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Implications of Approximate SU_3 Symmetry and Mass Formulas for the Mesons*

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The connection between the limit of perfect SU_3 symmetry and the zero mass approximation for the pseudoscalar meson octet is discussed. The well-known quadratic mass formula for the pseudoscalar meson octet and the linear mass formula for the baryons are derived. A simple model of strong interaction is presented which leads to two mass formulas for the nine spin-1 mesons:

$$2M_\phi+M_\rho+M_\omega=4M_{K^*} \quad\text{and}\quad M_\rho\cong M_\omega.$$

The general invariance property of this model which contains, among others, a baryon octet, a pseudoscalar octet, and a vector nonet is examined.

I. GENERAL DISCUSSION

THE question whether the strong interactions are approximately invariant under a SU_3 transformation has been discussed extensively in the recent literature.[1] There exists by now a rather impressive body of experimental evidence[2] supporting such an approximate invariance. Part of this evidence is based on the remarkably accurate "mass formulas" which are obtained[3] by treating, among other assumptions, the symmetry violating interactions as small. On the other hand, since the mass differences among, say, K, π, and η which are members of the same multiplet are not small compared to their actual masses, the violation of SU_3 symmetry is apparently not weak. The additional fact that these mass formulas are linear functions of masses for the baryons, but quadratic functions for the mesons, seems to further veil the foundation of such an approximate symmetry. In this paper, we shall attempt to clarify some of these questions.

The existence of an approximate symmetry under SU_3 for the strong interactions suggests that the total Hamiltonian H contains a part, called primary interaction, H_0, which is invariant under SU_3, and represents the main contribution to the interaction. The remaining part, called secondary interaction, $h=H-H_0$ is small compared to H_0 and has symmetry violating properties. The *typical* energy scale[4] (or level spacing) due to the primary interaction H_0 is $\sim M$ which is assumed to be of the order of a few BeV. The energy shift caused by the secondary interaction h is $\sim m$ which is of the order of a few hundred MeV's. The smallness of the phenomenological dimensionless constant

$$\lambda\equiv(m/M)\ll 1 \qquad (1)$$

makes possible the use of h as a perturbation.

We shall further specify the primary interaction H_0 by assuming that it contains eigenstates corresponding to an octet of pseudoscalar *mesons of zero mass.* The perturbation h can be written as

$$h=h_0+h_1, \qquad (2)$$

where we assume that h_0 is invariant[5] under SU_3 and h_1 transforms like the isotopic spin$=0$ component of the octet representation[1] of SU_3. Both H_0 and h transform in the usual way under the inhomogeneous Lorentz transformation and conserve charge, parity, strangeness, etc.

We now apply these considerations to the mass shifts of the pseudoscalar meson octet. Let η, π, K denote the familiar members of this octet and E_η, E_K, E_π their respective energies. By our assumption, these mesons have zero mass in the absence of h (i.e., $\lambda=0$). Therefore, the zeroth-order values of the energies $E_\eta(p)$, $E_K(p)$, $E_\pi(p)$ are given by

$$E_\eta{}^0(p)=E_K{}^0(p)=E_\pi{}^0(p)=p, \qquad (3)$$

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[1] M. Gell-Mann, California Institute of Technology Report CTSL-20, 1961 (unpublished). Y. Ne'eman, Nucl. Phys. **26**, 222 (1961). M. Gell-Mann, Phys. Rev. **125**, 1067 (1962). See references in these papers for earlier work on SU_3. Cf. also D. Speiser and A. Tarski, J. Math. Phys. **4**, 588 (1963). [An earlier reference to this work in connection with SU_3 symmetry can be found in footnote 14 of a paper by T. D. Lee and C. N. Yang, Phys. Rev. **122**, 1954 (1961).]

[2] See especially, S. L. Glashow and A. H. Rosenfeld, Phys. Rev. Letters **10**, 192 (1963). V. E. Barnes, P. L. Connolly, D. J. Crennell *et al.*, *ibid.* **12**, 204 (1963).

[3] M. Gell-Mann, Ref. 1, S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962); **28**, 64 (1962). See also J. J. Sakurai and S. Glashow, Nuovo Cimento **25**, 337 (1962).

[4] It is difficult to give a precise definition of the typical energy scale without a concrete picture of the dynamics. For a more detailed discussion on the meaning of this energy scale M associated with the primary interaction, see the special model given in Sec. II and the related example discussed in Appendix II.

[5] We are reminded of the analogy with the electromagnetic interaction which, while violating the isotopic spin conservation, contains a part that is invariant under the isotopic spin rotation. Similarly, the weak interactions, which do not conserve strangeness, nevertheless have a part, such as the β decay, that does conserve strangeness. In analogy with the relative magnitude of the isoscalar and the isovector terms of the electromagnetic interactions, we assume h_0 and h_1 to be of comparable magnitude.

where p is the magnitude of the three-momentum of the particles. The application of h conserves the three-momentum p but brings an energy shift to each of these particles.

To use perturbation theory we must choose p to be nonzero and fixed. Mathematically, to first order in h, as a result of its assumed transformation property, these energies satisfy the linear relation

$$3E_\eta(p)+E_\pi(p)=4E_K(p). \quad (4)$$

From Lorentz invariance, any first-order correction to $E_i(p)$ must be of the form[6]

$$\delta E_i(p)=E_i(p)-p=m_i^2/(2p), \quad (5)$$

where $i=\eta$, K or π and m_i is the corresponding mass of the particle. Therefore, we obtain

$$3m_\eta^2+m_\pi^2=4m_K^2, \quad (6)$$

which is valid also to the first order in h. Thus, we are led to the mass-squared formula by the requirement of Lorentz invariance and the assumption of zero mass for the pseudoscalar mesons in the unperturbed situation.

In contrast, the linear mass formula for the baryon octet follows if the corresponding baryon mass eigenvalue M_0 of the primary interaction H_0 is large compared to the mass shifts due to the perturbation h. In that case the zeroth-order energy of the baryon octet (N,Λ,Σ,Ξ) is given by

$$E_N{}^0(p)=E_\Lambda{}^0(p)=E_\Sigma{}^0(p)=E_\Xi{}^0(p)=(p^2+M_0{}^2)^{1/2}. \quad (7)$$

To first order in h, the perturbed energies satisfy the following relation:

$$3E_\Lambda(p)+E_\Sigma(p)=2[E_N(p)+E_\Xi(p)]. \quad (8)$$

By using $E_i=(p^2+M_i{}^2)^{1/2}$, the first-order correction in $E_i(p)$ must be of the form

$$\delta E_i=E_i-E_i{}^0=(p^2+M_0{}^2)^{-1/2}M_0\delta M_i, \quad (9)$$

where $M_i=M_0+\delta M_i$ and $i=N$, Λ, Σ, Ξ. Hence, Eq. (8) can be written as

$$3M_\Lambda+M_\Sigma=2(M_N+M_\Xi). \quad (10)$$

[Note that to first order in h, (10) is identical with $3M_\Lambda{}^2+M_\Sigma{}^2=2(M_\Xi{}^2+M_N{}^2)$.]

Similar considerations can be easily extended to other multiplets. The difference between a quadratic and linear mass formula lies solely in the different magnitude of the masses of these multiplets in the absence of the perturbation h.

In this connection it is interesting to note that zero mass pseudoscalar mesons appear also in certain theories with γ_5 invariance.[7] In the absence of h (i.e., $\lambda=0$), because of the zero mass character of pseudoscalar mesons, the axial vector currents in weak decays can be conserved.[8] This conservation law is broken due to the presence of the secondary interaction h. Since the scale of the primary interaction M is also much larger than the nucleon mass m_N, we expect the ratio of the Gamow-Teller coupling constant G_A to the Fermi coupling constant G_V to be of the form[9]

$$G_A/G_V=-1+O(m_N/M), \quad (11)$$

which makes it reasonable to assume the scale M to be of the order[10] of 10 BeV. Furthermore, in the absence of h, the well-known Goldberger-Treiman formula[11] holds.[9] Deviation from the Goldberger-Treiman formula occurs only as a result of the secondary interaction and is therefore relatively small.

II. A MODEL

It seems natural to explore the possibility that the large energy scale M of the primary Hamiltonian H_0 and its invariance character under SU_3 indicate the existence of some massive triplets with very strong interactions which can form compound systems corresponding to the known octets and decuplets.[12]

In this section we give a special model of the primary and the secondary interactions, partly to illustrate some of the discussions given in the previous section. We assume the existence of two massive triplets $(\alpha_0,\alpha_1,\alpha_2)$ and $(\beta_0,\beta_1,\beta_2)$ under the SU_3 transformations. The charges of $(\alpha_0,\alpha_1,\alpha_2)$, and $(\beta_0,\beta_1,\beta_2)$, are given, respectively, by $(q,\ q+1,\ q)$ in units of e. The baryon numbers of the α's and the β's are, respectively, n and $(n+1)$ where q and n are any integers including zero. One of these triplets, say, β_i consists of fermions while the α's represent bosons.[13] Under the isotopic spin rotations, α_0 and β_0 behave like states with $I=0$; (α_1,α_2), and (β_1,β_2) like states with $I=\frac{1}{2}$. Let $\bar{\alpha}_i$ and $\bar{\beta}_i$ be the antiparticles of α_i and β_i. The primary interaction H_0 can be written as

$$H_0=H_{\rm free}+H_{\rm int}, \quad (12)$$

[6] For the physical case, in order that Eqs. (4) and (5) be a good approximation we must choose states with $p\gg m_i$. The resulting formula (6) is obviously independent of p. In Appendix I we give a trivial but explicit example illustrating the use of the perturbation method.

[7] See, e.g., Y. Nambu, Phys. Rev. **122**, 345 (1961).

[8] J. C. Taylor, Phys. Rev. **110**, 1216 (1956), J. C. Polkinghorne, Nuovo Cimento **8**, 179 and 781 (1958). Y. Nambu, Ref. 7, Y. Nambu, Phys. Rev. Letters **4**, 380 (1960). F. Gürsey, Nuovo Cimento **16**, 230 (1960); Ann. Phys. (N.Y.) **12**, 91 (1061). M. Gell-Mann and M. M. Lévy, Nuovo Cimento **16**, 705 (1960).

[9] J. Bernstein, S. Fubini, M. Gell-Mann, and W. Thirring, Nuovo Cimento **17**, 757 (1960). F. Gürsey, Ref. 8, Y. Nambu and G. Jona-Lasinio, Phys. Rev. **122**, 345 (1961) and other previous references by Y. Nambu.

[10] Dr. C. S. Wu has kindly informed us that recently C. P. Bhalla (unpublished)has re-examined the *Ft* value of neutron and obtained $(G_A/G_V)=-1.15\pm0.02$. We wish to thank Dr. Wu for this information.

[11] M. L. Goldberger and S. B. Treiman, Phys. Rev. **110**, 1178 (1958).

[12] While this work was in progress, we received a reprint by J. Schwinger in which some similar ideas are discussed. We wish to thank Professor Schwinger for communicating his results to us before publication.

[13] An equally adequate model can be constructed if we exchange the roles of α's and β's.

where

$$H_{\text{free}}=\sum_{\mathbf{k},i}\{(M^2+k^2)^{1/2}[a^{\dagger i}(\mathbf{k})a_i(\mathbf{k})+a_i'^{\dagger}(\mathbf{k})a'^i(\mathbf{k})]$$
$$+(M'^2+k^2)^{1/2}[b^{\dagger i}(\mathbf{k})b_i(\mathbf{k})+b_i'^{\dagger}(\mathbf{k})b'^i(\mathbf{k})]\}\,, \quad (13)$$

M and M' are both of the order of a few BeV, representing the zeroth-order (i.e., $h=0$) masses of α_i and β_i. The $a_i(\mathbf{k})$, $a'^i(\mathbf{k})$, $b_i(\mathbf{k})$, $b'^i(\mathbf{k})$ are respectively the annihilation operators for α_i, $\bar{\alpha}_i$ and β_i, $\bar{\beta}_i$ with momentum $\mathbf{k}$, and $a^{\dagger i}(\mathbf{k})$, $a_i'^{\dagger}(\mathbf{k})$, $b^{\dagger i}(\mathbf{k})$, $b_i'^{\dagger}(\mathbf{k})$ are their Hermitian conjugates:

$$\begin{aligned}&[a_i(\mathbf{k})]^{\dagger}=a^{\dagger i}(\mathbf{k})\,,\quad [a'^i(\mathbf{k})]^{\dagger}=a_i'^{\dagger}(\mathbf{k})\,,\\ &[b_i(\mathbf{k})]^{\dagger}=b^{\dagger i}(\mathbf{k})\quad\text{and}\quad [b'^i(\mathbf{k})]^{\dagger}=b_i'^{\dagger}(\mathbf{k})\,,\end{aligned} \quad (14)$$

where † denotes Hermitian conjugation and $i=0$, 1, 2. In all these formulas, the spin dependence is suppressed.

The particular form of H_{int} is not too relevant at this point. We only require that there are strong attractive forces between α_i and $\bar{\alpha}_j$, β_i and $\bar{\beta}_j$, and also between $\bar{\alpha}_i$ and β_j. The forces between all other pairs such as (α_i,α_j) or (β_i,β_j) are assumed to be repulsive. As a result of these strong attractive forces the zero mass pseudoscalar meson octet of H_0 can be regarded as a composite system of α_i and $\bar{\alpha}_j$ and is denoted by $(\bar{\alpha}\alpha)_8$ where the subscript 8 indicates the dimension of the group representation. Similarly, we regard all other known strongly interacting particles also as composite systems of α_i and β_j. We denote $(\bar{\beta}\beta)_8$, $(\bar{\beta}\beta)_1$, and $(\bar{\alpha}\beta)_8$ to be the appropriate bound states of $\bar{\beta}_i$, β_j and of $\bar{\alpha}_i$, β_j which transform like the spin-1 meson octet, the spin-1 meson, and the baryon octet,[14] respectively. In contrast to the pseudoscalar meson states, the baryon octet has a large mass M_0 in the absence of the secondary interaction. Furthermore, we assume the force between α_i and $\bar{\alpha}_j$ depends on the representation; as a consequence, the pseudoscalar singlet $(\bar{\alpha}\alpha)_1$, if it exists, may have a different energy from the pseudoscalar octet $(\bar{\alpha}\alpha)_8$.

Let us first consider a particularly simple example for the secondary interaction h where the only violation of SU3 symmetry occurs through additional mass shifts m_i and m_i' (which may be of the order of a few hundred MeV) for the triplet particles α_i and β_i, respectively. We are led to the following simple expression for h:

$$h=h_\alpha+h_\beta\,, \quad (15)$$

where

$$h_\alpha=\sum_{\mathbf{k}}\sum_{i=0}^{2}\{(M^2+k^2)^{-1/2}Mm_i \times[a^{\dagger i}(\mathbf{k})a_i(\mathbf{k})+a_i'^{\dagger}(\mathbf{k})a'^i(\mathbf{k})]\} \quad (16)$$

and h_β is given by

$$\sum_{\mathbf{k}}\sum_{i=0}^{2}\{(M'^2+k^2)^{-1/2}M'm_i' \times[b^{\dagger i}(\mathbf{k})b_i(\mathbf{k})+b_i'^{\dagger}(\mathbf{k})b'^i(\mathbf{k})]\}\,, \quad (17)$$

where the isotopic spin conservation is insured by choosing

$$m_1=m_2\quad\text{and}\quad m_1'=m_2'\,. \quad (18)$$

Since we have already assumed that the primary SU3 invariant Hamiltonian H_0 contains, besides the free term (13), also an interaction term that lifts the degeneracy between singlet and octet states for the pseudoscalar mesons and the baryons, it now follows from the transformation property of (15) that the mass formulas such as (6) and (10) hold for the pseudoscalar meson octet $(\alpha\bar{\alpha})_8$ the baryon octet $(\beta\bar{\alpha})_8$, respectively.

To discuss the mass formula for the spin-1 meson states we impose a further condition on the primary interaction. We require that the strong attractive forces between β_i and $\bar{\beta}_j$ that result from H_0 are essentially *independent*[15] of i and j, very much like the Wigner forces for nucleons. Therefore, in the absence of h, the states $(\bar{\beta}\beta)_8$ and $(\bar{\beta}\beta)_1$ are degenerate. For definiteness, we assume the mass of these states to be fairly large.

Most of the degeneracies among these nine states are removed to the first order in h. Let ρ, ω, K^* and ϕ be the physical resonances and $(\bar{\beta}_i\beta_j)$ the bound-state wave function of the system $\bar{\beta}_i$ and β_j. Using the Eqs. (15) and (18) we make the following identifications for the vector meson states:

$$\begin{aligned}|\phi\rangle&=\bar{\beta}_0\beta_0\,,\\ |\omega\rangle&=2^{-1/2}(\bar{\beta}_1\beta_1+\bar{\beta}_2\beta_2)\,,\\ |\rho^+\rangle&=-\bar{\beta}_2\beta_1\,,\end{aligned} \quad (19)$$

and

$$|(K^*)^+\rangle=\bar{\beta}_0\beta_1\,,$$

where $(\bar{\beta}_i\beta_j)$ has the same transformation property as the state $b_i'^{\dagger}b^{\dagger j}|0\rangle$. The other five states ρ^-, ρ^0, $(K^*)^0$, etc., can be obtained from ρ^+ and $(K^*)^+$ through the use

[14] The choice of the baryon decuplet is rather free at this stage. In a model it is easy to invent selection rules such that in the absence of the secondary interaction h, the baryon decuplet is completely stable. As an example, we may arbitrarily assume the existence of an SU3 fermion singlet γ with baryon number 1, but charge -1. The compound state $(\alpha^3\gamma)_{10}$ can represent the baryon decuplet. By adding an appropriate term to Eq. (15) for the secondary interaction h, both the energy shifts and the transitions of the decuplet can be regarded as caused by h. To first order in h, we have the well-known equal level spacing formula for the decuplet. The width of these levels are second order in h. Therefore, the use of perturbation theory could be easily justified. (However, if we compare these results with their experimental values, the fact that the width of N^*, the $I=\frac{3}{2}$ member of the baryon decuplet, is not much smaller than the decuplet energy spacings makes it questionable whether this particular example of the decuplet has much value other than being a simple concrete mathematical example. We wish to thank K. Huang and F. E. Low for pointing out the importance of the large width to us.)

[15] As a model, we may assume the existence of an additional neutral (SU3) singlet field χ. The primary forces between $\bar{\beta}_i$ and β_j are generated by a Lagrangian of the form $\chi\sum_i b^{\dagger i}b_i$ plus a similar term for the antiparticle $\bar{\beta}_i$. Therefore, the primary forces between $\bar{\beta}_i$ and β_j are independent of i and j. On the other hand, because of the difference in spin and statistics the forces between, say, $\bar{\alpha}_i$ and α_j are due to a completely different origin and do depend on the particular singlet or octet representation of $(\bar{\alpha}\alpha)$.

of isotopic spin rotation and charge conjugation. In terms of the two $I=0$ states ω and ϕ, the unitary singlet $(\bar{\beta}\beta)_1$ becomes[16]

$$(\bar{\beta}\beta)_1=(\tfrac{1}{3})^{1/2}(\bar{\beta}_0\beta_0+\bar{\beta}_1\beta_1+\bar{\beta}_2\beta_2)=(\tfrac{1}{3})^{1/2}\phi+(\tfrac{2}{3})^{1/2}\omega\,, \quad (20)$$

while the $I=0$ member of the octet is given by

$$(\tfrac{1}{6})^{1/2}(2\bar{\beta}_0\beta_0-\bar{\beta}_1\beta_1-\bar{\beta}_2\beta_2)=(\tfrac{2}{3})^{1/2}\phi-(\tfrac{1}{3})^{1/2}\omega\,. \quad (21)$$

From (21) and the general transformation property of h [Eq. (2)], it follows (as will be proved under more general conditions in the next section) that the masses of these resonances satisfy[16a]

$$(2M_\phi+M_\omega)+M_\rho=4M_{K^*}\,, \quad (22)$$

which agrees very well with the known experimental values of these masses. [Note that similarly to the baryon mass formula Eq. (10), to first order in h Eq. (22) can also be written as $2M_\phi^2+M_\omega^2+M_\rho^2=4M_{K^*}^2$.] If we use (15) as the explicit form of h, then these masses can be explicitly calculated in terms of m_0' and m_1' of the triplet masses. We find, by using (19),

$$M_\phi=M_1+2m_0'\xi\,,$$
$$M_\omega=M_1+2m_1'\xi\,,$$
$$M_\rho=M_1+(m_1'+m_2')\xi\,,$$

and

$$M_{K^*}=M_1+(m_0'+m_1')\xi\,, \quad (23)$$

where M_1 is the zeroth-order mass of these nine states and ξ is the expectation value

$$\langle\textstyle\sum_{\mathbf{k}}(M'^2+k^2)^{-1/2}M'b^{\dagger 0}(\mathbf{k})b_0(\mathbf{k})\rangle \quad (24)$$

evaluated for the state $(\bar{\beta}_0\beta_0)$ in its center-of-mass system. The precise numerical value of ξ depends on the wave function $(\bar{\beta}_0\beta_0)$. (See Appendix II for an example of such a calculation.) From (23) and (18) we obtain another interesting relation:

$$M_\rho=M_\omega\,, \quad (25)$$

which agrees reasonably well with the experimental values.

In the above, the mass formulas (23) and (25) are derived from the simple form (17) for h_β. Actually these formulas are valid under a general class of secondary interactions. We may consider, in addition to (17), the h_β part of the secondary interaction to contain a general quartic dependence of the $b_i(\mathbf{k})$, $b'^i(\mathbf{k})$ and their Hermitian conjugates. Let h_β be given by

$$h_\beta=(17)+\lambda_1P_0{}^0+\lambda_2(P_1{}^1+P_2{}^2)+\lambda_3Q_0{}^0+\lambda^4(Q_1{}^1+Q_2{}^2)\,, \quad (26)$$

where

$$P_i{}^j=(b^{\dagger l}b_l)b_i'^{\dagger}b'^j+b^{\dagger j}b_i(b_l'^{\dagger}b'^l)\,, \quad (27)$$

and

$$Q_i{}^j=\epsilon_{ikl}\epsilon^{jmn}(b_m'^{\dagger}b'^k)(b^{\dagger l}b_n)\,, \quad (28)$$

in which we suppress the momentum dependence and sum over all repeated indices. The λ_1, λ_2, λ_3, λ_4 are constants and ϵ_{ijk}, ϵ^{ijk} are the third-rank antisymmetric tensors whose only nonvanishing components are ±1 depending on (ijk) being even or odd permutations of (012). It can be readily verified that the mass formula (22) holds if $\lambda_4=0$ and the mass formula (25) holds if, in addition, $\lambda_3=0$. Comparison with the experimental values of these masses indicate $\lambda_4=0$ and $\lambda_3\cong0$.

Physically, the mass formula (22) holds if the secondary forces between $(\bar{\beta}_i\beta_j)$ do not convert the pair $(\bar{\beta}_0\beta_0)$ into any other pairs such as $(\bar{\beta}_1\beta_1)$ and $(\bar{\beta}_2\beta_2)$. The mass formula (25) holds if, in addition, the secondary forces between $(\bar{\beta}_1\beta_1)$, $(\bar{\beta}_2\beta_2)$ and $(\bar{\beta}_2\beta_1)$ are approximately independent of the total isotopic spin $I=0$ or 1 of the compound system.

These massive triplets α_i and β_i, if they exist, can only be strongly produced in pairs by using the octet particles as the beam and the target. Their decays depend on the particular form of the secondary interaction h. If the SU_3 symmetry violating part of this secondary interaction transforms like the isotopic singlet member of an octet, then these two triplets would contain stable members with respect to strong interactions. If, in addition, the usual vector and axial vector currents of the strongly interacting particles that occur in weak interactions also transform like members of an octet, then the basic triplets would contain members that are absolutely stable.

III. FURTHER DISCUSSIONS OF THE MODEL

In this section we wish to investigate in the above model some general conditions under which the primary interaction H_0 should contain nine degenerate $(\bar{\beta}\beta)$ states but only eight degenerate $(\bar{\alpha}\alpha)$ states and eight degenerate $(\bar{\alpha}\beta)$ states. It is useful to introduce the following operators:

$$T_i{}^j=\textstyle\sum_{\mathbf{k}}[a^{\dagger j}(\mathbf{k})a_i(\mathbf{k})-a_i'^{\dagger}(\mathbf{k})a'^j(\mathbf{k})+b^{\dagger j}(\mathbf{k})b_i(\mathbf{k})-b_i'^{\dagger}(\mathbf{k})b'^j(\mathbf{k})]\,, \quad (29)$$

$$U_i{}^j=T_i{}^j-\tfrac{1}{3}\delta_i{}^jT_k{}^k\,, \quad (30)$$

$$S_i{}^j=\eta\textstyle\sum_{\mathbf{k}}[b^{\dagger j}(\mathbf{k})b_i(\mathbf{k})+b_i'^{\dagger}(\mathbf{k})b'^j(\mathbf{k})]\,, \quad (31)$$

and

$$V_i{}^j=S_i{}^j-\tfrac{1}{3}\delta_i{}^jS_k{}^k\,, \quad (32)$$

where

$$\eta=\tfrac{1}{2}[1+(-1)^{N_\alpha}]\,, \quad (33)$$

and

$$N_\alpha=\textstyle\sum_{\mathbf{k}}[a^{\dagger i}(\mathbf{k})a_i(\mathbf{k})-a_i'^{\dagger}(\mathbf{k})a'^i(\mathbf{k})]\,. \quad (34)$$

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[16] The ϕ and ω mesons are therefore mixtures of a unitary singlet and the $I=0$ member of an octet. The mixing angle obtained through (20) and (21) agrees numerically with the one calculated on semiempirical ground by J. J. Sakurai [Phys. Rev. **132**, 434 (1963)]. See also M. Gell-Mann (Ref. 1); S. Okubo, Phys. Letters **5**, 165 (1963).

[16a] *Note added in proof*. After this paper was submitted for publication, we received an unpublished report by G. Zweig (An SU_3 model for strong interaction symmetry and its breaking) in which a similar mass formula for the vector mesons is also discussed in the model of a single triplet with fractional charge and baryon number.

All repeated indices are to be summed over. The U_i^j are the generators of the SU$_3$ transformation and satisfy the commutation relation

$$[U_i^j,U_l^k]=\delta_i^k U_l^j-\delta_l^j U_i^k. \tag{35}$$

The invariance property of the primary interaction H_0 is given by the commutation relation

$$[H_0,U_i^j]=0. \tag{36}$$

For clarity, we will restrict our discussions to those eigenstates of H_0 which consist of only the free single-particle states of either α_i, or β_i (or $\bar{\alpha}_i$, $\bar{\beta}_i$) and the various bound states of these particles. Let the totality of these eigenstates span a subdomain R in the entire Hilbert space. The projection $O(R)$ of any operator O in R is defined by

$$\langle\nu|O(R)|\mu\rangle=\langle\nu|O|\mu\rangle \tag{37}$$

if both states $|\mu\rangle$ and $|\nu\rangle$ are in R; otherwise,

$$\langle\nu|O(R)|\mu\rangle=0. \tag{38}$$

Theorem 1. If

$$[H_0,V_0{}^0(R)]=0, \tag{39}$$

then under the primary interaction, $(\bar{\beta}\beta)_1$ must be degenerate with $(\bar{\beta}\beta)_8$. However, Eq. (39) does not imply any degeneracy between $(\bar{\alpha}\beta)_1$ and $(\bar{\alpha}\beta)_8$, nor between $(\bar{\alpha}\alpha)_1$ and $(\bar{\alpha}\alpha)_8$.

Proof: From (33), it follows that $\eta=0$ or 1 depending on N_α=odd or even. Let the states $|\phi\rangle$ and $|\omega\rangle$ be defined by Eq. (19). We find

$$V_0{}^0|\phi\rangle=\tfrac{4}{3}|\phi\rangle,$$
$$V_0{}^0|\omega\rangle=-\tfrac{2}{3}|\omega\rangle,$$
$$V_0{}^0|\rho\rangle=-\tfrac{2}{3}|\rho\rangle,$$

and

$$V_0{}^0|K^*\rangle=\tfrac{1}{3}|K^*\rangle. \tag{40}$$

From the definition Eq. (37), it is clear that identical equations are satisfied by $V_0{}^0(R)$. We note that the mass splittings given by Eq. (23) are proportional to the values of $V_0{}^0$ for these states. Since $V_0{}^0$ [therefore, also $V_0{}^0(R)$] does not commute with either $U_0{}^i$ or $U_i{}^0$, a necessary consequence of the commutation relations (36) and (39) is that $(\bar{\beta}\beta)_1$ must be degenerate with respect to $(\bar{\beta}\beta)_8$ under the primary interaction H_0.

In contrast, $(\bar{\alpha}\beta)_1$, $(\bar{\alpha}\beta)_8$, $(\bar{\alpha}\alpha)_1$, and $(\bar{\alpha}\alpha)_8$ are all eigenstates of $V_0{}^0$ and $V_0{}^0(R)$ with the same eigenvalue 0. Thus, Eq. (39) does not imply any additional degeneracy, and the only degeneracy is that required by the SU$_3$ invariance. Under the primary interaction H_0, there are eight degenerate spin-$\frac{1}{2}$ baryon states $(\bar{\alpha}\beta)_8$, eight degenerate pseudoscalar meson states $(\bar{\alpha}\alpha)_8$ but *nine* degenerate spin-1 meson states $(\bar{\beta}\beta)_1$ and $(\bar{\beta}\beta)_8$.

The introduction of the secondary interaction h splits these degeneracies. According to Eq. (2), under the SU$_3$ transformations generated by U_i^j, the secondary interaction $h=h_0+h_1$ transforms like a sum of an SU$_3$ invariant term h_0 and another term h_1 which transforms like isotopic spin=0 member of an octet. For the octets $(\bar{\alpha}\alpha)_8$ and $(\bar{\alpha}\beta)_8$ the mass formulas Eqs. (6) and (10) hold. The following theorem gives a condition under which the mass formula Eq. (22) for the nine spin-1 meson states holds:

Theorem 2. If

$$[h(R),V_0{}^0(R)]=0, \tag{41}$$

then to first order in h, the mass formula

$$2M_\phi+M_\omega+M_\rho=4M_{K^*}$$

holds.

Proof. By using Eqs. (37), (40), and (41), it follows that

$$\langle\omega|h|\phi\rangle=\langle\omega|h(R)|\phi\rangle=0. \tag{42}$$

Therefore, Eq. (21) remains correct. Theorem 2 is then established by using Eq. (21), together with the general transformation property of h under the SU$_3$ group generated by U_i^j.

So far, our discussions are restricted to the single-particle states and to the bound states of H_0. The extension of some of the above considerations to the multi-particle states leads to several unusual results. Suppose we replace the projection $V_0{}^0(R)$ in Eq. (39) by the entire operator $V_0{}^0$, and assume that

$$[H_0,V_0{}^0]=0 \tag{43}$$

is valid. Now, H_0 commutes with U_i^j which, however, does not commute with $V_0{}^0$. As proved in Appendix III, Eqs. (36) and (43) necessitate the invariance of the primary interaction H_0 under a group G=SU$_3$×SU$_3$×SU$_3$ which has, among other irreducible representations, a $(\bar{\beta}\beta)$ nonet, a $(\bar{\alpha}\alpha)$ octet and a $(\bar{\alpha}\beta)$ octet. The group G also contains the particular SU$_3$ group whose generators are the nonlocal operators V_i^j given by Eq. (32). It is also shown in Appendix III that several unphysical consequences, such as violation of the asymptotic condition and the violation of crossing symmetry occur as a result. Therefore, Eq. (43) cannot be strictly correct. Nevertheless, it is interesting to explore other physical consequences by assuming the approximate validity of (43). Let v be the eigenvalue of $V_0{}^0$. From (43), we have the approximate selection rule

$$\Delta v=0. \tag{44}$$

The $(\bar{\alpha}\alpha)_8$ states are eigenstates of $V_0{}^0$ with eigenvalue $v=0$. Thus, the decay of the $(\bar{\beta}\beta)$ nonet is forbidden under the primary interaction H_0. The decays of these particles such as $\rho\rightarrow 2\pi$, etc., occur through the secondary interaction h. Next, let us consider the compound state $(\bar{p}+p)$ which has $v=-\frac{2}{3}$. The conservation law (44) and Eq. (40) yield the following types of approximate selection rules:

$$\bar{p}+p \nleftrightarrow l\pi+mK+n\eta, \tag{45}$$

$$\bar{n}+p \nleftrightarrow K^*+l\pi+mK+n\eta, \tag{46}$$

and

$$\bar{p}+p \nrightarrow \phi+l\omega+mK+n\eta, \quad (47)$$

where $l\pi$, mK, $n\eta$ denote arbitrary numbers of π, K (or $\bar{K}$), and η. On the other hand, reactions

$$\bar{p}+p \rightarrow \rho+l\pi+mK+n\eta \quad (48)$$

and

$$\bar{p}+p \rightarrow \omega+l\pi+mK+n\eta \quad (49)$$

are allowed, provided other quantum numbers such as strangeness and isotopic spin are conserved. Similarly, we can consider the compound state $(\pi+p)$ and derive the approximate selection rule

$$\pi+p \nrightarrow (\bar{\beta}\beta)_9+(\bar{\alpha}\beta)_8+l\pi+mK+n\eta, \quad (50)$$

where $(\bar{\beta}\beta)_9$ and $(\bar{\alpha}\beta)_8$ represent, respectively, any member of the vector nonet and the baryon octet. The realistic value of such extension of the model is expected to be limited. As mentioned earlier, there are severe inherent difficulties of the commutation rule (43) which must throw doubt on the approximate validity of Eq. (44) for these collision processes. Furthermore, additional violation can occur through the secondary interaction h which can give sizeable contributions to the rates for these reactions. It is, therefore, rather surprising to find some of the above results, Eqs. (47)–(49), concerning the annihilation process $(\bar{p}+p)$ seem to be approximately correct.[17]

IV. CONCLUDING REMARKS

The above discussions on strong interactions several topics have been discussed, which are related but of rather different speculative nature. If the approximate SU_3 symmetry has a fundamental basis, there should exist two entirely different classes of strong interactions: the SU_3 symmetric primary interaction and the nonsymmetric secondary interaction, the former being much stronger than the latter. The fact that the masses of the known pseudoscalar mesons are of the same order of magnitude as their energy differences makes it attractive to regard all these meson masses as generated by the same secondary interaction. The zero mass requirement for the pseudoscalar meson octet gives a clear phenomenological definition of the primary interaction.

The connection between SU_3 symmetry and the approximate zero mass nature of pseudoscalar mesons leads to an understanding of the mass-squared formula for the pseudoscalar meson octet; it also ties in with work done in the last few years by many authors who consider the zero mass approximation independently of SU_3 symmetry (e.g., conserved axial vector current,[8] Goldberger-Treiman formula,[9,11] emission of soft pions,[18] etc.).

As already emphasized in Sec. II, the invariance character of the primary interaction under the SU_3 transformation and the existence of a large energy scale $\sim$ a few BeV's seems to lead rather naturally to a model containing some massive triplets with very strong interactions. In the model, so far as the transformation properties of the known meson and baryon states are concerned, they can be regarded as the same as the composite systems of these triplets. While many different models can be constructed,[19] the fact that there are nine vector states but only eight pseudoscalar states makes it attractive to consider a model which has two different triplets α_i and β_i. The derivation of the mass formula Eq. (22) for the nine vector states gives added support to the general character of this particular model. If the approximate degeneracy of these nine states is regarded as nonaccidental, then the primary interaction should be invariant under a group much larger than a simple SU_3.

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APPENDIX I: A TRIVIAL EXAMPLE

In order to illustrate the use of perturbation formula for particles which, in the absence of perturbation, have zero masses, we consider a totally trivial example of an octet of free spin-0 particles. Let ϕ_μ $(\mu=1, 2, \cdots 8)$ represent the Hermitian field operators of these particles. The Lagrangian L_0 for the primary interaction is

$$L_0=-\frac{1}{2}\int \sum_{\lambda=1}^{4}\sum_{\mu=1}^{8}\left(\frac{\partial\phi_\mu}{\partial x_\lambda}\right)d^3r \quad (I1)$$

and the perturbation Lagrangian is

$$-h=-\frac{1}{2}\int \sum_{\mu} m_\mu{}^2\phi_\mu{}^2 d^3r, \quad (I2)$$

where h is the Hamiltonian for the secondary interaction and x_λ are the usual space-time coordinates. It is convenient to expand the ϕ_μ in terms of the annihilation operator $a_\mu(k)$ and their Hermitian conjugates $a_\mu{}^\dagger(k)$:

$$\phi_\mu(\mathbf{r})=\sum_{\mathbf{k}}(2\Omega k)^{-1/2}[a_\mu(\mathbf{k})\exp(i\mathbf{k}\cdot\mathbf{r}) + a_\mu{}^\dagger(\mathbf{k})\exp(-i\mathbf{k}\cdot\mathbf{r})], \quad (I3)$$

[17] We wish to thank J. Steinberger for a discussion on these aspects.

[18] Y. Nambu and D. Lurié, Phys. Rev. **125**, 1429 (1962).

[19] M. Gell-Mann, Phys. Letters (to be published). In Appendix IV, we give a list of other simple models. For a formal treatment with two triplets, see also J. J. Sakurai, in *Varenna Lecture Notes, Proceedings of the International School of Physics, Course 26* (Academic Press Inc., New York, 1962).

where Ω is the volume of the system, $k=|\mathbf{k}|$ and $\mathbf{k}$ is the three-momentum conjugate to the space coordinate $\mathbf{r}$. The Hamiltonians H_0 and h for the primary and the secondary interactions become

$$H_0=\sum_{\mathbf{k}}\sum_{\mu}\tfrac{1}{2}k[a_\mu^\dagger(\mathbf{k})a_\mu(\mathbf{k})+a_\mu(\mathbf{k})a_\mu^\dagger(\mathbf{k})] \quad (\mathrm{I}4)$$

and

$$h=\sum_{\mathbf{k}}\sum_{\mu}(4k)^{-1}m^2[a_\mu^\dagger(\mathbf{k})a_\mu(\mathbf{k})+a_\mu(\mathbf{k})a_\mu^\dagger(\mathbf{k}) +a_\mu(\mathbf{k})a_\mu(-\mathbf{k})+a_\mu^\dagger(\mathbf{k})a_\mu^\dagger(-\mathbf{k})]. \quad (\mathrm{I}5)$$

Let $|\mathrm{vac}\rangle_0$ and $|\mu,\mathbf{p}\rangle_0$ be the ground state and the one (μth) particle state with three-momentum $\mathbf{p}$ for the primary interaction. The zeroth-order energy of the μth particle $E_\mu{}^0(\mathbf{p})$ is given by

$$E_\mu{}^0(\mathbf{p})=\langle\mu,\mathbf{p}|H_0|\mu,\mathbf{p}\rangle_0-\langle\mathrm{vac}|H_0|\mathrm{vac}\rangle_0. \quad (\mathrm{I}6)$$

By using (I4) we have

$$E_\mu{}^0(\mathbf{p})=p=|\mathbf{p}|. \quad (\mathrm{I}7)$$

The first-order energy shift produced by h is easily seen to be

$$\delta E_\mu(\mathbf{p})=m_\mu{}^2/(2p). \quad (\mathrm{I}8)$$

Therefore, a mass formula similar to Eq. (6) results if we identify the eight states $|\mu,\mathbf{p}\rangle$ to be the real 8×8 representation of the SU_3 group and if we choose the appropriate values m_μ such that h transforms according to Eq. (2) under SU_3. The exact energy $E_\mu(\mathbf{p})$ is given by

$$E_\mu(\mathbf{p})=\langle\mu,\mathbf{p}|H_0+h|\mu,\mathbf{p}\rangle-\langle\mathrm{vac}|H_0+h|\mathrm{vac}\rangle, \quad (\mathrm{I}9)$$

where $|\mathrm{vac}\rangle$ and $|\mu,\mathbf{p}\rangle$ are, respectively, the ground state and the one-particle state of the total Hamiltonian (H_0+h). The familiar formula

$$E_\mu(\mathbf{p})=p+\tfrac{1}{2}(m_\mu{}^2/p)-\tfrac{1}{8}(m_\mu{}^4/p^3)+\cdots =(p^2+m_\mu{}^2)^{1/2} \quad (\mathrm{I}10)$$

can be obtained either by applying the usual perturbation series[20] to (I9) or by using the canonical transformation,

$$a_\mu(\mathbf{k})=[\cosh\theta_\mu(k)]b_\mu(\mathbf{k})+[\sinh\theta_\mu(\mathbf{k})]b_\mu^\dagger(-\mathbf{k}), \quad (\mathrm{I}11)$$

where

$$\tanh[2\theta_\mu(k)]=-(2k^2+m_\mu{}^2)^{-1}m_\mu{}^2, \quad (\mathrm{I}12)$$

which changes the total Hamiltonian to its diagonal form

$$(H_0+h)=\tfrac{1}{2}\sum_{\mathbf{k}}\sum_{\mu}(k^2+m_\mu{}^2)^{1/2} \times[b_\mu^\dagger(\mathbf{k})b_\mu(\mathbf{k})+b_\mu(\mathbf{k})b_\mu^\dagger(\mathbf{k})]. \quad (\mathrm{I}13)$$

Identical results can be derived for particles of arbitrary spin by regarding the mass term as the perturbation h. If h is a linear function in the masses, then the lowest order energy shift δE must depend quadratically on h.

APPENDIX II: A NONTRIVIAL BUT UNREALISTIC EXAMPLE

In this Appendix, we shall take advantage of the known solutions of the Bethe-Salpeter equation obtained by Wick[21] and Cutkosky[22] and consider another less trivial but still unrealistic example which, however, will serve to illustrate further the origin of the "mass squared" formula for zero mass bound states. We consider a basic massive triplet (under SU_3) of spin-0 particles (A_0,A_1,A_2) with charge $(q,\ q+1,\ q)$ where q is any number including zero. Under the isotopic spin rotation A_0 behaves like $I=0$ and (A_1,A_2) like $I=\frac{1}{2}$. There exist very strong attractive forces between A_i and the antiparticles $\bar{A}_j$ which we assume to be of the same form as that generated through a zero mass zero spin field. Furthermore we suppose that the Bethe-Salpeter equation is the correct equation for the compound system A_i and $\bar{A}_j$.

In the absence of the secondary interaction h, the zeroth-order masses $M_i{}^0$ of the triplet A_i are all equal:

$$M_1{}^0=M_2{}^0=M_3{}^0=M. \quad (\mathrm{II}1)$$

Let $\psi_i{}^j$ be the wave function describing the compound system of A_i and $\bar{A}_j$. The Bethe-Salpeter equation for $\psi_i{}^j$ is given in this case by

$$(p_i{}^2+M^2)(p_j{}^2+M^2)\psi_i{}^j(p)=\left(\frac{f}{\pi^2}\right)\int\frac{d^4k}{(p-k)^2}\psi_i{}^j(k), \quad (\mathrm{II}2)$$

where i and j can independently be 0, 1 or 2, $p_i{}^2$, and $p_j{}^2$ are the (4-momentum)^2 of A_i and $\bar{A}_j$, respectively, p is the relative momentum of the system. As shown by Wick,[21] this equation has zero mass bound state if the coupling constant f is given by

$$f=2M^2. \quad (\mathrm{II}3)$$

There are altogether 9 such zero-mass zero-spin states corresponding to the different values of i and j. In the following, we assume f is indeed given by (II3).

Similar to Eq. (15), we assume the secondary interaction consists of *only* the mass shifts of the basic triplet:

$$m_i=M_i-M_i{}^0\geqq 0, \quad (\mathrm{II}4)$$

where the inequality is used to exclude bound states with imaginary mass which appear in this unrealistic model. In order to ensure isotopic spin rotation invariance we must have

$$m_1=m_2. \quad (\mathrm{II}5)$$

Using the explicit solutions given in Refs. 21 and 22 one can compute the corresponding mass shifts of these

[20] It is of interest to note that because of the totally trivial structure of this example, the zero-mass approximation for the mesons does not lead to any spurious infrared divergence. For a more complicated case the apparent infrared difficulty can be resolved by using the physical mass of the meson as an infrared cutoff. The only change in the validity of the perturbation series is that, instead of the condition $\lambda\ll 1$, we have $[\lambda\ln(M/m_\mu)]\ll 1$ where m_μ is the mass of the mesons and λ is given by Eq. (1).

[21] G. C. Wick, Phys. Rev. **96**, 1124 (1954).

[22] R. E. Cutkosky, Phys. Rev. **96**, 1135 (1954).

states ψ_i^j. We call these nine (zero spin) eigenstates $I_{1/2}$, I_1, I_0, and I_0', where the subscripts refer to the isotopic spin. To first order in m_i, the corresponding changes of masses of these bound states are given by

$$[m(I_1)]^2=5M[2m_1],$$
$$[m(I_{1/2})]^2=5M[m_0+m_1],$$
$$[m(I_0)]^2=5M[2m_1],$$

and

$$[m(I_0')]^2=5M[2m_0]. \tag{II6}$$

The above results are obtained from the relation

$$f=\tfrac{1}{2}(M_1+M_2)^2-(2/5)m^2, \tag{II7}$$

valid near $m^2=0$ and to first order in M_1-M_2. Here m is the mass of the bound state and M_1, M_2 are the masses of the bound particles. For $M_1=M_2$, this relation (II7) is readily derived from Eq. (54) of Wick (Ref. 21) by a perturbation expansion in m^2, and for $M_1\neq M_2$ by using an appropriate transformation. (See Sec. IV of Cutkosky's paper, Ref. 22.) It is instructive to observe that under the primary interaction ($M_i^0=M$), the octet and the singlet part of ψ_i^j are degenerate. Similar to Eqs. (19)–(21), we find the two isotopic spin=0 states are given by $(I_0)=(1/\sqrt{2})(\psi_1^1+\psi_2^2)$ and $(I_0')=\psi_0^0$, both of which are mixtures of the octet and the singlet representations under SU_3. Because of the zero mass nature of these states (in the absence of the secondary interaction) these masses satisfy the quadratic mass formulas

$$2[m(I_0')]^2+[m(I_0)]^2+[m(I_1)]^2=4[m(I_{1/2})]^2 \tag{II8}$$

and

$$[m(I_0)]^2=[m(I_1)]^2. \tag{II9}$$

APPENDIX III

In this Appendix it will be shown that if R, the restricted Hilbert space defined in Sec. III, is extended to the entire Hilbert space, then the degeneracy of the state $(\bar{\beta}\beta)_1$ with states $(\beta\bar{\beta})_8$, originally expressed by Eq. (39), and now replaced by Eq. (43), implies invariance of the primary Hamiltonian H_0 under a group $G=SU_3\times SU_3\times SU_3$.

The explicit expression of the operator V_0^0 appears in Eq. (43) is

$$V_0^0=\tfrac{1}{3}\eta\sum_{\mathbf{k}}[2b^{\dagger 0}(\mathbf{k})b_0(\mathbf{k})+2b_0'^{\dagger}(\mathbf{k})b'^0(\mathbf{k})-b^{\dagger 1}(\mathbf{k})b_1(\mathbf{k})$$
$$-b_1'^{\dagger}(\mathbf{k})b'^1(\mathbf{k})-b^{\dagger 2}(\mathbf{k})b_2(\mathbf{k})-b_2'^{\dagger}(\mathbf{k})b'^2(\mathbf{k})], \tag{III1}$$

where η is the projection operator given by (33). The fact expressed by (43) that the primary interaction H_0 commutes with V_0^0 and U_i^j, but V_0^0 does not commute with all the U_i^j implies that H_0 is invariant under a group G bigger than SU_3; otherwise $(\bar{\beta}\beta)_1$ would not be degenerate with $(\bar{\beta}\beta)_8$ under H_0.

To investigate the structure of G we first establish the commutation relations:

$$[U_i^j,V_k^l]=\delta_i^l V_k^j-\delta_k^j V_i^l \tag{III2}$$

which merely express the fact that V_k^l possesses tensor character under the group with generators U_i^j. Thus if U_i^j and V_0^0 are part of the generators for G, the latter must also contain all other V_i^j through the repeated operations such as $[U_i^j,V_0^0]$, $[U_l^k,[U_i^j,V_0^0]]$, etc.

For convenience we introduce the operators

$$B_i^j=\eta\sum_{\mathbf{k}}[b^{\dagger j}(\mathbf{k})b_i(\mathbf{k})-\tfrac{1}{3}\delta_i^j b^{\dagger l}(\mathbf{k})b_l(\mathbf{k})] \tag{III3}$$

and

$$B_i'^j=-\eta\sum_{\mathbf{k}}[b_i'^{\dagger}(\mathbf{k})b'^j(\mathbf{k})-\tfrac{1}{3}\delta_i^j b_l'^{\dagger}(\mathbf{k})b'^l(\mathbf{k})] \tag{III4}$$

in terms of which we can write

$$V_i^j=B_i^j-B_i'^j. \tag{III5}$$

Then, we find

$$[V_i^j,V_k^l]=\delta_i^l(B_k^j+B_k'^j)-\delta_k^j(B_i^l+B_i'^l) \tag{III6}$$

so that B_i^j and $B_i'^j$, are also contained in the set of generators for the group G. The analysis of the group structure is further facilitated by the introduction of the operators

$$A_i^j=U_i^j-(B_i^j+B_i'^j). \tag{III7}$$

The A_i^j, B_i^j, and $B_i'^j$ satisfy the commutation relations

$$[A_i^j,B_l^k]=[A_i^j,B_l'^k]=[B_i^j,B_l'^k]=0, \tag{III8}$$
$$[A_i^j,A_l^k]=\delta_i^k A_l^j-\delta_l^j A_i^k, \tag{III9}$$
$$[B_i^j,B_l^k]=\delta_i^k B_l^j-\delta_l^j B_i^k, \tag{III10}$$
$$[B_i'^j,B_l'^k]=\delta_i^k B_l'^j-\delta_l^j B_i'^k \tag{III11}$$

and

$$A_i^i=B_i^i=B_i'^i=0. \tag{III12}$$

Thus the A_i^j, B_i^j, $B_i'^j$ are generators of three independent SU_3 groups. Their direct product $SU_3\times SU_3\times SU_3$ defines the group G. It is important to note that if H_0 commutes with U_i^j and V_0^0, then it must also be invariant under the entire group G and we have

$$[H_0,A_i^j]=[H_0,B_i^j]=[H_0,B_i'^j]=0. \tag{III13}$$

In addition, H_0 transforms like the generator for time translations of the Lorentz group, is invariant under the rotation group, and satisfies the commutation relations

$$[H_0,C]=[H_0,P]=[H_0,N]=[H_0,Q]=0, \tag{III14}$$

where C, P, N, Q are, respectively, the operators for charge conjugation, parity, baryon number, and charge. The operators P, N, and Q commute with the entire group G. On the other hand, the charge-conjugation operator obeys the following relations:

$$CN+NC=0, \tag{III15}$$
$$CQ+QC=0, \tag{III16}$$
$$CB_i^jC^{\dagger}=-B_j'^i, \tag{III17}$$

and

$$CA_i^jC^{\dagger}=-A_j^i. \tag{III18}$$

The eigenstates of H_0 can be labeled by three pairs of numbers, each pair characterizing a definite representation of each independent SU_3 group. For representations with small dimensionality d, d is sufficient to define the irreducible representation apart from the conjugate representation. Hence, in this case, we may represent the eigenstates of H_0 by (x,y,z) where x, y, z are respectively the dimensionality of the irreducible representation under each of the SU_3 groups generated by A_i^j, B_i^j, and $B_i'^j$. Thus, the spin -1 multiplet is represented by (1,3,3), containing $1\times3\times3=9$ states with $N=0$. The pseudoscalar meson octet is represented by (8,1,1,) with $N=0$ and the baryon octet is (8,1,1) with $N=1$. It should be pointed out that only the sufficient condition that G contains the correct varieties of multiplets is established.

The group G has been constructed artificially to accommodate the presently known variety of multiplets. It has many unusual features which we now proceed to discuss.

(1) The separate invariance of H_0 under B_i^j and $B_i'^j$ is a necessary result if the primary force between $\bar{\beta}_i$ and β_j is independent of i and j. For example, in the case of two spin-$\frac{1}{2}$ particles, if the force between these two particles is spin-independent, then the triplet state is degenerate with the singlet. The force is invariant under separate SU_2 transformations for these two spins. Therefore, the relevant group is $(SU_2\times SU_2)$ which has a representation containing 4 states.

(2) The projection operator η has been included in the definition of B_i^j and $B_i'^j$ in order to ensure the compatibility of having nine degenerate vector mesons with the fact that apparently only eight spin-$\frac{1}{2}$ baryons states [identified with $(\bar{\alpha}\beta)_8$] exist. Otherwise η should be replaced by 1 in Eqs. (III3) and (III4).

(3) As far as the properties of the vector meson states $|\Phi_i^j\rangle$ are concerned ,we can take $\eta=1$ and consider only the group $G'=SU_3\times SU_3$, generated by B_i^j and $B_i'^j$. We shall presently show that crossing symmetry is violated in the coupling of $|\Phi_i^j\rangle$ with β_i and $\bar{\beta}_j$. Consider, for example, the particular element of the group G' defined by

$$T(\lambda)=\exp[\tfrac{3}{2}i\lambda V_0{}^0]. \qquad \text{(III19)}$$

The transformation laws for the states $|\phi\rangle$, $|\beta_{0k}\rangle$, and $|\bar{\beta}_0\rangle$ under $T(\lambda)$ are

$$T|\phi\rangle=\exp(2i\lambda)|\phi\rangle, \qquad \text{(III20)}$$

$$T|\beta_0\rangle=\exp(i\lambda)|\beta_0\rangle, \qquad \text{(III21)}$$

and

$$T|\bar{\beta}_0\rangle=\exp(i\lambda)|\bar{\beta}_0\rangle. \qquad \text{(III22)}$$

It follows that the matrix element $\langle\phi|\beta_0,\bar{\beta}_0\rangle$ is invariant under T while the matrix element $\langle\beta_0|\beta_0,\phi\rangle$ is not. Thus, the process

$$\phi\to\beta_0+\bar{\beta}_0, \qquad \text{(III23)}$$

which is allowed by G', does not lead to the process

$$\beta_0\to\phi+\beta_0, \qquad \text{(III24)}$$

which is forbidden by G', in contradiction with crossing symmetry. Time reversal invariance is however valid since both $\langle\phi|\beta_0,\bar{\beta}_0\rangle$ and $\langle\beta_0,\bar{\beta}_0|\phi\rangle$ are invariant under G'.

It may also be noted that because creation and annihilation operators for the particle ϕ transform differently under (III19), a local Hermitian field operator that one might try to introduce to represent the bound state ϕ, would not have a definite transformation property under G'. Thus the group G' (or G) cannot be applied to local field operators for the nonet.

(4) An extension of G to the many-particle continuum states may lead to paradoxical results. For example, the state of one baryon octet $(\bar{\alpha}\beta)_8$ and one antibaryon octet $(\bar{\beta}\alpha)_8$ at infinity should be represented by $(8,1,1)\times(8,1,1)$, with a multiplicity of 64, apart from other spin-momentum dependence. Yet, if we consider the two baryon octets as a single system, since $\eta=1$, the system is also represented by (x,y,z) where $x=1$ or 8 and $y=z=3$. The resulting multiplicity is no longer given by the product $(8,1,1)\times(8,1,1)$. This example illustrates the incompatibility of the group G with the asymptotic condition.

The paradox may be resolved by requiring the group G to be valid only for states which extend over a limited region in space where interaction takes place. Then, if we consider, for example, the collision of a baryon and an antibaryon with the production of a pseudoscalar meson and a vector meson we can write the sequence

$$(\bar{\alpha}\beta)_8+(\bar{\beta}\alpha)_8\to(x,y,z)_{\eta=1}\to(\bar{\alpha}\alpha)_8+(\bar{\beta}\beta)_9, \qquad \text{(III25)}$$

where the intermediate stage refers to the interaction region and $(\bar{\beta}\beta)_9$ denotes the vector meson nonet. In (III25) we must decompose both the initial and the final states into compound states $(x,y,z)_{\eta=1}$ which are representations of the group G. Clearly, only the compound states with $x=8$, $y=3$, and $z=3$ are involved in the process. Some of the particular results for $\bar{p}+p$ reactions have already been discussed in Sec. III.

(5) Restricting ourselves to the β particles and their bound state $|\Phi_i^j\rangle$ representing the nine vector mesons we may interpret the validity of the group $G'=SU_3\times SU_3$ as the mathematical expression of the stability of these bound states. Indeed, if we assume that β and $\bar{\beta}$ are bound by a neutral field χ transforming like a unitary singlet, then the annihilation and recombination process such as

$$\bar{\beta}_0+\beta_0\to\chi\to\bar{\beta}_0+\beta_0 \qquad \text{(III26)}$$

would contribute to the singlet state $(\bar{\beta}\beta)_1$ and not to the state $(\bar{\beta}\beta)_8$, thus lifting the degeneracy between the nine members of the vector meson multiplet. The Bethe-Salpeter type approximations for the two-body amplitude assume implicity that the contributions from the process (III26) are negligible, thus regarding the compound system $(\bar{\beta}_0\beta_0)$ as stable (independent of its mass value). Such an approximation leads immediately to the degeneracy of the nine mesons. The validity of the group G' (and G in general) thus emerges as an approximate dynamical group describing the independence of the $(\beta\bar{\beta})$ binding forces from the unitary spin together with the stability of the nonet.

APPENDIX IV

In this Appendix we list the simplest models involving one fundamental triplet. If no singlets are allowed, one is led to Gell-Mann's "quark" scheme[19] with noninteger charge q and baryon number n (unless the definitions of charge and baryon number are modified for each multiplet). The requirement of integer q and n leads one to consider schemes with at least one singlet α and one triplet β. The triplet components β_0, β_1 and β_2 are assumed to have, respectively, the charges $(q,\ q+1,\ q)$ and q is called the charge of the triplet. The case in which α and β have $q=0$, α being a baryon and β a $n=0$ fermion triplet has already been studied by Gell-Mann.[19] This is model I′ in our table. A variation on this model, in which the baryon number, the charge and the strangeness are all introduced once is the model I with β being a $n=0$ boson. Other simple schemes in which singlets, octets, and decuplets may arise from the direct product $3\times3\times3$ of the fundamental triplet with itself are provided by models II and III. Finally, in model IV, which involves one triplet and two singlets we give an example in which two kinds of octets and decuplets with the usual charge structure may arise, one set from $3\times\bar{3}$ and $(3\times\bar{3})(3\times\bar{3})$ and another set from $(3\times3\times3)$. We now have the possibility of a selection rule that prevents a decuplet belonging to the second set from decaying into two octets from the first set through SU_3 preserving interactions. In such a scheme we can have stable multiplets.

The selection rule is connected with the additional gauge transformation

$$\alpha\rightarrow\alpha,\quad \beta\rightarrow e^{iu}\beta \tag{IV1}$$

that becomes possible when a fundamental singlet exists as well as a triplet β. As long as u-invariant representations are considered, as in I and I′, this gauge group gives nothing new. However, when baryon multiplets having different "u" charge exist, as in model IV, a new selection rule arises.

Simple assumptions concerning the nature of the binding forces are shown in the table to illustrate how some of the representations that do not seem to occur for mesons and baryons can be eliminated on physical grounds. The other entries in the table are self-explanatory.

In connection with these simple models it should also be remarked that the most general Gell-Mann–Okubo mass splitting within a unitary multiplet is not obtained

TABLE I. A list of other models.

	Models	N	Spin	Q	Forces (R=repulsive, A=attractive)	Mesons ($N=0$)	Baryons ($N=1$)	Remarks
I	α^0 (singlet)	1	$\frac{1}{2}$	0	$\beta\beta(R)$ $\alpha^0\bar{\alpha}^0(A)$, $\beta\bar{\beta}(A)$ $\alpha^0\bar{\beta}(A)$, $\alpha^0\beta(A)$	$(\alpha^0\bar{\alpha}^0)_1$, $(\beta\bar{\beta})_1$ $(\beta\bar{\beta})_8$	$\alpha^0(\beta\bar{\beta})_8$, $\alpha^0(\beta\bar{\beta})_1$ $\alpha^0[(\beta\bar{\beta})(\beta\bar{\beta})]_{1,8,10,27}$	Representations are invariant under the "u" gauge. If $\beta\beta(A)$ is allowed new mesons states arise: $(\beta\beta)_{\bar{3}}$, $(\beta\beta)_6$, $(\beta\beta\beta)_{1,8,10}$
	β (triplet)	0	0	0				
I′	α^0 (singlet)	1	$\frac{1}{2}$	0	same	same	same	Discussed by Gell-Mann (Ref. 19)
	β (triplet)	0	$\frac{1}{2}$	0				
II	α^+ (singlet)	2	0	1	$\beta\beta(R)$ $\alpha^+\bar{\alpha}^+(A)$, $\bar{\alpha}^+\beta(A)$ $\beta\bar{\beta}(A)$	$(\alpha^+\bar{\alpha}^+)_1$, $(\beta\bar{\beta})_1$ $(\beta\bar{\beta})_8$	$\bar{\alpha}^+(\beta\beta\beta)_1$ $\bar{\alpha}^+(\beta\beta\beta)_8$ $\bar{\alpha}^+(\beta\beta\beta)_{10}$	Possible additional states: $(\alpha^+\bar{\beta})_3$: $N=1$, $Q=(1,0,1)$ $\bar{\alpha}^+(\beta\beta)_6$, $\bar{\alpha}^+(\beta\beta)_{\bar{3}}$: $N=0$, $Q=(1,\ 0,\ -1)$ containing a meson with $Q=-1$, $Y=-2$
	β (triplet)	1	$\frac{1}{2}$	0				
III	α^- (singlet)	1	$\frac{1}{2}$	-1	same	$(\alpha^-\bar{\alpha}^-)_1$, $(\beta\bar{\beta})_1$ $(\beta\bar{\beta})_8$	$\alpha^-(\beta\beta\beta)_1$ $\alpha^-(\beta\beta\beta)_8$ $\alpha^-(\beta\beta\beta)_{10}$	Possible additional states: $(\alpha^-\beta)_3$: $N=1$, $Q=(-1,\ 0,\ -1)$ $\alpha^-(\beta\beta)_6$, $\alpha^-(\beta\beta)_{\bar{3}}$: $N=1$
	β (triplet)	0	0	0				
IV	α^0, α^- (singlets)	1	$\frac{1}{2}$	0, -1	same	$(\alpha^0\bar{\alpha}^0)_1$, $(\alpha^-\bar{\alpha}^-)_1$ $(\beta\bar{\beta})_1$, $(\beta\bar{\beta})_8$	$\alpha^0(\beta\bar{\beta})_1$, $\alpha^0(\beta\bar{\beta})_8$ $\alpha^-(\beta\beta\beta)_1$, $\alpha^-(\beta\beta\beta)_{8,10}$	Example of a forbidden decay: $\alpha^-(\beta\beta\beta)_{10}\not\rightarrow\alpha^0(\beta\bar{\beta})_8+(\beta\bar{\beta})_8$
	β (triplet)	0	0	0				

by simply giving the $I=0$ member β_0 of the triplet a different mass than the $I=\frac{1}{2}$ members β_1 and β_2. An additional symmetry-breaking interaction Lagrangian is also needed. Furthermore, in any model which has only one triplet it is difficult to understand why there should be nine approximately degenerate spin-1 meson multiplets while there exist only eight approximately degenerate pseudoscalar meson states. For this reason, it appears that the special model discussed in Sec. II is a more realistic one.

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Explicit Construction of Asymptotic Fields*

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Examination of a separable potential model in field theory, when the interaction is attractive enough to produce bound states, shows that the $t=\pm\infty$ limits of the Heisenberg fields do not always have a particle interpretation but are superpositions of eigenfields. In this model the commutators of the in-fields of particles that are well enough localized to have a finite interaction energy are operators.

I. INTRODUCTION

ASYMPTOTIC fields play a central role in the axiomatic formulation of field theory given by Lehmann, Symanzik, and Zimmermann[1] and in many discussions of the analyticity of the S matrix based on their work. The properties of asymptotic fields have been extensively examined by Zimmermann, Haag, Nishigima, and Ruelle.[2] In order to provide an illustrative example that displays the Heisenberg fields for large times, the infields and their interrelation, we examined a separable potential model in field theory.[3] Within the framework of this model it is shown that: (a) The limits implied in the formal definition of infields[2] exist only after taking their matrix elements. (b) When the interaction is attractive enough to produce bound states, the Heisenberg field of a particle of momentum $\mathbf{k}$ has two terms which oscillate respectively with frequencies $\omega(k)$ and $\mu_n(<\mu)$, as $t\to\pm\infty$. The first term has the usual particle interpretation and reproduces the scattering states, whereas the second term cannot be interpreted as a particle since its energy is below the continuum. The latter term consists of an infinite product of fields and vanishes throughout a subspace that is free of heavy mesons (the target). (c) The commutator of the in-fields of those particles which are well enough localized to have a finite interaction energy is an operator.

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II. SOLUTION OF THE EQUATIONS OF MOTION

The Hamiltonian for a light boson that interacts via a separable potential with a static boson of mass M is

$$H=H_0+\lambda\varphi^\dagger\varphi G^\dagger G, \tag{1}$$

where

$$H_0=M\varphi^\dagger\varphi+\int \mathbf{dk}\omega(k)a^\dagger(k)a(k),$$

$$G=\int f(k)a(k)\mathbf{dk}, \tag{2}$$

$$[a(k),a^\dagger(k')]=\delta(k-k'),$$
$$[\varphi,\varphi^\dagger]=1,\ \omega(k)=(\mu^2+k^2)^{1/2}.$$

$a^\dagger(k)$ and $\varphi^\dagger$ are creation operators for a light boson of momentum k energy $\omega(k)$ and a static boson of mass M, respectively. From Eqs. (1) and (2)

$$[H,a^\dagger(k)]=\omega(k)a^\dagger(k)+\lambda\varphi^\dagger\varphi f(k)G^\dagger. \tag{3}$$

In terms of the quantities defined above, the Heisenberg fields are

$$e^{iHt}a^\dagger(k)e^{-iHt}=a^\dagger(k,t),$$
$$e^{iHt}\varphi^\dagger e^{-iHt}=\varphi^\dagger(t), \tag{4}$$
$$G(t)=\int f(k)a(k,t)\mathbf{dk}.$$

Since $\varphi^\dagger\varphi$ is a constant of the motion it follows from Eqs. (3) and (4) that

$$-i(d/dt)a^\dagger(k,t)=\omega(k)a^\dagger(k,t)+\lambda\varphi^\dagger\varphi f(k)G^\dagger(t). \tag{5}$$

This is a linear equation in $a^\dagger(k,t)$ that can be solved by