and negative energy states, then the solution for ϕ in (10a) indicates the relative numbers of particles which have been transformed by the interaction from a given spin and energy state at some initial time, into other states of spin and energy at some later time. In this sense, Eqs. (10) for the spinor ϕ yield more information, at least from an interpretive point of view, than Eqs. (1) for the bilinear forms v_μ and S_μ.

The classical equations may be determined in exactly the same manner from the asymptotic solutions of the Dirac equation (instead of the squared Dirac equation); for once we have found an asymptotic solution, $\Phi_{\rm WKB}$, in the form of Eq. (11), Eq. (3) then gives us the appropriate WKB approximation to the Dirac equation:

$$2mc\Psi_{\rm WKB}=m(v_\mu\gamma_\mu+ic)\Phi_{\rm WKB}. \tag{18}$$

If we write

$$2mc\Psi_{\rm WKB}=R\psi e^{iS/\hbar}, \tag{19}$$

ψ is easily shown to satisfy the relation $\bar\psi\psi=$ const, and the bilinear forms associated with ψ satisfy the following relations:

$$ic\bar\psi\gamma_\mu\psi=v_\mu, \tag{20a}$$

$$i\hbar\bar\psi\gamma_5\gamma_\mu\psi=S_\mu. \tag{20b}$$

In deriving (20), v_μ and S_μ are defined by Eqs. (14) and (16), and the identity (17) is invoked. If we interpret the left-hand sides of Eqs. (20) as the velocity and spin of a classical particle, we arrive at the classical equations.

III. CONCLUSION

In this note we have shown that the classical equations of Bargmann, Michel, and Telegdi may be derived from either the asymptotic solutions to the Dirac equation or the squared Dirac equation. In a future paper we shall discuss quantization of these WKB solutions, as well as the many analogies existing between the classical theory and the quantum theory of spinning particles.

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Classically Radiationless Motions and Possible Implications for Quantum Theory

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A simple general criterion is developed, using the retarded potentials of classical electromagnetic theory, for absence of radiation from arbitrary time-periodic charge-current distributions. The criterion is applied to rigid finitely extended distributions of charge which may undergo orbital motion with period T. It is found that, for this type of distribution, the condition for no radiation is that the extent b of the distribution be an integer multiple of cT. Some of these distributions may spin while orbiting. There exists at least one asymmetric spinning distribution which doesn't radiate under this condition; for this distribution, the (constant) spin angular velocity must be proportional to an integer ≥ 0 times c/b. This leads to the result that that part of the total (electromagnetic) angular momentum which is associated with the spin angular velocity must be an integer ≥ 0 times e^2/c times a numerical constant whose value depends on the details of the distribution. It is shown that, when such nonradiating distributions are considered as stable particles, there exists an intrinsic uncertainty relation of the same form and with almost the same meaning as that of quantum theory.

I. INTRODUCTION

IT still seems a fairly common belief that there exist no nontrivial charge-current distributions which do not radiate, according to classical electromagnetic theory retarded potential solutions. However, early in this century Sommerfeld,[1] Herglotz,[2] and Hertz[3] considered extended electron models, and established the existence of radiationless self-oscillations. In 1933, Schott[4] showed that a uniformly charged spherical shell will not radiate while in orbital motion with period T, provided the shell radius is an integral multiple of $cT/2$; the orbit need not be circular nor even planar. In 1948, Bohm and Weinstein[5] found several other rigid spherically symmetric distributions which can oscillate linearly without radiating.

In this paper we derive a simple exact criterion for absence of radiation, and apply it to moving rigid extended charge distributions.[6] We find that there are many such distributions, some of which may "spin," and others which need not be spherically symmetric.

[1] A. Sommerfeld, Nachr. Akad. Wiss. Goettingen, Math.-Physik. Kl. IIa Math.-Physik. Chem. Abt. **1904**, 99 and 363; **1905**, 201.
[2] G. Herglotz, Nachr. Akad. Wiss. Goettingen, Math.-Physik. Kl. IIa Math.-Physik. Chem. Abt. **1903**, 357; Math. Ann. **65**, 87 (1908).
[3] P. Hertz, Math. Ann. **65**, 1 (1908).
[4] G. A. Schott, Phil. Mag. Suppl. 7, **15**, 752 (1933).
[5] D. Bohm and M. Weinstein, Phys. Rev. **74**, 1789 (1948).
[6] In Bull. Am. Phys. Soc. **9**, 148 (1964), which I received while writing this paper, there appears an abstract by S. M. Prastein and T. Erber which implies that some of the content of this paper has been worked out independently by these authors.

The allowed types of such distributions are severely restricted by the condition of no radiation; further, it must be true that the finite radius of a rigid volume distribution be an integer multiple of cT, where T is the period of orbital motion. This last restriction implies that the perimeter of the orbit is less than the extent of the distribution.[4]

We show that there exists (at least) one periodic, orbiting, *spinning*, *asymmetric*, nonradiating distribution. The asymmetry requires that the constant spin angular velocity $\Omega=2\pi n/T$, n integer ≥ 0; the condition for no radiation is as before $b=lcT$, l integer >0, where b is the extent of the distribution. For such a distribution it then turns out that the electromagnetic spin angular momentum must be an integer times e^2/c times a numerical constant whose value depends on the details of the distribution. (The spin angular momentum is taken as that part which involves Ω.)

The postulate that all "particles" have finite size leads to an inherent lack of sharpness in the measured values of both the position and the velocity of any geometrical point associated with a distribution. For example, the minimum uncertainty in a position measurement must be of the order of the size of the smallest particles available for use in measuring devices. If the distribution is in a nonradiating orbit, our condition for no radiation provides a relation between the minimum uncertainties in simultaneously measured values of the position and momentum of the center of momentum of the distribution. This relation has the same form as the uncertainty relation of quantum theory.

Finally, we list and discuss a set of propositions which we feel must be at least partially verified before we could even hypothesize that existing particles (and aggregates) might be formed from nonradiating charge-current distributions.

II. FORMALISM

We consider a given current-charge distribution $(\mathbf{j},\rho)$ which executes periodic motion, period T, localized in a finite volume V of space.[7] We assume that a time and space Fourier decomposition is valid:

$$\mathbf{j}(\mathbf{x},t)=(2\pi)^{-3}\sum_{n=-\infty}^{\infty}\int d^3k\,\mathbf{J}(\mathbf{k},n)\times\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega_n t)], \quad (1a)$$

$$\rho(\mathbf{x},t)=(2\pi)^{-3}\sum_{n=-\infty}^{\infty}\int d^3k\,\omega_n^{-1}\mathbf{k}\cdot\mathbf{J}(\mathbf{k},n)\times\exp[-i(\mathbf{k}\cdot\mathbf{x}-\omega_n t)], \quad (1b)$$

where $\omega_n\equiv 2\pi n/T$, n integer. The Fourier coefficients for ρ follow from the continuity relation $\nabla\cdot\mathbf{j}+\partial\rho/\partial t=0$. Reality conditions imply that $\mathbf{J}^*(-\mathbf{k},-n)=\mathbf{J}(\mathbf{k},n)$. In (1b), the $n=0$ term must be taken equal to the (arbitrary) time-averaged value of $\rho(\mathbf{x},t)$; this is possible because generally $\mathbf{k}\cdot\mathbf{J}(\mathbf{k},0)=0$. In any event the $n=0$ terms of (1a), (1b) will not contribute to the radiation.

According to classical electromagnetic theory, the retarded potential solutions $(\mathbf{A},\varphi)$ of the Maxwell equations are[8] (Lorentz gauge, Gaussian units, $c=1$):

$$\mathbf{A}(\mathbf{x},t)=\int d^3y\,|\mathbf{x}-\mathbf{y}|^{-1}\mathbf{j}(\mathbf{y},t-|\mathbf{x}-\mathbf{y}|), \quad (2)$$

with φ the same except that $\mathbf{j}$ is replaced by ρ.

We take as definition of the rate of radiation R:

$$R\equiv\lim_{x\to\infty}\int d\Omega x^2\hat{x}\cdot\mathbf{S}, \quad (3)$$

where $\hat{x}$ is the outward radial unit vector, $d\Omega=\sin\theta d\theta d\varphi$ is the solid angle element, and $\mathbf{S}$ is the Poynting vector:

$$\mathbf{S}=(4\pi)^{-1}\mathbf{E}\times\mathbf{H}, \quad (4)$$

with $\mathbf{E}$, $\mathbf{H}$ the electric and magnetic fields,

$$\mathbf{E}=-\nabla\varphi-\partial\mathbf{A}/\partial t,\quad \mathbf{H}=\nabla\times\mathbf{A}.$$

To investigate the radiation we need only keep terms in $(\mathbf{A},\varphi)$ asymptotically proportional to x^{-1}. In view of the fact that $(\mathbf{j},\rho)$ is localized, these are[8]

$$\mathbf{A}(\mathbf{x},t)\sim x^{-1}\int d^3y\,\mathbf{j}(\mathbf{y},t-x+\hat{x}\cdot\mathbf{y}), \quad (5)$$

and similarly for φ. Inserting (1) in (5), we find

$$\mathbf{A}(\mathbf{x},t)\sim x^{-1}\sum\nolimits_n \mathbf{J}(\omega_n\hat{x},n)\exp i\omega_n(t-x), \quad (6)$$

and similarly for φ; combining (6), (4), and (3), we find

$$R=\sum\nolimits_n \exp i\omega_n(t-x)\left\{\sum\nolimits_l \omega_l\omega_{n-l}\times\int d\Omega\,\mathbf{J}(\omega_l\hat{x},l)\cdot(\hat{x}\hat{x}-\mathbf{I})\cdot\mathbf{J}(\omega_{n-l}\hat{x},n-l)\right\}, \quad (7)$$

where $\mathbf{I}$ is the idemfactor.

If R is to be zero for all t, the curly brackets above must be zero for all $n>0$. Certainly a *sufficient* condition for this is that

$$\mathbf{J}(\omega_n\hat{x},n)=0,\quad n>0, \quad (8)$$

although this condition may not be necessary. Another (weaker) sufficient condition is that $\mathbf{J}(\omega_n\hat{x},n)\propto\hat{x}$, $n>0$. [We need consider $n>0$ only, since $\mathbf{J}(-\omega_n\hat{x},-n)=\mathbf{J}^*(\omega_n\hat{x},n)$, and $\mathbf{J}(0,0)=0$. This last follows because $\mathbf{J}(0,0)$ is proportional to the time average of the *total* current, which we shall always take to be zero.]

[7] If the distribution translates as a whole with constant velocity, we can presumably transform to its "rest frame."

[8] L. Landau and E. Lifshitz, *The Classical Theory of Fields* (Addison-Wesley Publishing Company, Reading, Massachusetts, 1951), Chap. 9, p. 186.

We shall make use of (8) in what follows, since $\mathbf{J}(\omega_n\hat{x},n)\propto\hat{x}$ corresponds to the trivial case of spherically symmetric radial oscillations. It seems likely from the functional form of the integrand of (7) that one or the other of these conditions is also necessary for $R=0$; we have not investigated this in detail.

III. EXAMPLES

We shall confine our present study to a rigid extended $(\mathbf{j},\rho)$, exemplified by a functional form $\rho(\mathbf{z})$, $\mathbf{z}=\mathbf{x}-\mathbf{a}(t)$, where $\mathbf{a}(t)$ is the position vector of the "center" of the distribution, and is assumed periodic with period T. Also we insist that $(\mathbf{j},\rho)=0$ for $|\mathbf{z}|>b$; i.e., no charge or current outside a spherical surface radius b from the center. We assume that $\rho(\mathbf{z})$ is a continuous function[9] for all $\mathbf{z}$ with $|\mathbf{z}|<b$; if $\mathbf{j}(\mathbf{z})=\dot{\mathbf{a}}(t)\rho(\mathbf{z})$, the continuity relation will then hold even at $z=b$, if we take for this point

$$\lim_{\epsilon\to 0}\{[\boldsymbol{\nabla}\cdot\mathbf{j}+\partial\rho/\partial t]_{z=b\pm\epsilon}\}.$$

Using $\mathbf{j}(\mathbf{z})=\dot{\mathbf{a}}(t)\rho(\mathbf{z})$, $\mathbf{z}=\mathbf{x}-\mathbf{a}(t)$, we find

$$\mathbf{J}(\omega_n\hat{x},n)=T^{-1}\int_0^T dt\,\dot{\mathbf{a}}(t)[\exp i(\omega_n\hat{x}\cdot\mathbf{a}(t)-\omega_n t)]\times\int_{y=0}^{b}d^3y\,\rho(\mathbf{y})\exp i\omega_n\hat{x}\cdot\mathbf{y}. \quad (9)$$

The problem reduces to finding those $\rho(\mathbf{y})$ for which $I(\omega_n\hat{x})=0$, where

$$I(\mathbf{k})\equiv\int_{y=0}^{b}d^3y\,\rho(\mathbf{y})\exp i\mathbf{k}\cdot\mathbf{y}; \quad (10)$$

this is just the Fourier transform of $\rho(\mathbf{y})$.

A. Spherical Symmetry

By spherical symmetry we mean $\rho(\mathbf{z})=\rho(z)$. Then

$$I(\omega_n\hat{x})=I(\omega_n)=4\pi\int_0^b dy\,y\sin\omega_n y\rho(y),\quad (n>0). \quad (11)$$

1. Schott's Case

If e is the total charge on a spherical shell of radius $r<b$, then in this case $\rho(z)=(4\pi r^2)^{-1}e\delta(z-r)$. Inserting this in (11), we find

$$I(\omega_n)=er^{-1}\sin\omega_n r. \quad (12)$$

This will be zero if $\omega_1 r=2\pi r/T=l\pi$, l integer >0. Therefore, if $r=lcT/2$, there is no radiation. This condition represents the only restriction; e.g., the orbits need not be circular nor even planar.

[9] This assumption is in the spirit of classical field theory, but it could raise a large discussion. If all charge distributions must in the end be aggregates of point particles, our proof of existence of radiationless motions is not strictly valid. Cf. Sec. III A1.

Schott[4] derives this same result by an entirely different analytical procedure. Bohm and Weinstein[5] use a yet different analysis to achieve (11). Schott also gives a good bit of physical reasoning in an attempt to educate intuition up to the mathematics. It is worth repeating after Schott that (i) the spherical shell cannot "spin" (we also show this later in C of this section); (ii) the perimeter of the orbit is $\bar{v}T=2r\bar{v}/l<2\pi r$, where $\bar{v}=$average speed <1. This means that the spherical shell can only "wobble," but not execute a large orbit.

A further illustration might prove useful here. Suppose

$$\mathbf{j}(\mathbf{x},t)=\sum_{\alpha=1}^{N}e_\alpha\delta(\mathbf{x}-\mathbf{a}_\alpha(t))\dot{\mathbf{a}}_\alpha(t).$$

This corresponds to an aggregate of N point charges, each of charge e_α. Suppose the N point particles, each with $e_\alpha=e/N$, are distributed (perhaps nonuniformly) over a spherical surface of radius r. Then $\mathbf{a}_\alpha(t)=\mathbf{a}(t)+\mathbf{r}_\alpha$, where $\mathbf{a}(t)$ is as before the location of the center of the spherical surface. Then

$$\mathbf{J}(\omega_n\hat{x},n)=eT^{-1}\int_0^T dt\,\dot{\mathbf{a}}(t)[\exp i(\omega_n\hat{x}\cdot\mathbf{a}(t)-\omega_n t)]\times\sum_{\alpha=1}^{N}\exp i\omega_n\hat{x}\cdot\mathbf{r}_\alpha. \quad (13)$$

Now if θ_α is the angle between $\hat{x}$ and $\mathbf{r}_\alpha$, $\mathbf{J}(\omega_n\hat{x},n)$ will be zero if

$$\sum_\alpha\exp i\omega_n r\cos\theta_\alpha=0. \quad (14)$$

In the limit when $N\to\infty$ and the distribution is made uniform over the surface, we may replace the sum (14) by $(4\pi)^{-1}\int d\Omega\exp i\omega_n r\cos\theta$, and we get Schott's result (12) immediately.

This procedure illustrates that if we want the radiation to be *exactly* zero [by the criterion $\mathbf{J}(\omega_n\hat{x},n)=0$], we probably cannot built a distribution out of *point* charges. However, it seems that we can generally approach $\mathbf{J}(\omega_n\hat{x},n)=0$ as closely as we like by packing point charges closer and closer (in certain definite arrangements) throughout some *finite* region. A similar conclusion has been reached by Ksienski[10] in connection with antenna radiation patterns.

2. Volume Distributions

We develop here a rather general rigid spherically symmetric distribution which may orbit periodically without radiating.

Consider $\rho(y)=Ay^{-1}\cos\omega_q y$ for $y\leq b$, zero outside; q integer ≥ 0, $A=$constant. Then from (11)

$$I(\omega_n)=4\pi A\int_0^b dy\,\sin\omega_n y\cos\omega_q y.$$

[10] A. Ksienski, Can. J. Phys. **39**, 335 (1961).

This will be zero by the well-known circular function orthogonality provided $\omega_1 b=2\pi l$, l integer >0.

Therefore we may use any linear combination of terms of this type, and still $R=0$; i.e., if for $y\leq b$

$$\rho(y)=\sum_q y^{-1}A(q)\cos\omega_q y, \tag{15}$$

and $\rho(y)=0$ for $y>b$, where $A(q)$ are arbitrary real constants, and if $b=lcT$, l integer >0, there will be no radiation due to periodic orbital motion of this distribution.

We show that, as in Schott's case, this distribution cannot execute a large orbit. The amount of charge in a spherical shell of thickness dy at y is proportional to $y^2\rho(y)=\sum_q yA(q)\cos\omega_q y$. The function $\cos\omega_q y$ is symmetric about $y=b/2$. Therefore we could at best choose coefficients such that $y^2\rho(y)$ is large near $y=b/2$, nearly zero elsewhere. The perimeter of the orbit is $\bar{v}T=\bar{v}b/l=2(b/2)(\bar{v}/l)<2\pi(b/2)$, where $\bar{v}<1$ is the average speed. So the "radius" of the orbit is less than the apparent extent of the charge.

B. Nonspherically Symmetric Distributions

We extend formally to the case where $\rho(\mathbf{y})=0$ still outside a spherical surface of radius b, but has arbitrary angular dependence inside. The criterion for existence of radiationless orbits is then, from (10),

$$I(\omega_n\hat{x})=\int_0^b d^3y\,\rho(\mathbf{y})\exp i\omega_n\hat{x}\cdot\mathbf{y}=0. \tag{16}$$

We expand $\rho(\mathbf{y})$ in terms of the orthonormal spherical harmonics $Y_{lm}(\theta',\varphi')$:

$$\rho(\mathbf{y})=\sum_{l=0}^{\infty}\sum_{m=-l}^{l}R_{lm}(y)Y_{lm}(\theta',\varphi'). \tag{17}$$

Here,

$$Y_{lm}(\theta',\varphi')=\left[\frac{2l+1}{4\pi}\frac{l-|m|}{l+|m|}\right]^{1/2}P_l^m(\cos\theta')\exp im\varphi', \tag{18}$$

where $P_l^m(\cos\theta')$ are the associated Legendre polynomials.[11] The orthonormality is $\int d\Omega Y_{l'm'}^*(\theta,\varphi)Y_{lm}(\theta,\varphi)=\delta_{mm'}\delta_{ll'}$. Since $Y_{l,m}=Y_{l,-m}^*$, the reality requirement on ρ implies $R_{l,-m}^*=R_{l,m}$.

We express the plane wave $\exp i\mathbf{k}\cdot\mathbf{y}$ in terms of the spherical harmonics, where $\theta_{ky}\equiv$angle between $(\mathbf{k},\mathbf{y})$ [11]:

$$\exp i\mathbf{k}\cdot\mathbf{y}=\sum_{l=0}^{\infty}i^l(2l+1)j_l(ky)P_l(\cos\theta_{ky}), \tag{19}$$

where $j_l(ky)$ is the spherical Bessel function of order l. Now we use (18), and the addition theorem for Legendre polynomials,[12] and find

$$\exp i\mathbf{k}\cdot\mathbf{y}=4\pi\sum_{l=0}^{\infty}\sum_{m=-l}^{l}(1-\tfrac{1}{2}\delta_{m,0})i^lY_{l,m}(\theta,\varphi)\times Y_{lm}^*(\theta',\varphi')j_l(ky), \tag{20}$$

where (θ,φ) are polar angles of $\mathbf{k}$, (θ',φ') are polar angles of $\mathbf{y}$, with respect to z axis. Then we find for (16)

$$I(\omega_n\hat{x})=\sum_{l,m}i^l(1-\tfrac{1}{2}\delta_{m,0})Y_{lm}(\theta,\varphi)\times\int_0^b dy\,y^2j_l(\omega_n y)R_{lm}(y), \tag{21}$$

which implies that, for radiationless orbits,

$$I_{lm}(\omega_n)\equiv\int_0^b dy\,y^2j_l(\omega_n y)R_{lm}(y)=0. \tag{22}$$

This agrees with the spherically symmetric case, since $j_0(x)=\sin x/x$. It's easy to show that (22) is satisfied for $l=1$, $|m|\leq l$, if we choose $R_{1m}(y)=A_{1m}=$const for $y\leq b$, zero outside, and if $b=lcT$ as before. Therefore we may choose for $\rho(\mathbf{y})$ inside $y=b$

$$\rho(\mathbf{y})=\sum_{q=0}^{\infty}y^{-1}A(q)\cos\omega_q y+\sum_{m=-1}^{1}A_{1m}Y_{1m}(\theta,\varphi); \tag{23}$$

this distribution will orbit without radiating provided $\omega_1=2\pi b/T=2\pi l$, or $b=lcT$, l integer >0

It is tempting to assume that interesting solutions of (22) could be found for all l. This no doubt would be true if the spherical Bessel functions $j_l(x)$ had equally spaced zeros. Since they do not, we must at present conclude that our solutions for $l=0$, 1 are accidental. We have not tried to find solutions of (22) for $l>1$.

Possibly we could find nonradiative solutions for rigid charge distributions vanishing outside a spheroidal or some other nonspherical surface. Rather than pursue this, we plan in the future to search for mcuh more general nonradiating distributions, involving parameters describing several internal degrees of freedom.

C. Inclusion of Spin

We add to the current $\dot{\mathbf{a}}(t)\rho(\mathbf{x}-\mathbf{a}(t))$ a term

$$\mathbf{j}'(\mathbf{z})=(\mathbf{\Omega}\times\mathbf{z})g(z),\quad z\leq b, \tag{24}$$

zero outside, where $\mathbf{z}=\mathbf{x}-\mathbf{a}(t)$,$\mathbf{\Omega}$ is a constant angular velocity, and $g(z)$ is a spherically symmetric function having the dimension of charge density. [We use $g(z)$ instead of the actual charge density $\rho(z)$ to allow for nonrigid spinning.] Continuity is automatically satisfied, since $\nabla\cdot\mathbf{j}'=0$. As previously, we take $\mathbf{J}'(\omega_n\hat{x},n)=0$

[11] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., Chap. 4, p. 73; Chap. 5, p. 104.

[12] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, Cambridge, England, 1952), 4th ed., Chap. 15, p. 328.

as the criterion for no radiation. This is

$$J'(\omega_n\hat{x},n)=T^{-1}\int_0^T dt[\exp i(\omega_n\hat{x}\cdot\mathbf{a}(t)-\omega_n t)]$$
$$\times\left\{-i\mathbf{\Omega}\times\nabla_{\mathbf{k}}\int_{y=0}^{b} d^3y\, g(y)\exp i\mathbf{k}\cdot\mathbf{y}\right\}_{\mathbf{k}=\omega_n\mathbf{x}/x}. \quad (25)$$

The problem reduces to finding those $g(y)$ for which $\mathbf{I}(\omega_n\hat{x})=0$, where

$$\mathbf{I}(\mathbf{k})\equiv\nabla_k\int_{y=0}^{b} d^3y\, g(y)\exp i\mathbf{k}\cdot\mathbf{y}=4\pi k^{-3}\mathbf{k}$$
$$\times\int_0^b dy\, yg(y)(ky\cos ky-\sin ky). \quad (26)$$

It is easy to verify that the following expression for $g(y)$ makes $\mathbf{I}(\omega_n\hat{x})=0$, provided $b=lcT$ as before:

$$g(y)=By^{-1}, \quad (27)$$

with B an arbitrary real constant.

We may add together all those distributions we have found up to now, and the whole will not radiate, under the restriction that the radius of the distribution is an integer (times c) times the period of its orbital motion.

We note that it is immediately apparent from (26) that a spinning spherical shell distribution $g(y)\propto\delta(y-r)$, $r<b$, cannot satisfy $\mathbf{I}(\omega_n\hat{x})=0$ for all $n>0$.

D. Spin Plus Asymmetry

We show now that there exists at least one type of rigid nonspherically symmetric spinning distribution which will not radiate while in orbital motion, provided as before that the extent is an integer multiple of the period.

We consider a charge distribution

$$\rho(\mathbf{x},t)=h(z)+g(z)(z_1\cos\Omega t+z_2\sin\Omega t),\quad z\leq b, \quad (28)$$

and $\rho=0$ for $z>b$, where, as before, $\mathbf{z}=\mathbf{x}-\mathbf{a}(t)$, with $\mathbf{a}(t)$ the position of the "center." Here z_1 and z_2 are the x and y components of $\mathbf{z}$. As a suitable current density we take

$$\mathbf{j}(\mathbf{x},t)=[\dot{\mathbf{a}}(t)+\Omega(\hat{x}_3\times\mathbf{z})]\rho(\mathbf{x},t), \quad (29)$$

where $\hat{x}_3$ is a unit vector in the z direction. It's easily verified that this satisfies continuity.

These functional forms clearly represent a rigid non-symmetric distribution which spins with angular speed Ω about the z axis. We see from (28) that, because of the asymmetry, $2\pi/\Omega$ must be commensurate with the period of orbital motion T_0. Suppose that the distribution makes at least $1/q$ rotations per orbit, where $q=$ integer such that $0<q<\infty$. Then the necessarily finite period of motion $T=qT_0$. Then it follows that

$$\Omega=2\pi n/T,\quad n \text{ integer }\geq 0. \quad (30)$$

Since we have ensured that the distribution is both periodic and localized, we may apply the criterion $\mathbf{J}(\omega_n\hat{x},n)=0$ for no radiation. The current density (29) will be a sum of four terms, which we label $\mathbf{j}^{(i)}$, $i=1,\cdots,4$. We now analyze the corresponding $\mathbf{J}^{(i)}(\mathbf{k},n)$.

First term: $\mathbf{j}^{(1)}=\dot{\mathbf{a}}(t)h(z)$.

This is of the same form as the current considered in Sec. IIIA2. Therefore it will not radiate provided $b=lcT$, l integer >0, and

$$h(z)=\textstyle\sum_{q=0}^{\infty} z^{-1}H(q)\cos\omega_q z, \quad (31)$$

where $H(q)$ are arbitrary coefficients, and $\omega_q=2\pi q/T$.

Second term: $\mathbf{j}^{(2)}=\dot{\mathbf{a}}(t)g(z)(z_1\cos\Omega t+z_2\sin\Omega t)$.

This is of the same form as one of those considered in Sec. IIIB; in particular, this current may be written in terms of the spherical harmonics $Y_{1,1}$ and $Y_{1,-1}$. Here, the function $zg(z)$ corresponds to the radial functions $R_{1,1}$ or $R_{1,-1}$. Therefore this current distribution will not radiate provided $b=lcT$ and

$$g(z)=z^{-1}G,\quad G=\text{const}. \quad (32)$$

Third term: $\mathbf{j}^{(3)}=\Omega(\hat{x}_3\times\mathbf{z})h(z)$.

This is of the same form as that considered in Sec. IIIC. Therefore it will not radiate provided $b=lcT$, and

$$h(z)=Hz^{-1},\quad H=\text{const}. \quad (33)$$

This result means that we must set all the $H(q)=0$ in (31), except $H(0)$.

Fourth term: $\mathbf{j}^{(4)}=\Omega(\hat{x}_3\times\mathbf{z})g(z)(z_1\cos\Omega t+z_2\sin\Omega t)$.

This has not been treated before. We investigate

$$\mathbf{J}^{(4)}(\mathbf{k},n)=T^{-1}\int_0^T dt[\exp i(\mathbf{k}\cdot\mathbf{a}(t)-\omega_n t)]$$
$$\times\int_{y=0}^{b} d^3y(\exp i\mathbf{k}\cdot\mathbf{y})\mathbf{j}^{(4)}(\mathbf{y},t). \quad (34)$$

From (34), the expression of interest (which should be zero for $\mathbf{k}=\omega_n\hat{x}$, $n>0$), is

$$\mathbf{K}(\mathbf{k})\equiv\int_{y=0}^{b} d^3y\,\mathbf{yy}g(y)\exp i\mathbf{k}\cdot\mathbf{y}$$
$$=-\nabla_k\nabla_k\int_{y=0}^{b} d^3y\, g(y)\exp i\mathbf{k}\cdot\mathbf{y}. \quad (35)$$

We must use the form (32) for $g(y)$, and also use $b=lcT$; making these substitutions in $\mathbf{K}(\mathbf{k})$, we find

$$\mathbf{K}(\omega_n\hat{x})=\hat{x}\hat{x}(\pi/\omega_n^2)b^2G. \quad (36)$$

This is not zero by itself. However, we may add to the distribution (28) another distribution *concentric* with (28) [i.e., $\mathbf{a}(t)$ is the same for both], but generally of a *different* radius. If both have the same Ω, and $b_1=l_1cT$,

$b_2=l_2cT$, and both have the functional forms (32) and (33), from (36) there will be no radiation from the total distribution provided

$$l_1^2G_1+l_2^2G_2=0. \quad (37)$$

Therefore there will be no radiation from a charge-current distribution

$$\mathbf{j}(\mathbf{x},t)=[\dot{\mathbf{a}}(t)+\Omega(\hat{x}_3\times\mathbf{z})][\rho_1(\mathbf{x},t)+\rho_2(\mathbf{x},t)], \quad (38a)$$

where

$$\begin{aligned}\rho_1(\mathbf{x},t)&=z^{-1}[H_1+G_1(z_1\cos\Omega t+z_2\sin\Omega t)], \quad 0\leq z\leq b_1;\\ \rho_2(\mathbf{x},t)&=z^{-1}[H_2-(l_1^2/l_2^2)G_1(z_1\cos\Omega t+z_2\sin\Omega t), \quad 0\leq z\leq b_2;\end{aligned} \quad (38b)$$

with $\rho_1=0$ for $z>b_1$, $\rho_2=0$ for $z>b_2$, provided $b_i=l_icT$, and $\Omega=2\pi n/T$, n integer ≥ 0. Combining $\Omega=2\pi n/T$ with $b_i=l_icT$, we have for possible values of Ω

$$\Omega=(2\pi cnl_i/b_i)=(2\pi c/b_2)n, \quad (39)$$

where $b_2>b_1$, n integer ≥ 0.

From (38b) it is evident that if we choose $b_1=b_2$, the distribution reverts to a perfectly spherical one, and we lose the condition (39).

The procedure of adding together several different distributions, each of different extent, may help us to find other interesting nonradiating distributions. We plan to exploit this possibility in the future.

IV. CONSERVED ELECTROMAGNETIC QUANTITIES

It is not at all immediately evident just how to extract physical meaning from the above results. It just seems that these results *should* apply to the real world. Especially fascinating to us is the result in Sec. IIID for a spinning asymmetric distribution, that the spin angular velocity Ω must be $2\pi c/b$ times an integer ≥ 0, where b is the over-all extent of the distribution. In what follows we show, in the limit of negligible effect from the orbital motion, that the condition $\Omega=(2\pi c/b)n$ implies quite generally that the electromagnetic spin angular momentum associated with a given distribution must be a numerical constant times e^2/c times an integer, independent of the size or electromagnetic energy content of the distribution.

We consider the energy, angular momentum, and magnetic moment of a finite nonradiating distribution. We regard these as completely electromagnetic quantities; as such, they must be constants of the motion of the distribution. The general expressions for the electromagnetic energy density, magnetic moment density, and angular momentum density, in terms of the fields $(\mathbf{E},\mathbf{H})$ and $(\mathbf{j},\rho)$ are[13]

$$W=(8\pi)^{-1}(E^2+H^2)+\int dt(\mathbf{j}\cdot\mathbf{E}), \quad (40a)$$

$$\mathbf{m}=(2c)^{-1}\mathbf{x}\times\mathbf{j}, \quad (40b)$$

$$\mathbf{M}=(4\pi c)^{-1}\mathbf{x}\times(\mathbf{E}\times\mathbf{H})+\int dt\,\mathbf{x}\times(\rho\mathbf{E}+c^{-1}\mathbf{j}\times\mathbf{H}), \quad (40c)$$

where dt is the time differential element.

The total energy $\mathcal{E}$, the magnetic moment $\boldsymbol{\mu}$, and the total angular momentum $\mathbf{S}$, are the integrals over all space of these densities. These quantities must be expressible in terms of the three parameters e, c, b, if we neglect the influence of orbital motion, and replace Ω by $2\pi cn/b$. (Here e is either the total charge of the distribution, or, in case of total charge $=0$, e is some appropriate charge.) It is clear that, in this limit of negligible effect from orbital motion, $|\mathbf{j}|$ and $|\mathbf{H}|$ are proportional to $e\Omega/c$, and $|\mathbf{E}|\propto e$. Also since in this limit we expect very nearly

$$\mathbf{j}\propto\boldsymbol{\Omega}\times\mathbf{x}, \quad \mathbf{H}\propto\mathbf{x}\times(\boldsymbol{\Omega}\times\mathbf{x}), \quad \mathbf{E}\propto\mathbf{x},$$

we see that the second terms in (40a), (40c) will make negligible contribution to $\mathcal{E}$ and $\mathbf{S}$. Therefore by dimensional analysis, with the help of (40),

$$\mathcal{E}\propto e^2b^{-1}[1+(\text{const.})(b\Omega/c)^2] = e^2b^{-1}[1+(\text{const.})n^2]; \quad (41)$$

$$|\boldsymbol{\mu}|\propto eb(\Omega b/c)\propto ebn, \quad (42)$$

$$|\mathbf{S}|\propto e^2c^{-1}(\Omega b/c)\propto (e^2/c)n. \quad (43)$$

If we choose the "mass" of the distribution by the Einstein relation $\mathcal{E}=mc^2$ and solve (41a) for b in terms of m, we find from (41) the magnetogyric ratio $|\boldsymbol{\mu}|\propto(e/mc)|\mathbf{S}|$, as expected.

The striking feature here is the behavior of the spin angular momentum: It is proportional to e^2/c times an integer, independent of the mass or size of the distribution. The value of the proportionality constant depends of course on the details of the distribution; we should expect to find it in the neighborhood of $\hbar c/2e^2\approx 137/2$.

We have carried through a calculation of $\mathcal{E}$, $\boldsymbol{\mu}$, and $\mathbf{S}$ based on a steady (nonorbiting) current density $\mathbf{j}(\mathbf{x})=(\boldsymbol{\Omega}\times\mathbf{x})\rho(x)$, $\rho(x)=e/2\pi b^2x$, $x\leq b$, zero outside. This represents a distribution which would not radiate while in orbital motion, provided $b=lcT$. Of course, this one, being steady and spherical, will not radiate no matter what the values of Ω or b; however, the very slightest asymmetry and orbital motion would then require $\Omega=2\pi cn/b$, n integer ≥ 0.

We quote our results merely to show that there is no horrible order-of-magnitude disagreement with experience. Using the above $(\mathbf{j},\rho)$ and standard Maxwell theory, plus the known numerical values for the mass and charge of the electron, we found (with $\Omega=2\pi cn/b$)

$$\begin{aligned}&|\mathbf{S}|\approx 10^{-29}n\ \text{g-cm}^2\text{-sec}^{-1},\\ &|\boldsymbol{\mu}|\approx 6(e/mc)|\mathbf{S}|,\\ &b\approx 5\times10^{-13}\ \text{cm},\quad n=1.\end{aligned} \quad (44)$$

[13] See any standard treatise on electromagnetic theory; e.g., that of Ref. 8.

These values are certainly somewhat anomalous; in particular, "Planck's constant" is almost 2 orders of magnitude too small, and the magnetogyric ratio is 6 times too large for the electron. But factors of 6 or 10^2 ought to be amenable to change, provided we can find other asymmetric spinning distributions satisfying the same conditions for no radiation.

V. UNCERTAINTY RELATION

In this section we show that, for our nonradiating distributions, there exists an uncertainty relation between momentum and position which has the same form as that of quantum theory, and a similar meaning.

To begin, we must describe a measurement process on a distribution. Presumably, if at a given time we knew the electromagnetic fields exactly at all points of space surrounding the distribution, we could infer exactly both the instantaneous position and velocity of both the center of the distribution and the center of (electromagnetic) linear momentum. (In our previous calculations, the origin of coordinates has been taken at the center of momentum. This follows because we have always chosen the coordinate frame such that the total current is zero, and, since there is no radiation from the distribution, the total linear momentum must then be a constant, equal to zero in that frame.)

In order to measure the field pattern in actuality, we must use field probes of *finite* size. Naturally we would want to use the smallest ones available; in principle, these would be of an extent b_1, of the order of the size of the smallest particles available to us. (At present we must regard as a postulate the assumption that only finite sized particles exist in nature.) Then it seems clear that we must have an inherent uncertainty $\Delta x_i \gtrsim b_1$ in the measured value of the ith component of the position of any geometrical point associated with a distribution.

There must also be an inherent uncertainty in our simultaneous knowledge of the velocity of any point, since a value for velocity must be inferred from the same inaccurate field pattern. We assume that just before the measurement the distribution is in a radiationless state; i.e., the center of the distribution is orbiting about the center of momentum with period T. Then the measured ith component of the velocity of any point must have an uncertainty $\Delta v_i > b_1/T = (b_1/b_2)lc$. (The last equality follows from the condition for no radiation, $b_2 = lcT$, l integer >0, where b_2 is the radius of the distribution.) If the point being considered is the center of momentum, we have $\Delta p_i \approx m_2 \Delta v_i$, where m_2 is the mass of the distribution. Altogether,

$$\Delta p_i \Delta x_i \gtrsim m_2 b_1^2/T \geq m_2 b_1^2 c/b_2 . \tag{45}$$

As in the preceding section, we consider the mass to be electromagnetic. Then we have $m_2 = K'e^2/b_2c^2$, where K' is a numerical constant depending on the details of the distribution. Inserting this in (45), we get

$$\Delta p_i \Delta x_i \gtrsim K'(b_1^2/b_2^2)(e^2/c) . \tag{46}$$

If the distribution is itself of size b_1, ($b_2 \approx b_1$), then $\Delta p_i \Delta x_i \gtrsim K'(e^2/c)$. This is the case of interest in modern physics.

Suppose that the distribution is large compared to b_1. We consider roughly that the distribution is composed of N "particles" of size b_1. Then we may write

$$m_2 = f(N)m_1 = K'e^2/b_2c^2 = f(N)Ke^2/b_1c^2 ,$$

where $f(N) \lesssim N$, $f(1)=1$; we take $b_2 \approx N^{1/3}b_1$. Then from (46) we find

$$\Delta p_i \Delta x_i \gtrsim f(N)N^{-1/3}K(e^2/c) . \tag{47}$$

We expect that $f(N)$ increases with N faster than $N^{1/3}$ (but of course slower than N), so that $\Delta p_i \Delta x_i \gtrsim Ke^2/c$ in all cases. But (46) and (47) show that our uncertainty relation is not necessarily the same for all distributions. Actually we know that, for macroscopic distributions, the linear extent of a distribution is very much greater than the sum of the diameters of the constituents. It is more likely that $b_2 \lesssim f(N)b_1$, than that $b_2 \approx N^{1/3}b_1$. The former dependence would make the uncertainty relation (46) universal. We conclude only that the size dependence of (46) is bound to be weak.

Although the functional form of our uncertainty relation is identical with that of quantum mechanics, the interpretation is somewhat different. In particular, we wish to emphasize the implication of a *finite* minimum uncertainty in both position and velocity measurements. A fundamental lack of sharpness in position measurement was postulated and called an "inherent dispersion" in a theory recently developed by Ingraham.[14] The condition for no radiation makes an apparently fortuitous connection between inherent dispersions in simultaneous position and velocity measurements.

VI. DISCUSSION

The crucial result in the preceding sections was that, in the limit of negligible effect from the orbital motion, the (electromagnetic) spin angular momentum of an asymmetric, orbiting, spinning, nonradiating distribution must be a numerical constant times e^2/c times an integer. Naturally, it is very tempting to hypothesize from this that the existence of Planck's constant is implied by classical electromagnetic theory augmented by the conditions for no radiation. Such a hypothesis would be essentially equivalent to suggesting a "theory of nature" in which all stable particles (or aggregates) are merely nonradiating charge-current distributions whose mechanical properties are electromagnetic in origin. We certainly do not believe that this paper gives a sufficient foundation for hypothesizing a theory with

[14] R. L. Ingraham, Nuovo Cimento **24**, 1117 (1962).

such profound implications. Rather, we hope that this paper will serve as a foundation and as a stimulus for much further investigation of nonradiating distributions.

With this in mind, we offer a set of propositions which we feel must be at least partially verified before the above "theory of nature" could even be hypothesized. Following each proposition we give a short discussion of its implications.

Proposition (*i*). There exist many types of periodic, *nonrigid*, asymmetric, spinning, nonradiating volume distributions. In real life there exist no *stable* distributions which are perfectly symmetric about the spin axis.

This proposition is necessary to not violate the relativity principle (nonrigid) and to ensure that $\Omega=2\pi n/T$ (asymmetric). By "nonrigid" we mean "capable of changing shape under action of an external field." A nonrigid distribution must possess internal vibrational modes. Unless *all* those vibrational modes which are normally excited are axially symmetric, the distribution will not be.

Our present knowledge of the structure of macroscopic matter indicates that energetically stable perfectly axially symmetric distributions occur not at all. We believe it likely that the structural details of microscopic matter mirror those of macroscopic matter. (We also believe it likely that this is a philosophical sore point.) To investigate this, we propose to consider a distribution composed of N point-like charges and ask in the limit of very large N, "how must their motion be limited in order that there be no radiation from the aggregate?" We also plan to consider the "collective" approach, in which a relatively small number of parameters characterize the distribution. The total energy of a nonradiating distribution would be a function of these parameters; it might be a function with several local minima. We should try to identify the minimum energy nonradiating distributions with particles occurring in nature.

Proposition (*ii*). All physically reasonable distributions satisfy the *same* condition for no radiation. [For example, that of our distributions, whose extent must be a positive integer (times c) times the (finite) period of motion.]

This proposition is necessary in order that the allowed values of spin angular velocity Ω be related to the extent b through the relation $\Omega \propto cn/b$, n integer ≥ 0. It seems clear that a relation very close to this must be obeyed in general in order that the spin angular momentum spectrum be discrete.

The spin angular momentum spectrum which we have obtained ($|\mathbf{S}| \propto n$, n integer ≥ 0) differs from the intrinsic angular momentum spectrum predicted by quantum theory ($|\mathbf{J}| \propto [n(n+1)]^{1/2}$, n half-integer, or integer ≥ 0). However, we must remember that we foreordained $|\mathbf{S}| \propto n$ by choosing the spin axis fixed in space. If this were not required, it is not clear just what would be the details of the spin angular momentum spectrum. Nevertheless, it does seem clear that this spectrum would be discrete, provided propositions (i) and (ii) were satisfied.

A short discussion might be useful, concerning the interplay of orbital and spin angular momentum. (Remember that spin angular momentum here means that part associated with the spin angular velocity Ω.) If the spin axis were not fixed, neither the spin nor the orbital part of the angular momentum would be separately conserved. [In fact, from Eq. (40c) it is evident that in general they probably could not even be considered separately.] This would mean that the center of a *free* distribution might well orbit about a space-fixed or uniformly translating point. Recently, Corben[15] has found a similar behavior for free particles, according to relativistic classical mechanics.

Up to now we have been tentatively thinking of the spin angular momentum as "intrinsic." The above considerations seem to cast some doubt on this correlation. Whether we should regard as the intrinsic angular momentum the total, the spin, the time averaged value of the spin, or some other, remains for the moment an open question.

Proposition (*iii*). A "theory of nature" in which all stable particles (or aggregates of particles) are formed from nonradiating charge-current distributions agrees completely with present quantum theory, in domains where the latter is valid.

This is rather a "catch-all" proposition; however, it does not supersede propositions (i) and (ii), since it is not at all clear that present quantum theory can account for the structure of those fundamental particles which appear to be extended (or composite).

In particular, we would like to show an (at least approximate) isomorphism between quantum theory and this theory. Failing that, we must at least be able to explain atomic structure on this theory. We have as yet made no progress in these directions.

In conclusion: There certainly exist, in classical theory, nontrivial, nonradiating charge-current distributions. Some of their properties are suggestive of a possible connection with the quantal structure of nature. There remains a tremendous amount of investigation to be done before such a connection could be established.

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[15] H. C. Corben, Nuovo Cimento **20**, 529 (1961).