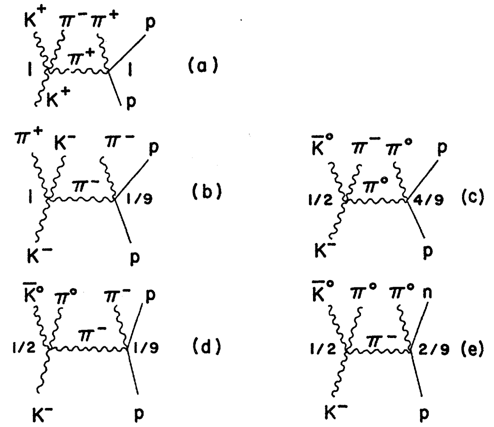

FIG. 7. One-pion-exchange diagrams leading to the three charge states under study.

appears in marked contrast to the high-energy K^+ data.[1] This can be easily understood because (a) the threshold for the K^*N^* final state is approximately at 1.7 BeV/c and (b) the charge states available here are not as favorable to K^*N^* production as in the K^+p interactions, if we think of the K^*N^* final state being produced through the OPE diagram, which appears to be the case for K^+p interactions.[1] The relative suppression of various charge states is illustrated in Fig. 7 where we show the lowest-order Feynman diagrams for the reactions in question. The number next to each vertex represents the relative strength of that vertex as compared with the corresponding vertex for the $K^+p\rightarrow K^+\pi^-\pi^+p$ reaction. We assume dominance of $T=1/2$ state for the $K\pi$ interaction and of $T=3/2$ for πN interaction.

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Vertex Functions and the Unitarity Relation*

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It is shown that a unitarity relation holds for vertex functions in a form analogous to the one for form factors and that the one-particle irreducible parts of scattering amplitudes satisfy unitarity by themselves. The second half of the present work considers the case of nonrelativistic S-wave scattering with one bound state. Interrelations among the S matrix, the denominator function, and functions for the bound state, such as the form factor, the propagator, and the vertex function are discussed under certain general restrictions.

1. INTRODUCTION

THE S-matrix theory of strong interactions considers physical quantities on the mass shell. By contrast, most fundamental in the Green's function approach are such functions as propagators and vertex functions which require knowledge of quantities off the mass shell. The connection between the two approaches has not been well understood, although it would be very desirable to see if an S-matrix theory could incorporate any new principle which is absent in Green's function theory. In some processes, such as electron-nucleon scattering and weak decay of strongly interacting particles, it becomes necessary to know about form factors. In S-matrix theory a link between scattering amplitudes and form factors is provided by the unitarity relation for the latter, although its solutions are known to have the ambiguity of Omnès.

S-matrix theory does not directly deal with propagators and vertex functions, but the single-dispersion parts in Mandelstam's representation for scattering amplitudes are closely related to them. Because of their importance it seems worthwhile to ask to what extent an S matrix can determine these functions. The main purpose of the present work is to study the problem for the case of nonrelativistic scattering under certain general restrictions. Properties of the form factor, the propagator, and the vertex function for a bound state are also discussed. It will be seen that there exists some kind of correspondence between the scattering amplitude and its one-particle irreducible part and between the form factor and the vertex function.

We shall begin with a relativistic case. After a brief summary in Sec. 2 of the main properties of a propagator and a vertex function, it is shown in Sec. 3 that a unitarity relation holds for the pion vertex functions

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in a form analogous to the one for the pion form factors. It is also shown that the one-particle irreducible parts of the full scattering amplitudes satisfy the unitarity relation by themselves.

In the remaining sections we shall discuss nonrelativistic S-wave scattering with one bound state. In Sec. 4 we discuss how to construct the denominator function from the phase shift. We also derive a general effective-range formula, which may be useful for phenomenological analysis. These are discussed under the requirement that the S matrix have a pole with a positive residue at the point corresponding to the bound-state energy.

In the last section we study the problem of constructing the form factor, the propagator, and the vertex function of the bound state from the denominator function. Levinson's theorem is assumed and the Omnès ambiguity is discarded. It is shown that one free parameter enters in[1] due to the inevitability of a CDD pole.[2] This pole does not correspond to an elementary particle, contrary to what is frequently said about CDD poles. Finally we discuss whether the phase of the one-particle irreducible part of the full amplitude can determine the binding energy and/or the coupling constant of the bound state. Our answer is negative. In a certain case, however, we can determine one of the two parameters when the other is known. Explicit examples to illustrate the discussion of Sec. 5 are given in the Appendix.

2. SUMMARY OF LSZ's RESULTS

For later convenience and for fixing our notation we summarize the main results of Lehmann[3] and of Lehmann-Symanzik-Zimmermann[4] (LSZ). The pion propagator can be represented under general conditions as

$$i\Delta_F'(s)=\frac{1}{\mu^2-s}+\frac{1}{\pi}\int_{9\mu^2}^{\infty}\frac{\sigma(s')}{s'-s-i\epsilon}ds', \qquad (2.1)$$

with the spectral function given by

$$\sigma(s)=\mathbf{K}^+\boldsymbol{\varrho}\mathbf{K}/(s-\mu^2)^2. \qquad (2.2)$$

$\mathbf{K}(s)$ stands for the matrix elements of the pion source operator between the vacuum state and states which can be produced by a virtual pion, and $\boldsymbol{\varrho}(s)$ is the phase-volume factor. We define a function related to the pion renormalization constant by

$$Z_\pi^{-1}(s)\equiv\Delta_F'(s)/\Delta_F(s)$$
$$=1-\frac{s-\mu^2}{\pi}\int_{9\mu^2}^{\infty}\frac{\sigma(s')}{s'-s-i\epsilon}ds'. \qquad (2.3)$$

As the propagator is a Herglotz function,[2] $Z_\pi(s)$ can be expressed as

$$Z_\pi(s)=1+\frac{s-\mu^2}{\pi}\int_{9\mu^2}^{\infty}\frac{|Z_\pi(s')|^2\sigma(s')}{s'-s-i\epsilon}ds' +(s-\mu^2)\sum_n\frac{c_n}{s_n-s}, \qquad (2.4)$$

where $c_n\geq 0$, $s_n>\mu^2$, and there can be at most one such pole in the interval $\mu^2<s<9\mu^2$. We find

$$\mathrm{Im}Z_\pi(s)=(s-\mu^2)\left\{\frac{\mathbf{\Gamma}^+\boldsymbol{\varrho}\mathbf{\Gamma}}{(s-\mu^2)^2}+\pi\sum_n c_n\delta(s-s_n)\right\}, \qquad (2.5)$$

where

$$\mathbf{\Gamma}(s)\equiv Z_\pi(s)\mathbf{K}(s). \qquad (2.6)$$

Taking the limit $s\to\infty$, we obtain the sum rule

$$1=\lim_{s\to\infty}Z_\pi(s)+\frac{1}{\pi}\int_{9\mu^2}^{\infty}\frac{\mathbf{\Gamma}^+\boldsymbol{\varrho}\mathbf{\Gamma}}{(s-\mu^2)^2}ds+\sum_n c_n. \qquad (2.7)$$

Since we know that

$$1>\lim_{s\to\infty}Z_\pi(s)\geq 0$$

and all the c_n's are nonnegative, we obtain LSZ's inequality

$$1\geq\frac{1}{\pi}\int_{9\mu^2}^{\infty}\frac{\mathbf{\Gamma}^+\boldsymbol{\varrho}\mathbf{\Gamma}}{(s-\mu^2)^2}ds. \qquad (2.8)$$

Since we have

$$\Gamma_{N\bar{N}}^*\rho_{N\bar{N}}\Gamma_{N\bar{N}}\leq\mathbf{\Gamma}^+\boldsymbol{\varrho}\mathbf{\Gamma},$$

where

$$\rho_{N\bar{N}}(s)\equiv\frac{s}{8\pi}\left(1-\frac{4m^2}{s}\right)^{1/2}, \qquad (2.9)$$

we are led to write

$$1\geq\frac{1}{\pi}\int_{4m^2}^{\infty}\frac{|\Gamma_{N\bar{N}}|^2\rho_{N\bar{N}}}{(s-\mu^2)^2}ds. \qquad (2.10)$$

From this it follows that

$$\lim_{s\to\infty}\Gamma_{N\bar{N}}(s)=0. \qquad (2.11)$$

3. THE UNITARITY RELATION FOR VERTEX FUNCTIONS

The scattering amplitude $\mathbf{T}$ defined from the S matrix by

$$\mathbf{S}=\mathbf{1}+2i\boldsymbol{\varrho}^{1/2}\mathbf{T}\boldsymbol{\varrho}^{1/2} \qquad (3.1)$$

satisfies the unitarity relation

$$(\mathrm{Im}\mathbf{T})_{ij}=(\mathbf{T}^+\boldsymbol{\varrho}\mathbf{T})_{ij} \qquad (3.2)$$

for the center-of-mass energy s larger than its respective physical threshold s_{ij}. By the physical threshold s_i

[1] We cannot, however, exclude the possibility that a knowledge of the phase shifts in other angular momentum states may uniquely determine the parameter.

[2] L. Castillejo, R. H. Dalitz, and F. J. Dyson, Phys. Rev. **101**, 453 (1955).

[3] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[4] H. Lehmann, K. Symanzik, and W. Zimmermann, Nuovo Cimento **2**, 425 (1955).

for the state i we mean the square of the sum of the rest masses of the particles in the state. s_{ij} is then given by the larger of s_i and s_j. We confine ourselves only to states which can be produced by a virtual pion, and we choose ϱ so as to be identical with the phase-volume factor which appeared in the expression, (2.2), for the pion propagator. The form factor, $\mathbf{K}(s)$, satisfies the unitarity relation

$$(\text{Im}\mathbf{K})_i = (\mathbf{T}^+\varrho\mathbf{K})_i = (\mathbf{K}^+\varrho\mathbf{T})_i \tag{3.3}$$

for s larger than its respective physical threshold s_i.

We shall derive a unitarity relation, analogous to Eq. (3.3), for the vertex function $\mathbf{\Gamma}(s)$ defined by Eq. (2.6). For the imaginary part of $\Gamma_i(s)$ we have

$$\begin{aligned}\text{Im}\Gamma_i(s) &= \text{Im}[Z_\pi(s)K_i(s)]\\ &= Z_\pi(s)\,\text{Im}K_i(s) + K_i^*(s)\,\text{Im}Z_\pi(s).\end{aligned}$$

We note here that if $Z_\pi^{-1}(s)$ has a simple zero at a point larger than s_i, then its imaginary part and, hence, $K_i(s)$, should vanish at the same point. $\Gamma_i(s)$ can thus have no poles for $s>s_i$, and we need not take into account possible CDD poles of $Z_\pi(s)$ when we consider $\text{Im}\Gamma_i(s)$ for $s>s_i$. From Eqs. (3.3) and (2.5) it follows that

$$\begin{aligned}\text{Im}\Gamma_i(s) &= Z_\pi(s)(\mathbf{T}^+\varrho\mathbf{K})_i + K_i^*(s)\frac{(\mathbf{\Gamma}^+\varrho\mathbf{\Gamma})}{s-\mu^2}\\ &= \left\{\left(\mathbf{T}^+ - \mathbf{\Gamma}^*\frac{Z_\pi^{-1*}(s)}{\mu^2-s}\mathbf{\Gamma}^+\right)\varrho\mathbf{\Gamma}\right\}_i, \quad \text{for} \quad s>s_i.\end{aligned}$$

If we put

$$\mathbf{T} = \mathbf{\Gamma} i\Delta_F'\mathbf{\Gamma}^T + \mathbf{U}, \tag{3.4}$$

where $\mathbf{\Gamma}^T$ stands for the transposed matrix of $\mathbf{\Gamma}$, we can write

$$\text{Im}\Gamma_i = (\mathbf{U}^+\varrho\mathbf{\Gamma})_i, \quad \text{for} \quad s>s_i. \tag{3.5}$$

It is easy to see that we also have

$$\text{Im}\Gamma_i = (\mathbf{\Gamma}^+\varrho\mathbf{U})_i, \quad \text{for} \quad s>s_i. \tag{3.5$'$}$$

It is to be noted here that our argument given above does not forbid $\Gamma_i(s)$ from having poles for $s<s_i$. As for the pion-nucleon vertex $\Gamma_{N\bar{N}}(s)$, for instance, we have seen that it can have no poles for $s>4m^2$. It has no poles for $s<\mu^2$ when the Lehmann representation for the pion propagator holds without subtraction. However, it could have poles at the points in the interval $\mu^2<s<4m^2$, at which the pion propagator vanishes.

The first term in the expression (3.4) for the full amplitude $\mathbf{T}$ is equal to the Born contribution with all the radiative corrections included. The second term $\mathbf{U}$ is what is sometimes called the one-particle irreducible part of the full amplitude.[5] In order to show that $\mathbf{U}$ satisfies the unitarity relation by itself, we rewrite both sides of Eq. (3.4):

$$\begin{aligned}\text{Im}\mathbf{T} &= \text{Im}\mathbf{U} + \text{Im}\left[\mathbf{\Gamma}\frac{Z_\pi^{-1}}{\mu^2-s}\mathbf{\Gamma}^T\right]\\ &= \text{Im}\mathbf{U} + \text{Im}\mathbf{\Gamma}\frac{Z_\pi^{-1}}{\mu^2-s}\mathbf{\Gamma}^T + \mathbf{\Gamma}^*\frac{(\mathbf{K}^+\varrho\mathbf{K})}{(s-\mu^2)^2}\mathbf{\Gamma}^T\\ &\qquad + \mathbf{\Gamma}^*\frac{Z_\pi^{-1*}}{\mu^2-s}\,\text{Im}\mathbf{\Gamma}^T,\end{aligned}$$

$$\begin{aligned}\mathbf{T}^+\varrho\mathbf{T} &= \left(\mathbf{U}^+ + \mathbf{\Gamma}^*\frac{Z_\pi^{-1*}}{\mu^2-s}\mathbf{\Gamma}^+\right)\varrho\left(\mathbf{U} + \mathbf{\Gamma}\frac{Z_\pi^{-1}}{\mu^2-s}\mathbf{\Gamma}^T\right)\\ &= \mathbf{U}^+\varrho\mathbf{U} + (\mathbf{U}^+\varrho\mathbf{\Gamma})\frac{Z_\pi^{-1}}{\mu^2-s}\mathbf{\Gamma}^T + \mathbf{\Gamma}^*\frac{(\mathbf{K}^+\varrho\mathbf{K})}{(s-\mu^2)^2}\mathbf{\Gamma}^T\\ &\qquad + \mathbf{\Gamma}^*\frac{Z_\pi^{-1*}}{\mu^2-s}(\mathbf{\Gamma}^+\varrho\mathbf{U}).\end{aligned}$$

From Eqs. (3.2), (3.5), and (3.5)$'$ it thus follows that

$$(\text{Im}\mathbf{U})_{ij} = (\mathbf{U}^+\varrho\mathbf{U})_{ij}, \quad \text{for} \quad s>s_{ij}. \tag{3.6}$$

This kind of unitarity relation was first noted by Blankenbecler *et al.*[6] for the case of nonrelativistic potential scattering when no subtraction is necessary with respect to the momentum transfer variable in the Mandelstam representation for the scattering amplitude. It should be emphasized that in our derivation of Eqs. (3.5) and (3.6) we have used expressions only for the imaginary parts of $K_i(s)$, $Z_\pi^{-1}(s)$. Therefore, our results are independent from possible necessity of subtractions for these functions.

As an example of an application of our results, we consider the elastic scattering of a nucleon-antinucleon pair in the ${}^1S_0$ state with isotopic spin one. The unitarity relations, (3.2) and (3.6), give an upper limit on $|T_{N\bar{N}}(s)|$ and $|U_{N\bar{N}}(s)|$, respectively. We find

$$|T_{N\bar{N}}(s)|\rho_{N\bar{N}}(s) \leq 1, \tag{3.7}$$

$$|U_{N\bar{N}}(s)|\rho_{N\bar{N}}(s) \leq 1, \tag{3.7$'$}$$

for $s>4m^2$, where $\rho_{N\bar{N}}(s)$ has been given by Eq. (2.9). Since

$$\Gamma_{N\bar{N}}(s)\frac{Z_\pi^{-1}(s)}{\mu^2-s}\Gamma_{N\bar{N}}(s) = T_{N\bar{N}}(s) - U_{N\bar{N}}(s),$$

it follows from the inequalities (3.7) and (3.7)$'$ that for $s>4m^2$ we have

$$\begin{aligned}|\Gamma_{N\bar{N}}(s)|^2/|Z_\pi(s)| &= (s-\mu^2)|T_{N\bar{N}}(s) - U_{N\bar{N}}(s)|\\ &\leq \frac{2(s-\mu^2)}{\rho_{N\bar{N}}(s)} = 16\pi\frac{1-(\mu^2/s)}{[1-(4m^2/s)]^{1/2}}.\end{aligned}$$

[5] K. Symanzik, *Lectures in Theoretical Physics* (Federal Nuclear Energy Commission of Yugoslavia, Belgrade, 1961), p. 485.

[6] R. Blankenbecler, M. L. Goldberger, N. N. Khuri, and S. B. Treiman, Ann. Phys. (N. Y.) **10**, 62 (1960).

If we introduce a vertex-renormalization function

$$Z_{N\bar{N}}(s)\equiv\Gamma_{N\bar{N}}(s)/\Gamma_{N\bar{N}}(\mu^2)\,, \tag{3.8}$$

where

$$\Gamma_{N\bar{N}}(\mu^2)\equiv\sqrt{2}g\,, \tag{3.9}$$

the above inequality becomes

$$\frac{|Z_\pi(s)|}{|Z_{N\bar{N}}(s)|^2}\geq\frac{g^2}{8\pi}\frac{[1-(4m^2/s)]^{1/2}}{1-(\mu^2/s)}\,,\quad\text{for}\quad s>4m^2. \tag{3.10}$$

When s goes to infinity, we find

$$\lim_{s\to\infty}\frac{|Z_\pi(s)|}{|Z_{N\bar{N}}(s)|^2}\geq\frac{g^2}{8\pi}. \tag{3.11}$$

As another application, we discuss decay of the π meson into a lepton pair through weak interaction. The weak form factor $K_w(s)$, which was written as $F(s)$ in a previous work,[7] satisfies the unitarity relation

$$\mathrm{Im}K_w(s)=\mathbf{T}_w{}^+\boldsymbol{\varrho}\mathbf{K}\,,\quad\text{for}\quad s>9\mu^2\,, \tag{3.12}$$

where $\mathbf{T}_w$ is the scattering amplitude from an initial lepton pair to final states of strongly interacting particles. By an argument similar to that used to derive Eq. (3.5), we find the unitarity relation for the weak vertex,

$$\mathrm{Im}\Gamma_w(s)=\mathbf{U}_w{}^+\boldsymbol{\varrho}\boldsymbol{\Gamma}\,,\quad\text{for}\quad s>9\mu^2\,, \tag{3.13}$$

where the weak vertex is defined by

$$\Gamma_w(s)\equiv Z_\pi(s)K_w(s)\,, \tag{3.14}$$

and $\mathbf{U}_w$ is the one-particle irreducible part of $\mathbf{T}_w$; that is,

$$\mathbf{T}_w=\boldsymbol{\Gamma}i\Delta_F{}'\Gamma_w+\mathbf{U}_w. \tag{3.15}$$

In general, one must add delta functions to the right-hand side of Eq. (3.13) corresponding to possible poles of $\Gamma_w(s)$ because the argument which excluded poles of $\Gamma_i(s)$ for $s>s_i$ cannot be used in this case. In order to avoid such poles, $T_w(s)$ was divided in the previous work into two parts in a somewhat different way:

$$\mathbf{T}_w(s)=\mathbf{K}(s)\frac{F}{\mu^2-s}+\frac{1}{s}\mathbf{L}_w(s)\,, \tag{3.16}$$

where F is the decay constant defined by

$$F\equiv K_w(\mu^2). \tag{3.17}$$

The poles at $s=0$ are of kinematical origin. From Eqs. (3.15) and (3.16) and from

$$\Gamma_w(s)=F+\frac{s-\mu^2}{\pi}\int_{9\mu^2}^\infty\frac{\mathrm{Im}\Gamma_w(s')}{(s'-\mu^2)(s'-s-i\epsilon)}ds'\,, \tag{3.18}$$

it follows that

$$\frac{1}{s}\mathbf{L}_w(s)=\mathbf{U}_w(s)-\mathbf{K}\frac{1}{\pi}\int_{9\mu^2}^\infty\frac{\mathrm{Im}\Gamma_w(s')}{(s'-\mu^2)(s'-s-i\epsilon)}ds'. \tag{3.19}$$

When the pion-renormalization constant vanishes, the assumption of no subtraction for $K_w(s)$ led us to the requirement

$$F=\lim_{s\to\infty}\frac{\mathbf{L}_w{}^+\boldsymbol{\varrho}\mathbf{K}}{\mathbf{K}^+\boldsymbol{\varrho}\mathbf{K}}. \tag{3.20}$$

Equation (3.20) can also be expressed as

$$\lim_{s\to\infty}\frac{s\mathbf{U}_w{}^+\boldsymbol{\varrho}\mathbf{K}}{\mathbf{K}^+\boldsymbol{\varrho}\mathbf{K}}=0\,, \tag{3.21}$$

since we have

$$F=\frac{1}{\pi}\int_{9\mu^2}^\infty\frac{\mathrm{Im}\Gamma_w(s)}{s-\mu^2}ds. \tag{3.22}$$

The implication of Eq. (3.20) has been studied by Nishijima[8] more explicitly for a simplified model.

4. THE S MATRIX AND THE DENOMINATOR FUNCTION

Because of the many-body character of the relativistic scattering problem, it seems very difficult to investigate it without a drastic approximation. In the remainder of the present work we shall deal with the much simpler case of nonrelativistic scattering, which will enable us to discuss some questions untouched in the previous section.

For simplicity, we consider the S-wave scattering of two spinless particles, which we call "nucleons," with one bound state. Our problem is to construct such functions as the denominator function, the form factor, the propagator, and the vertex function when the S-wave phase shift is given. It has long been known that the phase shift of an angular momentum state cannot uniquely determine the denominator function, or the Jost function, when there is a bound state in the same angular momentum state.[9,10] Our problem should then be to study whether, and if possible how, we can reduce ambiguities under general restrictive conditions. We begin with the denominator function.

According to van Kampen[10] and Omnès,[11] the denominator function is expressed as

$$D(s)=C(s+s_B)\times\exp\left[-\frac{s+s_B}{\pi}\int_0^\infty\frac{\delta(s')ds'}{(s'+s_B)(s'-s-i\epsilon)}\right], \tag{4.1}$$

[7] M. Ida, Phys. Rev. **132**, 401 (1963).

[8] K. Nishijima, Phys. Rev. **133**, B1092 (1964).

[9] R. Jost, Helv. Phys. Acta **20**, 256 (1947).

[10] N. G. van Kampen, Phil. Mag. **42**, 851 (1951).

[11] R. Omnès, Nuovo Cimento **8**, 316 (1958).

where s denotes the center-of-mass momentum squared and $s_B \equiv mB$, with B the binding energy. The phase shift is normalized so that $\delta(0)=0$, and C is a constant.

Since the S matrix can be written in the form

$$S(s)=D^{II}(s)/D(s)\,, \tag{4.2}$$

where $D^{II}(s)$ stands for the denominator function in the unphysical sheet, it would not have a pole at $s=-s_B$ if $D^{II}(s)$ should vanish at the same point. A state corresponding to the "lost pole" cannot be regarded as a bound state of the nucleons, for its wave function vanishes identically. In order to exclude this kind of "neutral" interaction we require that:

(I) the S matrix have a pole at $s=-s_B$.

Thus, the exponential factor of Eq. (4.1), or the Omnès integral, should have a pole at $s=-s_B$ when analytically continued into the unphysical sheet. In addition to this pole it may have other poles in the unphysical sheet, which make redundant poles[12] in $S(s)$. We then present the second restriction:

(II) the residue G^2 of the pole of $S(s)$ at $s=-s_B$ be positive.

When there is only one pole in $S(s)$ with a positive residue, (II) is enough to single out a bound-state pole. If there are more than one with positive residues, each of them is eligible to be a bound-state pole.

It should be noted here that analytic continuation of $D(s)$ into the unphysical sheet cannot be performed unless the phase shift is given by means of an analytic expression. When the phase shift is known approximately, one can determine the binding energy with some confidence only for the case of a loosely bound state. The well-known correlation in this case of the low-energy behavior of the phase shift with s_B and G^2 will be discussed in a way more general than is usually done.

From what we have seen under the restriction (I), it follows that $D(s)$ should be written in the form

$$D(s)=[s_B^{1/2}+is^{1/2}][A(s)+is^{1/2}B(s)]\,, \tag{4.3}$$

where both $A(s)$ and $B(s)$ are real on the positive real axis, and their left-hand singularities cancel each other for $D(s)$. We thus find

$$S(s)=\frac{s_B^{1/2}-is^{1/2}}{s_B^{1/2}+is^{1/2}}\,\frac{s_B^{1/2}-is^{1/2}\varphi(s)}{s_B^{1/2}+is^{1/2}\varphi(s)}\,, \tag{4.4}$$

where the function $\varphi(s)$ is defined by

$$\varphi(s)\equiv s_B^{1/2}B(s)/A(s)\,, \tag{4.5}$$

and is real for $s>0$. We have normalized the coupling constant G so that the scattering amplitude

$$T(s)\equiv(1/2is^{1/2})[S(s)-1] \tag{4.6}$$

has a pole at $s=-s_B$ of the form $G^2/(-s_B-s)$. From Eq. (4.4) we find

$$G^2=2s_B^{1/2}\frac{1+\varphi(-s_B)}{1-\varphi(-s_B)}\,, \tag{4.7}$$

and the restriction (II) now becomes

$$\text{(II}'\text{)}\quad |\varphi(-s_B)|<1\,. \tag{4.8}$$

It is now easy to express the phase shift in terms of s_B and $\varphi(s)$. The phase shift can most conveniently be written in the form

$$s^{1/2}\cot\delta(s)=-\frac{s_B^{1/2}}{1+\varphi(s)}+\frac{\varphi(s)}{1+\varphi(s)}\frac{s}{s_B^{1/2}}\,, \tag{4.9}$$

which is reminiscent of the ordinary effective-range formula,

$$s^{1/2}\cot\delta(s)=\frac{1}{a}+\frac{r}{2}s\,. \tag{4.10}$$

From Eq. (4.9), $\varphi(s)$ can be written as

$$\varphi(s)=\frac{s+s_B}{s-s_B^{1/2}s\cot\delta(s)}-1\,. \tag{4.11}$$

By calculating the differential coefficient of $\varphi(s)$ at $s=0$, we find

$$-\frac{1}{a}+\frac{r}{2}s_B=s_B^{1/2}\left\{1+\frac{1}{a^2}\frac{d\varphi(0)}{ds}\right\}\,, \tag{4.12}$$

where a and r are now defined as the coefficient of the effective-range expansion,

$$s^{1/2}\cot\delta(s)=\frac{1}{a}+\frac{r}{2}s+\cdots\,. \tag{4.10$'$}$$

Thus, the condition

$$d\varphi(0)/ds=0 \tag{4.13}$$

is equivalent to the familiar relation[13]

$$-\frac{1}{a}+\frac{r}{2}s_B=s_B^{1/2}\,. \tag{4.14}$$

From Eq. (4.9) we see that the scattering length can be expressed as

$$a=-(1+\varphi)/s_B^{1/2}\,, \tag{4.15}$$

where $\varphi\equiv\varphi(0)$. When Eq. (4.13) holds, the effective range is given by

$$r=[2\varphi/(1+\varphi)](1/s_B^{1/2})\,. \tag{4.16}$$

[12] S. T. Ma, Phys. Rev. **69**, 668 (1946); **71**, 195 (1947).

[13] J. M. Blatt and V. F. Weisskopf, *Theoretical Nuclear Physics* (John Wiley & Sons, Inc., New York, 1952), Chap. 2.

It follows that

$$-2r/a=4\varphi/(1+\varphi)^2, \tag{4.17}$$

and that $0<-(2r/a)<1$ for $\varphi>0$ and $-(2r/a)<0$ for $\varphi<0$. φ thus has two real roots. One is smaller than unity in the absolute magnitude, and the other larger. Under Eq. (4.13) and the assumption of a small binding energy, $\varphi(s)$ is almost constant in the interval $0>s>-s_B$, and we may discard the root larger in absolute magnitude by the requirement (4.8). The scattering length and the effective range can then uniquely determine φ and, hence, s_B. The coupling constant G^2 is obtained by replacing $\varphi(-s_B)$ in Eq. (4.7) with φ.[14] For the $^3S_1$ n-p scattering we find in this way that

$$\varphi=\frac{|a|}{r}-1-\left\{\left(\frac{|a|}{r}-1\right)^2-1\right\}^{1/2}\simeq 0.25,$$

$$1/s_B{}^{1/2}=a/(1+\varphi)\simeq 4.3\text{ F},\quad (\text{or } B\equiv s_B/m\simeq 2.2\text{ MeV}),$$

which is to be compared with the experimental value of B, 2.226 MeV, and

$$G^2/m=2[(1+\varphi)/(1-\varphi)](B/m)^{1/2}\simeq 0.16.$$

The good agreement for B and Eq. (4.12) indicate that $|\varphi'(0)|\ll a^2$.

If the pole of the S matrix moves downward through the origin along the imaginary axis of the k plane, where $k=s^{1/2}$, it no longer represents a bound state, but corresponds to a virtual state. When $S^{II}(s)$ has a pole at $s=-s_V$, an expression analogous to Eq. (4.9) is obtained by replacing $s_B{}^{1/2}$ in Eq. (4.9) with $-s_V{}^{1/2}$:

$$s^{1/2}\cot\delta(s)=\frac{s_V{}^{1/2}}{1+\varphi(s)}-\frac{\varphi(s)}{1+\varphi(s)}\frac{s}{s_V{}^{1/2}}. \tag{4.18}$$

Equation (4.17) is still true when Eq. (4.13) holds. For the $^1S_0$ n-p scattering we find in a way similar to that for $^3S_1$ n-p scattering that

$$\varphi=-\left(\frac{a}{r}+1\right)+\left\{\left(\frac{a}{r}+1\right)^2-1\right\}^{1/2}\simeq -0.05,$$

$$1/s_V{}^{1/2}=a/(1+\varphi)\simeq 25\text{ F}\quad(\text{or } s_V/m\simeq 67\text{ keV}).$$

The sign of the scattering length discards one of the two roots for φ.

Recently Geshkenbein and Ioffe[15] (GI) obtained an interesting upper bound on coupling constants, under the assumption that vertex functions have no poles. In the nonrelativistic case their bound is given by[16]

$$G^2<4s_B{}^{1/2}. \tag{4.19}$$

It can also be expressed in terms of the function $\varphi(s)$ as

$$\varphi(-s_B)<\tfrac{1}{3}. \tag{4.20}$$

We see that the constant φ for the $^3S_1$ n-p scattering is smaller than $\frac{1}{3}$. In other words, the deuteron coupling constant satisfies the GI bound[15]:

$$G^2/m<4(B/m)^{1/2}=0.19.$$

It should be emphasized, however, that the GI bound is a necessary, but not sufficient, condition for the vertex function to have no poles. Indeed, in the nonrelativistic models the vertex for a bound state must have a pole under certain general conditions.

5. THE FORM FACTOR, THE PROPAGATOR, AND THE VERTEX FUNCTION

We next study the form factor, the propagator, and the vertex function when the denominator function is known. From the unitarity relation (3.3) for $K(s)$, its phase is given by $\delta(s)$. We restrict ourselves to the case in which

(III) $K(s)$ and $Z_B{}^{-1}(s)$ have no poles.

We do not require that $\Gamma(s)$ and $Z_B(s)$ have no poles. The reason for this is simply that in field theory the functions corresponding to the former are expressed as matrix elements of Heisenberg operators, while those corresponding to the latter are not. $K(s)$ can now be written in the form

$$K(s)=GP(s)\exp\left[\frac{s+s_B}{\pi}\int_0^\infty\frac{\delta(s')ds'}{(s'+s_B)(s'-s-i\epsilon)}\right], \tag{5.1}$$

where $P(s)$ is an arbitrary polynomial normalized so that $P(-s_B)=1$. From (I) it follows that $K^{II}(s)$ has a zero at $s=-s_B$. $Z_B{}^{-1}(s)$ is defined, in analogv to Eq. (2.3), by

$$Z_B{}^{-1}(s)=1-\frac{s+s_B}{\pi}\int_0^\infty\frac{|K(s')|^2(s')^{1/2}ds'}{(s'+s_B)^2(s'-s-i\epsilon)}, \tag{5.2}$$

except for further subtractions, if necessary.

In order to fix the asymptotic behavior of the phase shift, we also require that:

(IV) the phase shift satisfy Levinson's theorem.[17]

Since we are concerned with the case of one bound state and the phase shift is normalized by $\delta(0)=0$, it follows from (IV) that $\delta(s)$ tends to $-\pi$ as $s\to\infty$. The exponential factor in Eq. (5.1) increases linearly with s, apart from a possible logarithmic factor and

[14] Incidentally, the coefficient N of the asymptotic form of the normalized radial wave function for a loosely bound state,

$$u(|\mathbf{r}|)\to Ne^{-s_B{}^{1/2}|\mathbf{r}|},\quad\text{as}\quad|\mathbf{r}|\to\infty,$$

is approximately given by $N^2\simeq 2s_B{}^{1/2}[1-(s_B)^{1/2}r]^{-1}$. (See Ref. 13, Chap. 12.) By substituting Eq. (4.16) into the above expression and comparing it with Eq. (4.7), we are led to the familiar result, $N^2\simeq G^2$.

[15] B. V. Geshkenbein and B. L. Ioffe, Phys. Rev. Letters **11**, 55 (1963).

[16] C. J. Goebel and B. Sakita, Phys. Rev. Letters **11**, 293 (1963).

[17] N. Levinson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **25**, No. 9 (1949).

Eq. (5.2) requires at least one more subtraction. Here we confine ourselves to the simplest solution by further requiring that:

(V) $K(s)$ have no zeros;

that is, $P(s)\equiv 1$. Equation (5.2) should now be replaced by

$$Z_B^{-1}(s)=1-d(s+s_B)-\frac{(s+s_B)^2}{\pi}\times\int_0^\infty \frac{|K(s')|^2(s')^{1/2}ds'}{(s'+s_B)^3(s'-s-i\epsilon)}, \quad (5.3)$$

where d is an arbitrary constant. The vertex function is then given by

$$\Gamma(s)=K(s)/Z_B^{-1}(s), \quad (5.4)$$

and the phase of $U(s)$ is, by Eq. (3.5), equal to that of $\Gamma(s)$. $\Gamma(s)$ has no zeros because of the conditions (V) and (III). We also note that $Z_B^{-1}(s)$ increases like $s^{3/2}$ and $\Gamma(s)$ decreases like $s^{-1/2}$, again apart from possible logarithmic factors, as s goes to infinity.

We have seen that we have one free parameter d in determining the propagator and the vertex function from the denominator function under our restrictive conditions. This situation may be related to the well-known ambiguity[18–20] one encounters in determining the potential from the denominator function when there is a bound state.

Equation (5.3) means that the Lehmann representation for the propagator needs one subtraction[16]:

$$i\Delta_F'(s)=\frac{1}{-s_B-s}+d+\frac{s+s_B}{\pi}\times\int_0^\infty \frac{|K(s')|^2(s')^{1/2}ds'}{(s'+s_B)^2(s'-s-i\epsilon)}. \quad (5.5)$$

The subtraction should be made at a *finite* point, in spite of a contrary statement in Ref. 16. The propagator still is a Herglotz function, and since it tends to $-\infty$ as s goes to $-\infty$ it must have a zero below $-s_B$. We call its position $-s_1$, where $s_1>s_B$. It follows that $\Gamma(s)$ has a pole at this point and, hence, the one-particle reducible part of $T(s)$ or the first term of Eq. (3.4) does too. Thus, the one-particle irreducible part, $U(s)$, should have a pole at the same point to cancel the pole of the one-particle reducible part unless the zero of the propagator happens to coincide with a redundant pole of $S(s)$. It is to be mentioned that the CDD zero at $s=-s_1$ is inevitable and has nothing to do with the introduction of an elementary particle. Since d is determined when s_1 is given, we may regard s_1 as a free parameter to represent the ambiguity we have encountered.

For completeness we shall discuss the case in which there is an elementary particle coupled to the S-wave channel. Applying Levinson's theorem to this case,[21] we find that the phase shift tends to zero as $s\to\infty$. Therefore, $K(s)$ becomes constant at infinity and the Lehmann representation for the propagator needs no subtraction. We also note that the renormalization constant is nonvanishing because

$$\lim_{s\to\infty} Z_B^{-1}(s)=1+\frac{1}{\pi}\int_0^\infty \frac{|K(s)|^2 s^{1/2}}{(s+s_B)^2}ds<\infty\,. \quad (5.6)$$

Thus, for the simplest solution, in which $K(s)$ has neither zeros nor poles, $Z_B^{-1}(s)$ and $\Gamma(s)$ can be uniquely determined from the denominator. Since $\Gamma(s)$ has no poles, G^2 should satisfy the GI bound (4.19).

Finally we consider the inverse problem of constructing the vertex function, the propagator, and the form factor, when the phase $\eta(s)$ of the function $U(s)$ is given. This has been discussed in a different context by Blankenbecler *et al.*[6] We encounter various ambiguities similar to those we had before. We shall consider only the cases in which the restrictions (III) to (V) are satisfied. Therefore, $\Gamma(s)$ has at least one pole but no zeros. When $\eta(s)$, normalized by $\eta(0)=0$, tends to $-(1/2+n)\pi$ as s goes to infinity, where n is a nonnegative integer, $\Gamma(s)$ can be expressed in the form

$$\Gamma(s)=\frac{G}{Q_n(s)}\frac{s_1-s_B}{s_1+s}\times\exp\left[\frac{s+s_B}{\pi}\int_0^\infty \frac{\eta(s')ds'}{(s'+s_B)(s'-s-i\epsilon)}\right]. \quad (5.7)$$

Here $Q_n(s)$ is an arbitrary polynomial of order n which has at most one zero in the interval, $-s_B<s<0$, and no zeros for $s<-s_B$, and is normalized to unity at $s=-s_B$.

The relation between $\eta(s)$ and $-s_1$ is similar to the one between $\delta(s)$ and $-s_B$. In Sec. 4 we could narrow the possibilities by (I) and (II), but here we do not have any physically plausible reason to make corresponding restrictions. If $U(s)$ has a pole or poles, one may choose one of them as $-s_1$. If this is the case, s_1 can be regarded as essentially given by $\eta(s)$. Of course, one need not do so. $U(s)$ may even have no poles. Then, $-s_1$ is completely independent of $\eta(s)$.

Let us confine ourselves to the solution with as small a number of arbitrary parameters as possible, and assume hereafter that $-s_1$ is the only pole of $\Gamma(s)$ or $Q_n(s)\equiv 1$. From this assumption and (V) it follows that $Z_B(s)$ has no CDD poles other than the one at

[18] V. Bargmann, Rev. Mod. Phys. **21**, 488 (1949).
[19] I. M. Gel'fand and B. M. Levitan, Doklady Akad. Nauk SSSR **77**, 55 (1951).
[20] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952).
[21] See, for instance, M. Ida, Progr. Theoret. Phys. (Kyoto) **21**, 625 (1959).

$s=-s_1$. $Z_B(s)$ can now be written as

$$Z_B(s)=\frac{1}{\pi}\int_0^\infty \frac{|\Gamma(s')|^2(s')^{1/2}ds'}{(s'+s_B)(s'-s-i\epsilon)}+\frac{s_1-s_B}{s_1+s}c. \quad (5.8)$$

From the normalization condition we have

$$1=Z_B(-s_B)=\frac{1}{\pi}\int_0^\infty \frac{|\Gamma(s)|^2 s^{1/2}ds}{(s+s_B)^2}+c, \quad (5.9)$$

which is a nonrelativistic analog of Eq. (2.7). Since we know that $Z_B(s)$ decreases like $s^{-3/2}$ for large s, we also get

$$(s_1-s_B)c=\frac{1}{\pi}\int_0^\infty \frac{|\Gamma(s)|^2 s^{1/2}ds}{s+s_B}. \quad (5.10)$$

By substituting Eq. (5.10) into Eq. (5.9), we obtain[22]

$$1=\frac{1}{\pi}\int_0^\infty \frac{|\Gamma(s)|^2 s^{1/2}ds}{(s+s_B)^2}+\frac{1}{s_1-s_B}\frac{1}{\pi}\int_0^\infty \frac{|\Gamma(s)|^2 s^{1/2}ds}{s+s_B}$$
$$=\frac{1}{\pi}\int_0^\infty \frac{(s_1+s)|\Gamma(s)|^2 s^{1/2}}{(s_1-s_B)(s+s_B)^2}ds. \quad (5.11)$$

The form factor is now given by $K(s)=\Gamma(s)/Z_B(s)$.

The sum rule, (5.11), establishes a relation among s_B, G^2, and s_1. We are thus led to the conclusion that either s_B or G^2 can be chosen independently from $\eta(s)$, when $-s_1$ is a pole of $U(s)$ and, hence, can be regarded as essentially given by $\eta(s)$ and when $\Gamma(s)$ has no poles other than $-s_1$. If s_1 is independent from $\eta(s)$, then both s_B and G^2 can be given arbitrarily. Our procedure discussed above corresponds to that of Blankenbecler *et al.*[6] The inevitable introduction of one CDD pole, however, has made our result different from theirs. The difference comes from our requirement that Levinson's theorem be satisfied.

Some simple but explicit examples given in the Appendix will serve to illustrate the discussions in this section.

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APPENDIX

Simple examples are given here to facilitate the understanding of the general discussion of Sec. 5.

Let us consider S-wave scattering with one bound state, the phase shift of which is exactly given by the effective-range formula, (4.10). By Levinson's theorem, the scattering length a and the effective-range r must have opposite signs. The function $\varphi(s)$ is a constant in our model and we see that it is positive. Therefore, a and r must be such that $a<0$, $r>0$ and $-a/2r>1$. The constant φ is given by

$$\varphi=\frac{|a|}{r}-1-\left[\left(\frac{|a|}{r}-1\right)^2-1\right]^{1/2}<1. \quad (A1)$$

We then have

$$s_B^{1/2}=(1+\varphi)/|a|, \quad (A2)$$

$$G^2=2s_B^{1/2}(1+\varphi)/(1-\varphi). \quad (A3)$$

If we put

$$\varphi=(s_B/s_2)^{1/2}, \quad (A4)$$

where $s_2>s_B$, the S matrix is expressed as

$$S(s)=\frac{(s_B^{1/2}-is^{1/2})(s_2^{1/2}-is^{1/2})}{(s_B^{1/2}+is^{1/2})(s_2^{1/2}+is^{1/2})}. \quad (A5)$$

It has a redundant pole at $s=-s_2$. For the form factor given by Eq. (5.1) with $P(s)\equiv 1$, we have

$$K(s)=G\frac{(s_B^{1/2}-is^{1/2})(s_2^{1/2}-is^{1/2})}{2s_B^{1/2}(s_2^{1/2}+s_B^{1/2})}. \quad (A6)$$

$Z_B^{-1}(s)$ is then given by Eq. (5.3) as

$$Z_B^{-1}(s)=\frac{1}{2s_B^{1/2}(s_2-s_B)}(s_B^{1/2}-is^{1/2})(s_2+s)+\left\{\frac{1}{4s_B}-\frac{1}{s_2-s_B}-d\right\}(s+s_B). \quad (A7)$$

It has a zero below $-s_B$, which we call $-s_1$ as before. (A7) can then be written as

$$Z_B^{-1}(s)=\frac{(s_B^{1/2}-is^{1/2})(s_1^{1/2}+is^{1/2})}{2s_B^{1/2}(s_1^{1/2}-s_B^{1/2})}\times\frac{s_2-s_1+(s_1^{1/2}-s_B^{1/2})(s_1^{1/2}-is^{1/2})}{s_2-s_B}. \quad (A8)$$

The constant d, or equivalently s_1, is quite arbitrary. As the S matrix of our model has a redundant pole, however, we can fix the constant by making the zero of $Z_B^{-1}(s)$ coincide with the redundant pole of $S(s)$, that is, by putting $s_1=s_2$. d is then given a unique value:

$$d=(1/4s_B)-[1/(s_1-s_B)]. \quad (A9)$$

It should be emphasized that the only reason to do this here is to simplify the solution. $Z_B^{-1}(s)$ then becomes

$$Z_B^{-1}(s)=\frac{1}{2s_B^{1/2}(s_1-s_B)}(s_B^{1/2}-is^{1/2})(s_1+s) \quad (A10)$$

[22] This LSZ sum rule should not be identified with the one obtained from the normalization of the wave function for a bound state. For the latter sum rule, see S. Weinberg, Phys. Rev. **130**, 776 (1963).

and the vertex function is

$$\Gamma(s)=G\frac{s_1^{1/2}-s_B^{1/2}}{s_1^{1/2}+is^{1/2}}. \tag{A11}$$

We also find

$$U(s)=-1/(s_1^{1/2}+is^{1/2}). \tag{A12}$$

For the sum rule, (5.11), we have

$$1=\frac{G^2}{2s_B^{1/2}}\left(\frac{s_1^{1/2}-s_B^{1/2}}{s_1^{1/2}+s_B^{1/2}}\right)^2+\frac{2s_B^{1/2}}{s_1^{1/2}+s_B^{1/2}}, \tag{A13}$$

where the second term of the right-hand side of (A13) represents the contribution from the CDD pole of the propagator.

When there is an elementary particle coupled to the S-wave channel, a and r must have the same sign. The constant φ is now negative, and given by

$$\varphi=-\left(\frac{a}{r}+1\right)+\left[\left(\frac{a}{r}+1\right)^2-1\right]^{1/2}>-1. \tag{A14}$$

Therefore, both a and r must be negative. s_B and G^2 are expressed by Eqs. (A2) and (A3), respectively, with φ given by (A14). If we put

$$\varphi=-(s_B/s_2)^{1/2}, \tag{A15}$$

where $s_2>s_B$, the S matrix is

$$S(s)=\frac{(s_B^{1/2}-is^{1/2})(s_2^{1/2}+is^{1/2})}{(s_B^{1/2}+is^{1/2})(s_2^{1/2}-is^{1/2})}. \tag{A16}$$

It has a zero at $s=-s_2$, instead of a redundant pole. The form factor is then given by

$$K(s)=G\frac{s_2^{1/2}+s_B^{1/2}}{2s_B^{1/2}}\frac{s_B^{1/2}-is^{1/2}}{s_2^{1/2}-is^{1/2}}. \tag{A17}$$

From Eq. (5.2) we find

$$Z_B^{-1}(s)=\frac{s_2^{1/2}+s_B^{1/2}}{2s_B^{1/2}}\frac{s_B^{1/2}-is^{1/2}}{s_2^{1/2}-is^{1/2}}. \tag{A18}$$

We thus have

$$\Gamma(s)\equiv G, \tag{A19}$$

$$U(s)\equiv 0. \tag{A20}$$

The sum rule, (5.11), is

$$1=\frac{2s_B^{1/2}}{s_2^{1/2}+s_B^{1/2}}+\frac{G^2}{2s_B^{1/2}}, \tag{A21}$$

where the first term of the right-hand side of (A21) represents the propagator renormalization constant.

We finally consider the inverse problem, in which the phase $\eta(s)$ of $U(s)$ is given by the scattering length formula,

$$s^{1/2}\cot\eta(s)=a_0. \tag{A22}$$

From what we have seen in the text, it follows that a_0 must be negative. $U(s)$ is then given by

$$U(s)=-1/(|a_0|^{-1}+is^{1/2}). \tag{A23}$$

As $U(s)$ has a pole at $s=-a_0^{-2}$, we choose this point as the location of the only pole of $\Gamma(s)$, just to simplify the solution. We call this point $-s_1$ as before. We thus have $s_1=a_0^{-2}$, and $\Gamma(s)$ is then given by Eq. (A11). $Z_B(s)$ is easily calculated, by Eq. (5.8), to be

$$Z_B(s)=2s_B^{1/2}(s_1-s_B)/(s_B^{1/2}-is^{1/2})(s_1+s), \tag{A24}$$

and the form factor is given by (A6). The sum rule, (5.11), now becomes

$$\begin{aligned}1&=(G^2/2s_B^{1/2})[(s_1^{1/2}-s_B^{1/2})/(s_1^{1/2}+s_B^{1/2})]^2\\&\quad+G^2(s_1^{1/2}-s_B^{1/2})/(s_1^{1/2}+s_B^{1/2})^2\\&=(G^2/2s_B^{1/2})[(s_1^{1/2}-s_B^{1/2})/(s_1^{1/2}+s_B^{1/2})].\end{aligned} \tag{A25}$$

Therefore, it determines G^2 only if s_B is given.