zeta=\frac{p_1+p_2+i\lambda}{p_1+p_2+k+i\lambda}$$

and $(a,b)=\Gamma(a+b)/\Gamma(a)$. This series converges for $|\bar{x}|>1$, $|\bar{y}|<1$, and $|\bar{z}/2|<1$.

IV. NUMERICAL RESULTS

The formula for $d^3\sigma$ was programmed for the University of Notre Dame's UNIVAC–1107 Computer for an incident energy $W_1=1.25m$, for a photon energy $k=0.75(W_1-m)$, and for $Z=79$. From the numerical results for the radial integrals the sums over k_1, k_2, and l were terminated at $k_1=10$, $k_2=6$, and $l=3$. All other parameters were determined in terms of these through the selection rules contained in the angular momentum

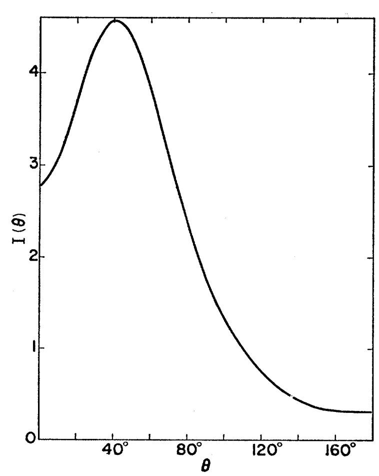

FIG. 1. Spin-independent part of the bremsstrahlung cross section for $W_1=1.25\,m$, $k=0.75\,(W_1-m)$, and $Z=79$, where θ is the angle between $\mathbf{p}_1$ and $\mathbf{k}$.

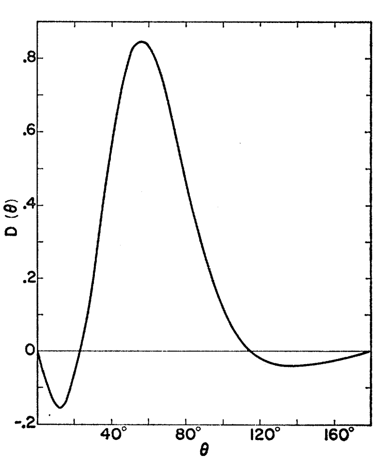

FIG. 2. Spin-dependent part of the bremsstrahlung cross section.

coupling coefficients. The series in Legendre polynomials was terminated at $L=8$.

Expressing

$$d^3\sigma=4\pi r_0^2\alpha^2Z^3(dk/k)d\Omega_\kappa(I(\theta)+\hat{n}\cdot\hat{\zeta}_1D(\theta))$$

and $P(\theta)=\hat{n}\cdot\hat{\zeta}_1S(\theta)$ where r_0 is the classical electron radius, $I(\theta)$, $D(\theta)$, and $S(\theta)$ are shown in Figs. 1, 2, and 3. Writing

$$d\sigma=4\pi r_0^2\alpha^2Z^3(dk/k)a_0\,,$$

one found $a_0=2.14$.

Comparing the results shown in Fig. 3 for the asymmetry with those given in the earlier paper, one finds that the Born approximation can give order of magnitude results but not the detailed shape of the curve.[1] This is not surprising since the previous work was the lowest order contribution from an expansion in powers of ν_1, and here $\nu_1=0.96$.

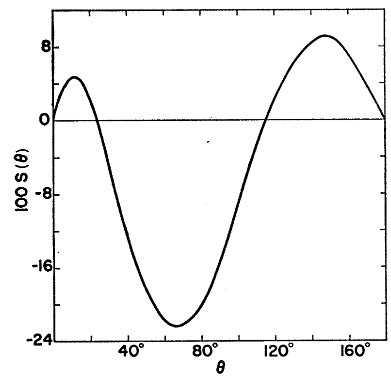

FIG. 3. Azimuthal asymmetry in the bremsstrahlung distribution.

V. SPECIAL CASES

Because of its complexity, two checks were made on the expression for the differential cross section. The first case considered was that of using a plane wave as the wave function for the incident electron. The wave function for the outgoing electron was taken to be the limit of the exact wave function as p_2 went to zero, neglecting terms of order α^2Z^2.

For this case, since one is neglecting terms of order α^2Z^2 in the outgoing wave function, only $\kappa_2=-1$ contributes, and the radial integrals reduce to the following:

$$I_{\kappa_1-1l}=\frac{\alpha ZB^*}{2p_1^2k}\left(\frac{W_1+m}{2W_1}\right)^{1/2}\{\delta_{l_1,l+1}\mathcal{B}_l-\delta_{l_1,l-1}\mathcal{A}_l\}\,,$$

$$J_{\kappa_1-1l}=-\frac{\alpha ZB^*}{2p_1k^2}\left(\frac{W_1-m}{2W_1}\right)^{1/2}\delta_{l_1',l}\mathcal{G}_l\,,$$

where

$$\mathcal{A}_l=\frac{1}{2}\left[lQ_l(W_1/p_1)+\frac{m}{p_1}Q_l'(W_1/p_1)\right],$$

$$\mathcal{B}_l=\frac{1}{2}\left[(l+1)Q_l(W_1/p_1)-\frac{m}{p_1}Q_l'(W_1/p_1)\right],$$

$$\mathcal{G}_l=-\frac{m}{p_1}Q_l'(W_1/p_1)\,,$$

with $B^*=(2\pi m\alpha Z/p_2)^{1/2}e^{3\pi i/4+i\nu_2-i\nu_2\ln\nu_2}$ and the $Q_l(x)$ are the Legendre functions of the second kind with $|x|>1$, and $Q_l'(x)=(d/dx)Q_l(x)$.[8] Integrating over the angles of $\mathbf{k}$ and summing over all remaining indices except l, one obtains:

$$d\sigma=4\pi r_0^2\alpha^2Z^3\frac{dk}{k}\frac{2m^4}{p_1^3k}\sum_l\Bigg\{(2l+1)\mathcal{G}_l^2+l\mathcal{A}_l^2+(l+1)\mathcal{B}_l^2$$
$$+\frac{2(l+1)}{(2l+3)}[(l+1)\mathcal{G}_l+(l+2)\mathcal{G}_{l+2}]\mathcal{B}_l$$
$$-\frac{2l}{(2l-1)}[l\mathcal{G}_l+(l-1)\mathcal{G}_{l-2}]\mathcal{A}_l\Bigg\}\,.$$

Using the formulas for summing Q_l's, one can carry out this l sum analytically, and the resulting expression agrees with that obtained by a direct calculation for this case. This case was also checked numerically, thus guaranteeing the computer program for the computation of A_L.

Since the $Q_l(x)$ functions have a singularity at $x=1$, the series representation will not converge for the relativistic case. This method is thus suitable only to the lower energy region, and in particular it is well suited to the calculation of asymmetries.

The second case considered was that of using a distorted plane wave as the wave function of the incident electron with the same outgoing wave function as that used for the preceding case. Again only $\kappa_2=-1$ contributed. The angular distribution was numerically computed for the spin-dependent and spin-independent parts of the cross section and compared with the results from a direct calculation of this case, and both agreed. This case served as a check on the numerical computation of B_L. In both of these cases the sum over k_1, was terminated at $k_1=10$.

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[8] See, for example, W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), p. 55.