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Recoupling Effects in the Isobar Model. I. General Formalism for Three-Pion Scattering*

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The consequence of elastic unitarity and analyticity for the isobar model of isocalar three-pion scattering are derived. All recoupling effects are retained and a careful discussion of real pion exchange is presented. In this way we show how the isobar model may be made to satisfy Watson's theorem. We display the common origin of both these phenomena and obtain the modifications they generate in the partial-wave discontinuity equations. In preparation for a forthcoming treatment of one and two-pion exchange forces, the N/D equations are derived for the $J=1^-$ isoscalar amplitude. The effect of recoupling in the narrow width approximation is demonstrated by a calculation of the effective $\rho-\pi$ phase space and suggests that recoupling is important.

1. INTRODUCTION

THIS is, hopefully, the first of a series of papers in which a careful treatment of the isobar model is applied to various reactions involving three-particle initial and/or final states. The isobar model[1] is characterized by the assumption that the amplitude in question may be written as a sum of terms, each one of which involves a definite pair of particles from the three-particle state appearing in a state of definite angular momentum and isotopic spin [Eqs. (2.7), (2.8)]. The sum is an approximation because it is finite.

Experimentally, the model is motivated by the dominance of two-particle resonances in three or more particle final states of production processes.[2] Theoretically the model has been studied by several authors[3–7] in the past but always with an assumed form of the dependence on the energy variables of the resonating pairs. This form violates the unitarity equation in the principal channel and is usually claimed to be a good approximation if the two-particle resonances are "narrow." The ρ meson and the $\frac{3}{2}$, $\frac{3}{2}$ pion-nucleon N^* are not narrow, however, and the approximation in these most important cases seems unjustified. Regardless of how broad the resonances are, their appearance in only one angular momentum state and one isotopic spin state is very nearly, if not exactly, true. Hence we retain the first characterization of the isobar model and seek the form of the dependence on the resonance energies which will not violate the unitarity equation [Eqs. (3.17), (3.19)]. It is simple to show that this form also satisfies Watson's theorem[8] which amounts to the unitarity equations for the two particle-resonance channels. It is true that, in this paper at least, we set up the N/D equations for an approximation to the form we have derived. Nevertheless we now know the extent to which the approximation violates Watson's theorem and the direction to proceed should we wish to improve upon it.

In Sec. 2 we develop the formalism of the isobar model for isoscalar three-pion elastic scattering. The absence of spinning particles and the existence of a pure elastic region make the calculations comparatively simple. The unitarity equation and assumed analyticity of the isobar amplitudes are discussed and discontinuity equations derived in Sec. 3. In particular, we note that the contributions from the real exchange of one pion (Fig. 1)[9] and the recoupling terms in the discontinuity equations have a common origin. Mandelstam *et al.*[3] have discussed the phenomenon of real particle exchange for the $2\pi+N\rightarrow 2\pi+N$ amplitude but they neglect all the recoupling terms. Harrington's[6] more recent study of three-pion scattering proceeds similarly. Frazer and Wong,[7] on the other hand, included the recoupling terms in the discontinuity equation for the

FIG. 1. One-pion exchange.

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[1] The name "isobar model" has been applied to a wide variety of schemes for calculating reaction amplitudes. The only justification is that all these schemes assume that the amplitudes being computed are dominated by the excitation and subsequent decay of two-particle resonances, commonly called isobars.

[2] M. Roos, Rev. Mod. Phys. **35**, 314 (1963); this paper contains a large collection of further references.

[3] We are referring here to the more recent calculations aimed at determining the form of the reaction amplitudes via the unitarity constraint and not the calculations of momentum distributions in final states as done by R. M. Sternheimer and S. J. Lindenbaum, Phys. Rev. **123**, 333 (1961).

[4] P. Carruthers, Nuovo Cimento **22**, 867 (1961); P. G. Federbush, M. T. Grisaru, and M. Tausner, Ann. Phys. (N.Y.) **18**, 23 (1962).

[5] S. Mandelstam, J. E. Paton, R. F. Peierls, and A. Q. Sarker, Ann. Phys. (N. Y.) **18**, 198 (1964).

[6] D. Harrington, Phys. Rev. **127**, 2235 (1962).

[7] W. R. Frazer and D. Wong, Phys. Rev. **128**, 1927 (1962).

[8] K. M. Watson, Phys. Rev. **88**, 1163 (1952).

[9] This phenomenon has been neglected in the otherwise more ambitious calculations of L. F. Cook, Jr., and B. W. Lee, Phys. Rev. **127**, 183, 297 (1962); and J. S. Ball. W. R. Frazer, and M. Nauenberg, *ibid.* **128**, 478 (1962).

unitary cut but made no mention of real pion exchange. We believe this is the first work which has attempted to treat both effects on an equivalent basis.[10] In Sec. 4 the $J=1^-$ amplitude is projected out with an eye to a subsequent study of the influence of one- and two-pion exchange forces on the appearance of the ω meson. In carrying out the projection the angular integration sweeps over the real pion exchange (R.P.E.) pole, at certain energies, and thereby destroys the real analytic character of the partial-wave amplitude. This fact and its consequences for the unitary cut were first pointed out to the present writer by Paton.[11] The result is that the discontinuity across the unitary cut is proportional to

$$M(\cdots s_+\cdots)^* M(\cdots s_+\cdots)$$

rather than

$$M(\cdots s_-\cdots) M(\cdots s_+\cdots)$$

as Harrington assumes. We also claim, although the tedious algebraic derivation is not presented, that the R.P.E. cut into which the R.P.E. pole is projected must be approached from below in evaluating the physical amplitude. Paton has suggested the following argument for understanding this result; starting with Eq. (2.16),

$$s+u+t=\sigma+\sigma'+2\mu_\pi^2,$$

we may regard t as being essentially the variable of integration in the partial-wave projection. It is therefore forced to remain real. Holding s, σ, and σ' fixed and real we notice that if the variation of t brings u up to the R.P.E. pole, then the physical amplitude is obtained by letting u pass *over* the pole, i.e.,

$$u\to u+i\epsilon \quad \epsilon>0.$$

To maintain Eq. (2.16) we are forced to give s a small negative imaginary part which means that in the projected amplitude s must pass under the branch points of the R.P.E. cut. As it stands this argument is not foolproof since it is not t, but some $\cos\theta$ which is constrained to be real in the partial-wave projection. Nevertheless, the result is the same. Finally in Sec. 5 we introduce the "resonance approximation," widely used in calculations of this sort, to reduce the problem of solving the unitary cut equation to a one-dimensional one. Deriving the "reflection properties" that *are* true of the partial-wave amplitude, we cast the discontinuity equation in a form amenable to the N/D method. The general equations of that method are then derived for this particular problem.

A short appendix indicates the modification suffered by the effective phase space, $R(s)$, from recoupling effects in the extreme approximation of replacing the π-π scattering amplitudes in the integrals by appropriately normalized delta functions.

The discussion of anomalous, or structure singularities, has been postponed to the second paper where it will complement a "solution" of the N/D equations developed here with the insertion of unphysical singularities from one- and two-pion exchange.

[10] The study of singularities on unphysical sheets by R. Hwa, Phys. Rev. **130**, 2580 (1963), considers real particle exchange but assumes a form for the discontinuity equations which is inconsistent with the existence of the R.P.E. diagram.

[11] J. Paton (private discussion). See also J. E. Paton, Princeton University, 1962 (to be published).

2. THE ISOBAR MODEL

As a matter of convenience we restrict our considerations to isoscalar states from the outset. The three-pion isoscalar "in" or "out" state satisfies the symmetry,

$$|p_1p_2p_3\pm\rangle=-|p_2p_1p_3\pm\rangle=|p_3p_1p_2\pm\rangle, \qquad (2.1)$$

required by the generalized Pauli principle. Consequently, this state vector does not facilitate a discussion of scattering mechanisms which require an asymmetric treatment of the pions at intermediate states of calculation. Such a mechanism is provided by the isobar model which assumes that each scattering event is initiated by the formation of a two-pion resonance, involving a particular pair of incident pions, and terminated by the decay of such a resonance, involving a particular pair of final pions. For this reason we introduce extensions of the physical, isoscalar, three-pion Hilbert spaces,

$$\mathcal{H}_3^\pm\sim|p_1p_2p_3\pm\rangle,$$

to Hilbert spaces of lower symmetry. We denote these new vector spaces by $\tilde{\mathcal{H}}_3^\pm$ and write[12]

$$|(p_1p_2)p_3\pm\rangle=-|(p_2p_1)p_3\pm\rangle, \qquad (2.2)$$

for the general element of the corresponding asymptotic basis. We interpret Eq. (2.2) as that state which in the distant (future, past) consists of three pions, the bracketed pair of which interacts (last, first) in the isovector state. The relation between $\mathcal{H}_3$ and $\tilde{\mathcal{H}}_3$ is defined by

$$|p_1p_2p_3\rangle=\tfrac{1}{3}[|(p_1p_2)p_3\rangle+|(p_2p_3)p_1\rangle+|(p_3p_1)p_2\rangle], \qquad (2.3)$$

where we have suppressed the "in" and "out" signatures. In other words, $\mathcal{H}_3$ is the fully symmetrized subspace of $\tilde{\mathcal{H}}_3$. Notice that Eq. (2.1) is a consequence of Eqs. (2.2) and (2.3). The factor $\frac{1}{3}$ in Eq. (2.3) is chosen so that the normalization,

$$\langle(p_1'p_2')p_3'|(p_1p_2)p_3\rangle=\tfrac{1}{2}\tilde{\delta}(p_3,p_3')[\tilde{\delta}(p_1,p_1')\tilde{\delta}(p_2,p_2')-\tilde{\delta}(p_1,p_2')\tilde{\delta}(p_2,p_2')], \qquad (2.4)$$

yields for the unit operator in $\mathcal{H}_3$

$$I_3=\int\tilde{d}p_1\tilde{d}p_2\tilde{d}p_3|p_1p_2p_3\rangle\langle p_1p_2p_3|, \qquad (2.5)$$

[12] In the language of wave mechanics, the state vectors (2.2) correspond to the isoscalar projection of a product of a single-pion wave function and an isovector two-pion wave function.

where

$$\tilde{\delta}p = d^3\mathbf{p}/2p_0;\quad \tilde{\delta}(p,p') = 2p_0\delta^3(\mathbf{p}-\mathbf{p}').$$

Let T be the scattering operator. Since the ρ meson is a vector particle[13] the isobar model is characterized by the equation

$$\langle (p_1'p_2')p_3' | T | (p_1p_2)p_3\rangle = \sum_{\lambda'=-1}^{1}\sum_{\lambda=-1}^{1} Y_{1\lambda'}(\beta_3',\alpha_3') \times \langle (q';1\lambda')p_3' | T | (q;1\lambda)p_3\rangle Y_{1\lambda}^*(\beta_3,\alpha_3), \quad (2.6)$$

where λ, λ' and q, q' are the helicities and four momenta of the initial and final ρ mesons, respectively. The angles β_3 and α_3 are the polar and azimuthal angles of $\mathbf{p}_1$ in the center-of-mass system of the pair (p_1p_2). The z axis of the coordinate frame points along $-\mathbf{p}_3$. Similar considerations apply to the angles β_3' and α_3'.[14] Through Eq. (2.6) the three-particle scattering problem presented by the matrix element

$$\langle p_1'p_2'p_3' | T | p_1p_2p_3\rangle$$

is reduced to an effective two-particle problem with the complication that one of the "particles" has a variable mass

$$q^2 = \sigma;\quad q'^2 = \sigma'.$$

Let (j,j_2,j_3) denote that even permutation of (1,2,3) which starts with j. Using this notation we can write

$$\langle p_1'p_2'p_3' | T | p_1p_2p_3\rangle = \tfrac{1}{9}\sum_{i,j} \langle (p_{j_2}'p_{j_3}')p_j' | T | (p_{i_2}p_{i_3})p_i\rangle \quad (2.7)$$

and

$$\langle (p_{j_2}'p_{j_3}')p_j' | T | (p_{i_2}p_{i_3})p_i\rangle = \sum_{\lambda'=-1}\sum_{\lambda=-1} Y_{1\lambda'}(\beta_j',\alpha_j') \times \langle (q_j';1\lambda')p_j' | T | (q_i;1\lambda)p_i\rangle Y_{1\lambda}^*(\beta_i,\alpha_i), \quad (2.8)$$

where

$$q_i = p_{i_2} + p_{i_3}\quad q_j' = p_{j_2}' + p_{j_3}'. \quad (2.9)$$

Now among the scattering events there are those in which one of the pions never interacts but passes through as a free particle while the other two pions scatter (Fig. 2). Although these events do contribute to the scattering amplitude, they do not involve the three-particle interaction and hence are not interesting. Furthermore, we eventually wish to work with amplitudes which are analytic functions of their variables and these disconnected events contribute three-momentum delta functions which conflict with the desired analyticity. Writing T_D for that part of T contributing

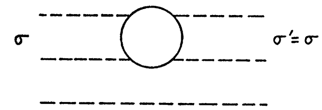

FIG. 2. Disconnected part.

the disconnected events, we have (from the interpretation of $\mathfrak{H}_3$)

$$\langle (p_{j_2}'p_{j_3}')p_j' | T_D | (p_{i_2}p_{i_3})p_i\rangle = 3\tilde{\delta}(p_i,p_j')\langle p_{j_2}'p_{j_3} | T | p_{i_2}p_{i_3}\rangle, \quad (2.10)$$

where the factor of 3 is necessary to yield

$$\langle p_1'p_2'p_3' | T_D | p_1p_2p_3\rangle = \tfrac{1}{3}\sum_{i,j}\tilde{\delta}(p_i,p_j') \times \langle p_{j_2}'p_{j_3}' | T | p_{i_2}p_{i_3}\rangle. \quad (2.11)$$

Writing $T_C = T - T_D$ and factoring out the total four-momentum delta function, we have

$$\langle (q_j';1\lambda')p_j' | T_C | (q_i;1\lambda)p_i\rangle = \delta^4(P-P')\langle (q_j';1\lambda')p_j' | t_c | (q_i;1\lambda)p_i\rangle, \quad (2.12)$$
$$P = q_i + p_i;\quad P' = q_j' + p_j'$$

and we assume $\langle (q';1\lambda')p' | t_c | (q;1\lambda)p\rangle$ to be an analytic function of its scalar variables.

What are the scalar variables? If the squared "masses"

$$q^2 = \sigma;\quad q'^2 = \sigma' \quad (2.13)$$

are held fixed, then the scattering process described by Eq. (2.12) is kinematically identical to the familiar two-particle case[15] with the variables

$$s = (q+p)^2 = (q'+p')^2, \quad (2.14a)$$
$$t = (q'-q)^2 = (p'-p)^2, \quad (2.14b)$$
$$u = (q'-p)^2 = (q-p')^2. \quad (2.14c)$$

Thus we can write

$$\langle (q';1\lambda')p' | t_c | (q;1\lambda)p\rangle = t_c^{\lambda'\lambda}(\sigma';s,t,u;\sigma). \quad (2.15)$$

These variables are not all independent. They are connected by

$$s + t + u = 2\mu_\pi^2 + \sigma + \sigma'. \quad (2.16)$$

Finally, introducing the subscripts which refer to specific pions we write

$$\langle (q_j';1\lambda')p_j' | t_c | (q_i;1\lambda)p_i\rangle = t_c^{\lambda'\lambda}(\sigma_j';s t_{ij} u_{ij};\sigma_i), \quad (2.17)$$

where the notation is clear.

3. UNITARITY AND ANALYTICITY

The scattering operator T satisfies

$$T - T^\dagger = 2iT^\dagger T \quad (3.1)$$

in the physical Hilbert space. Operating in the three-pion isoscalar subspaces $\mathfrak{H}_3^\pm$ the approximation to

[13] J. Anderson, V. Bang, P. G. Burke, D. Carmons, and N. Schmitz, Phys. Rev. Letters 6, 365 (1961); A. R. Erwin, R. March, W. D. Walker, and E. West, *ibid.* 6, 628 (1961).

[14] We have followed the conventions of M. Jacob and G. C. Wick, Ann. Phys. (N. Y.) 7, 404 (1959) in choosing angular variables.

[15] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

elastic unitarity may be written in operator form as

$$T-T^{\dagger}=2iT^{\dagger}I_3{}^{\pm}T, \tag{3.2}$$

where I_3 is defined by Eq. (2.5). The extension of Eq. (3.2) throughout $\tilde{\mathcal{H}}_3{}^{\pm}$ can at most involve additional terms which vanish in the fully symmetrized subspaces $\mathcal{H}_3{}^{\pm}$. Since the form of the unitarity equation is significant only in the physical subspace, we adopt Eq. (3.2) throughout $\tilde{\mathcal{H}}_3{}^{\pm}$. Note that since I_3 is not the unit operator in $\tilde{\mathcal{H}}_3$, Eq. (3.2) is not the same as Eq. (3.1) even for matrix elements with

$$P^2<25\mu_\pi{}^2,$$

which defines the pure elastic region.

In taking matrix elements of Eq. (3.2) in $\tilde{\mathcal{H}}_3$, we will find the following relations useful:

$$|(p_1p_2)p_3\rangle=\sum_{l,\lambda}|(q_3;l\lambda)p_3\rangle Y_{l\lambda}(\alpha_3,\beta_3)^*, \tag{3.3}$$

$$\bar{I}_3=\int \tilde{d}p_1\tilde{d}p_2\tilde{d}p_3|(p_1p_2)p_3\rangle\langle(p_1p_2)p_3|$$
$$=\sum_{l,\lambda}\int d^4q_3\rho(\sigma_3)\tilde{d}p_3|(q_3;l\lambda)p_3\rangle\langle(q_3;l\lambda)p_3|, \tag{3.4}$$

and

$$|(p_1p_2)p_3\rangle\langle(p_2p_3)p_1|=\sum_{\substack{l',\lambda',\\ l'',\lambda''}}\int d^4q\rho(\sigma')dp' \times\int d^4q''\rho(\sigma'')\tilde{d}p''|(q';l'\lambda')p'\rangle \times\langle(q';l'\lambda')p'|(p_1p_2)p_3\rangle\langle(p_2p_3)p_1|(q'';l''\lambda'')p''\rangle \times\langle(q'';l''\lambda'')p''|, \tag{3.5}$$

where

$$\rho(\sigma)=\tfrac{1}{8}[(\sigma-4\mu_\pi{}^2)/\sigma]^{1/2}. \tag{3.6}$$

Employing Eqs. (3.2)–(3.6) we obtain

$$\langle(q';1\lambda')p'|T-T^{\dagger}|(q;1\lambda)p\rangle=\tfrac{2}{3}i\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''\langle(q';1\lambda')p'|T^{\dagger}|(q'';1\lambda'')p''\rangle\langle(q'';1\lambda'')p''|T|(q'1\lambda)p\rangle$$
$$+\tfrac{4}{3}i\sum_{\lambda''\lambda'''}\int d^4q''\rho(\sigma'')\tilde{d}p''\int d^4q'''\rho(\sigma''')\tilde{d}p'''\langle(q';1\lambda')p'|T^{\dagger}|(q'';1\lambda'')p''\rangle C^{1\lambda'';1\lambda'''}(q''p'';q'''p''')\delta^4(P''-P''')$$
$$\times\delta((q'''-p'')^2-\mu_\pi{}^2)\langle(p''';1\lambda''')p'''|T|(q;1\lambda)p\rangle, \tag{3.7}$$

where

$$\delta^4(P''-P''')\delta((q'''-p'')^2-\mu_\pi{}^2)C^{l''\lambda'';l'''\lambda'''}(q''p'';q'''p''')=\int\tilde{d}p_1\tilde{d}p_2\tilde{d}p_3\langle(q'';l''\lambda'')p''|(p_1p_2)p_3\rangle \times\langle(p_2p_3)p_1|(q''';l'''\lambda''')p'''\rangle. \tag{3.8}$$

We call the function C the recoupling coefficient.[16] Factoring out four-momentum delta functions to obtain the reduced amplitudes and separating off the matrix elements of t_c, we get

$$\langle(q';1\lambda')p'|t_c-t_c{}^{\dagger}|(q;1\lambda)p\rangle=2i\rho(\sigma')f(\sigma_-')\Big[\langle(q';1\lambda')p'|t_c|(q;1\lambda)p\rangle$$
$$+2\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''C^{1\lambda';1\lambda''}(q'p';q''p'')\delta^4(P''-P)\delta((q''-p')^2-\mu_\pi{}^2)\langle(q'';1\lambda'')p''|t_c|(q;1\lambda)p\rangle\Big]$$
$$+\tfrac{2}{3}i\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''\Big[\langle(q';1\lambda')p'|t_c{}^{\dagger}|(q'';1\lambda'')p''\rangle\delta^4(P''-P)\langle(q'';1\lambda'')p''|t_c|(q;1\lambda)p\rangle$$
$$+2\sum_{\lambda'''}\int d^4q'''\rho(\sigma''')\tilde{d}p'''\langle(q';1\lambda')p'|t_c{}^{\dagger}|(q'';1\lambda'')p''\rangle\delta^4(P''-P''')C^{1\lambda'';1\lambda'''}(q''p'';q'''p''')$$
$$\times\delta((q'''-p'')^2-\mu_\pi{}^2)\delta^4(P'''-P)\langle(q''';1\lambda''')p'''|t_c|(q;1\lambda)p\rangle\Big]+2i\Big[\langle(q';1\lambda')p'|t_c{}^{\dagger}|(q;1\lambda)p\rangle$$
$$+2\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''\langle(q';1\lambda')p'|t_c{}^{\dagger}|(q'';1\lambda'')p''\rangle C^{1\lambda'';1\lambda}(q''p'';qp)\delta^4(P''-P)\delta((q''-p)^2-\mu_\pi{}^2)\Big]$$
$$\times\rho(\sigma)f(\sigma_+)+12i\rho(\sigma')f(\sigma_-')\delta((q'-p)^2-\mu_\pi{}^2)C^{1\lambda';1\lambda}(q'p';qp)f(\sigma_+)\rho(\sigma), \tag{3.9}$$

[16] The recoupling coefficient is closely related to a quantity of the same name appearing in the reference of footnote 14.

where

$$f(\sigma_+)=\langle q;1\lambda|t|q;1\lambda\rangle \tag{3.10}$$

is the isovector, P-wave, π-π scattering amplitude and

$$\langle(q';1\lambda')p'|T_D|(q;1\lambda)p\rangle=3\tilde{\delta}(p,p')\delta^4(q'-q)\delta_{\lambda'\lambda}f(\sigma_+),$$

as a consequence of Eqs. (2.10) and (2.8).

In the previous section we assumed that

$$\langle(p_1'p_2')p'|t_c|(p_1p_2)p\rangle=t_c(\Phi'\sigma';stu;\sigma\Phi) \tag{3.11}$$

was an analytic function of its scalar variables, where $\Phi'=(\alpha'\beta')$ and $\Phi=(\alpha,\beta)$. To extract information from Eq. (3.9) concerning the analytic structure of the helicity amplitudes, $t_c^{\lambda'\lambda}$, we must relate the left-hand side of Eq. (3.9) to the process of analytic continuation. For this we assert a generalized reflection principle which has been claimed to hold in perturbation theory.[17] Holding the angular variables fixed and letting the scalar variables assume complex values, we write

$$t_c(\Phi'\sigma';stu;\sigma\Phi)^*=t_c(\Phi'\sigma'^*;s^*t^*u^*;\sigma^*\Phi). \tag{3.12}$$

From Eq. (2.6) the corresponding relation for the helicity amplitudes is

$$t_c^{\lambda'\lambda}(\sigma'stu\sigma)^*=t_c^{-\lambda'-\lambda}(\sigma'^*s^*t^*\sigma^*). \tag{3.13}$$

We also need time reversal invariance in the form

$$t_c^{\lambda'\lambda}(\sigma'stu\sigma)=t_c^{-\lambda-\lambda'}(\sigma stu\sigma'). \tag{3.14}$$

Combining Eqs. (3.13) and (3.14), the left-hand side of Eq. (3.9) becomes

$$t_c^{\lambda'\lambda}(\sigma_+'s_+t_+u_+\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_-t_-u_-\sigma_-), \tag{3.15}$$

where the signatures indicate the manner of approach to the real axis in the customary fashion.

Notice that we have allowed for the possibility of singularities in the momentum transfer variables t and u, notwithstanding our concentration on the physical region of the s channel. This allowance is superfluous in the case of t but the one-pion exchange pole (Fig. 1) in the u variable does, in fact, lie in the physical region.

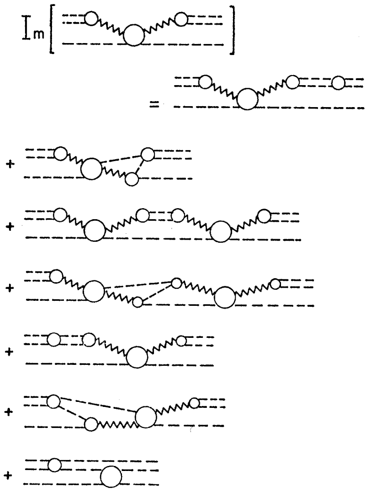

FIG. 3. Contributions of elastic unitarity.

This is a consequence of the possibility of exchanging the pion as a real particle in the physical region. Indeed a little calculation shows that the discontinuity upon crossing the pole is exactly given by the last term on the right of Eq. (3.9). Notice that from Eq. (2.14c) the delta function appearing in that term may be written as $\delta(u-\mu_\pi^2)$. If we were concerned with higher energies, we would also have to consider the elastic branch cut in u since the real exchange of three pions would be possible.

We can expand Eq. (3.15) to separate the singularities associated with individual variables:

$$\begin{aligned}&t_c^{\lambda'\lambda}(\sigma_+'s_+tu_+\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_-tu_-\sigma_-)\\&\quad=[t_c^{\lambda'\lambda}(\sigma_+'s_+tu_+\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_+tu_+\sigma_+)]\\&\qquad+[t_c^{\lambda'\lambda}(\sigma_-'s_+tu_+\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_-tu_+\sigma_+)]\\&\qquad+[t_c^{\lambda'\lambda}(\sigma_-'s_-tu_+\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_-tu_-\sigma_+)]\\&\qquad+[t_c^{\lambda'\lambda}(\sigma_-'s_-tu_-\sigma_+)-t_c^{\lambda'\lambda}(\sigma_-'s_-tu_-\sigma_-)].\end{aligned} \tag{3.16}$$

Since each of the terms on the right of Eq. (3.9) have obvious diagrammatic representations (Fig. 3) we can follow the prescriptions of S-matrix theory for associating singularities with intermediate states and write

$$\begin{aligned}d_{\sigma'}[t_c^{\lambda'\lambda}(\sigma_+'s_+tu_+\sigma_+)]=2i\rho(\sigma')f(\sigma')\Big[&\langle(q';1\lambda')p'|t_c|(q;1\lambda)p\rangle+2\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''C^{1\lambda';1\lambda''}(q'p';q''p'')\\&\times\delta((q''-p')^2-\mu_\pi^2)\delta^4(P''-P)\langle(q'';1\lambda'')p''|t_c|(q;1\lambda)p\rangle\Big],\end{aligned} \tag{3.17}$$

$$\begin{aligned}d_s[t_c^{\lambda'\lambda}(\sigma_-'s_+tu_+\sigma_+)]=\tfrac{2}{3}i\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''\langle(q'';1\lambda''|t_c|(q';1\lambda')p\rangle^*\Big[&\delta^4(P''-P)\langle(q'';1\lambda'')p''|t_c|(q;1\lambda)p\rangle\\&+2\sum_{\lambda'''}\int d^4q'''\rho(\sigma''')\tilde{d}p'''\delta^4(P''-P''')C^{1\lambda'';1\lambda'''}(q''p'';q'''p''')\delta((q'''-p'')^2-\mu_\pi^2)\\&\times\delta^4(P'''-P)\langle(q''';1\lambda''')p'''|t_c|(q;1\lambda)p\rangle\Big],\end{aligned} \tag{3.18}$$

[17] See the references in footnote 9.

$$d_\sigma[t_c^{\lambda'\lambda}(\sigma_-'s_-tu_-\sigma_+)]=2i\Big[\langle(q;1\lambda)p|t_c|(q';1\lambda')p'\rangle^*+2\sum_{\lambda''}\int d^4q''\rho(\sigma'')\tilde{d}p''\langle(q'';1\lambda'')p''|t_c|(q';1\lambda')p'\rangle^* \times C^{1\lambda'';1\lambda}(q''p'';qp)\delta^4(P''-P)\delta((q''-p)^2-\mu_\pi^2)\Big]\rho(\sigma)f(\sigma_+), \quad (3.19)$$

where

$$d_x[F(x_+)]=(F(x_+)-F(x_-)).$$

To the best of the present writer's knowledge, previous applications[4–7] of the isobar model to three-particle channels have neglected the second terms on the right of Eqs. (3.17) and (3.19) while the corresponding terms in Eq. (3.18) have been included by some[7] and neglected by others. Our derivation points out the common origin of these terms and in the case of Eqs. (3.17) and (3.19) these terms suffice to make the full amplitude $\langle p_1'p_2'p_3'|t_c|p_1p_2p_3\rangle$ satisfy Watson's theorem. For our purposes Watson's theorem amounts to the equation

$$d\sigma_j'[\langle p_1'p_2'p_3'|t_c|p_1p_2p_3\rangle] =2i\int \tilde{d}p_{j_2}''\tilde{d}p_{j_3}''\langle p_{j_2}'p_{j_3}'|t|p_{j_2}''p_{j_3}''\rangle^* \times\langle p_{j_2}''p_{j_3}''p_j'|t_c|p_1p_2p_3\rangle,$$

for the final state variables and a similar equation for the initial state variables. The recoupling terms in (3.17), (3.19) can be shown to yield these simple *crossed-channel unitarity equations* for the full connected amplitude. It is also clear from the appearance of the delta functions of the form, $\delta((q'-p)^2-\mu_\pi^2)$, in the "recoupling" terms of Eqs. (3.17)–(3.19) that these terms contribute in the same region of phase space as R.P.E. Therefore a treatment of R.P.E. which ignores these terms seems inconsistent.

In the next section we project out the $J=1^-$ amplitude so we can focus attention on the state in which the ω meson should appear. The existence of the R.P.E. pole, which lies in the physical region, requires some care in carrying out the calculations.

4. THE $J=1^-$ AMPLITUDE

The states with definite total angular momentum J, in the c.m. frame, and $J_z=M$ satisfy[14]

$$|PJM(\sigma l\lambda)\rangle=\mathfrak{N}_J\int d\Omega\mathfrak{D}_{M\lambda}{}^J(\theta,\phi,0)|(q;l\lambda)p\rangle, \quad (4.1)$$

where

$$\mathfrak{N}_J=|(2J+1)/4\pi|^{1/2}, \quad (4.2)$$
$$\sigma=q^2,$$

and θ, ϕ are the c.m. polar and azimuthal angles of $\mathbf{q}$ relative to the z axis along which M is defined. We therefore define

$$t_J{}^{\lambda'\lambda}(\sigma_+'s_+\sigma_+)=\mathfrak{N}_J{}^2\int d\Omega'd\Omega\mathfrak{D}_{M\lambda'}{}^J(\Omega')^* \times t_c{}^{\lambda'\lambda}(\sigma_+'s_+t(\Omega',\Omega)u_+(\Omega'\Omega)\sigma_+)\mathfrak{D}_{M\lambda}{}^J(\Omega), \quad (4.3)$$

paying due regard to the signature of the variable u. It follows from Eq. (4.3) that the partial-wave amplitude does not satisfy a simple reflection property like Eq. (3.13) since

$$t_J{}^{\lambda'\lambda}(\sigma_-'s_-\sigma_-)=\mathfrak{N}_J{}^2\int d\Omega'd\Omega\mathfrak{D}_{M\lambda'}{}^J(\Omega')^* \times t_c{}^{\lambda'\lambda}(\sigma_-'s_-t(\Omega',\Omega)u_+(\Omega'\Omega)\sigma_-)\mathfrak{D}_{M\lambda}{}^J(\Omega). \quad (4.4)$$

Consequently, when the discontinuity equations for the partial-wave amplitude are calculated, it will not be possible to replace expressions like

$$t_J{}^{\lambda''\lambda'}(\sigma_+''s_+\sigma_+')^*t_J{}^{\lambda''\lambda}(\sigma_+''s_+\sigma_+),$$

by

$$t_J{}^{\lambda'\lambda''}(\sigma_-'s_-\sigma_-'')t_J{}^{\lambda''\lambda}(\sigma_+''s_+\sigma_+).$$

These considerations have been pointed out by Mandelstam *et al.*[5] and more recently by Paton[11] but they have not, heretofore, been applied to the three-pion problem.

Bearing in mind the precautions we have discussed, the derivation of the discontinuity equations for the partial-wave amplitudes is tedious but straightforward. Finally, for the case $J=1$, the negative parity amplitude can be constructed according to

$$t_{1^-}(\sigma's\sigma)=\tfrac{1}{2}[t_1{}^{1,1}(\sigma's\sigma)-t_1{}^{-1,1}(\sigma's\sigma) +t_1{}^{-1,-1}(\sigma's\sigma)-t_1{}^{1,-1}(\sigma's\sigma)] =M(\sigma's\sigma) \quad (4.5)$$

and the resulting discontinuity equations are

$$d_{\sigma'}[M(\sigma_+'s_+\sigma_+)]=2if(\sigma_-')\rho(\sigma')\Big[M(\sigma_+'s_+\sigma_+)+2\int d\sigma''\rho(s,\sigma'')C(\sigma's\sigma'')M(\sigma_+''s_+\sigma_+)\Big], \quad (4.6)$$

$$d_s[M(\sigma_-'s_+\sigma_+)]=\tfrac{2}{3}i\int d\sigma''\rho(s,\sigma'')M(\sigma_+'s_+\sigma_+'')^*M(\sigma_+''s_+\sigma_+)+\tfrac{4}{3}i\int d\sigma''\rho(s,\sigma'')\int d\sigma'''\rho(s,\sigma''') \times M(\sigma_+'s_+\sigma_+'')^*C(\sigma''s\sigma''')M(\sigma_+'''s_+\sigma_+), \quad (4.7)$$

$$d_\sigma[M(\sigma_+'s_+\sigma_+)]=2i\left[M(\sigma_+'s_+\sigma_+)+2\int d\sigma''\rho(s,\sigma'')M(\sigma_+'s_+\sigma_+'')C(\sigma''s\sigma)\right]\rho(\sigma)f(\sigma_-)\,, \tag{4.8}$$

where

$$\rho(s,\sigma)=[Q(s,\sigma)/4s^{1/2}]\rho(\sigma)\,, \tag{4.9}$$

$$Q(s,\sigma)=(1/2s^{1/2})\{[s-(\sigma^{1/2}+\mu_\pi)^2][s-(\sigma^{1/2}-\mu_\pi)^2]\}^{1/2}\,, \tag{4.10}$$

and

$$C(\sigma's\sigma)=\frac{-3}{8\pi}\sum_{\substack{|\lambda'|=1\\|\lambda|=1}}\int d\Omega'd\Omega\,\mathfrak{D}_{M\lambda'}(\Omega')^*(-1)^{\frac{1}{2}(\lambda+\lambda')}C^{1\lambda';1\lambda}(q'p';qp)\delta((q'-p)^2-\mu_\pi^2)\mathfrak{D}_{M\lambda}(\Omega)\,. \tag{4.11}$$

Notice that to obtain Eq. (4.8) from Eq. (3.19) we must first take the complex conjugate of both sides of Eq. (3.19) and then project partial waves. Note also that as a consequence of the definition Eq. (4.5), $M(\sigma's\sigma)$ is symmetric

$$M(\sigma's\sigma)=M(\sigma s\sigma')\,. \tag{4.12}$$

The equations (4.6) and (4.8) which describe final- and initial-state interactions can be simplified somewhat if we factor π-π scattering amplitudes and threshold factors out of M. We write

$$M(\sigma's\sigma)=\frac{Q(s,\sigma')}{P(\sigma')}f(\sigma')F(\sigma's\sigma)f(\sigma)\frac{Q(s,\sigma)}{P(\sigma)}\,, \tag{4.13}$$

and using

$$d_\sigma\left[\frac{Q(s,\sigma)}{P(\sigma)}f(\sigma_+)\right]=\frac{Q(s,\sigma)}{P(\sigma)}d_\sigma[f(\sigma_+)]=2i\frac{Q(s,\sigma)}{P(\sigma)}f(\sigma_+)\rho(\sigma)f(\sigma_-)\,, \tag{4.14}$$

which holds in the physical region $4\mu_\pi^2\leq\sigma\leq(s^{1/2}-\mu_\pi)^2$, we find for $F(\sigma's\sigma)$,

$$d_{\sigma'}[F(\sigma_+'s_+\sigma_+)]=4i\rho(\sigma')\frac{P(\sigma')}{Q(s,\sigma')}\int d\sigma''C(\sigma's\sigma'')\rho(s,\sigma'')\frac{Q(s,\sigma'')}{P(\sigma'')}f(\sigma_+'')F(\sigma_+''s_+\sigma_+)\,, \tag{4.15}$$

$$\begin{aligned}d_s[F(\sigma_-'s_+\sigma_+)]=&\tfrac{2}{3}i\int d\sigma''\rho(s,\sigma'')\frac{Q^2(s,\sigma'')}{P^2(\sigma'')}|f(\sigma'')|^2F(\sigma_+'s_+\sigma_+'')^*F(\sigma_+''s_+\sigma_+)+\tfrac{4}{3}i\int d\sigma''\rho(s,\sigma'')\int d\sigma'''\rho(s,\sigma''')\\&\times F(\sigma_+'s_+\sigma_+'')^*\frac{Q(s,\sigma'')}{P(\sigma'')}f(\sigma_-'')C(\sigma''s\sigma''')f(\sigma_+''')\frac{Q(s,\sigma''')}{P(\sigma''')}F(\sigma_+'''s_+\sigma_+)\,,\end{aligned} \tag{4.16}$$

and

$$d_\sigma[F(\sigma_+'s_+\sigma_+)]=4i\int d\sigma''F(\sigma_+'s_+\sigma_+'')f(\sigma_+'')\frac{Q(s,\sigma'')}{P(\sigma'')}\rho(s,\sigma'')C(\sigma''s\sigma)\frac{P(\sigma)}{Q(s,\sigma)}\rho(\sigma)\,. \tag{4.17}$$

If the recoupling terms had been neglected in Eq. (4.6) and Eq. (4.8) then $F(\sigma's\sigma)$ would have no elastic cut in σ' and σ. The simplification represented by Eqs. (4.15)–(4.17) could have been effected without factoring the threshold behavior. We assume, however, that these factors contain the extra kinematical singularities generated by the partial wave projection, as is the case with two-particle scattering. This notion is supported by the appearance of just these factors in the projection of the one-pion exchange amplitude.

The singularities explicitly represented by Eqs. (4.15)–(4.17) are not the only ones lying in the physical region. The partial-wave projection transforms the O.P.E. pole into two branch cuts in the s plane for fixed σ and σ'.[6] One of these branch cuts lies on the positive real axis in the physical region and extends between $s^{(-)}$ and $s^{(+)}$ where

$$s^{(\pm)}=\frac{\sigma'\sigma+2\mu^4}{2\mu^2}\pm\frac{1}{2\mu^2}\{\sigma'\sigma(\sigma'-4\mu^2)(\sigma-4\mu^2)\}^{1/2}\,. \tag{4.18}$$

This cut corresponds to real pion exchange. The other branch cut lies on the real axis in the unphysical region between $s=0$ and $s=-\infty$ and corresponds to virtual pion exchange. Since the first cut lies in the physical region of the real s axis, why doesn't it contribute to Eq. (4.16)? The answer is that a careful treatment of the partial-wave projection of the O.P.E. amplitude shows that the physical amplitude must be evaluated by approaching the R.P.E. cut from below. Since Eq.

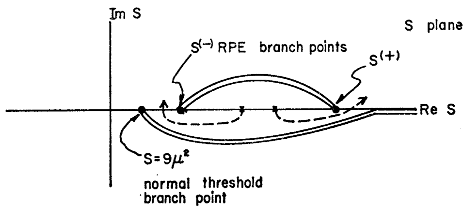

FIG. 4. Distortion of branch cuts to permit continuation from within interval, $S(-)<S<S(+)$.

(4.16) gives the discontinuity upon jumping from above the unitary cut to below it, we receive no contribution from the R.P.E. cut which we are below from the start. The evaluation of the physical amplitude "between" two branch cuts, both of which lie on the real axis, does not violate the assumed analyticity of the amplitude since the branch cuts can be distorted to allow continuations from within the interval defined by Eq. (4.18); see Fig. 4.

To determine the discontinuity in $F(\sigma_-'s_+\sigma_+)$ upon jumping the R.P.E. cut, we refer back to Eqs. (3.9) and (3.16)–(3.18). These yield

$$d_u[t_c^{\lambda'\lambda}(\sigma_-'s_-tu_+\sigma_+)]=12if(\sigma_-')\rho(\sigma')C^{1\lambda';1\lambda}(q'p';qp)\times\delta(u-\mu_\pi^2)\rho(\sigma)f(\sigma_+),\quad(4.19)$$

where $C^{1\lambda';1\lambda}(q'p';qp)$ is given by Eq. (3.8). From Eq. (3.14) and (3.15), we have

$$\begin{aligned}d_u[t_c^{\lambda'\lambda}(\sigma_-'s_+tu_+\sigma_+)]&=d_u[t_c^{-\lambda-\lambda'}(\sigma_+s_+tu_+\sigma_-')]\\&=[t_c^{\lambda\lambda'}(\sigma_-s_-tu_-\sigma_+')-t_c^{\lambda\lambda'}(\sigma_-s_-tu_+\sigma_+')]^*\\&=-d_u[t_c^{\lambda\lambda'}(\sigma_-s_-tu_+\sigma_+')]^*=+12if(\sigma_+)\rho(\sigma)\\&\quad\times C^{1\lambda;1\lambda'}(qp;q'p')^*\delta(u-\mu_\pi^2)\rho(\sigma')f(\sigma_-'),\end{aligned}$$

where Eq. (4.19) was used for the last step. Finally from Eq. (3.8),

$$C^{1\lambda;1\lambda'}(qp;q'p')^*=C^{1\lambda';1\lambda}(q'p';qp),\quad(4.20)$$

so that

$$d_u[t_c^{\lambda'\lambda}(\sigma_-'s_+tu_+\sigma_+)]=12if(\sigma_-')\rho(\sigma')\,C^{1\lambda';1\lambda}(q'p';qp)\delta(u-\mu_\pi^2)\rho(\sigma)f(\sigma_+).\quad(4.21)$$

But the partial-wave projection of the left side of Eq. (4.21) is just what we mean by the discontinuity of $M(\sigma_-s_+\sigma_+)$ across the R.P.E. cut. Hence from Eqs. (4.3) and (4.11) and (4.21) we find

$$M(\sigma_-'s_+\sigma_+)-M(\sigma_-'s_{++}\sigma_+)=12if(\sigma_-')\rho(\sigma')C(\sigma's\sigma)\rho(\sigma)f(\sigma_+),\quad(4.22)$$

where the signature $++$ on the left side of Eq. (4.22) denotes evaluation above both the unitary and R.P.E. cuts. The corresponding discontinuity for $F(\sigma_-'s_+\sigma_+)$ is obtained from Eq. (4.13) as

$$F(\sigma_-'s_+\sigma_+)-F(\sigma_-'s_{++}\sigma_+)=12i\frac{P(\sigma')}{Q(s,\sigma')}\rho(\sigma')C(\sigma's\sigma)\times\rho(\sigma)\frac{P(\sigma)}{Q(s,\sigma)}=2i\gamma(\sigma's\sigma).\quad(4.23)$$

5. N/D EQUATIONS FOR THE RESONANCE APPROXIMATION

If we introduce the "kernel"

$$K(\sigma's\sigma)=\tfrac{1}{3}\delta(\sigma'-\sigma)\rho(s,\sigma)\frac{Q^2(2,\sigma)}{P(\sigma)^2}|f(\sigma)|^2+\tfrac{2}{3}\rho(s,\sigma')\frac{Q(s,\sigma')}{P(\sigma')}f(\sigma')C(\sigma's\sigma)f(\sigma_+)\frac{Q(s,\sigma)}{P(\sigma)}\rho(s,\sigma),\quad(5.1)$$

then Eq. (4.16) becomes

$$d_s[F(\sigma_-'s_+\sigma_+)]=2i\int d\sigma''d\sigma'''F(\sigma_+'s_+\sigma_+'')^*\times K(\sigma''s\sigma''')F(\sigma_+'''s_+\sigma_+).\quad(5.2)$$

Notice that Eq. (5.2) relates the discontinuity in s of $F(\sigma_-'s_+\sigma_+)$ to the function $F(\sigma_+'s_+\sigma_+)$, i.e., to the F function with all signatures positive. Hence, a consequence of the existence of elastic σ and σ' cuts in F is that we cannot apply the N/D method to Eq. (5.2) as it stands. We must replace the F's on the right of Eq. (5.2) with F functions having one σ signature negative. This requires us to solve the integral equations,

$$F(\sigma_+'s_+\sigma_+'')=F(\sigma_+'s_+\sigma_-'')+2\int d\sigma'''F(\sigma_+'s_+\sigma_+''')H(\sigma'''s\sigma'),\quad(5.3)$$

and

$$F(\sigma_+'''s_+\sigma_+)=F(\sigma_-'''s_+\sigma_+)+2\int d\sigma''H(\sigma''s\sigma''')F(\sigma_+''s_+\sigma_+),\quad(5.4)$$

which come from Eqs. (4.17) and (4.15), respectively, and have the kernel,

$$H(\sigma's\sigma)=2if(\sigma_+')\frac{Q(s\sigma')}{P(\sigma')}\rho(s,\sigma')C(\sigma's\sigma)\frac{P(\sigma)}{Q(s,\sigma)}\rho(\sigma).\quad(5.5)$$

In this paper we will not attempt an exact solution of these difficult equations but instead introduce the resonance approximation. This approximation is effected by writing in Eq. (5.3)

$$\int d\sigma'''F(\sigma_+'s_+\sigma_+''')H(\sigma'''s\sigma'')=F(\sigma_+'s_+m_{\rho+}^2)\int d\sigma'''H(\sigma'''s\sigma'')\quad(5.6)$$

and correspondingly for Eq. (5.4). One argues that the resonance behavior of $f(\sigma_+'')$ results in the interval around $\sigma''\sim m_\rho^2$ dominating the integrand and in this interval one hopes the variation of F to be unimportant.

The solutions of Eqs. (5.3) and (5.4) become

$$F(\sigma_+{}'s_+\sigma_+{}'')=F(\sigma_+{}'s_+\sigma_-{}'')+2\alpha(s,\sigma'')\frac{F(\sigma_+{}'s_+m_{\rho-}{}^2)}{1-2\alpha(s,m_\rho{}^2)}, \quad (5.7)$$

$$F(\sigma_+{}'''s_+\sigma_+)=F(\sigma_-{}'''s_+\sigma_+)+2\alpha(s,\sigma''')\frac{F(m_{\rho-}{}^2s_+\sigma_+)}{1-2\alpha(s,m_\rho{}^2)}, \quad (5.8)$$

where

$$\alpha(s,\sigma)=\int d\sigma'''H(\sigma'''s\sigma). \quad (5.9)$$

Upon substitution into Eq. (5.2) with yet another application of the resonance approximation we obtain

$$d_s[F(\sigma_-{}'s_+\sigma_+)]=2iF(\sigma_+{}'s_+m_{\rho-}{}^2)^*\times R(s)F(m_{\rho-}{}^2s_+\sigma_+), \quad (5.10)$$

where

$$R(s)=\frac{\int d\sigma''\int d\sigma'''K(\sigma''s\sigma''')}{|1-2\alpha(s,m_\rho{}^2)|^2} \quad (5.11)$$

is real as a consequence of the Hermiticity of K. Finally, to apply the N/D method we must eliminate the operation of complex conjugation appearing on the right-hand side of Eq. (5.10).[18] To do this we write first

$$M(\sigma_+{}'s_+\sigma_-)=-(3/4\pi)\sum_{|\lambda'|,|\lambda|=1}(-1)^{\frac{1}{2}(\lambda+\lambda')}\times\int d\Omega'd\Omega\mathfrak{D}_{M\lambda'}{}'(\Omega')^*t_c{}^{\lambda'\lambda}(\sigma_+{}'s_+t(\Omega',\Omega)u_+(\Omega',\Omega)\sigma_-)\times\mathfrak{D}_{M\lambda}{}'(\Omega), \quad (5.12)$$

which follows from Eqs. (4.3) and (4.5). Next, using Eq. (3.13) and properties of the rotation matrices, we find

$$M(\sigma_+{}'s_+\sigma_-)^*=-(3/4\pi)\sum_{|\lambda'|,|\lambda|=1}(-1)^{\frac{1}{2}(\lambda+\lambda')}\times\int d\Omega'd\Omega\mathfrak{D}_{M\lambda'}{}'(\Omega')^*t_c{}^{\lambda'\lambda}(\sigma_-{}'s_-t(\Omega'\Omega)u_-(\Omega'\Omega)\sigma_+)\times\mathfrak{D}_{M\lambda}{}'(\Omega). \quad (5.13)$$

But

$$t_c{}^{\lambda'\lambda}(\sigma_-{}'s_-tu_-\sigma_+)=t_c{}^{\lambda'\lambda}(\sigma_-{}'s_-tu_+\sigma_+)-d_u[t_c{}^{\lambda'\lambda}(\sigma_-{}'s_-tu_+\sigma_+)],$$

and from Eqs. (4.19) and (4.21)

$$d_u[t_c{}^{\lambda'\lambda}(\sigma_-{}'s_-tu_+\sigma_+)]=d_u[t_c{}^{\lambda'\lambda}(\sigma_-{}'s_+tu_+\sigma_+)].$$

Therefore, making the appropriate substitutions in Eq. (5.13), we find

$$M(\sigma_+{}'s_+\sigma_-)^*=M(\sigma_-{}'s_-\sigma_+)-[M(\sigma_-{}'s_{++}\sigma_+)-M(\sigma_-{}'s_+\sigma_+)]. \quad (5.14)$$

From Eqs. (4.13) and (4.23) we then find

$$F(\sigma_+{}'s_+\sigma_-)^*=F(\sigma_-{}'s_-\sigma_+)+2i\gamma(\sigma's\sigma). \quad (5.15)$$

This enables us to replace Eq. (5.10) by

$$d_s[F(\sigma_-{}'s_+\sigma_+)]=2iF(\sigma_-{}'s_-m_{\rho+}{}^2)R(s)F(m_{\rho-}{}^2s_+\sigma_+)-4\gamma(\sigma'sm_\rho{}^2)R(s)F(m_{\rho-}{}^2s_+\sigma_+), \quad (5.16)$$

which is susceptible to the N/D method.

Let $D(\sigma_-{}',s_+)$ be an analytic function of σ' and s possessing as its only singularity in s a branch point at $s=9\mu_\pi{}^2$ with the branch cut running along the positive real axis. We leave open, for the time being, the dependence on σ'. Defining $N(\sigma's\sigma)$ by

$$F(\sigma_-{}'s_+\sigma_+)-D(\sigma_-{}'s_+)F(m_{\rho-}{}^2s_+\sigma_+)=N(\sigma_-{}'s_+\sigma_+), \quad (5.17)$$

we will show that it is possible to choose the discontinuity of $D(\sigma',s)$ across the s cut so that

$$d_s[N(\sigma_-{}'s_+\sigma_+)]=0. \quad (5.18)$$

Bear in mind that we are talking about the cut in D or N arising from the unitary cut in F, i.e., from Eq. (5.16), and we cannot conclude that N has no s cuts in the physical region. Indeed, since we desire D to carry only the unitary cut, it follows that N must contain the R.P.E. cut, as well as all unphysical singularities. Thus in the physical region we will have to supplement Eq. (5.18) by

$$N(\sigma_-{}'s_{++}\sigma_+)-N(\sigma_-{}'s_+\sigma_+)\neq 0, \quad (5.19a)$$

$$D(\sigma_-{}'s_{++}\sigma_+)-D(\sigma_-{}'s_+\sigma_+)=0. \quad (5.19b)$$

The solution of Eq. (5.17) is

$$F(\sigma_-{}'s_+\sigma_+)=N(\sigma_-{}'s_+\sigma_+)+D(\sigma_-{}'s_+)N(m_{\rho-}{}^2s_+\sigma_+)/[1-D(m_{\rho-}{}^2s_+)]. \quad (5.20)$$

Calculating the discontinuity across the unitary cut on both sides of Eq. (5.17), substituting Eq. (5.16) on the left and Eq. (5.18) on the right and finally factoring out an over-all factor of $F(m_{\rho-}{}^2s_+\sigma_+)$ we get

$$d_s[D(\sigma_-{}'s_+)]=2iR(s)N(\sigma_-{}'s_+m_{\rho+}{}^2)-4R(s)[\gamma(\sigma'sm_\rho{}^2)-D(\sigma_-{}'s_-)\gamma(m_\rho{}^2sm_\rho{}^2)]. \quad (5.21)$$

For the discontinuity across the R.P.E. cut, we find

$$N(\sigma_-{}'s_{++}\sigma_+)-N(\sigma_-{}'s_+\sigma_+)=-[2i\gamma(\sigma's\sigma)-2iD(\sigma_-{}'s_+)\gamma(m_\rho s\sigma)] \quad (5.22)$$

and a similar equation holds for all unphysical singularities. Equation (5.21) can be simplified by introducing a function, $g(s)$, such that

$$d_s[g(s_+)]=4R(s)g(s_-)\gamma(m_\rho{}^2sm_\rho{}^2). \quad (5.23)$$

Defining

$$h(\sigma_-{}'s_+)=D(\sigma_-{}'s_+)/g(s_+), \quad (5.24)$$

[18] In the simpler two-body scattering problems, if it were not for the reflection property, $f(s_+)^*=f(s_-)$, of the partial-wave amplitudes, the N/D method could not be applied to the unitarity equation

$$2i\,\mathrm{Im}f(s_+)=2i\rho(s)|f(s_+)|^2.$$

we get from Eqs. (5.21), (5.23)

$$d_s[h(\sigma_-'s_+)]=2i\frac{R(s)}{g(s_+)}[N(\sigma_-'s_+m_{\rho+}^2)+2i\gamma(\sigma'sm_\rho^2)]. \quad (5.25)$$

Now that we have separated the unitary cut and the R.P.E. cut by placing them in different functions, we can display the sense in which $F(\sigma_-'s_+\sigma_+)$ is evaluated "between" two colinear cuts in a very picturesque manner. We simply note that

$$D(\sigma_-'s_+)=\lim_{\epsilon\to 0+} D(\sigma', s+i\epsilon),$$

while

$$N(\sigma_-'s_+\sigma_+)=\lim_{\epsilon\to 0+} N(\sigma', s-i\epsilon, \sigma_+). \quad (5.26)$$

Hence restricting ourselves to functions defined only on the physical sheet it is *not* possible to write

$$F(\sigma_-'s_+\sigma_+)=\lim_{\epsilon\to 0+} F(\sigma_-', s+i\epsilon, \sigma_+).$$

This does not contradict the analytic character of F. It simply says that F is not everywhere the boundary value of a continuation of F into the physical sheet. In the vicinity of the R.P.E. cut we must first cross over the real axis into the second sheet, bypass the first R.P.E. branch point, and then approach from below the real axis and the boundary of the physical sheet.

In the sequel to this paper we will look for the ω-meson resonance as a consequence of one- and two-pion exchange forces. It follows from Eq. (5.20) that we need only concern ourselves with $D(m_{\rho-}^2s_+)$ for that problem. Hence, as we would expect from an exact treatment[19] the position and width of the ω will not depend on σ' or σ.

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[19] R. Blankenbecler, Phys. Rev. **122**, 983 (1961); see also the reference of footnote 9.

APPENDIX: EFFECTIVE PHASE SPACE IN THE NARROW WIDTH LIMIT

If in Eqs. (5.1) and (5.9) we make the substitutions

$$f(\sigma_\pm)=\frac{\gamma P^2(\sigma)}{m_\rho^2-\sigma-i\gamma[P^3(\sigma)/4\sigma^{1/2}]} \quad (A1)$$

and

$$\frac{\gamma P^3/4\sigma^{1/2}}{(m_\rho^2-\sigma)^2+\gamma^2P^6/16\sigma}\sim\pi\delta(m_\rho^2-\sigma) \quad (A2)$$

then $R(s)$ is easy to calculate. This approximation is more justified the narrower the width of the ρ meson, i.e., the smaller γ is. Hence, if recoupling effects are

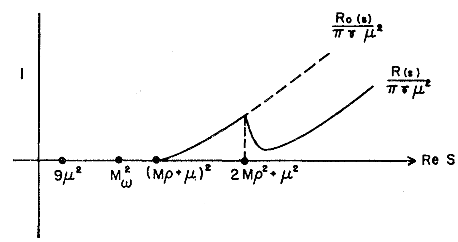

FIG. 5. ρ-π phase space with and without recoupling.

negligible for the case of ρ-π scattering,[20] this calculation should yield an effective phase space, $R(s)$, which deviates only slightly from the result of neglecting recoupling

$$R_0(s)=3\pi\gamma Q^3(s,m_\rho^2)/4s^{1/2}. \quad (A3)$$

The calculated result is shown in Fig. 5.

[20] Dr. E. Abers, California Institute of Technology, has informed me that an attempt to treat real pion exchange in ρ-π scattering while treating the ρ meson as a stable particle (no recoupling) violates unitarity.