exhibits the ϕ correlations, and ϕ dependence in each c.m. separately, which no SZE or MSZE process can produce.

We may conclude that an analysis of polarization correlations can distinguish three cases:

(1) Exchange of one spin-zero meson (SZE): No correlations of C and D polarizations are possible.

(2) Multiple spin-zero meson exchange (MSZE): There may be correlations between components of polarization along the momentum transfer direction in the C c.m. with components of polarization along the momentum transfer direction in the D c.m.

(3) Higher spin exchange: Violations of the two SZE criteria of Treiman and Yang, as well as correlations of the Treiman-Yang angles of directions in the two c.m.'s, may occur.

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Vector Harmonics for Three Identical Fermions*

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Orthonormal vector harmonics for the three-nucleon system are presented.

INTRODUCTION

THE wave function of a system consisting of three nucleons depends on the coordinates τ_i, σ_i, $\mathbf{r}_i$, $i=1, 2, 3$, where τ_i is the two-valued isospin coordinate, σ_i is the two-valued spin coordinate, and the space coordinate $\mathbf{r}_i$ ranges over three-dimensional Euclidian space. The possible wave functions $\psi(\tau_i,\sigma_i,\mathbf{r}_i)$ are classified according to their transformation properties under translations, rotations, and reflections of the coordinate system and under permutations of the particle coordinates. The functions that belong to a definite representation $\mathbf{K}$ of the translation group are

$$\psi_{\mathbf{K}}=\exp(i\mathbf{K}\cdot\mathbf{R})\psi(\boldsymbol{\lambda},\boldsymbol{\varrho},\sigma_i,\tau_i)\,,\quad i=1,\,2,\,3 \tag{1}$$

where

$$\begin{aligned}\mathbf{R}&=\tfrac{1}{3}(\mathbf{r}_1+\mathbf{r}_2+\mathbf{r}_3)\,,\\ \boldsymbol{\lambda}&=6^{-1/2}(\mathbf{r}_1+\mathbf{r}_2-2\mathbf{r}_3)\,,\\ \boldsymbol{\varrho}&=2^{-1/2}(\mathbf{r}_1-\mathbf{r}_2)\,.\end{aligned} \tag{2}$$

The vectors $\boldsymbol{\lambda}$ and $\boldsymbol{\varrho}$ are invariant under translations. The internal wave functions ψ are chosen to have definite parity and to belong to definite irreducible representations of the quantum-mechanical rotation group SU_2, and the permutation group $S_3^{\tau\sigma r}$ where the Pauli principle specifies that ψ must belong to the antisymmetric representation of $S_3^{\tau\sigma r}$. The problem that will be considered here is the construction and parametrization of the functions ψ.

A function ψ that belongs to the antisymmetric representation of $S_3^{\tau\sigma r}$ can be split into parts that belong to definite irreducible representations of S_3^{τ}, S_3^{σ}, and S_3^{r} separately, where S_3^{τ}, S_3^{σ}, and S_3^{r} are the groups consisting of permutations of isospin, spin, and space coordinates only, respectively. Since these groups do not leave the Hamiltonian invariant, the three-nucleon wave function cannot belong to a single irreducible representation of one of these groups, but must be a linear combination of functions, each of which belongs to a single irreducible representation of each of the groups S_3^{τ}, S_3^{σ}, S_3^{r}. Since the irreducible representations of S_3^{σ} have definite total spin, the rotational classification is completed by requiring that the space part of the function have definite orbital angular momentum L and definite parity, besides belonging to a definite irreducible representation of S_3^{r}.

THE GROUP S_3 AND ITS REPRESENTATIONS

The group S_3 has three irreducible representations: a one-dimensional symmetric representation R_S, a one-dimensional antisymmetric representation R_A, and a two-dimensional mixed representation R_M. If φ belongs to R_S, then $P^i\varphi=\varphi$, where P^i is any permutation in S_3. Similarly, if φ belongs to R_A, then $P^i\varphi=\epsilon_i\varphi$ where $\epsilon_i=\pm1$ is the sign of the permutation P^i. If φ_1

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and φ_2 belong to R_M, then $P^i\varphi_1=P_{11}{}^i\varphi_1+P_{21}{}^i\varphi_2$ and $P^i\varphi_2=P_{12}{}^i\varphi_1+P_{22}{}^i\varphi_2$. By choosing a linear combination of φ_1 and φ_2, the matrices $P_{rs}{}^i$ can be put into a standard form. Since S_3 is generated by $P^{(12)}$ and $P^{(123)}$, it is sufficient to specify standard forms for $P_{rs}{}^{(12)}$ and $P_{rs}{}^{(123)}$ in the mixed representation. These are taken to be

$$P^{(12)}=\begin{pmatrix}1 & 0\\ 0 & -1\end{pmatrix},\quad P^{(123)}=\begin{pmatrix}-\frac{1}{2} & \frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & -\frac{1}{2}\end{pmatrix}. \tag{3}$$

Then, for example, the standard form for $P^{(13)}$ is

$$P^{(13)}=P^{(123)}P^{(12)}=\begin{pmatrix}-\frac{1}{2} & -\frac{1}{2}\sqrt{3}\\ -\frac{1}{2}\sqrt{3} & \frac{1}{2}\end{pmatrix}.$$

Here the permutation $P^{(123)}$ is understood to mean: where $\mathbf{r}_1$ appears in φ, replace it by $\mathbf{r}_2$: where $\mathbf{r}_2$ appears, replace it by $\mathbf{r}_3$, etc.

As an example, consider $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$:

$$\begin{aligned}
P^{(12)}\boldsymbol{\lambda}&=6^{-1/2}(\mathbf{r}_2+\mathbf{r}_1-2\mathbf{r}_3)=\boldsymbol{\lambda}\,,\\
P^{(12)}\boldsymbol{\rho}&=2^{-1/2}(\mathbf{r}_2-\mathbf{r}_1)=-\boldsymbol{\rho}\,,\\
P^{(123)}\boldsymbol{\lambda}&=6^{-1/2}(\mathbf{r}_2+\mathbf{r}_3-2\mathbf{r}_1)=-\tfrac{1}{2}\boldsymbol{\lambda}-\tfrac{1}{2}\sqrt{3}\boldsymbol{\rho}\,,\\
P^{(123)}\boldsymbol{\rho}&=2^{-1/2}(\mathbf{r}_2-\mathbf{r}_1)=\tfrac{1}{2}\sqrt{3}\boldsymbol{\lambda}-\tfrac{1}{2}\boldsymbol{\rho}\,.
\end{aligned}$$

Hence for $\varphi_1=\boldsymbol{\lambda}$, $\varphi_2=\boldsymbol{\rho}$, $P^{(12)}$ and $P^{(123)}$ take the forms (3), and $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ are the correct linear combinations to give the standard representation. The notation $R_M(\boldsymbol{\lambda},\boldsymbol{\rho})$ is used to indicate that $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ generate R_M in standard form. In general, if χ_1 and χ_2 are known to belong to R_M, then it is necessary to find linear combinations of χ_1 and χ_2 that are even and odd under $P^{(12)}$. One off-diagonal matrix element of $P^{(123)}$ is needed to determine the correct relative factor.

Table I gives the rules for forming the direct product of two irreducible representations of S_3.

SPIN AND ISOSPIN FUNCTIONS

The classification of isospin and spin functions according to irreducible representations of $S_3{}^\tau$ and $S_3{}^\sigma$ is well known. Since definite transformation properties under SU_2 are required, vector-coupling coefficients are used in forming products:

$$\{s(1),s(2)\}_M{}^S=\textstyle\sum_m C_m{}^{1/2}\;{}_{M-m}{}^{1/2}{}^S s_m(\sigma_1)s_{M-m}(\sigma_2)\,, \tag{4}$$

etc., where the left-hand side of (4) will be written simply $\{s(1),s(2)\}^S$, since the M value is immaterial.

The three-particle isospin functions are

$$\begin{aligned}
T^{3/2}&=\{\{t(1),t(2)\}^1t(3)\}^{3/2}\,,\\
T_1{}^{1/2}&=\{\{t(1),t(2)\}^1t(3)\}^{1/2}\,,\\
T_2{}^{1/2}&=\{\{t(1),t(2)\}^0t(3)\}^{1/2}\,.
\end{aligned} \tag{5}$$

The function $T^{3/2}$ belongs to R_S of S_3, while $T_1{}^{1/2}$ and $T_2{}^{1/2}$ belong to R_M in standard form. The spin functions $\Sigma^{3/2}$, $\Sigma_1{}^{1/2}$, and $\Sigma_2{}^{1/2}$ are defined analogously. The

Table I. Products of representations of S_3.

$R_S(\varphi)\otimes R_S(\chi)=R_S(\varphi\chi)$
$R_S(\varphi)\otimes R_M(\chi_1,\chi_2)=R_M(\varphi\chi_1,\varphi\chi_2)$
$R_S(\varphi)\otimes R_A(\chi)=R_A(\varphi\chi)$
$R_M(\varphi_1,\varphi_2)\otimes R_M(\chi_1,\chi_2)=R_S(\varphi_1\chi_1+\varphi_2\chi_2)$ $\oplus R_M(\varphi_2\chi_2-\varphi_1\chi_1,\ \varphi_1\chi_2+\varphi_2\chi_1)$ $\oplus R_A(\varphi_1\chi_2-\varphi_2\chi_1)$
$R_A(\varphi)\otimes R_M(\chi_1,\chi_2)=R_M(\varphi\chi_2,\ -\varphi\chi_1)$
$R_A(\varphi)R_A\otimes(\chi)=R_S(\varphi\chi)$

isospin-spin $(TS)_R$ functions can now be constructed with the aid of Table I:

$$\begin{aligned}
(\tfrac{3}{2},\tfrac{3}{2})_S&=T^{3/2}\Sigma^{3/2}\,,\\
(\tfrac{1}{2},\tfrac{1}{2})_S&=2^{-1/2}(T_1{}^{1/2}\Sigma_1{}^{1/2}+T_2{}^{1/2}\Sigma_2{}^{1/2})\,,\\
(\tfrac{3}{2},\tfrac{1}{2})_{M1,2}&=(T^{3/2}\Sigma_1{}^{1/2},T^{3/2}\Sigma_2{}^{1/2})\,,\\
(\tfrac{1}{2},\tfrac{3}{2})_{M1,2}&=(T_1{}^{1/2}\Sigma^{3/2},T_2{}^{1/2}\Sigma^{3/2})\,,\\
(\tfrac{1}{2},\tfrac{1}{2})_{M1,2}&=(2^{-1/2}[T_2{}^{1/2}\Sigma_2{}^{1/2}-T_1{}^{1/2}\Sigma_1{}^{1/2}]\,,\\
&\qquad 2^{-1/2}[T_1{}^{1/2}\Sigma_2{}^{1/2}+T_2{}^{1/2}\Sigma_1{}^{1/2}])\,,\\
(\tfrac{1}{2},\tfrac{1}{2})_A&=2^{-1/2}(T_1{}^{1/2}\Sigma_2{}^{1/2}-T_2{}^{1/2}\Sigma_1{}^{1/2})\,.
\end{aligned} \tag{6}$$

The subscripts S, M, A indicate the representation of $S_3{}^{\tau\sigma}$ to which the isospin-spin functions belong. With the factors $2^{-1/2}$ as shown, the functions are orthonormal.

Thus, if space function(s) $f_R{}^L(\boldsymbol{\rho},\boldsymbol{\lambda})$ is given, construction of possible antisymmetric functions $\psi_R{}^{TSLJ}$ with space part $f_R{}^L$ follows from Table I and is given in Table II. In Table II, the vector coupling is between the spin part of $(TS)_R$ and the space functions $f_{RM_L}{}^L$ making up $f_R{}^L$.

SPACE FUNCTIONS

From these preliminaries, it is clear that what is required now is a classification of functions $f_R{}^L(\boldsymbol{\lambda},\boldsymbol{\rho})$ according to irreducible representations of the rotation group O_3 (specified by L) and $S_3{}^r$ (specified by subscript S, M, A). The materials available for these functions are the three vectors $\boldsymbol{\lambda}$, $\boldsymbol{\rho}$, and $\boldsymbol{\lambda}\times\boldsymbol{\rho}$. Hence there are three independent scalars λ^2, ρ^2, and $\boldsymbol{\lambda}\cdot\boldsymbol{\rho}$. Any $L=0$ function must be a function of these three scalars. It follows immediately that there are no odd-parity $L=0$ functions.

Table II. Wave functions associated with space functions belonging to a definite representation of $S_3{}^r$.

Space functions	Wave functions
$R_S(f_S{}^L)$	$\psi_S{}^{\frac{1}{2}\,\frac{1}{2}LJ}=\{(\frac{1}{2},\frac{1}{2})_A,f_S{}^L\}^J$
$R_M(f_1{}^L,f_2{}^L)$	$\psi_M{}^{\frac{3}{2}\,\frac{1}{2}LJ}=\{(\frac{3}{2},\frac{1}{2})_{M2},\ f_1{}^L-(\frac{3}{2},\frac{1}{2})_{M1},\ f_2{}^L\}^J$
	$\psi_M{}^{\frac{1}{2}\,\frac{3}{2}LJ}=\{(\frac{1}{2},\frac{3}{2})_{M2},\ f_1{}^L-(\frac{1}{2},\frac{3}{2})_{M1},\ f_2{}^L\}^J$
	$\psi_M{}^{\frac{1}{2}\,\frac{1}{2}LJ}=\{(\frac{1}{2},\frac{1}{2})_{M2},\ f_1{}^L-(\frac{1}{2},\frac{1}{2})_{M1},\ f_2{}^L\}^J$
$R_A(f_A{}^L)$	$\psi_A{}^{\frac{3}{2}\,\frac{3}{2}LJ}=\{(\frac{3}{2},\frac{3}{2})_S,f_A{}^L\}^J$
	$\psi_A{}^{\frac{1}{2}\,\frac{1}{2}LJ}=\{(\frac{1}{2}\ \frac{1}{2})_S,f_A{}^L\}^J$

The $L=0$ even-parity functions will now be classified according to irreducible representations of S_3. Since $(\boldsymbol{\lambda},\boldsymbol{\rho})$ belong to R_M in standard form, Table I gives for the bilinear scalars:

$$R_M(\boldsymbol{\lambda},\boldsymbol{\rho})\otimes R_M(\boldsymbol{\lambda},\boldsymbol{\rho}) = R_S(\lambda^2+\rho^2)\oplus R_M(\rho^2-\lambda^2,\, 2\boldsymbol{\lambda}\cdot\boldsymbol{\rho}). \quad (7)$$

It is now time to note the special role played by symmetric scalars (space functions with $L=0$ belonging to R_S). If a function $\psi_R{}^{TSLJ}$ is multiplied by any function of symmetric scalars, the resulting product has the same quantum numbers $TSLJR$. It is therefore convenient to consider only "S-independent" space functions: The functions $f_1, f_2, \cdots, f_N$ are "S-independent" if and only if

$$\sum_{i=0}^{N} g_i(s_1,s_2,\cdots)f_i=0 \quad (8)$$

implies $g_i=0$ for all i, where the $g_i(s_1,s_2\cdots)$ are functions of the symmetric scalars $s_1, s_2\cdots$. All space functions with definite L can be generated from the S-independent space functions with that value of L and the symmetric scalars. The value of this procedure is due to the fact that, as will be seen, the number of independent symmetric scalars is finite and equal to three, and the number of S-independent space functions with a given L is also finite.[1]

It follows that the symmetric scalar $r=\lambda^2+\rho^2$ may be ignored in constructing further $L=0$ functions. Because of the special role played by $\rho^2-\lambda^2$ and $2\boldsymbol{\lambda}\cdot\boldsymbol{\rho}$, it is convenient to introduce s and φ by

$$\begin{aligned} \rho^2-\lambda^2 &= rs\cos\varphi, & 0\leq s\leq 1 \\ 2\boldsymbol{\lambda}\cdot\boldsymbol{\rho} &= rs\sin\varphi, & -\pi\leq\varphi\leq\pi \end{aligned} \quad (9)$$

so that the bilinear mixed representation is S-equivalent to R_M $(s\cos\varphi,\ s\sin\varphi)$. Further irreducible representations can only be generated by the direct product of this mixed representation with itself, which gives (Table I)

$$R_S(s^2)\oplus R_M(-s^2\cos2\varphi,\, s^2\sin2\varphi), \quad (10)$$

so that it is clear that s is a symmetric scalar and $R_M(\cos\varphi,\sin\varphi)$ and $R_M'(\cos2\varphi,-\sin2\varphi)$ are mixed representations in standard form. Their product is

$$R_M\otimes R_M' = R_S(\cos3\varphi) \oplus R_M(-\cos\varphi,-\sin\varphi)\oplus R_A(-\sin3\varphi). \quad (11)$$

The independent symmetric scalars thus far are r, s, and $\cos3\varphi$, while the S-independent scalars are $R_S(1)$, $R_M(\cos\varphi,\sin\varphi)$, $R_M'(\cos2\varphi,-\sin2\varphi)$, and $R_A(\sin3\varphi)$. Further,

$$R_M'\otimes R_M' = R_S(1)\oplus R_M''(-\cos4\varphi,-\sin4\varphi), \quad (12)$$

but

$$R_M''(\cos4\varphi,\sin4\varphi) = 2\cos3\varphi R_M \ominus R_M' \quad (13)$$

and is therefore not S-independent of R_M and R_M'. Similarly,

$$\begin{aligned} R_A\otimes R_M &= R_M{}^{(4)}(\sin3\varphi\sin\varphi,\, -\sin3\varphi\cos\varphi) \\ &= -\cos3\varphi R_M\oplus R_M', \quad (14) \\ R_A\otimes R_M' &= R_M{}^{(4)}(\sin3\varphi\sin2\varphi,\, -\sin3\varphi\cos2\varphi) \\ &= \cos3\varphi R_M'\ominus R_M, \end{aligned}$$

and therefore these products give rise to no further S-independent scalars.

Thus, there are in all three independent symmetric scalars: r, s, and $\cos3\varphi$; one antisymmetric scalar: $\sin3\varphi$; and two S-independent mixed scalars, which will be taken to be $R_M(\cos\varphi,\sin\varphi)$ and $R_M(\sin3\varphi\sin\varphi, -\sin3\varphi\cos\varphi)$. It is convenient to divide $\sin3\varphi$ by the symmetric scalar $(1-\cos^2 3\varphi)^{1/2}$, so that the final notation for the S-independent scalars is

$$\begin{aligned} S_S &= 1, \\ S_M &= (\cos\varphi,\sin\varphi), \\ S_N &= \frac{\sin3\varphi}{(1-\cos^2 3\varphi)^{1/2}}(\sin\varphi,-\cos\varphi), \quad (15) \\ S_A &= \sin3\varphi/(1-\cos^2 3\varphi)^{1/2}. \end{aligned}$$

The most general symmetric scalar is an arbitrary function of r, s, and $\cos3\varphi$, or, equivalently, a function $g(r,s,\varphi)$ that is an arbitrary function of r and s, and is an even function of φ with period $2\pi/3$. The notation $g(r,s,\varphi)$ will be used exclusively for such symmetric scalars.

In order to construct functions with $L>0$, it is convenient to replace $\boldsymbol{\lambda}$ and $\boldsymbol{\rho}$ first by the vectors

$$\begin{aligned} \mathbf{r}_e &= \boldsymbol{\lambda}\cos\varphi+\boldsymbol{\rho}\sin\varphi, \\ \mathbf{r}_o &= \boldsymbol{\lambda}\sin\varphi-\boldsymbol{\rho}\cos\varphi, \end{aligned} \quad (16)$$

belonging to R_S and R_A of S_3, respectively. The representation $R_M(\boldsymbol{\lambda},\boldsymbol{\rho})$ can be obtained from (16) and (15):

$$R_M(\boldsymbol{\lambda},\boldsymbol{\rho}) = [R_S(\mathbf{r}_e)\otimes S_M]_M \oplus [R_A(\mathbf{r}_o)\otimes S_M]_M. \quad (17)$$

Then $\mathbf{r}_e$ and $\mathbf{r}_o$ are replaced by the equivalent vectors

$$\begin{aligned} \mathbf{R}_1 &= \frac{\mathbf{r}_e(1+\cos3\varphi)+\mathbf{r}_o\sin3\varphi}{[r(1-s)(1+\cos3\varphi)]^{1/2}}, \\ \mathbf{R}_2 &= \frac{\mathbf{r}_e(1-\cos3\varphi)-\mathbf{r}_o\sin3\varphi}{[r(1+s)(1-\cos3\varphi)]^{1/2}}, \end{aligned} \quad (18)$$

which both belong to R_S of S_3 and satisfy

$$R_1{}^2 = R_2{}^2 = 1, \quad \mathbf{R}_1\cdot\mathbf{R}_2 = 0. \quad (19)$$

Since $\mathbf{r}_e$ is S equivalent to a linear combination of $\mathbf{R}_1$ and $\mathbf{R}_2$ and $\mathbf{r}_o$ is S equivalent to a linear combination

[1] E. Jezak, Ph.D. thesis, University of Minnesota, 1962 (unpublished).

of $S_A\otimes\mathbf{R}_1$ and $S_A\otimes\mathbf{R}_2$, it follows that the most general space functions can be formed by using $\mathbf{R}_1$ and $\mathbf{R}_2$ to form space functions belonging to R_S of S_3 and then using the functions in (15) to generate one symmetric representation, one antisymmetric representation and two mixed representations from each of the symmetric space functions. For example, the odd-parity $L=1$ space functions are generated by $\mathbf{R}_1$ and $\mathbf{R}_2$:

$$\begin{aligned}&R_S(\mathbf{R}_1\otimes S_S)\,,\quad R_S(\mathbf{R}_2\otimes S_S)\,,\\&R_A(\mathbf{R}_1\otimes S_A)\,,\quad R_A(\mathbf{R}_2\otimes S_A)\,,\\&R_M(\mathbf{R}_1\otimes S_M)\,,\quad R_M(\mathbf{R}_1\otimes S_N)\,,\\&R_M(\mathbf{R}_2\otimes S_M)\,,\quad R_M(\mathbf{R}_2\otimes S_N)\,.\end{aligned}\tag{20}$$

All $L=1$ odd-parity functions are S-dependent on the S-independent functions (20).

The only symmetric even parity $L=1$ function is

$$\mathbf{R}_3=\mathbf{R}_1\times\mathbf{R}_2=\frac{\sin3\varphi}{(1-\cos^2 3\varphi)^{1/2}}\cdot\frac{2\boldsymbol{\lambda}\times\boldsymbol{\varrho}}{r(1-s^2)^{1/2}}\,,\tag{21}$$

$$(\mathbf{R}_3)_m{}^1=2^{1/2}\{\mathbf{R}_1,\mathbf{R}_2\}_m{}^1\,.$$

When a vector is to be vector-coupled, its components are understood to be

$$\begin{aligned}R_0&=-iR_z\,,\\R_{\pm1}&=2^{-1/2}(\pm iR_x-R_y)\,.\end{aligned}\tag{22}$$

The choices give

$$R_m{}^*=(-)^{1-m}R_{-m}\,,\tag{23}$$

for $\mathbf{R}_1$, $\mathbf{R}_2$, and $\mathbf{R}_3$.

The independent 2+ functions are

$$\{\mathbf{R}_1,\mathbf{R}_1\}^2,\quad\{\mathbf{R}_1,\mathbf{R}_2\}^2,\quad\text{and}\quad\{\mathbf{R}_2,\mathbf{R}_2\}^2\tag{24}$$

and the 2− functions are

$$\{\mathbf{R}_1,\mathbf{R}_3\}^2\quad\text{and}\quad\{\mathbf{R}_2,\mathbf{R}_3\}^2\,.\tag{25}$$

In general, for $\pi=(-)^L$, there are $L+1$-independent $L\pi$ functions:

$$f_n{}^{EL}=\{\cdots\{\{\mathbf{R}_1,\mathbf{R}_1\}^2,\mathbf{R}_1\}^3\cdots\mathbf{R}_1\}^n,\ \mathbf{R}_2\}^{n+1},\mathbf{R}_2\}^{n+2}\cdots\mathbf{R}_2\}^L\,,\tag{26}$$

with $n=0,\,1,\,\cdots L$. For $\pi=(-)^{L+1}$, there are L-independent functions:

$$f_i{}^{ML}=\{f_i{}^{EL-1},\mathbf{R}_3\}^L\quad i=0,\,1,\,\cdots L-1\,.\tag{27}$$

The choices (26) and (27) give

$$f_m{}^{L*}=(-)^{L-m}f_{-m}{}^L\tag{28}$$

for all functions.

It follows from Table II and the preceding that a wave function $\psi_{R\gamma}{}^{TSLJ}$ is constructed from a symmetric space function $f_\gamma{}^L$ according to

$$\begin{aligned}\psi_{R\gamma}{}^{TSLJ}&=\{(TS)_{R'}\otimes,f_\gamma{}^LS_R\}_A{}^J\\&=\{((TS)_{R'}\otimes S_R)_A,\,f_\gamma{}^L\}^J\\&=(\{(TS)_{R'},f_\gamma{}^L\}^J\otimes S_R)_A\,,\end{aligned}\tag{29}$$

where R' is the representation adjoint to R, namely, $R_S'=R_A$, $R_A'=R_S$, $R_M'=R_N'=R_M$.

ORTHONORMALITY

Consider

$$\sum_{t_i\sigma_i}\int d\boldsymbol{\varrho}d\boldsymbol{\lambda}g_{R_2\gamma'}{}^{T'S'L'J'\pi'*}\psi_{R_2\gamma'M_{T'}M_{J'}}{}^{T'S'L'J'\pi'*}g_{R_1\gamma}{}^{TSLJ\pi}\psi_{R_1\gamma M_TM_J}{}^{TSLJ\pi}=I\left({}^{T'S'L'J'\pi'}_{R_2\gamma'M_{T'}M_{J'}},{}^{TSLJ\pi}_{R_1\gamma M_TM_J}\right),\tag{30}$$

where g is a symmetric function and $\psi_{R\gamma}{}^{TSLJ}$ is constructed from the space function $f_\gamma{}^{L\pi}S_R$ and the isospin-spin function $(TS)_{R'}$ according to (29). The labels M and N are considered *different* values of R. The index γ is used to distinguish different space functions with the same L, π, R.

Summation over the isospin and spin coordinates gives

$$\begin{aligned}I\left({}^{T'S'L'J'\pi'}_{R_2\gamma'M_{T'}M_{J'}},{}^{TSLJ\pi}_{R_1\gamma M_TM_J}\right)&=\delta_{R_1'R_2'}\delta_{TT'}\delta_{SS'}\delta_{M_TM_{T'}}\sum_m C_{mM_{L'}M_{J'}}{}^{SL'J'}C_{mM_LM_J}{}^{SLJ}\\&\quad\cdot\int d\boldsymbol{\varrho}d\boldsymbol{\lambda}g_{R_2\gamma'}{}^{TSL'J'\pi'*}g_{R_1\gamma}{}^{TSLJ\pi}f_{\gamma'M_{L'}}{}^{L'\pi'*}f_{\gamma M_L}{}^{L\pi}(S_{R_2}\otimes S_{R_1})_S\,,\end{aligned}\tag{31}$$

where Table I has been used. It follows from (15) that

$$(S_{R_2}\otimes S_{R_1})_S=\delta_{R_1R_2}\,.\tag{32}$$

Use of (28) together with rotation and inversion invariance gives

$$I\left({}^{T'S'L'J'\pi}_{R_2\gamma'M_{T'}M_{J'}},{}^{TSLJ\pi}_{R_1\gamma M_TM_J}\right)=\delta_{R_1R_2}\delta_{TT'}\delta_{SS'}\delta_{M_TM_{T'}}\delta_{LL'}\delta_{JJ'}\delta_{\pi\pi'}\delta_{M_JM_{J'}}\cdot J_{\gamma'\gamma}{}^{TSLJ\pi R}\,,\tag{33}$$

$$J_{\gamma'\gamma}{}^{TSLJ\pi R}=(2L+1)^{-1}\sum_{M_L}\int d\boldsymbol{\varrho}d\boldsymbol{\lambda}g_{R\gamma'}{}^{TSLJ\pi*}g_{R\gamma}{}^{TSLJ\pi}(-)^{L-M_L}f_{\gamma'-M_L}{}^{L\pi}f_{\gamma M_L}{}^{L\pi}$$
$$=(2L+1)^{-1/2}\int d\boldsymbol{\varrho}d\boldsymbol{\lambda}g_{R\gamma'}{}^{TSLJ\pi*}g_{R\gamma}{}^{TSLJ\pi}\{f_{\gamma'}{}^{L\pi},f_{\gamma}{}^{L\pi}\}^0. \quad (34)$$

The integrand depends only on r, s, and φ, so that the integrations over $d\Omega_\rho$ and the azimuthal angle of $\boldsymbol{\lambda}$ about $\boldsymbol{\varrho}$ can be performed immediately; they yield

$$J_{\gamma'\gamma}{}^{TSLJ\pi R}=8\pi^2\int_0^\infty \rho^2d\rho\int_0^\infty\lambda^2d\lambda\int_{-1}^1 d\mu g_{R\gamma'}{}^{TSLJ\pi*}g_{R\gamma}{}^{TSLJ\pi}(2L+1)^{-1/2}\{f_{\gamma'}{}^{L\pi},f_\gamma{}^{L\pi}\}^0, \quad (35)$$

where μ is the cosine of the angle between $\boldsymbol{\lambda}$ and $\boldsymbol{\varrho}$. The Jacobian $\partial(\rho\lambda\mu)/\partial(rs\varphi)$ can be obtained from

$$\rho^2=(r/2)(1+s\cos\varphi),$$
$$\lambda^2=(r/2)(1-s\cos\varphi), \quad (36)$$
$$\mu=\frac{s\sin\varphi}{(1-s^2\cos^2\varphi)^{1/2}},$$

and it is

$$\partial(\rho\lambda\mu)/\partial(rs\varphi)=-r^2s/16\lambda^2\rho^2, \quad (37)$$

so that

$$J_{\gamma'\gamma}{}^{TSLJ\pi R}=\frac{\pi^2}{2}\int_0^\infty r^2dr\int_0^1 sds\int_{-\pi/3}^{\pi/3}d\varphi g_{R\gamma'}{}^{TSLJ\pi*}g_{R\gamma}{}^{TSLJ\pi}\frac{1}{(2L+1)^{1/2}}\{f_{\gamma'}{}^{L\pi},f_\gamma{}^{L\pi}\}^0$$
$$=\frac{3\pi^2}{4}\int_0^\infty r^2dr\int_0^1 ds^2\int_{-\pi/3}^{\pi/3}d\varphi g_{R\gamma'}{}^{TSLJ\pi*}g_{R\gamma}{}^{TSLJ\pi}\frac{1}{(2L+1)^{1/2}}\{f_{\gamma'}{}^{L\pi},f_\gamma{}^{L\pi}\}^0, \quad (38)$$

where the fact that g and $\{f,f\}^0$ are functions of 3φ with period 2π has been used.

The functions $f_\gamma{}^{L\pi}$ will be chosen so that

$$\frac{3\pi^2}{4}\frac{1}{(2L+1)^{1/2}}\{f_{\gamma'}{}^{L\pi},f_\gamma{}^{L\pi}\}^0=\delta_{\gamma'\gamma}. \quad (39)$$

Then

$$I\left(\begin{smallmatrix}T'S'L'J'\pi' \\ R'\gamma'M_T'M_J'\end{smallmatrix}, \begin{smallmatrix}TSLJ\pi \\ R\gamma M_TM_J\end{smallmatrix}\right)=\delta_{\text{all labels}}\cdot\int_0^\infty r^2dr\int_0^1 ds^2\int_{-\pi/3}^{\pi/3}d\varphi|g_{R\gamma}{}^{TSLJ\pi}|^2. \quad (40)$$

Clearly,

$$f^{0+}=2/\pi\sqrt{3}. \quad (41)$$

For the functions with $L\neq0$, the relations

$$(1/\sqrt{3})\{\mathbf{A},\mathbf{B}\}^0=\tfrac{1}{3}\mathbf{A}\cdot\mathbf{B},$$
$$[1/(5)]^{1/2}\{\{\mathbf{A},\mathbf{B}\}^2,\{\mathbf{C},\mathbf{D}\}^2\}^0$$
$$=-(1/15)(\mathbf{A}\cdot\mathbf{B})(\mathbf{C}\cdot\mathbf{D})$$
$$+\tfrac{1}{10}(\mathbf{A}\cdot\mathbf{C}\mathbf{B}\cdot\mathbf{D}+\mathbf{A}\cdot\mathbf{D}\mathbf{B}\cdot\mathbf{C}), \quad (42)$$

which are easily proved, are needed. The 1+ function is

$$f^{1+}=C_{1+}\mathbf{R}_3, \quad (43)$$

and (39) and (42) give

$$(3\pi^2/4)\cdot\tfrac{1}{3}R_3{}^2\cdot C_{1+}{}^2=1=(\pi^2/4)C_{1+}{}^2. \quad (44)$$

Hence,

$$f^{1+}=(2/\pi)\mathbf{R}_3. \quad (45)$$

For the 2+ functions, let

$$\frac{2}{\pi}\{\mathbf{R}_1,\mathbf{R}_1\}^{(2)}=f_1,\quad \frac{2}{\pi}\{\mathbf{R}_1,\mathbf{R}_2\}^{(2)}=f_2,$$
$$\frac{2}{\pi}\{\mathbf{R}_2,\mathbf{R}_2\}^{(2)}=f_3. \quad (46)$$

Then the matrix

$$(3\pi^2/4)\cdot5^{-1/2}\{f_i,f_j\}^{(0)}=M_{ij} \quad (47)$$

is, according to (42)

$$M=\begin{bmatrix}\frac{2}{5} & 0 & -\frac{1}{5}\\ 0 & \frac{3}{10} & 0\\ -\frac{1}{5} & 0 & \frac{2}{5}\end{bmatrix}. \quad (48)$$

It follows that f_1+f_3, f_2, and f_1-f_3 are mutually orthogonal; hence, choose

$$f_\alpha^{2+}=\frac{(10)^{1/2}}{\pi}[\{\mathbf{R}_1,\mathbf{R}_1\}^{(2)}+\{\mathbf{R}_2,\mathbf{R}_2\}^{(2)}]$$

$$f_\beta^{2+}=\frac{1}{\pi}\left(\frac{10}{3}\right)^{1/2}[\{\mathbf{R}_1,\mathbf{R}_1\}^{(2)}-\{\mathbf{R}_2,\mathbf{R}_2\}^{(2)}] \quad (49)$$

$$f_\gamma^{2+}=(2/\pi)(10/3)^{1/2}\{\mathbf{R}_1,\mathbf{R}_2\}^{(2)}.$$

A similar procedure can be used to orthogonalize the set of $f_\gamma^{L\pi}$ for any $L\pi$ and hence satisfy (39).

WAVE FUNCTION OF THE TRITON

The triton and He[3] have $T=\frac{1}{2}$, $J\pi=\frac{1}{2}+$. Hence the possible states ${}^{2S+1}L_R$ are, according to Table II, ${}^2S_{S,M,N,A}$, ${}^2P_{S,M,N,A}$, ${}^4P_{M,N}$, ${}^4D_{M,N}$. There are one f^{0+}, one f^{1+}, and three f^{2+} functions, and thus sixteen vector harmonics occur, corresponding to ${}^2S_{S,M,N,A}$, ${}^2P_{S,M,N,A}$, ${}^4P_{M,N}$, and ${}^4D_{M,N}{}^{\alpha,\beta,\gamma}$. There are sixteen coupled partial differential equations in r, s, and $\cos3\varphi$ for the sixteen symmetric scalar functions g in

$$\psi^{1/2;1/2+}=\sum g_{R\gamma}{}^{1/2SL1/2+}\psi_{R\gamma}{}^{1/2SL1/2+}. \quad (50)$$

RELATIONSHIP TO PREVIOUS CLASSIFICATIONS

According to (15) and the discussion preceding, the most general mixed scalar is

$$(g_1\cos\varphi+g_2\sin3\varphi\sin\varphi,\ g_1\sin\varphi - g_2\sin3\varphi\cos\varphi)=(f_1,f_2),$$

where

$$f_1(-\varphi)=f_1(\varphi),\quad f_2(-\varphi)=-f_2(\varphi)$$

and

$$f_1\left(\varphi+\frac{2\pi}{3}\right)=g_1\cos\left(\varphi+\frac{2\pi}{3}\right)+g_2\sin3\varphi\sin\left(\varphi+\frac{2\pi}{3}\right)$$
$$=(g_1\cos\varphi+g_2\sin3\varphi\sin\varphi)\cos\frac{2\pi}{3} -(g_1\sin\varphi-g_2\sin3\varphi\cos\varphi)\sin\frac{2\pi}{3}$$
$$=f_1(\varphi)\cos\frac{2\pi}{3}-f_2(\varphi)\sin\frac{2\pi}{3}$$

$$f_2\left(\varphi+\frac{2\pi}{3}\right)=f_1(\varphi)\sin\frac{2\pi}{3}+f_2(\varphi)\cos\frac{2\pi}{3}.$$

Hence, it is possible to specify the mixed representation $M(f_1,f_2)$ as consisting of functions f_1 and f_2 satisfying the above restrictions. This halves the "number" of mixed representations; it also complicates the orthonormality relations and confuses the situation as to the number of independent functions. It has proved useful in the past.[2]

The results of the present work are the same as those given by Clapp[3] for the triton. However, the present work generalizes these to all three-nucleon states and introduces a simplified notation.

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Prediction of p-, d-, and f-Wave Pion-Nucleon Scattering*

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We develop a peripheral method for predicting $\pi-N$ phase shifts up to moderate energies. Precise values are given for the p-, d-, and f-wave phase shifts (with the exception of p_{11}) up to 400 MeV, and the general behavior up to around 1 BeV is also predicted. The 600- and 900-MeV $\pi^- - p$ resonances are clearly identified with the D_{13} and F_{15} amplitudes, respectively, and it is probable that the 1.35 BeV $\pi^+ - p$ resonance is in F_{37}. The predictions at 310 MeV select the phase shift set *spdf* II of Vik and Rugge. The method consists in evaluating the dispersion relation for $F_{l\pm}(s)=f_{l\pm}(s)/q^{2l}$ where $f_{l\pm}(s)$ is the partial-wave amplitude. The factor q^{-2l} suppresses the unknown shorter range parts of the $\pi-N$ interaction. Various means are used to avoid the difficulties arising from lack of knowledge of the inelasticity. The symmetries in spin and isospin of the dispersion relation calculations of the various interactions are examined, together with equivalent model potentials.

1. INTRODUCTION

THE various parts of the pion-nucleon interaction have been studied in detail.[1] The parts of longest range are the long-range Born term (i.e., nucleon exchange), and the exchange of a low-energy s-wave pion pair. Shorter in range are the crossed physical cut term (which is mainly nucleon isobar exchange) and the exchange of a ρ meson. In addition there is a very-short-range interaction (range $<2\ 10^{-14}$ cm) about which

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[1] J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, Phys. Rev. **128**, 1881 (1962) (and earlier papers cited there). This paper will be referred to as HMOV.