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Self-Consistent Calculation of the ρ-Meson Regge Pole

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The left-hand discontinuities in the partial-wave amplitudes for π-π scattering are assumed to be dominated by the exchange of the ρ meson in a form suggested by the Regge representation for a resonance. This Regge behavior provides the necessary high-energy cutoff and allows the N/D equations to be solved. The partial-wave $I=1$ amplitudes are calculated for noninteger angular momenta $l<1$ as well as $l=1$. The trajectory $\alpha_\rho(s)$ as well as the residue $\beta_\rho(s)$ of the ρ-meson Regge pole are evaluated. An attempt is made to obtain a self-consistent solution for the relevant parameters, namely the position and width of the ρ resonance and $\alpha_\rho(0)$. The results of this calculation give $\alpha_\rho(0)\gtrsim 0.9$. The $I=0$ vacuum trajectory is also discussed.

I. INTRODUCTION

THERE have been a number of papers written on the problem of determining the position and width of the ρ meson self-consistently.[1,2] In essence, these bootstrap calculations of the ρ used the exchange of this $I=1$, $l=1$ resonance in the crossed channels to provide the force necessary to produce the ρ meson in the direct channel. The $l=1$ part of the interaction is projected out and the partial-wave dispersion relations are solved by the N/D method. The hope is that the solution yields a resonance having the same position and width as that of the exchanged one.

A major difficulty is due to the divergence arising from the exchange of a massive vector particle, with sufficiently large coupling, which necessitates the use of a cutoff. Instead of considering the ρ to be a vector particle even when the energy of the exchanged ρ is not close to the resonant energy, Wong[2] employed a form suggested by the Regge representation for a resonance. This then provides a cutoff at high energy, the relevant parameter being the angular momentum of the ρ trajectory at zero energy, $\alpha_\rho^{\mathrm{In}}(0)$.

The purpose of this article is to carry Wong's ρ (bootstrap) calculation with a "Regge cutoff" a step further. For $l=1$ we carry out a calculation similar to his but then continue the N/D equations for noninteger angular momenta and calculate $\alpha_\rho(s)$, comparing $\alpha_\rho(0)$ with the input parameter $\alpha_\rho^{\mathrm{In}}(0)$. In other words, this is an attempt to bootstrap not only the position and width of the ρ resonance, but the slope of its Regge trajectory. The residue function $\beta_\rho(s)$ is also determined. The sensitivity of our results to some of the approximations made is examined. For example, the above calculation is compared to a similar one in which we take the exchanged ρ to have constant angular momentum and employ a straight cutoff. The $I=0$ vacuum trajectory is also calculated.

Section II is devoted to a presentation of the relevant formalism. The results of the numerical calculations are given and discussed in Sec. III.

The results may be summarized as follows: In the same sense that the usual bootstrap calculations of the ρ are not self-consistent, i.e., the output width of the ρ (for reasonable values of the position of the ρ) is larger than the input width of the exchanged ρ,[1,2] so the calculated $\alpha_\rho(0)$ is larger than the input parameter $\alpha_\rho^{\mathrm{In}}(0)$. For all cases, both $\alpha_\rho(0)$ are $\gtrsim 0.9$, in agreement with results of Foley *et al.*[3] and the calculation of Chang and Sharp[4]; however, in disagreement with other determinations of $\alpha_\rho(0)\sim 0.5$.[5] The residue of the ρ Regge trajectory, after removal of a threshold factor, turns out to be nearly constant in the scattering region ($s<0$) and very close to the input β. The calculations of the $I=0$ vacuum pole trajectory give a small slope: $\alpha_P'(0)\lesssim 1/500$.[6]

II. FORMULATION OF THE INTEGRAL EQUATIONS

We shall obtain amplitudes for pion-pion scattering by the familiar N/D solution[7] of the partial-wave dispersion relations. The usual expressions for the scalar variables s, t, and u in terms of the momentum k and scattering angle θ in the center-of-mass system of the direct or s channel are[6] $s=4(k^2+1)$, $t=-2k^2(1-\cos\theta)$, and $u=4-s-t$. The invariant partial-wave amplitude A_l is defined in terms of the S matrix by

$$A_l(s)\equiv(1/2i\rho)(S_l-1)\equiv B_l(s)+{}^RA_l(s)\,, \quad (1)$$

where

$$\rho=((s-4)/s)^{1/2}\,, \quad (2)$$

and B_l is regular for $s>0$ and ${}^RA_l(s)$ has only a right-

* Supported by the U. S. Atomic Energy Commission.

† Supported in part by the U. S. Air Force through Air Force Office of Scientific Research Grant AF-AFOSR-62-452.

[1] F. Zachariasen, Phys. Rev. Letters **7**, 112 (1961); F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962).

[2] D. Wong, Phys. Rev. **126**, 1220 (1962).

[3] K. Foley, S. Lindenbaum, W. Love, S. Ozaki, J. Russell, and L. Yuan, Phys. Rev. Letters **10**, 376 (1963).

[4] H. Cheng and D. Sharp, Phys. Rev. **132**, 1854 (1963).

[5] I. Muzinich, Phys. Rev. Letters **11**, 88 (1963).

[6] We use units $\hbar=c=m_\pi=1$.

[7] G. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960).

hand cut. The right-hand discontinuity in $A_l(s)$ is given by unitarity: We make the approximation that elastic unitarity holds for all physical k^2:

$$A_l(s)=B_l(s)+\frac{1}{\pi}\int_4^\infty \frac{ds'}{s'-s}|A_l(s')|^2\left(\frac{s'-4}{s'}\right)^{1/2}. \quad (3)$$

The left-hand discontinuity or generalized potential[8] is derived from application of an approximate form of crossing symmetry. We will first determine $B_l(s)$ and then discuss the N/D equations and their solution.

Using crossing symmetry, $B_l(s)$ is calculated from the scattering amplitude in the crossed t and u channels. We will consider *only* the exchange of the $I=1$ ρ resonance in the t and u channels. Then in the s channel for $I=1$ and l equal to an *integer* we obtain

$$B_l^{I=1}(s)=\frac{1}{2}\int_{-1}^{1} P_l(\cos\theta)d\cos\theta \times[\tfrac{1}{2}A_R^{I=1}(t,s)-\tfrac{1}{2}A_R^{I=1}(u,s)], \quad (4)$$

which for l odd becomes

$$B_l^1(s)=\frac{1}{(s-4)}\int_{-(s-4)}^{0} P_l\left(1+\frac{2t}{s-4}\right)dtA_R^1(t,s), \quad (5)$$

where $A_R^1(t,s)$ is the part of the scattering amplitude in the t channel, $A^1(t,s)$, which has no singularities for $s>0$, i.e., $t<4$.

Taking a Breit-Wigner form for the ρ resonance, we have

$$A^1(t,s)\approx\frac{3\Gamma(t-4)}{m_\rho^2-t-i\Gamma(t-4)^{3/2}/t^{1/2}}P_1\left(1+\frac{2s}{t-4}\right). \quad (6)$$

Further making the narrow width approximation, so that $A_R^1(t,s)=A^1(t,s)$, we have the simple form for l equal to an odd integer[9]:

$$B_l^1(s)=\frac{6\Gamma}{s-4}(m_\rho^2-4+2s)Q_l\left(1+\frac{2m_\rho^2}{s-4}\right). \quad (7)$$

Equation (7) has an acceptable behavior in the l plane as $|l|\to\infty$ and thus can be continued for noninteger l even though both (4) and (5) cannot.[10] However, $B_l(s)$ as given by (7) diverges like $\log(s)$ as $s\to\infty$ and the resulting N/D equations do not have a unique solution.

A mechanism that damps this singular high-energy behavior is provided by the Regge motion of resonance poles. In the Regge description for the ρ resonance we take

$$A^1(t,s)=\frac{b_\rho(t)}{\sin\pi\alpha_\rho(t)}\frac{1}{2}\left[P_{\alpha_\rho(t)}\left(-1-\frac{2s}{t-4}\right)-P_{\alpha_\rho(t)}\left(1+\frac{2s}{t-4}\right)\right]. \quad (8)$$

We are interested in B_l for $s\geq4$ and hence in the region $t\leq0$ where $\alpha_\rho(t)$ is real and <1. For large s, (8) is or order $s^{\alpha_\rho(t)}$ and hence an acceptable input to the N/D equations.

Since we do not know the behavior of $b_\rho(t)$ or $\alpha_\rho(t)$ except in the immediate vicinity of the ρ resonance, we will take a very simple form for (8) which reduces to the correct Breit-Wigner form (6) near $t=m_\rho^2$, yields the same $B_{l=1}^1(s=4)$ as Eq. (7), and gives the same high-energy behavior in s (for small t) as the Regge pole:

$$A_R^1(t,s)\approx\frac{3\Gamma(t-4)}{(m_\rho^2-t)}\left(1+\frac{2s}{t-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho'(0)(t-m_\rho^2)}. \quad (9)$$

With this approximation, $A_{l=1}^1(s)$ is readily calculated numerically.[11] However we are interested in continuing the partial-wave amplitude for noninteger l. Eq. (5) *cannot* be continued; there are alternate formulations for $B_l(s)$ which can be continued.[10] From the point of making the computations manageable, we again note that *expression (7) can be continued in the l plane.* Thus we are led to make the further approximation that using (5) in making the partial-wave projection $B_l^1(s)$ of (9) we evaluate the last factor $(s/4)^{\alpha_\rho'(0)(t-m_\rho^2)}$ at $t=0$ (where it gives the maximum contribution). Hence our "Reggeized" $B_l^1(s)$ becomes[12]

$$B_l^1(s)=\frac{6\Gamma}{(s-4)}(m_\rho^2-4+2s)Q_l\left(1+\frac{2m_\rho^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1}. \quad (10)$$

This expression which is our approximate form for the left-hand cut for the partial wave $\pi-\pi$ amplitude in the $I=1$ state and odd integer l has acceptable behavior for large l and *can* be continued in the l plane.

Now in order to insure that $A_l^1(s)$ has the proper threshold behavior, i.e., $(s-4)^l$ and also remove this additional cut from $B_l^1(s)$ for noninteger l, we define new amplitudes

$$\mathbf{A}_l^1(s)\equiv1/(s-4)^lA_l^1(s)\equiv1/2i\rho_l(S_l-1) \equiv\mathbf{B}_l^1(s)+{}^R\mathbf{A}_l^1(s), \quad (11)$$

where

$$\rho_l=((s-4)/s)^{1/2}(s-4)^l, \quad (12)$$

[8] G. Chew and S. Frautschi, Phys. Rev. **124**, 264 (1961).

[9] If we look at $I=0$ and even angular momenta, the relevant Born term is of the same form as (7) with Γ replaced by 2Γ.

[10] E. Squires, Nuovo Cimento **25**, 242 (1962). A continuation of Eq. (5) based on the lines discussed in this reference will yield the same result.

[11] Equation (9) and other more complicated approximations to (8) were considered and used to calculate $A_{l=1}^1(s)$ even though these could not be continued to noninteger l simply.

[12] Thus the input cutoff parameter $\alpha_\rho^{\rm In}(0)$ should be considered as some average value. Using form given by (8) would have necessitated a somewhat smaller $\alpha_\rho^{\rm In}(0)$.

and

$$\mathbf{B}_l{}^1(s)=\frac{6\Gamma}{(s-4)^{l+1}}(m_\rho{}^2-4+2s)\times Q_l\left(1+\frac{2m_\rho{}^2}{s-4}\right)\left(\frac{s}{4}\right)^{\alpha_\rho(0)-1}. \quad (13)$$

Now define

$$\mathbf{A}_l{}^1(s)=N_l(s)/D_l(s), \quad (14)$$

where N has only a left-hand cut and D has only a right-hand cut. Then in terms of the generalized potential $\mathbf{B}_l{}^1(s)$ which is regular in the physical region, the N and D equations are [2,13]

$$D_l(s)=1-(s-s_0)\frac{P}{\pi}\int_4^\infty \rho_l(s')N_l(s')\frac{ds'}{(s'-s)(s'-s_0)}-i\rho_l(s)N_l(s)\Theta(s-4), \quad (15)$$

$$N_l(s)=\mathbf{B}_l{}^1(s)+\frac{1}{\pi}\int_4^\infty\left(\mathbf{B}_l{}^1(s')-\frac{(s-s_0)}{(s'-s_0)}\mathbf{B}_l{}^1(s)\right)\times\rho_l(s')N_l(s')\frac{ds'}{s'-s}. \quad (16)$$

Note that the solutions $\mathbf{A}_l{}^1(s)$ are independent of the subtraction point s_0. As long as $0<l<2-\alpha_\rho(0)<2$, these equations have unique solutions. The Fredholm integral Eq. (16) for $N_l(s)$ was solved by matrix inversion on the Stanford 7090 computer.

For given input parameters $m_\rho{}^{\text{In}}$, Γ^{In} and $\alpha_\rho{}^{\text{In}}(0)$, which determine $B_l{}^1(s)$ [$\alpha_\rho{}^{\text{In}}(0)$ being fixed by the requirement that we get an $l=1$ resonance at $m_\rho{}^{\text{In}}$, i.e., $\text{Re}D_{l=1}(s=(m_\rho{}^{\text{In}})^2)=0$], we calculate the width of the $l=1$ resonance. Then we solve (15) and (16) for non-integer $l<1$ in order to determine the properties of the ρ trajectory. For a given l, we look for the value of $s(\equiv s_l)$ for which $\text{Re}D_l(s)=0$:

$$\text{Re}D_l(s_l)=0. \quad (17)$$

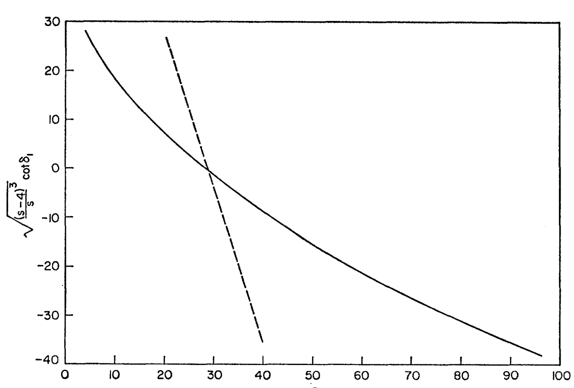

FIG. 1. Phase shift for $I=1$, $l=1$ amplitude versus s. The solid curve corresponds to the output, whereas the dashed curve comes from our input Breit-Wigner form. $\Gamma^{\text{In}}=0.145$ and $\alpha_\rho{}^{\text{In}}(0)=0.949$. For Figs. 1–4, $(m_\rho{}^{\text{In}})^2=29$ and the "cutoff parameter," i.e., $\alpha_\rho{}^{\text{In}}(0)$ is adjusted to force an $l=1$ resonance at $m_\rho{}^{\text{In}}$.

[13] J. Uretsky, Phys. Rev. **123**, 1459 (1961).

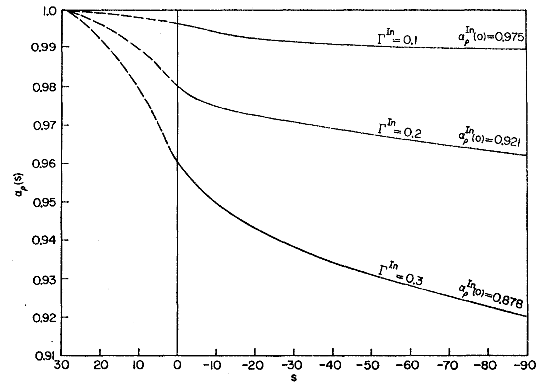

FIG. 2. $\alpha_\rho(s)$ for various input parameters. The dashed lines for $s>4$ in Figs. 2–4 emphasize that we only investigated the vanishing of the real part of $D_l(s)$.

For $s_l<4$ this gives directly the Regge trajectory $\alpha_\rho(s)$, whereas for $s_l>4$, in the limit of a narrow resonance, it gives approximately $\text{Re}\alpha_\rho(s)$. The residue $b_\rho(s)$ is determined as follows: Since $\text{Re}D_l(s_l)=0$, in the vicinity of s_l we have (for $s_l<4$)

$$\mathbf{A}_l{}^1(s)=\left(N_l(s)\Big/\frac{\partial\,\text{Re}D_l(s)}{\partial s}\right)_{s_l}\Big/(s-s_l). \quad (18)$$

This residue is real since $N_l(s_l)$ is simply given by

$$N_l(s_l)=\frac{P}{\pi}\int_4^\infty \mathbf{B}_l{}^1(s')\rho(s')N_l(s')\frac{ds'}{s'-s_l}. \quad (19)$$

The partial-wave projection of the ρ Regge pole of "odd j parity"[10] divided by the threshod factor $(s-4)^l$,

$$\frac{b_\rho(s)}{\sin\pi\alpha_\rho(s)(s-4)^l}P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right)\equiv\frac{\beta_\rho(s)\pi(2\alpha_\rho(s)+1)}{\sin\pi\alpha_\rho(s)}P_{\alpha_\rho(s)}\left(-1-\frac{2t}{s-4}\right), \quad (20)$$

then must be compared with (18).[14] Now

$$\frac{1}{2}\int_{-1}^1 P_l(\cos\theta)P_{\alpha_\rho(s)}(-\cos\theta)d\cos\theta\beta_\rho(s)\frac{\pi(2\alpha_\rho(s)+1)}{\sin\pi\alpha_\rho(s)}=\frac{\beta_\rho(s)(2\alpha_\rho(s)+1)}{(\alpha_\rho(s)-l)(\alpha_\rho(s)+l+1)}\underset{s\approx s_l}{\approx}\frac{\beta_\rho(s_l)}{\alpha_\rho{}'(s_l)(s-s_l)}. \quad (21)$$

Thus for a given l, we find $\alpha'(s_l)$ from $\alpha(s)$ [as found from (17)] and hence the residue is given by

$$\beta_\rho(s_l)=\left(N_l(s)\Big/\frac{\partial\,\text{Re}D_l(s)}{\partial s}\right)_{s_l}\alpha_\rho{}'(s_l). \quad (22)$$

[14] For the evaluation of the residue function $b_\rho(s)$ we have factored out the threshold factor $(s-4)^l$. The partial-wave projection of a single Regge pole does not have this (correct) threshold behavior but goes as $(s-4)^{\alpha_\rho(s)}$. As $\alpha_\rho(s)$ varies little over a large range of s, the above definition of β is adequate. A representation in which each pole has the correct partial-wave threshold behavior has been given by N. Khuri, Phys. Rev. **130**, 429 (1963).

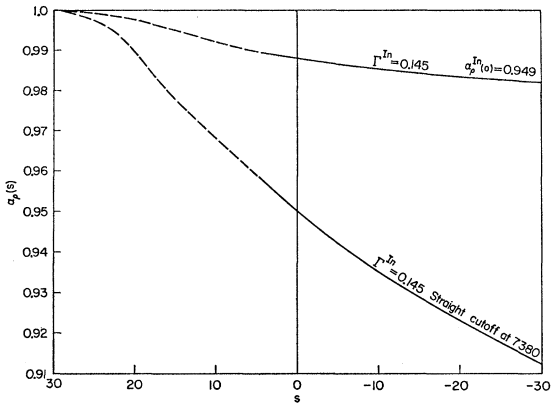

FIG. 3. Comparison of $\alpha_\rho(s)$ for a "straight cutoff" and a "Regge cutoff."

III. RESULTS AND CONCLUSIONS

As discussed earlier, in addition to evaluating the $I=1$, $l=1$ $\pi-\pi$ scattering amplitude in an attempt to "bootstrap" the ρ meson, we calculate the ρ's Regge pole parameters for noninteger $l<1$. We computed both the position α_ρ and residue β_ρ of the pole as functions of s.

We investigated the problem for several values of the input coupling constant $\Gamma^{\rm In}$ (or input width of the ρ) and for several input masses $(m_\rho^{\rm In})^2$ ranging from 10 to the experimental value of 29. No self-consistent solution was obtained. The procedure was to evaluate the $I=1$, $l=1$ amplitude for many values of $\alpha_\rho^{\rm In}(0)$ until the mass of the input ρ was reproduced by a zero of $\mathrm{Re}D_{l=1}(s)$ at $s=(m_\rho^{\rm In})^2$, i.e., we always forced the mass of the produced ρ to be the same as that of the exchanged ρ. The output width could be determined either by evaluating the quantity $[N_{l=1}(s)/\partial D_{l=1}(s)/\partial s]$ at the position of the resonance (which is a correct procedure for a narrow resonance), or by actually looking at the $l=1$ phase shift as a function of s. In either the former case or the latter looking below the resonant energy the output width was larger than the input one by a factor of 3–6. Looking at the phase shift itself on the high-energy side of the resonance situation is even worse. The function $((s-4)^3/s)^{1/2}\cot\delta_1(s)$ is plotted in Fig. 1 together with the input value for this function. For energies larger than the position of the ρ resonance the function decreases too slowly for a resonant behavior. The input values for the exchanged ρ were $(m_\rho^{\rm In})^2=29$ and $\Gamma^{\rm In}=0.145$ (which corresponds to a full width at half-maximum of 110 MeV).

Hence for given $m_\rho^{\rm In}$ and $\Gamma^{\rm In}$, $\alpha_\rho^{\rm In}(0)$ is determined from the self-consistency requirement on m_ρ in the $l=1$ calculation. Thus the generalized potential $\mathbf{B}_l^1(s)$ is determined and we solve the full N_l/D_l Eqs. (15) and (16) to determine the Regge trajectory and residue for the ρ. In Figs. 2 to 4 we present some of the results for $(m_\rho^{\rm In})^2=29$. As the width of the produced ρ meson is rather large, the imaginary parts of the ρ trajectory will be large above $s=4$. Since we have only looked for the zero of the real part of D_l, we have obtained the actual trajectory only for $s<4$. We emphasize this by plotting dashed curves for $s>4$, e.g., the dashed $\alpha_\rho(s)$ curves correspond to an approximation to the real part of $\alpha_\rho(s>4)$.

For $\Gamma^{\rm In}=0.145$ we show in Fig. 3 a comparison of α_ρ for a calculation as mentioned above to one in which a pure $l=1$ ρ exchange [as given by Eq. (7)] was considered as a straight cutoff used in solving Eqs. (15) and (16) (again the self-consistency requirement of the output ρ position equaling $m_\rho^{\rm In}$ determined the value of the cutoff). We see that although there is some quantitative difference, both trajectories have $\alpha_\rho(0)$ larger than 0.9. These calculations with the straight cutoff and other calculations specifically for $A_{l=1}{}^1(s)$, e.g., using (9) to calculate $\mathbf{B}_{l=1}{}^1(s)$,[11] all gave very similar results for the $l=1$ partial wave. We felt this was a fairly good test for a number of the approximations made in obtaining Eq. (10).

In addition to obtaining the output width larger than the input one, the output $\alpha_\rho(0)$ was larger than $\alpha_\rho^{\rm In}(0)$.[12] The two discrepancies are correlated. Near the resonance, we have from (21), $(d\alpha_\rho/ds)=(\beta_\rho/\Gamma)$ so that a large Γ corresponds to a small slope for α and thus $\alpha_\rho(0)$ is larger at $s=0$ than $\alpha_\rho^{\rm In}(0)$. It is interesting to note that the output β_ρ, as shown in Fig. 4, is almost constant in the relevant scattering region ($s<0$) and is very close in magnitude to $\beta_\rho^{\rm In}=(d\alpha_\rho^{\rm In}/ds)\Gamma^{\rm In}$.

We have also calculated the scattering amplitude in $I=0$ channel again using only ρ exchange in the crossed channels. If we use the same parameters as for the $I=1$ calculation[9] we find that there is a vacuum trajectory but that for $s=0$ it has an $l>1$; specifically for $l=1$ the pole occurs for a very large negative s. Therefore we adjusted the cutoff parameters to force the $I=0$ trajectory to cross[15] 1 at $s=0$ and calculated the vacuum trajectory $\alpha_P(s)$. A typical curve is shown in Fig. 5. Note that the slope is quite small; $(d\alpha_P(s)/ds)_{s=0}\approx 10^{-3}$

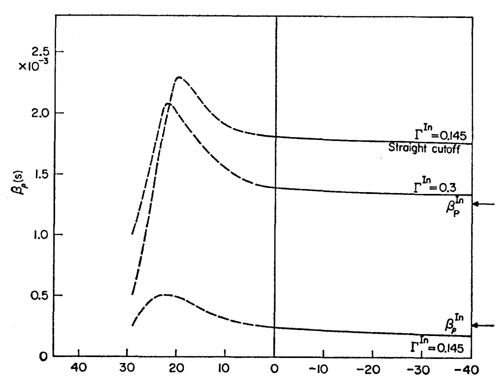

FIG. 4. The residue $\beta_\rho(s)$ for various input parameters. The arrows indicate the input $\beta_\rho^{\rm In}=(d\alpha_\rho^{\rm In}/ds)\Gamma^{\rm In}$.

[15] If we then recalculate the $I=1$, $l=1$ amplitude, no ρ resonance occurs.

and hence our results would not be consistent with the f^0 [16] being on the vacuum trajectory. We also calculated the residue of the vacuum pole at $s=0$. The residue corresponding to the trajectory shown in Fig. 5 gave an asymptotic total $\pi-\pi$ cross section of 3 mb as compared to a value of the 15 mb obtained using the factorization theorem[17] and the asymptotic πN and NN cross sections.

We feel that both the problem (a) that the output ρ width is larger than the input ρ width and the problem (b) that using the input ρ parameters which yield a ρ resonance to calculate the $(I=0)$ vacuum trajectory give $\alpha_P(0)>1$ are largely due to the one channel approximation. The effect of an inelastic channel below its threshold is to, (i) always act as an attraction, and (ii) tend to narrow a resonance. Hence if we include the inelastic effects in the $I=1$ channel, which we expect to be due largely to the $\pi\omega$ channel,[1] this would narrow the output ρ width, and increase the attraction so that a somewhat smaller $\alpha_\rho^{\text{In}}(0)$ would be required.[18] On the other hand, the $\pi\omega$ channel does not couple to the $I=0$ channel so that this additional attraction would not be present and hence we would have a smaller $\alpha_P(0)$.

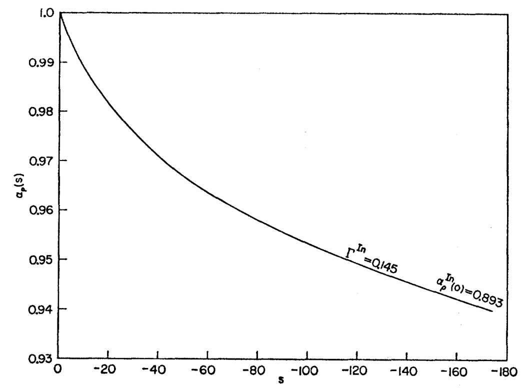

FIG. 5. The $I=0$ vacuum trajectory $\alpha_P(s)$ which has been adjusted to cross $s=0$ at $l=1$.

[16] W. Selove, V. Hagopian, H. Broad, A. Baker, and E. Leboy, Phys. Rev. Letters **9**, 277 (1962).

[17] M. Gell-Mann, Phys. Rev. Letters **8**, 263 (1962); V. Gribov and I. Pomeranchuk, *ibid.* **8**, 343 (1962).

[18] A relatively small change in $\alpha_\rho^{\text{In}}(0)$ produces a large shift in the output resonance position.

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Method of the Self-Consistent Field in General Relativity and its Application to the Gravitational Geon*

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Concentrations of radiation held together for a long time by their own gravitational attraction ("geons") have been studied for nearly a decade. We extend the previous analyses to the case where gravitational waves are the source of the geon's mass energy. To analyze these solutions of the free-space Einstein equations with persistent features, we develop an approximation method to treat small ripples on a strongly curved background metric. The background metric describes the large-scale persistent features of the geon and is taken to be spherically symmetric. The waves superimposed on this background have an amplitude small enough so that their dynamics can be analyzed in the linear approximation; however, their wavelength is so short, and their time dependence so rapid that their energy is appreciable and produces the strongly curved background metric in which they move. The Einstein equations are investigated in this limit of short wavelength. It is found that the large-scale features of thin-shell spherical gravitational geons—in fact, of thin-shell spherical geons constructed from *any* field of zero rest mass—are identical to those of the spherical electromagnetic geons analyzed previously.

I. INTRODUCTION

THE existence of nonsingular, asymptotically Euclidian solutions of the free-space Maxwell-Einstein equations with persistent large-scale features was first pointed out by Wheeler.[1] He gave the name "geons" to these and similar concentrations of zero rest-mass fields which are held together for a long time by gravitational attraction. In addition to Wheeler's electromagnetic geons,[1,2] neutrino geons have been dis-

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[1] J. A. Wheeler, Phys. Rev. **97**, 511 (1955); also see J. A. Wheeler, *Geometrodynamics* (Academic Press Inc., New York, 1962).

[2] F. J. Ernst, Phys. Rev. **105**, 1662 and 1665 (1957).