in KH that the 78.69-keV and ground states probably arise from different intrinsic levels ($g_{7/2}$ and $d_{5/2}$, respectively). As previously noted in KH, the relative strengths of the $E2$ components of the 54.84- and 133.54-keV transitions (see Table III) seem consistent with this interpretation, as does our nonobservance of a transition between the 133.54- and 123.73-keV states.

With the exception of the 373.15-keV state, the remaining levels in Cs^{131} seem to decay preferentially to the ground and 123.73-keV states. Since the 216.01- and 92.25-keV transitions are mainly $M1$, it appears that the 216.01-keV state cannot be interpreted as a collective level based upon the ground state. A definitive statement as to the origin of this state therefore cannot be made at this time.

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Electron Screening Corrections to Beta-Decay Spectra*

LOYAL DURAND, III
Physics Department, Yale University, New Haven, Connecticut

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The corrections to the Fermi function $F(Z, W)$ which arise from the screening of the Coulomb field of the nucleus by the atomic electrons have been investigated using a Hulthén model for the screened field. The resulting problem is exactly solvable for the Schrödinger and Klein-Gordon equations. The results agree with those obtained by Rose and by Longmire and Brown using a modification of the WKB method, and disagree markedly with those obtained by Reitz by numerical integration of the Dirac equation. The latter results appear to be incorrect. The screening corrections are sufficiently small for light nuclei as not to affect materially present tests for the universal Fermi interaction and conserved vector current hypotheses for beta decay, but may become significant for low-energy beta transitions in heavy nuclei.

I. INTRODUCTION

THE wave function for a free electron of moderate energy is well approximated in the vicinity of an atomic nucleus by the wave function appropriate to a pure Coulomb field. This approximation leads to the appearance of the Fermi factor $F^{\pm}(Z,W)$ for a Coulomb field in the electron spectrum for allowed beta decay,

$$dN(W)=\tfrac{1}{2}m^5\pi^{-3}|M|^2F^{\pm}(Z,W)pW(W_0-W)^2dW\,, \quad (1)$$

where

$$F^{\pm}(Z,W)=2(1+s)(2pR)^{2s-2} \times e^{\mp\pi\eta}|\Gamma(s+i\eta)|^2[\Gamma(2s+1)]^{-2}. \quad (2)$$

In these expressions, W is the total energy and $p=[W^2-m^2]^{1/2}$ is the momentum of the electron, and W_0 is the maximum electron energy possible in the decay. The Coulomb parameter $Z\alpha W/p$ is denoted by η, while $s=[1-Z^2\alpha^2]^{1/2}$. The units are such that $\hbar=c=1$. This result for the electron spectrum is subject to many small corrections, including the effects of forbidden transitions, the finite spacial extension of the wave function of the decaying nucleon, the finite electromagnetic size of the nucleus, radiative electromagnetic corrections, and the effects of the screening of the Coulomb field of the nucleus by the outer electrons. Most of these corrections are well understood for light nuclei.[1] However, the electron screening corrections calculated by different methods are not consistent. These corrections have been investigated by Rose[2] and by Longmire and Brown[3] using a modified WKB approximation. The corrections were found to be rather small at moderate energies for light nuclei. Quite disparate results were obtained by Reitz[4] by numerical integration of the Dirac equation using a Thomas-Fermi-Dirac model for the interaction between the electron and the residual ion. The discrepancies are especially large in the high-energy, low-Z region in which the WKB method should

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[1] L. Durand, III, L. F. Landovitz, and R. B. Marr, Phys. Rev. **130**, 1188 (1963). The known corrections to the f values for the $0+\to 0+$ transitions in light nuclei are summarized in Table I of this paper.
[2] M. E. Rose, Phys. Rev. **49**, 727 (1936).
[3] C. Longmire and H. Brown, Phys. Rev. **75**, 264, 1102E (1949).
[4] J. R. Reitz, Phys. Rev. **77**, 10 (1950).

be most reliable. Although the screening corrections are generally small, the accuracy of recent experiments[5] on $0+\rightarrow 0+$ transitions in light nuclei is such that the corrections should be considered. For example, the ft value for the $0+\rightarrow 0+$ transitions $O^{14}(\beta^+)N^{14*}$, $Al^{26*}(\beta^+)Mg^{26}$, and $Cl^{34}(\beta^+)S^{34}$ have been determined with accuracies of 0.3–0.7%. Calculations using the method of Rose yield screening corrections to f of +0.09, +0.11, and +0.13% for these three transitions. However, at the highest momenta considered in his paper, the screening corrections to $F^+(Z,W)$ obtained by Reitz are an order of magnitude larger than those obtained by the foregoing method. The numerical calculations unfortunately did not cover the regions of high momenta and low Z in which we are primarily interested. It is nevertheless possible to estimate the screening corrections to f using reasonable extrapolations of the tabulated results. The corrections so obtained are much larger than those quoted above, and could be as large as 1–2% for O^{14}.[6] However, corrections of this magnitude would be inconsistent with a rigorous bound on the ratio $F_{sc}(Z,W)/F(Z,W)$ obtained recently by Dr. Lowell Brown.[7] The bound is in fact violated by the results quoted by Reitz for high momenta and low Z. These inconsistencies, and the relevance of the indicated $0+\rightarrow 0+$ transitions to experimental tests of the conserved vector current and universal Fermi interaction hypotheses for the weak interactions,[8] clearly necessitate a re-examination of the screening corrections to the Fermi factor. It is with this problem that we shall be concerned.

The screening corrections to the Fermi function were estimated by Rose[2] using a modification of the WKB method designed to overcome a well-known difficulty, that the value of the WKB wave function at the origin is accurate only for large values of the orbital angular momentum quantum number. The WKB wave functions for the screened Coulomb field were consequently assumed to be accurate only for electron-nucleus separations larger than some minimum value r_0. If r_0 can be made small compared to the radius of the atom, the interaction potential associated with the atomic electrons will be essentially constant for $r<r_0$, and the wave functions in this inner region can be approximated by Coulomb wave functions for a shifted energy. The necessary conditions are satisfied if $r_0\sim p^{-1}\ll a_0$. When the inner wave functions are properly joined to the WKB wave functions for the exterior region, the Fermi factor for the screened Coulomb field is found to be

$$F_{sc}^{\pm}(Z,W)=(p'W'/pW)F^{\pm}(Z,W'),\quad W'=W\pm D_0,\quad (3)$$

where D_0 is the value of the electronic potential of the parent atom at the nucleus, and $F^{\pm}(Z,W)$ is defined in Eq. (2).

Although the argument which leads to Eq. (3) is plausible, it is very difficult to estimate the errors in the WKB wave functions in a convincing manner, and the accuracy of the approximation is consequently difficult to assess. We have therefore chosen to study the screening corrections of $F(Z,W)$ using the exact S-state solutions to the Schrödinger and Klein-Gordon equations which may be obtained for a Hulthén model of a completely screened Coulomb field,[9]

$$V(r)=Z\alpha\lambda e^{-\lambda r}[1-e^{-\lambda r}]^{-1}\rightarrow (Z\alpha/r)-\tfrac{1}{2}Z\alpha\lambda+\cdots,\ r\rightarrow 0.\quad (4)$$

It is found that the exact results for the screened Fermi function reduce to the appropriate modified WKB expressions in the limit $p/\lambda\gg 1$ for which the latter are valid. The Dirac equation cannot be solved for a Hulthén potential, but there is no reason to expect any peculiar behavior in this case. Furthermore, the very smallness of the correction to $F(Z,W)$ associated with the transition from an unscreened Coulomb field to a completely screened field, indicates that little error has been made by ignoring the odd charge of the residual ion in $V(r)$. The correct result for $p/\lambda\gg 1$, or more generally, for $WD_0/p^2\ll 1$, is clearly given by Eq. (3). The results obtained by Reitz[4] are undoubtedly in error at the higher momenta considered, but do not differ too greatly from those of Rose[2] for small momenta.

The main uncertainty in the screening corrections arises from the uncertainty in D_0. For a Thomas-Fermi-Dirac model of the atom, D_0 ranges from $1.91Z^{4/3}\alpha^2 m$ for light nuclei, to $1.82\,Z^{4/3}\alpha^2 m$ for heavy nuclei.[10] However, the electron charge density is too large at small radii for this model, diverging as $r^{-3/2}$ for $r\rightarrow 0$, and the resultant values of D_0 are undoubtedly too large. Perhaps the most reliable values of D_0 are those derived from atomic potentials calculated by the Hartree-Fock self-consistent-field method; for light to medium weight nuclei, the best value of D_0 appears to

[5] R. K. Bardin, C. A. Barnes, W. A. Fowler, and P. A. Seeger, Phys. Rev. **127**, 583 (1962); D. L. Hendrie and J. B. Gerhart, *ibid.* **121**, 846 (1961); J. W. Butler and R. O. Bondelid, *ibid.* **121**, 1770 (1961); J. M. Freeman, J. H. Montague, D. West, and R. E. White, Phys. Letters **3**, 136 (1962); J. M. Freeman, J. H. Montague, G. Murray, R. E. White, and W. E. Burcham, *ibid.* **8**, 115 (1964). The present results on the $0+\rightarrow 0+$ transitions are summarized in the last paper.

[6] Private communication from Dr. Joan M. Freeman. The author is indebted to Dr. Freeman for calling this problem to his attention.

[7] L. S. Brown, following paper, Phys. Rev. **135**, B314 (1964).

[8] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958); M. Gell-Mann, *ibid.* **111**, 362 (1958).

[9] The electron charge density corresponding to this potential diverges as r^{-1} for $r\rightarrow 0$, but the charge in the neighborhood of the origin is finite. The divergent term is readily removed by adding to $V(r)$ a term $\frac{1}{12}Z\alpha\lambda e^{-\lambda r}$, but the resulting Schrödinger equation is apparently not solvable. The divergence is less severe than that associated with the Thomas-Fermi-Dirac model of the atom, $\rho(r)\sim r^{-3/2}$, $r\rightarrow 0$. In any case, the screening corrections to $F(Z,W)$ will be shown to depend to an excellent degree of approximation only on the value at $r=0$ of the interaction potential associated with the atomic electrons. It is consequently important only that a model for the potential be reasonable, and reproduce this term correctly.

[10] R. P. Feynman, N. Metropolis, and E. Teller, Phys. Rev. **75**, 1561 (1949).

be $1.45Z^{4/3}\alpha^2 m$,[11] with an uncertainty of a few percent. The corrections are in any case small for energetic decays of light nuclei, and the uncertainty in the value of D_0 does not affect materially the tests of the universal Fermi interaction and conserved vector current hypotheses.

II. CALCULATION OF ELECTRON SCREENING CORRECTIONS TO THE FERMI FUNCTION

The Fermi function appropriate to a screened Coulomb field of the Hulthén form is readily obtained by solving the Schrödinger equation for the potential given in Eq. (4). We shall in fact consider a slightly generalized problem, and seek a solution to the equation

$$d^2u/dr^2+[p^2-ae^{-\lambda r}(1-e^{-\lambda r})^{-1}+be^{-2\lambda r}(1-e^{-\lambda r})^{-2}]u=0,$$

with the boundary conditions $u(r)\to 0$, $r\to 0$, and $u(r)\to e^{i\Phi}\sin(pr+\Phi)$, $r\to\infty$. The particular choice of parameters, $a=2mZ\alpha\lambda$, $b=0$, yields the Schrödinger equation for a positron in the screened Coulomb field of a point nucleus of charge Z. A second choice, $a=2WZ\alpha\lambda$, $b=(Z\alpha\lambda)^2$, leads to the Klein-Gordon equation for the same potential.[12] With the present normalization for $u(r)$, the Fermi function is defined for these cases as the value of $|u(r)/pr|^2$ at $r=0$ (Schrödinger equation), or as the value of $|u(r)/pr|^2$ at the nuclear radius (Klein-Gordon equation).

The differential equation for $u(r)$ is readily converted into an equation of the hypergeometric form by changing the independent variable from r to t,

$$t=1-e^{-\lambda r}.$$

Upon writing $u(t)$ in the form

$$u(t)=N(1-t)^{-i\kappa}\,v(t),\quad \kappa=p/\lambda,$$

we obtain a modified differential equation for the function $v(t)$,

$$\frac{d^2v}{dt^2}+\frac{1-2i\kappa}{t-1}\frac{dv}{dt}+\left[\frac{a/\lambda^2}{t(t-1)}+\frac{b/\lambda^2}{t^2}\right]v=0.$$

Solution of this equation in terms of the hypergeometric function is straightforward; we shall give only the properly normalized solution,

$$u(t)=\kappa\frac{\Gamma(\sigma+i\nu)\Gamma(\sigma-i\zeta)}{\Gamma(2\sigma)\Gamma(1-2i\kappa)}\times t^{\sigma}(1-t)^{-i\kappa}\,{}_2F_1(\sigma+i\nu,\sigma-i\zeta;2\sigma;t),\qquad(5)$$

where

$$\zeta=\lambda^{-1}[p^2+a+b]^{1/2}+p/\lambda,$$
$$\nu=\lambda^{-1}[p^2+a+b]^{1/2}-p/\lambda,$$

and

$$\sigma=\tfrac{1}{2}+\tfrac{1}{2}[1-(4b/\lambda^2)]^{1/2}.$$

The function $u(t)$ clearly satisfies the proper boundary condition at $r=0\,(t=0)$. That it also satisfies the proper boundary condition for $r\to\infty$ $(t\to 1)$ may be verified by using the well known relation between the hypergeometric functions with arguments t and $1-t$ to write[13]

$$u(t)=\frac{1}{2i}\frac{\Gamma(\sigma+i\nu)\Gamma(\sigma-i\zeta)}{\Gamma(1-2i\kappa)}t^{\sigma}\left[\frac{\Gamma(1+2i\kappa)}{\Gamma(\sigma-i\nu)\Gamma(\sigma+i\zeta)}(1-t)^{-i\kappa}\times{}_2F_1(\sigma+i\nu,\sigma-i\zeta;1-2i\kappa;1-t)-\frac{\Gamma(1-2i\kappa)}{\Gamma(\sigma+i\nu)\Gamma(\sigma-i\zeta)}(1-t)^{i\kappa}\times{}_2F_1(\sigma-i\nu,\sigma+i\zeta;1+2i\kappa;1-t)\right].$$

For $t\to 1$, the hypergeometric functions approach unity, and $u(t)$ approaches its asymptotic form, equivalent to

$$u(r)\to e^{i\Phi}\sin(pr+\Phi),\ \lambda r\gg 1,$$

$$\Phi=\arg\left[\frac{\Gamma(\sigma+i\nu)\Gamma(\sigma-i\zeta)}{\Gamma(1-2i\kappa)}\right].$$

The nonrelativistic Fermi function for the screened Coulomb field is readily obtained from Eq. (5) by considering the limit of the function $|u(r)/pr|^2$ for $r\to 0$ with the choice of parameters $a=2mZ\alpha\lambda$, $b=0$. Denoting this (Schrödinger) function by $F_S(Z,W)$, we obtain

$$F_S^+(Z,W)=|\Gamma(1+i\nu)\Gamma(1-i\zeta)/\Gamma(1-2i\kappa)|^2.\qquad(6)$$

The function $F_S^-(Z,W)$ may be obtained from $F_S^+(Z,W)$ by changing the sign of Z wherever it appears. The absolute squares of the gamma functions can be evaluated in terms of hyperbolic functions. For electron energies such that $p/\lambda\gg 1$, κ and ζ are large, and the results simplify somewhat:

$$F_S^+(Z,W)=(p'/p)F^+_{NR}(Z,W').\qquad(7)$$

Here, $F_{NR}^+(Z,W')$ is the nonrelativistic Fermi factor,

$$F_{NR}^+(Z,W')=e^{-\pi mZ\alpha/p'}|\Gamma(1+imZ\alpha/p')|^2\qquad(8)$$

[11] This result was derived from the numerical results for the atomic potentials given for the indicated atoms by: F. W. Brown, Phys. Rev. **44**, 214 (1933) [F,Ne]; E. H. Kennard and E. Ramberg, *ibid.* **46**, 1034 (1934) [Na]; D. R. Hartree, R. deL. Kronig, and H. Petersen, Physica **1**, 895 (1933–34) [Cl]; D. R. Hartree, Proc. Roy. Soc. (London) **A143**, 506 (1933–34) [K^+,Cu^+,Cs^+].

[12] It is amusing to note that the term quadratic in the potential may be used to approximate the centrifugal barrier for $l>0$. Provided that p is large, the barrier term is needed only for small values of r, and the approximation is very good. The appropriate choice of coefficients is given by $a+b=2mZ\alpha\lambda$, $b=-l(l+1)\lambda^2$. For $p/\lambda\gg 1$, the resulting expression for the Fermi function [Eq. (11)] reduces to the familiar nonrelativistic form. The actual potential in this case varies as r^{-2} for $r\to\infty$.

[13] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, 1952), p. 291.

evaluated for a modified momentum p'

$$p'=\tfrac{1}{2}[p^2+2mZ\alpha\lambda]^{1/2}+\tfrac{1}{2}p. \quad (9)$$

The usual Fermi factor is obtained in the limit $\lambda\rightarrow 0$.[12] The S-wave scattering phase shift is given for $p/\lambda\gg 1$ by

$$\Phi=-(mZ\alpha/p')\ln(2p'/\lambda)+\arg\Gamma(1+imZ\alpha/p').$$

This result differs from that for a pure Coulomb field only by the replacement of p by p', and of the usual radius-dependent term $\ln 2pr$ by $\ln(2p'/\lambda)$.

Except for the different definitions of the effective momenta, the function in Eq. (7) is the exact nonrelativistic analog of the modified WKB result given by Rose, Eq. (3). However, the effective momenta are in fact equal to within terms of order $\frac{1}{8}(mZ\alpha\lambda/p^2)^2$ relative to unity; to this accuracy, p' may be approximated as

$$p'\approx[p^2+mZ\alpha\lambda]^{1/2}, \quad (10)$$

and the corresponding nonrelativistic energy, as

$$W'\approx W+\tfrac{1}{2}Z\alpha\lambda. \quad (11)$$

The energy shift in this approximation is equal, as expected, to the negative of the potential energy of the positron in the field of the atomic electrons, evaluated at the origin; and the exact and modified WKB results are consequently identical in the low-Z, high-energy limit. The approximation which connects Eqs. (6) and (7) is valid for positron energies above a few kilovolts even for the heaviest nuclei. On the other hand, the approximation in Eq. (10) fails at low energies for heavy nuclei because of the appearance of powers of $Z\alpha/v$ in the correction terms, and the exact value of the effective momentum p' should be used. That p' should have a dependence on p other than that indicated by the modified WKB approximation is not unexpected. The variation of the effective energy shift over distances on the order of the electron wavelength was neglected in that approximation, yet this variation may be quite large for small momenta and large values of Z. The Hulthén model of the screened Coulomb field should give reasonable results for the screening corrections for electron energies sufficiently high that the electron wave length is smaller than the mean radius of the atomic K shell. For lower energies, the detailed structure of the atom may become important, and at very low energies, the concept of a static atomic potential will cease to be valid. It is readily shown using the WKB method that the difficulty with the Hulthén potential noted in footnote 9 does not materially affect the screening corrections.

Relativistic corrections to the screened Fermi function may be examined by considering the solutions of the Klein-Gordon equation for the Hulthén potential, Eq. (4). The wave function $u(r)/pr$, obtained from Eq. (5) with the choice of parameters $a=2WZ\alpha\lambda$, $b=(Z\alpha\lambda)^2$, diverges weakly for $r\rightarrow 0$. We shall therefore follow the procedure which is customary in the Dirac case, and define the Fermi function for the Klein-Gordon equation as the value of the quantity $|u(r)/pr|^2$ at the nuclear surface,

$$F_{KG}^{+}(Z,W) =(\kappa R)^{2\sigma-2}|\Gamma(\sigma+i\nu)\Gamma(\sigma-i\zeta)/\Gamma(2\sigma)\Gamma(1-2i\kappa)|^2. \quad (12)$$

The gamma functions which involve κ and ζ may again be replaced by their asymptotic forms if $p/\lambda\gg 1$. In this approximation, neglecting terms of order $(Z\alpha\lambda/p)^2$ relative to unity,

$$F_{KG}^{+}(Z,W)\rightarrow(p'/p)(2p'R)^{2\sigma-2} \times e^{-\pi\eta'}|\Gamma(\sigma+i\eta')/\Gamma(2\sigma)|^2, \quad (13)$$

where $\sigma=\frac{1}{2}+\frac{1}{2}[1-4Z^2\alpha^2]^{1/2}$ and $\eta'=Z\alpha W'/p'$. The shifted energy W' is again given by $W'=W+\frac{1}{2}Z\alpha\lambda$, while $p'=[W'^2-m^2]^{1/2}$. The approximation is excellent for light nuclei for positron energies greater than a few kilovolts, but the exact expression in Eq. (12) should be used for heavy nuclei and low energies. The result in Eq. (13) is identical to that obtained by the modified WKB method, and reduces to the exact result for a Coulomb field in the limit $\lambda\rightarrow 0$.

The Dirac equation unfortunately cannot be solved exactly for a Hulthén potential. However, since the modified WKB method reproduces the correct results for the screened Fermi function as calculated for the Schrödinger and Klein-Gordon equations, it is unlikely to be seriously in error in the Dirac case so long as the relevant parameter is small, $Z\alpha\lambda W/p^2<1$. It is clear, furthermore, that the energy-shift $\frac{1}{2}Z\alpha\lambda$ characteristic of the (completely screened) Hulthén potential should properly be replaced by D_0, the value of the atomic potential of the parent atom, evaluated at the nucleus. The results for the shielding correction to $F(Z,W)$ obtained from Eq. (3) differ markedly from the numerical results obtained by Reitz,[4] and we are forced to conclude that the latter are incorrect.

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