Upper Bounds for the Scattering Amplitude at High Energy

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Constraints imposed by unitarity and analyticity on the high-energy behavior of scattering amplitudes are studied in detail. The Froissart bound at nonforward (and nonbackward) angle is improved by two different methods: One is based on the analyticity in complex $\cos\theta$ plane, and the other on the analyticity of the partial-wave amplitude in the complex angular-momentum plane. It is also shown that the Froissart bound at $\theta=0$ or π cannot be improved insofar as one does not impose restrictions on the scattering amplitude other than unitarity and the analyticity in the $\cos\theta$ plane. The scattering of particles with spin is discussed briefly.

I. INTRODUCTION

A FEW years ago, Froissart[1] has shown that the combination of the unitarity condition and the Mandelstam representation with finite subtraction imposes rather strong constraints on the growth of the scattering amplitude at high energy. Namely, the scattering amplitude $f(s,\cos\theta)$ (for neutral scalar particles of equal mass) satisfies the inequalities

$$|f(s,\cos\theta)|<C_1 s(\ln s)^2 \qquad \text{for } \theta=0 \text{ or } \pi\,, \qquad (\text{I.1})$$

$$|f(s,\cos\theta)|<C_2 s^{3/4}(\ln s)^{3/2} \qquad \text{for } \theta\neq 0 \text{ or } \pi\,, \qquad (\text{I.2})$$

for very large s, where s and θ are the square of the total energy and the scattering angle in the center-of-mass system. f is normalized here in a relativistic way so that $d\sigma_{\text{el}}/d\Omega=4|f|^2/s$, $\sigma_{\text{tot}}=(8\pi/k\sqrt{s})\,\text{Im}f(s,1)$ (k is the center-of-mass momentum).

It was recognized later by Greenberg and Low[2] and by Martin[3] that it is not necessary to make use of the full analyticity assumed in the Mandelstam representation to obtain the bounds (I.1) and (I.2). It is sufficient to assume that $f(s,\cos\theta)$ is analytic in an ellipse in complex $\cos\theta$ plane, with foci at $+1$ and -1 and semimajor axis of length $1+\alpha/k^2$ (α: positive constant), and that f is uniformly bounded in this ellipse by some power of s. Conversely, it was found that the bounds (I.1) and (I.2) cannot be improved insofar as one assumes no more than the analyticity in the ellipse.[4,5] (This is proved by constructing counter examples, as is discussed in detail in Ref. 4.) These results have led us to examine whether or not it is possible to improve the Froissart bounds if we assume analyticity in a larger domain than the ellipse, for instance the entire cut z plane. For the case $\theta\neq0$ or π, we have been able to improve the bound (I.2) considerably. Our result is[4–6]

$$|f(s,\cos\theta)|<C_3\frac{(\ln s)^{3/2}}{\sin^2\theta} \quad \text{for} \quad \theta\neq0 \quad \text{or} \quad \pi\,. \qquad (\text{I.3})$$

For the case $\theta=0$ or π, it is found that the forward bound (I.1) cannot be improved as far as one does not impose restrictions on $f(s,\cos\theta)$ other than the unitarity and the analyticity in $\cos\theta$ plane.[7]

We should like to emphasize here that these results, although we have not been able to improve them further, may not be the best possible ones because we did not take full account of analyticity and unitarity. In particular, we disregarded analyticity with respect to energy and also unitarity in crossed channels. It will be very interesting to see whether the upper bounds can be improved further by a proper consideration of these important features of Mandelstam representation.

The purpose of this paper is to present the results of our investigation on the upper bounds of scattering amplitude in as complete and detailed a manner as possible. We will not only supply mathematical details to the results published already in our preliminary

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[1] M. Froissart, Phys. Rev. **123**, 1053 (1961).
[2] O. W. Greenberg and F. E. Low, Phys. Rev. **124**, 2047 (1961).
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[5] A. Martin, lecture at the Scottish Summer School of Theoretical Physics, Edinburgh, Scotland, 1963 (to be published).
[6] T. Kinoshita, J. J. Loeffel, and A. Martin, Phys. Rev. Letters **10**, 460 (1963).
[7] T. Kinoshita, J. J. Loeffel, and A. Martin, Proceedings of the Sienna International Conference on Elementary Particles, Sienna, Italy, 1963 (to be published).

reports, but also discuss alternative approaches which widen our scope and give us a better understanding of our results. In Sec. II we give mathematical details of the method used in Ref. 6. In Sec. III we study the properties of the partial-wave amplitude as a function of the angular momentum and show how the results of Ref. 6 can be derived by an entirely different method. The scattering of particles with spin is discussed in Sec. IV. In Sec. V we show why we cannot improve the forward Froissart bound (I.1) if we assume only unitarity and analyticity in $\cos\theta$ plane. Section VI contains general remarks about our results. We also give examples to illustrate how the upper bounds may be lowered further if the scattering amplitude is subject to more specific restrictions. Some of the mathematical questions encountered in our investigation are described in five appendices.

II. COMPLEX COSθ PLANE APPROACH

This section does not contain any new result. It gives merely mathematical details and general comments on the method used in Ref. 6 to improve the Froissart bound (I.2) at fixed angle different from 0 or π. In Sec. IIA we give a detailed derivation of some properties of Legendre series. In IIB we call attention of the reader to some general properties of holomorphic functions which turned out to be decisive in our work.[8] In IIC we comment on the method used in Ref. 6. We give also proofs of some assertions already contained there.

A. Some Properties of Functions Defined by Legendre Series

Throughout Sec. II, C_r (r: positive number) will denote the disk $|z|<r$ in the complex z plane. E_r (for $r>1$) will denote the open set $\{z|z=(t+t^{-1})/2,\ r^{-1}<|t|<r\}$. This is the elliptical disk with foci at $+1$ and -1, and semimajor axis $(r+r^{-1})/2$.

If the sequence of complex numbers $a_0, a_1, \cdots$ satisfies the condition

$$\limsup_{n\to\infty}|a_n|^{1/n}=r^{-1}, \quad (r>1), \qquad \text{(II.1)}$$

the Legendre series $\sum a_nP_n(z)$ converges in E_r, and only in E_r, and defines there a holomorphic function

$$f(z)=\sum_{n=0}^{\infty} a_nP_n(z). \qquad \text{(II.2)}$$

Similarly, the Taylor series $\sum a_nz^n$ converges in C_r, and defines there a holomorphic function

$$g(z)=\sum_{n=0}^{\infty} a_nz^n. \qquad \text{(II.3)}$$

[8] We are greatly indebted to Professor M. Zerner (Marseilles University, France) for having directed our attention to these properties.

Mathematicians have studied the relation between the property of the coefficients of a power series like (II.3) and the property of its analytic continuation to the exterior of its convergence circle.[9] In order to be able to use their results in the analogous problem for the Legendre series, we study in this subsection the relation between the analytic continuation of the functions (II.2) and (II.3).

(1) Let Γ be the circle $|u|=R$ with $1<R<r$. The Cauchy formula for the coefficients of the Taylor series (II.3) gives

$$a_n=\frac{1}{2\pi i}\int_\Gamma u^{-n-1}g(u)du.$$

Thus, for z in E_r,

$$f(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty} P_n(z)\int_\Gamma u^{-n-1}g(u)du. \qquad \text{(II.4)}$$

For each z, consider the holomorphic function of u

$$K(z,u)=(1-2uz+u^2)^{-1/2}, \qquad \text{(II.5)}$$

defined by the condition $uK(z,u)\to 1$ for $|u|\to\infty$ in the u plane cut along the straight line segment from $z-(z^2-1)^{1/2}$ to $z+(z^2-1)^{1/2}$. Now, if z is in E_R, we have

$$|z\pm(z^2-1)^{1/2}|<R.$$

Thus, for each z in E_R, the expansion of $K(z,u)$ in powers of u^{-1}, which is known to be

$$\sum_{n=0}^{\infty} u^{-n-1}P_n(z),$$

converges uniformly for u on Γ. Consequently we can interchange the order of integration and summation in (II.4), and obtain

$$f(z)=\frac{1}{2\pi i}\int_\Gamma K(z,u)g(u)du, \quad z\in E_R. \qquad \text{(II.6)}$$

Let F be the domain $|u|>1$ and G be the complement of the closed straight line segment $I\equiv[-1,1]$. We shall define the one-to-one mapping of G onto F:

$$a(z)=z+(z^2-1)^{1/2} \qquad \text{(II.7)}$$

by the condition that $z^{-1}(z^2-1)^{1/2}\to 1$ as $|z|\to\infty$. The inverse mapping is given by

$$b(u)=\tfrac{1}{2}(u+u^{-1}). \qquad \text{(II.8)}$$

We shall now prove the following:

Theorem 1. Let V be a simply connected and bounded domain which contains a disk C_R with $r>R>1$ and has a smooth closed boundary ∂V. Let g be holomorphic in V and continuous on $V\cup\partial V$. Then f is holomorphic in the simply connected domain $W=I\cup b(V\cap F)$ and can be

[9] See, for example, L. Bieberbach, *Analytische Fortsetzung* (Springer Verlag, Berlin-Goettingen-Heidelberg, 1955); P. Dienes, *The Taylor Series* (Dover Publications, New York, 1957), Chap. X.

expressed as

$$f(z)=\frac{1}{2\pi i}\int_{\partial V} K(z,u)g(u)du\,,\quad z\in W. \tag{II.9}$$

Proof: For every z in E_R, we can make use of Cauchy's theorem and deform the integration path Γ of (II.6) into ∂V since g is holomorphic in V. On the other hand, for u on ∂V, the finite branch point $b(u)$ of $K(z,u)$, considered as a function of z, is on $\partial W=b(\partial V)$. Thus, for every u on ∂V, $K(z,u)$ is holomorphic and one-valued for z in W. The holomorphy of f in W follows. Q. E. D.

For every z in W, the two branch points $a(z)$ and $a^{-1}(z)$ of $K(z,u)$ as a function of u are both in V. Let γ be any cut running inside V between the two branch points. $K(z,u)$, which is one-valued in the u plane cut in this manner, should satisfy $uK(z,u)\to 1$ for $|u|\to\infty$. The value of K at opposite sides of the cut differ only by the sign. Thus, if we deform the integration path ∂V of (II.9) continuously into a closed path along γ, we obtain the formula given earlier[6]

$$f(z)=\frac{1}{\pi i}\int_{\gamma} K(z,u)g(u)du\,, \tag{II.10}$$

where the integration is along the curve γ connecting the points $a(z)$ and $a^{-1}(z)$. This may also be written as

$$f(z)=\frac{1}{\pi}\int_{-1}^{1} g(z+(z^2-1)^{1/2}\cos t)dt\,, \tag{II.11}$$

by a suitable change of variable. [See Ref. 6 for an alternative derivation of (II.11).]

(2) To derive an inverse formula that gives g in terms of f, we proceed along similar lines. Put $\Gamma'=b(\Gamma)$; Γ' is the boundary of E_R and thus contained in E_r. We have[10]

$$a_n=\frac{2n+1}{2\pi i}\int_{\Gamma'} Q_n(u)f(u)du\,,$$

where $Q_n(u)$ denotes the Legendre function of the second kind. If we define

$$h(z)=\frac{1}{2\pi i}\sum_{n=0}^{\infty} z^n\int_{\Gamma'} Q_n(u)f(u)du\,, \tag{II.12}$$

we have

$$g(z)=(2z\partial_z+1)h(z)\,. \tag{II.13}$$

As is shown in Appendix A, for every z in C_R, $\sum z^nQ_n(u)$ converges uniformly in the complement of E_R, defining there a holomorphic function $L(z,u)$. We can thus interchange summation and integration in (II.12), obtaining

$$h(z)=\frac{1}{2\pi i}\int_{\Gamma'} L(z,u)f(u)du\,,\quad z\in C_R. \tag{II.14}$$

[10] E. T. Whittaker and G. N. Watson, *Modern Analysis* (Cambridge University Press, New York, 1952), p. 322.

Let W be a simply connected and bounded domain which contains E_R and has a smooth boundary ∂W. Let f be holomorphic in W and continuous on $W\cup\partial W$. We have then

$$h(z)=\frac{1}{2\pi i}\int_{\partial W} L(z,u)f(u)du\,,\quad z\in C_R.$$

Put $V=\{z|\,|z|\leq 1\}\cup a(W\cap G)$. For every u on ∂W, $L(z,u)$ is holomorphic for z in V (see Appendix A), so that we get the following:

Theorem 2. If f is holomorphic in W and continuous in $W\cup\partial W$, g is holomorphic in the simply connected domain V defined above. For every z in V, we have

$$g(z)=(2z\partial_z+1)h(z)\,, \tag{II.15}$$

where

$$h(z)=\frac{1}{2\pi i}\int_{\partial W} L(z,u)f(u)du\,,\quad z\in V. \tag{II.16}$$

The formula of our previous paper[6]

$$h(z)=\frac{1}{2}\int_{-1}^{1}(1-2uz+z^2)^{-1/2}f(u)du\,,$$

valid for $|z|<1$, follows easily from (II.16) making use of (A4). One can also derive, with the help of (A16), analogous formulas that hold in regions containing points z with $|z|\geq 1$.

B. Subharmonic Property of $|f(z)|$ and $\ln|f(z)|$

We shall now quote some general properties of holomorphic functions that play important roles in our work.

(1) If $f(z)$ is holomorphic in a domain D, $|f(z)|$ is subharmonic in the same domain. Furthermore, if f is not identically zero in D, $\ln|f(z)|$ is also subharmonic in D.[11]

By definition,[12] a real function u which is subharmonic in D has the property:

(2) Let D' be any bounded domain which is contained in D together with its boundary $\partial D'$. Let h be harmonic in D', continuous in $D'\cup\partial D'$, and $h\geq u$ on $\partial D'$. Then $h\geq u$ everywhere in D'.

Property (2) allows us to derive upper bounds for the modulus $|f|$ and for the logarithm of the modulus $\ln|f|$ of a holomorphic function f in a domain D when bounds for these quantities are known on subsets of D. For example, the maximum modulus principle[13] may be derived in this way. As an illustration, let us give a proof of Hadamard's three circle theorem[14]: Let f be holomorphic and one-valued in the domain $r_1<|z|<r_2$, and continuous in $r_1\leq|z|\leq r_2$. Assume that $|f|\leq m_1$

[11] T. Rado, *Subharmonic Functions* (Chelsea Publishing Company, New York, 1949), p. 22.
[12] T. Rado, Ref. 11, p. 1.
[13] See, for example, E. C. Titchmarsh, *The Theory of Functions* (Oxford University Press, New York, 1939), 2nd ed., p. 165.
[14] E. C. Titchmarsh, Ref. 13, p. 172.

for $|z|=r_1$ and $|f|\leq m_2$ for $|z|=r_2$. Then

$$\ln|f(z)|\leq\frac{(\ln|z|-\ln r_1)\ln m_2+(\ln r_2-\ln|z|)\ln m_1}{\ln r_2-\ln r_1}. \tag{II.17}$$

Proof. The right-hand side of (II.17) is a harmonic function in $r_1<|z|<r_2$ and is equal to $\ln m_1$ for $|z|=r_1$ and to $\ln m_2$ for $|z|=r_2$. Since the inequality (II.17) is valid for $|z|=r_1$ and r_2, it holds also for $r_1<|z|<r_2$. Q. E. D.

We quote another consequence of (1) and (2): Let $f(s,z)$ be continuous for z in a domain D and for real and positive s. Suppose that, for each s, f is holomorphic in D, and that

$$\alpha(z)=\limsup_{s\to+\infty}\left[\ln|f(s,z)|/\ln s\right]$$

has a finite upper bound in D. Then $\alpha(z)$ satisfies the property (2) in D (Zerner's lemma[15]). As an (unpublished) application, Zerner finds for the holomorphy domain of the Regge interpolation in Mandelstam representation a result quite similar to that of Bardakci.[16]

The result published in Ref. 6 is obtained by a method which makes essential use of (1) and (2). We shall discuss it in more detail in the next subsection. Further use of (1) and (2) will be found in the following sections of this paper as well as in Ref. 17.

C. Comments on a Derivation of Upper Bounds

We shall discuss here the assumptions and the method used in Ref. 6 to find upper bounds on the growth of elastic differential cross section at high energy.

We consider the elastic scattering of two spin-zero particles of equal mass m. To take account of the unitarity condition, we expand the elastic-scattering amplitude f in a Legendre series as follows:

$$f(s,z)=\frac{\sqrt{s}}{2k}\sum_{l=0}^{\infty}(2l+1)a_l(s)P_l(z), \tag{II.18}$$

where $z=\cos\theta$. The unitarity condition can then be expressed as

$$\operatorname{Im}a_l\geq|a_l|^2,\quad l=0,1,2,\cdots. \tag{II.19}$$

In the discussion of this subsection, the bound for the cross section is obtained under the following assumptions on f:

(*a*) *Analyticity.* For every real $s>4m^2$, $f(s,z)$ is holomorphic for z in a domain $D(s)$ of the following shape: $D(s)$ is the intersection of an elliptical disk E_x (see Sec. IIA), where x (>1) is independent of s, and the complex z plane cut along the real axis from $-\infty$ to $-\rho$ and from ρ to $+\infty$, where $\rho=1+2m^2/k^2$.

(*b*) *Temperedness.* There are positive numbers N and m_0^2, independent of z, such that

$$|f(s,z)|<(s/m^2)^N \tag{II.20}$$

for $s>m_0^2$ and for z in $D(s)$.

A few words about these assumptions. We are primarily interested in the amplitudes satisfying the Mandelstam representation. Such amplitudes fulfill the assumptions (*a*) and (*b*). But these assumptions are obviously less restrictive than the Mandelstam representation: No analyticity is postulated with respect to s, no crossing, and so on. The very point we wish to emphasize by formulating our assumptions in this form is that we do not know how to use the analyticity in s or the crossing (more precisely, unitarity in the crossed channel, for example). As to the point that in this subsection we do not even assume holomorphy in the whole cut z plane, our motivation is of more technical nature. Having restricted z to a bounded domain, we do not have to worry about big $|z|$ values in the formulation of the temperedness property (*b*). The only essential feature of the peculiar form we chose for $D(s)$ is that x is independent of s. Other details are motivated only by considerations of convenience. We may add that the limitation to bounded domains seems to be unessential for our purpose of deriving the asymptotic behavior of upper bounds for $s\to\infty$. In fact, in the following sections the assumption that f is holomorphic in the whole cut z plane is used in an essential way. Nevertheless, we have not been able to improve the results of this section by the use of this more restrictive assumption.

If we define now (see Sec. IIA)

$$g(s,z)=(\sqrt{s}/2k)\sum_{l=0}^{\infty}(2l+1)a_l(s)z^l, \tag{II.21}$$

it follows immediately from (*a*) and from Theorem 2 that

(*a'*) *Analyticity.* For every real $s>4m^2$, $g(s,z)$ is holomorphic in $D'(s)$, which is the intersection of the disk C_x $(x>1)$ with the z plane cut along the real axis from $-\infty$ to $-r$ and from r to $+\infty$, where $r=\rho+(\rho^2-1)^{1/2}=a(\rho)$ [see (II.7)].

Furthermore, we can deduce the following property of $g(s,z)$:

(*b'*) *Temperedness.* There are positive numbers N' and m_1^2, independent of z, such that

$$|g(s,z)|<(s/m^2)^{N'} \tag{II.22}$$

for $s>m_1^2$ and for z in $D''(s)$, where $D''(s)$ is a closed subdomain of $D'(s)$ whose boundary has the following properties: (1) it is outside the unit circle $|z|=1$; (2) its distance to the unit circle is larger than $C_1m/\sqrt{s}$, where C_1 is some positive number; (3) the distance of each of

[15] This lemma was shown to us by Professor M. Zerner. See M. Zerner, Bull. Soc. Math. France **90**, 165 (1962).

[16] K. Bardakci, Phys. Rev. **127**, 1832 (1962).

[17] F. Cerulus and A. Martin, Phys. Letters **8**, 80 (1964).

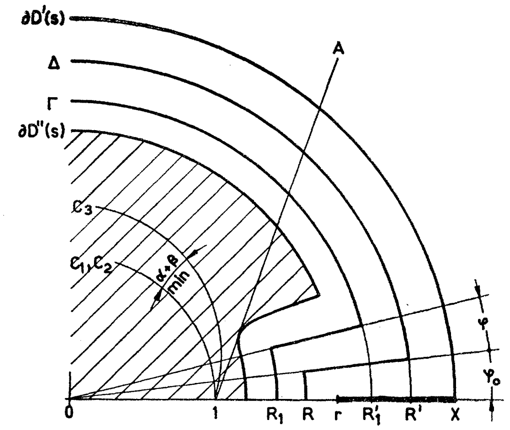

FIG. 1. The portion of $D''(s)$ in the first quadrant is represented by the shaded area. C_1, C_2 and C_3 are the circles of Fig. 2, Ref. 6, in their limiting position for $\epsilon=0$ and $s=\infty$. The line (1, A), which is tangential to both the boundary of $D''(s)$ and the circle C_3 in its limiting position, has a slope equal to $(4+\sqrt{7})/3+O(m/\sqrt{s})$. This fact illustrates the assertion made in Ref. 6 (p. 462) about the existence of nonzero minimum value for $\alpha+\beta$. For other symbols, see the text.

its points to the boundary of $D'(s)$ is smaller than $C_2m/\sqrt{s}$, where C_2 is some positive number.

The (admittedly clumsy) proof of this statement is based on the inequalities derived in Appendix B. The geometrical situation encountered in the course of proof is shown in Fig. 1. We have

$$r=1+4m/\sqrt{s}+O(m^2/s).$$

We choose now a closed curve Δ in the z plane which is made up of circular arcs of radius R and R' centered at the origin and of segments of rays through the origin at angles $\pm\varphi_0$, $\pi\pm\varphi_0$ with the real axis. (See also Fig. 3.) Here $R=1+3m/\sqrt{s}$ so that $r-R$ is positive for sufficiently large s, and R' and φ_0 $(<\frac{1}{2}\pi)$ are defined by

$$(x-R')/x=\sin\varphi_0=(r-R)/r.$$

Thus

$$\varphi_0=m/\sqrt{s}+O(m^2/s)$$

and

$$x-R'=xm/\sqrt{s}+O(m^2/s).$$

Δ is inside $D'(s)$ but outside the unit circle for sufficiently large values of s. Since $b(\Delta)$ [see (II.8)] is inside $D(s)$ when Δ is inside $D'(s)$, we can write

$$h(s,z)=\frac{1}{2\pi i}\int_{b(\Delta)}L(z,u)f(u)du$$

for sufficiently large s and for z inside Δ [see (II.12) and Theorem 2 for definition of h and its relation to g]. Define now the closed domain $\Delta(\mu)$ with the boundary curve Γ as in Appendix B, Fig. 3, where $\mu=m/\sqrt{s}$. Here $\varphi=m/\sqrt{s}+O(m^2/s)$ and $x-R_1'=2xm/\sqrt{s}+O(m^2/s)$. On the other hand, $R_1-1=2m/\sqrt{s}+O(m^2/s)$. Applying the inequality (B4) to $|h|$, we find that there are positive numbers C_3, C_4, and m_2^2, independent of s and z, such that

$$|h(s,z)|<C_3\left(\frac{s}{m^2}\right)^{N+\frac{1}{2}}\ln\left(\frac{s}{m^2}\right)<C_4\left(\frac{s}{m^2}\right)^{N+1} \quad \text{(II.23)}$$

for $s>m_2^2$ and z in $\Delta(\mu)$. Let us call $D''(s)$ the set of those points of $\Delta(\mu)$ whose distance to the boundary of $\Delta(\mu)$ is larger or equal to $m/\sqrt{s}$. For z in $D''(s)$, we write

$$g(s,z)=\frac{1}{2\pi i}\int_\Gamma[2z(u-z)^{-2}+(u-z)^{-1}]h(s,u)du,$$

according to (II.15), where Γ is the boundary of $\Delta(\mu)$. Using (II.23) together with this expression, we see that there is a positive number m_1^2, independent of z, such that (II.22) is valid for every $s>m_1^2$ and every z in $D''(s)$. We have only to choose $N'=N+3$. As to the asserted properties of $D''(s)$, they are obvious from the construction. This concludes the proof of (b').

The advantage we gain by the introduction of the function g in our problem of finding upper bounds for the scattering amplitude f derives from the fact that the unitarity condition (II.19) gives the following decisive inequality for every $s>4m^2$ and for every z with[6] $|z|<1$

$$|g(s,z)|\leq\frac{\sqrt{s}}{k}\frac{1}{(1-|z|)^2}. \quad \text{(II.24)}$$

In order to make clear what sort of conclusion we can draw from the facts recalled in Sec. IIB on one hand, and from analyticity [property (a) or (a')], temperedness [property (b) or (b')], and unitarity [in the form (II.24)] on the other, we want to make the following remark: For every s, $g(s,z)$ is holomorphic in $D'(s=\infty)$. For every z in $D'(\infty)$, we can define

$$\alpha(z)=\limsup_{s\to+\infty}[\ln|g(s,z)|/\ln(s/m^2)].$$

$\alpha(z)$ is smaller than or equal to N' for z in $D'(\infty)$ according to (II.22). For every z with $|z|<1$, $\alpha(z)\leq 0$ according to (II.24). Using Zerner's lemma (Sec. IIB), one can deduce from these inequalities that $\alpha(z)\leq 0$ for z in $D'(\infty)$ with $|z|=1$ (note that $D'(\infty)$ contains neither $=1$ or -1). In other words, for a given $z=\exp(i\theta)$ $(\theta\neq 0$ or $\pi)$, and for every positive ϵ, there is a positive number m_3^2 such that

$$|g(s,e^{i\theta})|<(s/m^2)^\epsilon \quad \text{(II.25)}$$

for $s>m_3^2$ (which depends on θ and ϵ). Now a formula like (II.10) or (II.11) will relate the asymptotic behavior of the scattering amplitude f for physical values of $\cos\theta$ to the asymptotic behavior of g on the path γ, which can be chosen to stay *inside* C_1 (except for the end points $e^{-i\theta}$ and $e^{i\theta}$) where the inequality (II.24) is valid. However, the preliminary result (II.25) is not sufficient to carry over the asymptotic behavior of g to f by integration along γ. For this purpose we need a bound which exhibits a uniformity property with respect to the integration variable, which is lacking in (II.25).

The way in which we solve this technical difficulty is given in detail in Ref. 6 and therefore will not be discussed here. The result we found for g is more precise than (II.25). It enables us to derive not only upper bounds of f at fixed angles but also at fixed momentum transfer $t<0$ [$t=2k^2(\cos\theta-1)$; when t is fixed, $k|\theta|\rightarrow|t|^{1/2}$ as $s\rightarrow\infty$]. It reads as follows: There are positive numbers m_4^2 and C', independent of s and θ, such that

$$|g(s,re^{i\theta})|<\frac{C'}{(1-r)^2} \qquad \text{for} \quad r\leq 1-\frac{|\sin\theta|}{\ln(s/m^2)},$$
$$|g(s,re^{i\theta})|<\frac{C'(\ln(s/m^2))^2}{\sin^2\theta} \qquad \text{for} \quad 1-\frac{|\sin\theta|}{\ln(s/m^2)}<r\leq 1, \tag{II.26}$$

where $s>m_4^2$ and $\theta\neq 0$ or π. From this it follows, using (II.10) with a suitably chosen path γ, that there are positive numbers M^2 and C, independent of s and θ, such that

$$|f(s,\cos\theta)|<C(\ln(s/m^2))^{3/2}/\sin^2\theta \tag{II.27}$$

for $s>M^2$ and $\theta\neq 0$ or π.

Thus far we have been unable to improve the s dependence of the upper bound (II.27) further. As far as the θ dependence is concerned, however, we will not claim that (II.27) is the best bound one could derive from our assumptions. We simply note that it is good enough to reproduce the upper bound for fixed negative momentum transfer t, which is implicitly contained in Ref. 3: There are positive numbers C_1 and M_1^2, independent of s and t, such that

$$\left|f\left(s,1-\frac{|t|}{2k^2}\right)\right|<C_1\frac{s}{|t|}\left(\ln\left(\frac{s}{m^2}\right)\right)^{3/2} \tag{II.28}$$

holds for $s>M_1^2$ and fixed negative t.

For the differential cross section, we obtain, for sufficiently large s, upper bounds

$$d\sigma_{\text{el}}/d\Omega<C'(\ln(s/m^2))^3/s\sin^4\theta \tag{II.29}$$

at fixed $\theta\neq 0$ or π, and

$$d\sigma_{\text{el}}/dt<(C''/|t|^2)(\ln(s/m^2))^3 \tag{II.30}$$

at fixed negative t, respectively.

We close this section with the following remarks:

(1) The proof of the inequalities (II.27)–(II.30) is essentially an existence proof. We have shown that there must be positive numbers M^2, C, C_1, $\cdots$ with the asserted properties. The emphasis is put on the fact that they are independent of s, θ, or t. Of course these constants depend on the geometrical dimensions of the holomorphy domain of f, and of the numbers N and m_0^2 that appear in the formulation of our assumptions. But we did not attempt to work out this dependence.

(2) The limitation to spin-zero particles does not seem to be essential. In two further cases, upper bounds having the same high-energy behavior as (II.27) have been derived, one for the scattering of a particle of spin 0 by a particle of spin $\frac{1}{2}$ (see Sec. IV), and the other for the scattering of two spin-$\frac{1}{2}$ particles.[18]

III. COMPLEX l-PLANE APPROACH

The upper bound (II.27) for the scattering amplitude $f(s,z)$ is much lower than what one would obtain by majorizing $f(s,z)$ by the sum of the moduli of partial-wave amplitudes

$$(\sqrt{s}/2k)\sum(2l+1)|a_l(s)||P_l(z)|,$$

which gives the upper bound (I.2) previously obtained by Froissart. Presumably this means that there is a strong cancellation among various partial waves owing to the fact that Legendre polynomials oscillate with respect to l at each fixed angle θ ($\neq 0$ or π). However, as was pointed out already,[4] the oscillatory behavior of Legendre polynomials does not necessarily lead to such a cancellation unless $a_l(s)$ itself has some nice property as a function of l. In fact, the result of Sec. II may be understood most easily if $a_l(s)$ is a very slowly varying function of l (at least for large l). If this is indeed the case, such a property should be a consequence of analyticity and unitarity satisfied by the scattering amplitude $f(s,z)$. In this section we shall therefore study what kind of constraints are imposed on the l dependence of $a_l(s)$ by unitarity and analyticity. We show in Sec. IIIA that $a_l(s)$, or rather its unique analytic continuation into the complex l plane, has a finite bound for all *real* (not necessarily integer) l greater than $C\ln s$ (C: a constant). This result is extended to a small neighborhood of the real axis ($l>C\ln s$) making use of the Phragmén-Lindelöf theorem. In IIIB, we give an estimate of smoothness of $a_l(s)$ as a function of real l and also describe alternative methods for deriving the fixed angle upper bound (II.27) for the scattering amplitude at high energy.

A. Behavior of Analytically Continued Partial-Wave Amplitudes

As is well known,[19] if the scattering amplitude $f(s,z)$ satisfies fixed energy dispersion relation, which we now assume explicitly, the partial-wave amplitude $a_l(s)$ can be interpolated by two functions $a^+(l,s)$ and $a^-(l,s)$ which are holomorphic in the half-plane Re$l>N$ (N is the number of subtractions to be made). $a^+(l,s)$ coincides

[18] H. Cornille, Nuovo Cimento (to be published).

[19] M. Froissart, Report to the International Conference on Weak and Strong Interactions, La Jolla, California, June, 1961 (unpublished); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1962 (1961) [English transl.: Soviet Phys.—JETP **14**, 1395 (1962)]; A. Martin, Phys. Letters **1**, 72 (1962); E. J. Squires, Nuovo Cimento **25**, 242 (1962); G. Prosperi, *ibid.* **24**, 957 (1962).

with $a_l(s)$ for even integer $l>N$ and $a^-(l,s)$ coincides with $a_l(s)$ for odd integer $l>N$. If $a^\pm(l,s)$ can be continued to $l=0$, we may write $f(s,z)$ as

$$f(s,z)=\tfrac{1}{2}(f^+(s,z)+f^+(s,-z))+\tfrac{1}{2}(f^-(s,z)-f^-(s,-z)), \quad \text{(III.1)}$$

where

$$f^+(s,z)=\frac{\sqrt{s}}{2k}\sum_{l=0}^{\infty}(2l+1)a^+(l,s)P_l(z), \quad \text{(III.2)}$$

the summation being extended to *all* non-negative integers. $f^-(s,z)$ is defined in an analogous way.

The unitarity condition imposes the inequality

(i) $|a^\pm(l,s)|\leq 1$ for $\begin{pmatrix}\text{even}\\ \text{odd}\end{pmatrix}$ integer l,

while the fixed energy dispersion relation and the temperedness condition gives the inequality[1]

(ii) $|a^\pm(l,s)|<|(l+c')^M s^N e^{-Cl/\sqrt{s}}|$ for $\mathrm{Re}\,l>N$, (III.3)

where c, c', M are constants. The latter condition may be derived from the formula

$$a^\pm(l,s)=\frac{1}{\pi k\sqrt{s}}\int_{t_0}^{\infty}Q_l\left(1+\frac{t}{2k^2}\right)(A_3(s,t)\pm A_2(s,t))dt,$$

where A_2 and A_3 are the absorptive parts of the scattering amplitude $f(s,z)$ in u and t channels. In the special case where A_2 and A_3 are just ordinary functions (not tempered distributions), (III.3) reduces to

$$|a^\pm(l,s)|<|s^N e^{-Cl/\sqrt{s}}| \quad \text{for} \quad \mathrm{Re}\,l>N.$$

It is seen from (III.3) that $|a^\pm(l,s)|<\text{const}$ for $\mathrm{Re}\,l>N'\sqrt{s}\ln s$ $(N'>N+\frac{1}{2}M)$. However, for $\mathrm{Re}\,l<N'\sqrt{s}\ln s$, the bound (III.3) may go to infinity as $s\to+\infty$ except at (even or odd) integer l where it is bounded by 1 by unitarity. Thus it is not at all obvious whether $a^\pm(l,s)$ stays finite or oscillates violently as we decrease l continuously along the real l axis into the region $\mathrm{Re}\,l<N'\sqrt{s}\ln s$. Our first problem is thus to find out to what extent this possible oscillation of $a^\pm(l,s)$ along real l axis may be suppressed because of the analyticity of $a^\pm(l,s)$ in l.

In order to answer this question, let us first state and prove:

Theorem 3. Let $f(z)$ be analytic in the right half-plane $\mathrm{Re}z>0$. Assume that $f(z)$ satisfies the inequalities

$$\begin{aligned}|f(z)|&\leq 1 \quad \text{for} \quad z=1,2,3,\cdots,\\ |f(z)|&<M|z+1|^p \quad \text{for} \quad \mathrm{Re}z>0,\end{aligned} \quad \text{(III.4)}$$

where M and p are positive constants. (M will be identified later with s^N.) Then $f(z)$ is bounded by a constant independent of M for all real values of z satisfying $z>(1+\epsilon)\ln M$, where ϵ is a positive number.

Proof.[20] Let us consider the integral

$$\begin{aligned}I(z_0,z_1)&=\frac{1}{2\pi i}\int_{-i\infty}^{i\infty}k(z_0,z_1,z)dz,\\ k(z_0,z_1,z)&=\frac{f(z)e^{\Lambda z}}{(z+1)^p(z-z_0)(z-z_1)}\frac{\Gamma(-\frac{1}{2}z)}{\Gamma(\frac{1}{2}z)},\end{aligned} \quad \text{(III.5)}$$

where z_0, z_1, and Λ are real and positive quantities to be determined later. From (III.4) we find

$$|I(z_0,z_1)|<\frac{M}{2\pi}\int_{-\infty}^{\infty}\frac{dx}{[(x^2+z_0^2)(x^2+z_1^2)]^{1/2}}<\frac{M}{2(z_0z_1)^{1/2}}. \quad \text{(III.6)}$$

On the other hand, $I(z_0,z_1)$ can be shown to be the limit of the closed path integral

$$\frac{1}{2\pi}\int_{-R}^{R}k(z_0,z_1,iy)dy+\frac{R}{2\pi}\int_{+\frac{1}{2}\pi}^{-\frac{1}{2}\pi}k(z_0,z_1,Re^{i\theta})e^{i\theta}d\theta,$$

when R increases indefinitely taking only odd integer values. Thus, using the identity

$$\Gamma(-\tfrac{1}{2}z)=-\pi/((\tfrac{1}{2}z)\Gamma(\tfrac{1}{2}z)\sin(\tfrac{1}{2}\pi z)),$$

we can write $I(z_0,z_1)$ as

$$\begin{aligned}I(z_0,z_1)=&\frac{f(z_0)}{(z_0+1)^p(z_0-z_1)}\frac{\pi e^{\Lambda z_0}}{\frac{1}{2}z_0[\Gamma(\frac{1}{2}z_0)]^2\sin(\frac{1}{2}\pi z_0)}\\ &+\frac{f(z_1)}{(z_1+1)^p(z_1-z_0)}\frac{\pi e^{\Lambda z_1}}{\frac{1}{2}z_1[\Gamma(\frac{1}{2}z_1)]^2\sin(\frac{1}{2}\pi z_1)}\\ &+2\sum_{n=1}^{\infty}\frac{(-1)^n f(2n)e^{2\Lambda n}}{(2n+1)^p(z_0-2n)(z_1-2n)n[\Gamma(n)]^2}.\end{aligned} \quad \text{(III.7)}$$

To put an upper bound on the last term of (III.7), let us examine the maximum of

$$\frac{e^{\Lambda z}}{\frac{1}{2}z(z+1)^p[\Gamma(\frac{1}{2}z)]^2} \quad \text{(III.8)}$$

for real positive z. It can be shown that (III.8) has a unique maximum for $\exp(2\Lambda)>3^p p^2$ (this is a very conservative estimate). Let us adjust Λ so that this maximum occurs at $z=z_0$, which is certainly possible as far as $z_0>p$. For $z_0\gg p$, we obtain

$$e^\Lambda\simeq\tfrac{1}{2}z_0. \quad \text{(III.9)}$$

Making use of the Stirling formula, we find that the value of (III.8) at $z=z_0$ is approximately equal to

$$(1/2\pi)e^{z_0}(z_0+1)^{-p}. \quad \text{(III.10)}$$

[20] An alternative proof may be obtained starting from the Lagrange interpolation method. See Ref. 5.

We now want to estimate the magnitude of $f(z_0)$ using (III.6) and (III.7). For this purpose we choose z_1 to be an odd integer so that $|f(z_1)/\sin(\pi z_1/2)|$ is less than one by (III.4). Furthermore, we require $|z_0-z_1|\leq 1$ just for convenience. Then, using $|f(2n)|<1$ and (III.10), we obtain

$$|f(z_0)|<1+\frac{2}{\pi}\left|\sin\left(\frac{\pi z_0}{2}\right)\right|\sum_{n=1}^{\infty}\frac{1}{|z_0-2n|\,|z_1-2n|}+M(z_0z_1)^{-1/2}(z_0+1)^pe^{-z_0}. \quad \text{(III.11)}$$

We notice that the second term of (III.11) can be bounded by a fixed number independent of z_0 and z_1 since z_1 is an odd integer close to z_0. The last term is also bounded by a constant C for $z_0>(1+\epsilon)\ln(M/C)$, where ϵ is positive. This completes the proof of Theorem 3.

In order to apply Theorem 3 to $a^{\pm}(l,s)$ we have only to change the scale by a factor of 2 to take account of the fact that $a^{\pm}(l,s)$ satisfies the unitarity bound at every other integer. If we ignore the exponential factor in (III.3) temporarily, we can identify M with s^N. Thus we find that $a^{\pm}(l,s)$ is bounded by a constant independent of s for all real l greater than $C\ln s$.

This result for real l can be easily extended to a small wedge-shaped domain of complex l plane making use of the Phragmén-Lindelöf theorem.[21] For this purpose, let us define $\phi=\arg(l-C\ln s)$. Then we find from (III.3) that $a^{\pm}(l,s)$ satisfies the inequality

$$|a^{\pm}(l,s)<\text{const}(\cos\phi)^{-p}s^{N+(p/2)}. \quad \text{(III.12)}$$

Note that the exponential factor in (III.3) plays an essential role in deriving (III.12). Thus $a^{\pm}(l,s)$ is bounded by $s^{N+(p/2)}$ on the ray $|\phi|=\frac{1}{2}\pi-\epsilon$ ($\epsilon>0$) and bounded by a constant on the ray $\phi=0$ (by Theorem 3). From this we obtain

$$|a^{\pm}(l,s)|<C' \quad \text{for} \quad |\phi|=|\arg(l-C\ln s)|<\text{const}/\ln s, \quad \text{(III.13)}$$

using the Phragmén-Lindelöf theorem, where C' is a constant independent of s.

B. Upper Bounds for the Scattering Amplitude

From the inequality (III.13) we can derive qualitative feature of the partial-wave amplitude suggested at the beginning of Sec. III. Namely, using the Cauchy's inequalities inside the region (III.13) we can obtain, for l real and greater than $C\ln s$, upper bounds for the derivatives

$$\left|\frac{d}{dl}a^{\pm}(l,s)\right|<\frac{C'\ln s}{l-C\ln s},\qquad \left|\frac{d^2}{dl^2}a^{\pm}(l,s)\right|<\frac{C''(\ln s)^2}{(l-C\ln s)^2},\ \text{etc.} \quad \text{(III.14)}$$

[21] E. C. Titchmarsh, Ref. 13, p. 183.

Integrating (III.14) with respect to l, we obtain

$$|a^{\pm}(l,s)-a^{\pm}(l+2,s)|<2C'\ln s/(l-C\ln s). \quad \text{(III.15)}$$

Inequalities involving higher differences can be derived in a similar manner. All these results show that as l increases $a^+(l,s)$ and $a^-(l,s)$ become smoother and smoother, as was conjectured.

We shall now proceed to evaluate the full scattering amplitude $f(s,z)$ with the help of the inequalities (III.14) and (III.15). To exploit the smoothness of $a^{\pm}(l,s)$ in l, let us rewrite, using the Abel summation procedure, the partial wave expansion of $f(s,z)$ as follows:

$$f(s,z)=\frac{\sqrt{s}}{2k}\frac{1}{1-z^2}\sum_{l=0}^{\infty}(2l+1)b_l(s)P_l(z),$$

where

$$b_l(s)=\frac{l(l-1)}{(2l+1)(2l-1)}(a_l-a_{l-2})-\frac{(l+2)(l+1)}{(2l+3)(2l+1)}(a_{l+2}-a_l). \quad \text{(III.16)}$$

Noting that $b_l(s)$ is essentially the second difference of $a_l(s)$, we get from (III.14)

$$|b_l(s)|<\frac{C'(\ln s)^2}{(l-C\ln s)^2}\quad \text{for}\quad l>C\ln s. \quad \text{(III.17)}$$

From unitarity of $a_l(s)$, we also have

$$|b_l(s)|\leq 1 \quad \text{for all integer } l. \quad \text{(III.18)}$$

In addition the b_l's are subject to the constraint

$$\sum_{l=0}^{\infty}(2l+1)b_l(s)=0. \quad \text{(III.19)}$$

Now we can majorize $f(s,z)$ for fixed z ($\neq\pm1$) using the bound $|P_l(z)|<2(\pi l\sin\theta)^{-1/2}$, and using (III.18) for $0\leq l\leq 2C\ln s$ and (III.17) for $2C\ln s<l<\infty$. By a simple calculation we find

$$|f(s,z)|<C(\ln s)^{3/2}/(1-z^2)^{5/4}. \quad \text{(III.20)}$$

This inequality exhibits the same energy dependence as our previous result (II.27). However, its angular dependence is slightly more singular at small angles. If we try to obtain a bound by the same method for forward angles corresponding to fixed momentum transfers, we find an energy dependence somewhat worse than (II.28). We believe that this is because we have disregarded some interference effects between terms of (III.16) which are certainly present as is seen for instance from the condition (III.19).

It is possible to avoid these difficulties by using a somewhat different approach which makes use of the Watson-Sommerfeld transformation.[22] We represent the

[22] T. Regge, Nuovo Cimento 14, 951 (1959).

scattering amplitude as a finite sum plus a line integral in the complex l plane as follows:

$$f(s,z)=\frac{\sqrt{s}}{2k}\sum_{l=0}^{C\ln s+L_0(\theta,s)}(2l+1)a_l(s)P_l(z)$$
$$-\frac{\sqrt{s}}{8ik}\int_\Gamma\frac{(2l+1)dl}{\sin\pi l}[a^+(l,s)(P_l(-z)+P_l(z))$$
$$+a^-(l,s)(P_l(-z)-P_l(z))],\quad \text{(III.21)}$$

where $L_0(\theta,s)$ is a positive quantity to be adjusted later ($\theta=\cos^{-1}z$), and Γ is the integration path along a straight line parallel to the imaginary axis, crossing the real axis at the smallest half-integer that exceeds $C\ln s+L_0(\theta,s)$. We note that in (III.21) the contribution from the big semicircle is ignored because the integrand decreases exponentially in all directions in the right half-plane, excluding the neighborhood of each integer point. This follows from the exponential decrease of $a^\pm(l,s)$ for $\text{Re}l\to\infty$ [see (III.3)], and from the following inequality which will be proved in Appendix D:

$$|P_l(\cos\theta)|<\frac{4\exp(|\text{Im}l|\theta)}{(|2l+1|\sin\theta)^{1/2}}.\quad \text{(III.22)}$$

We will now majorize the integral in (III.21). We shall first note that $a^\pm(l,s)$ satisfies the bound

$$|a^\pm(l,s)|<\text{const}\times\exp((A+B\ln s)|\tan\phi|),\quad \text{for } |\phi|<\tfrac{1}{2}\pi,\quad \text{(III.23)}$$

where A and B are suitably chosen positive constants and $\phi=\arg(l-C\ln s)$. This inequality is obtained from Theorem 3 and (III.12) in the following manner: Applying the Phragmén-Lindelöf theorem to the region $0<\phi<\phi_1(<\frac{1}{2}\pi)$, we obtain

$$|a^\pm(l,s)|<C'(\cos\phi_1)^{-p}\exp[(N+\tfrac{1}{2}p)(\phi/\phi_1)\ln s].$$

If we choose $\phi_1=\frac{1}{2}(\phi+\frac{1}{2}\pi)$, we find

$$|a^\pm(l,s)|<C'2^p(\cos\phi)^{-p}\exp\left[(N+\tfrac{1}{2}p)\frac{\phi}{\frac{1}{4}\pi+\frac{1}{2}\phi}\ln s\right].$$

Inequality (III.23) is obtained if we majorize $(\cos\phi)^{-1}$ by $\exp(\tan\phi)$ and $\phi/(1+2\phi/\pi)$ by $\tan\phi$.

From (III.22) we see easily that for $0<\theta<\pi/2$ the dominant contribution to the integrand of (III.21) comes from the terms containing $P_l(-\cos\theta)$. Hence, using both (III.22) and (III.23), we get the following upper bound for the integral along the path Γ:

$$\frac{\text{const}}{(\sin\theta)^{1/2}}\int_{-\infty}^{\infty}(|2l+1|)^{1/2}$$
$$\times\exp\left[-\left(\theta-\frac{A+B\ln s}{L_0(\theta,s)}\right)|\text{Im}l|\right]d(\text{Im}l).\quad \text{(III.24)}$$

A convenient choice for $L_0(\theta,s)$ will be

$$L_0(\theta,s)=2[(A+B\ln s)/\theta].\quad \text{(III.25)}$$

Using the inequality

$$(|2l+1|)^{1/2}<[2(C\ln s+L_0(\theta,s))+1]^{1/2}+(2|\text{Im}l|)^{1/2},$$

we thus find that (III.24) is less than

$$\frac{C_1(\ln s)^{1/2}}{\theta^{3/2}(\sin\theta)^{1/2}}+\frac{C_2}{\theta^{3/2}(\sin\theta)^{1/2}}<\frac{\text{const}(\ln s)^{1/2}}{\sin^2\theta}.\quad \text{(III.26)}$$

Obviously the final result holds also for $\frac{1}{2}\pi<\theta<\pi$.

Using the choice (III.25) for $L_0(\theta,s)$, the unitarity bound for $a_l(s)$, and the bound (III.22), we find that the sum in (III.21) contributes to $f(s,z)$ at most

$$\text{const}(\ln s)^{3/2}/\sin^2\theta.\quad \text{(III.27)}$$

Thus, combining (III.26) and (III.27), we obtain an upper bound for $f(s,z)$ which agrees completely with the previous result (II.27). However we now have something more because this new approach shows that the first partial waves with angular momentum $l<C\ln s/\sin\theta$ give the dominant contribution to the upper bound of the scattering amplitude. In this approach we cannot say that the contribution from the higher partial waves is less than $(\ln s)^{1/2}$ because each individual contribution for $l\sim\ln s$ is itself of the order of $(\ln s)^{1/2}$. However, if we introduce a smooth cutoff procedure in summing up the partial waves, we might be able to reduce the upper bound for the contribution of higher partial waves. In fact, making use of such a device, Yamamoto[23] has shown that the knowledge of the first $C\ln^2 s$ (instead of $C\ln s$) partial waves determine the fixed angle scattering amplitude within an error of the order of s^{-N}, where N can be made arbitrarily large by choosing C big enough. It would be interesting to see whether this $C\ln^2 s$ could be reduced to $C\ln s$.

IV. EXTENSION TO HIGHER SPINS

So far we have assumed that the particles have no spin and obtained the upper bound for the differential cross section

$$d\sigma_{\text{el}}/d\Omega<\text{const}(\ln s)^3/s(\sin\theta)^4.\quad \text{(IV.1)}$$

We want now to consider the case of higher spins. The first question that arises is the energy dependence of the upper bound for the fixed angle differential cross section. For this purpose the simplest approach might be that of Yamamoto[24] which leads us to the same s dependence for fixed angles as (IV.1) for arbitrary spins. However, it turns out that a more detailed investigation of the angular dependence is necessary if we want to exhibit the smooth transition between the fixed angle bound and the fixed momentum transfer bound.

[23] K. Yamamoto, Phys. Rev. 135, B567 (1964).
[24] K. Yamamoto, Nuovo Cimento 27, 1277 (1963).

We shall first treat the spin-0–spin-$\frac{1}{2}$ scattering in detail. In this case the cross section may be written as

$$d\sigma_{el}/d\Omega=(4/s)(|f(s,z)|^2+|g(s,z)|^2), \quad (IV.2)$$

with

$$f(s,z)=(\sqrt{s}/2k)\sum_l((l+1)a_{l^+}(s)+la_{l^-}(s))P_l(z), \quad (IV.3)$$

and

$$g(s,z)=(\sqrt{s}/2k)\sum_l(a_{l^+}(s)-a_{l^-}(s))\times(1-z^2)^{1/2}P_l'(z), \quad (IV.4)$$

where $a_{l^\pm}$ is the partial-wave amplitude with the total angular momentum $J=l\pm\frac{1}{2}$ satisfying the unitarity requirement

$$|a_{l^\pm}(s)|\leq 1.$$

Obviously the arguments of Secs. II and III can be applied without modification to $f(s,z)$, giving

$$|f(s,z)|<\text{const}(\ln s)^{3/2}/\sin^2\theta. \quad (IV.5)$$

To derive a bound for g using the methods of Sec. III, we need the following inequality (see Appendix D)

$$|\sin\theta P_l'(\cos\theta)|<Ce^{|\text{Im}l|\theta}(1/\sin\theta+(|l|/\sin\theta)^{1/2}), \quad (IV.6)$$

where θ is real satisfying $0\leq\theta\leq\pi$ and $\text{Re}l>0$. Then, using the Watson-Sommerfeld transformation, one obtains in a straightforward way

$$|g(s,z)|<\text{const}(\ln s)^{3/2}/\sin^2\theta. \quad (IV.7)$$

Thus, combining (IV.5) and (IV.7), we conclude that inequality (IV.1) holds also for spin-0–spin-$\frac{1}{2}$ case.

The case of spin-$\frac{1}{2}$–spin-$\frac{1}{2}$ scattering has been investigated by Cornille[18] using a method which is a generalization of the method of Sec. II. His result agrees essentially with (IV.1).

In principle, there seems to be no difficulty in extending these results to the scattering of particles with arbitrary spins, although we have not found a general method suitable for this purpose. Of course, the only unsolved problem is that of the angular dependence of the upper bound for the differential cross section, since the s dependence is already known. We wish to make it plausible, by the following argument, that (IV.1) will hold quite generally.

We consider the scattering of particles with arbitrary spins. Then the differential cross section can be written in the form[24]

$$d\sigma(s,z)/d\Omega=\frac{4}{s}\sum_{l=1}^{n}C_i(z)f_i(s,z)f_i^*(s,z^*), \quad (IV.8)$$

where C_i's are analytic in z. We shall assume for definiteness that the f_i's are analytic in the domain $D(s)$ defined in Sec. II. Then $d\sigma(s,z)/d\Omega$, as written in the form (IV.8), has a unique analytic continuation in the domain $D(s)$. Let us consider the function

$$\phi(s,z)=\int_{-1}^{z}\frac{d\sigma(s,z')}{d\Omega'}dz'. \quad (IV.9)$$

Again this function is analytic in $D(s)$, and, if we assume the polynomial boundedness for the amplitudes, we have $|\phi(s,z)|<s^N$ for $z\in D(s)$. In addition, $\phi(s,z)$ is subject to a very weak form of unitarity condition:

$$|\phi(s,z)|<\sigma_{\text{el}}(s)\leq\sigma_{\text{tot}}(s)<\text{const}(\ln s)^2 \quad \text{for} \quad -1\leq z\leq 1, \quad (IV.10)$$

where we have used the Froissart bound for the forward scattering amplitude.[24]

Let us consider the domain defined by

$$|\arg((1-z)/(1+z))|\leq\alpha. \quad (IV.11)$$

This domain, limited by two arcs of circle going through the points $z=1$ and $z=-1$, is inside $D(s)$ for sufficiently small positive α. Then $\phi(s,z)$ is bounded by s^N in this domain. Now, noting that (IV.10) holds for real z in the interval $-1\leq z\leq 1$, we obtain the bound

$$|\phi(s,z)|<\text{const}(\ln s)^2\exp\left(\frac{N}{\alpha}(\ln s)\left|\arg\left(\frac{1-z}{1+z}\right)\right|\right)$$

for z satisfying (IV.11), where we have made use of the general technique of subharmonic functions (see Sec. IIB). In particular, for z in the domain

$$|\arg((1-z)/(1+z))|<\text{const}/\ln s, \quad (IV.12)$$

we obtain the bound

$$|\phi(s,z)|<\text{const}(\ln s)^2. \quad (IV.13)$$

Applying Cauchy's inequality to a circle inside the domain (IV.12) centered at the real point z, we obtain

$$\left|\frac{d\phi(s,z)}{dz}\right|=\frac{d\sigma(s,z)}{d\Omega}<\frac{\text{const}(\ln s)^3}{1-z^2}. \quad (IV.14)$$

Although this bound has a better s dependence than (I.2), it is much poorer than (IV.1) for fixed z ($|z|<1$). (This is not surprising because we have used unitarity only in a very weak form.) However, for $0<1-|z|\leq\text{const}/s$, it is as good as (IV.1).

V. UPPER BOUND FOR FORWARD SCATTERING AMPLITUDE

In the preceding sections we have seen that unitarity combined with analyticity and temperedness in the complex z plane leads us to a considerable improvement of the Froissart upper bound for fixed nonforward (and nonbackward) angles. We may naturally ask whether the forward bound of Froissart can also be improved. The purpose of this section is to give an answer to this question. We shall show that, under the assumptions

(i) analyticity in the cut t plane for fixed real positive s,

(ii) temperedness in s and t for real positive s,

(iii) unitarity in the s channel, or more precisely: $1\geq \mathrm{Im}a_l(s)\geq |a_l(s)|^2\geq 0$ for integer l,

it is *impossible* to improve the forward bound (I.1).[7]

We shall prove this by constructing counter examples. Our approach is based on the property of the analytically continued partial wave amplitudes $a^{\pm}(l,s)$ obtained in Sec. III that it is bounded by a constant for real l greater than $C\ln s$. It is clear from (III.14) that this constant may be chosen very close to unity if we consider only those l which are much larger than $C\ln s$ for very large s. Since we are looking for a negative result, we may simplify the problem by making the more restrictive assumption

$$|a^{\pm}(l,s)|\leq 1 \quad \text{for all real } l>0. \tag{V.1}$$

Similarly we shall replace (III.3) by a simpler condition[25]

$$|a^{\pm}(l,s)|\leq |s\exp(-l/\sqrt{s})| \quad \text{for} \quad \mathrm{Re}l>0. \tag{V.2}$$

Since the main contribution to the bound (I.1) comes from partial waves with an angular momentum of order $\sqrt{s}\ln s$, it is convenient to change the scale from l to

$$x=l/(\sqrt{s}\ln s).$$

Then (V.2) and (V.1) become

$$|a(x,s)|\leq |s^{1-x}| \quad \text{for} \quad \mathrm{Re}x>0,$$
$$|a(x,s)|\leq 1 \quad \text{for} \quad x \text{ real and positive}.$$

(For simplicity we shall drop the superscripts $\pm$ from now on.) We also introduce the function

$$\phi(x,s)=a(x,s)s^{x-1}. \tag{V.3}$$

Then ϕ is analytic in $\mathrm{Re}x>0$ and satisfies the conditions

$$|\phi(x,s)|\leq 1 \quad \text{for} \quad \mathrm{Re}x>0, \tag{V.4}$$

$$|\phi(x,s)|\leq s^{x-1} \quad \text{for} \quad 0\leq x\leq 1. \tag{V.5}$$

Our problem is to find an analytic function $\phi(x,s)$ that satisfies (V.4), (V.5) and gives the largest possible forward amplitude. If $|\phi|=s^{x-1}$ (or more generally $cs^{x-1}<|\phi|<s^{x-1}$, $0<c<1$) holds for all x in the interval $0\leq x\leq 1$, our problem is solved because it gives an example of scattering amplitude that saturates the Froissart bound. However, it may turn out that, because of the analyticity requirement on ϕ, $|\phi|=s^{x-1}$ cannot be satisfied everywhere in $0\leq x\leq 1$. If this is the case, we might be able to improve the Froissart bound. Our first task will therefore be to see whether the condition (V.5) can be sharpened without altering our problem in any way.

The technical difficulty here is that the condition (V.5) is imposed *inside* the analyticity domain of $\phi(x,s)$. To resolve this trouble, let us first look for an analytic function $B_{x_0}(x,s)$ which satisfies the conditions

(a) $B_{x_0}(x,s)$ is analytic in the half-plane $\mathrm{Re}x>0$ with a cut running from $x=0$ to $x=x_0$, where $0\leq x_0\leq 1$,

(b) $|B_{x_0}(x,s)|\leq 1$ for $\mathrm{Re}x>0$ and $|B_{x_0}(x,s)|=1$ for $\mathrm{Re}x=0$,

(c) $|B_{x_0}(x,s)|=s^{x-1}$ for $0\leq x\leq x_0$.

This problem can be reduced to a typical boundary value problem by the mapping $x\rightarrow y=(x^2-x_0^2)^{1/2}$, which maps the domain (a) onto the right half-plane $\mathrm{Re}y>0$. Then, making use of the Poisson formula for a half-plane,[26] we can construct an analytic function

$$B_{x_0}(x,s)=s^{-(2/\pi)\cos^{-1}(1-(x_0/x)^2)^{1/2}+x-(x^2-x_0^2)^{1/2}}, \tag{V.6}$$

which satisfies all the properties (a), (b), and (c).

Now the modulus of the ratio ϕ/B_{x_0} is less than unity on the boundary of the domain (a), as is seen from (b), (c), (V.4), and (V.5). Thus, applying the Phragmén–Lindelöf theorem to ϕ/B_{x_0} we obtain the inequality

$$\ln|\phi(x,s)|\leq \ln|B_{x_0}(x,s)|\equiv V_{x_0}(x)\ln s$$

in the domain (a), where

$$V_{x_0}(x)=-\mathrm{Re}[(2/\pi)\cos^{-1}(1-(x_0/x)^2)^{1/2}-x+(x^2-x_0^2)^{1/2}]. \tag{V.7}$$

Obviously $V_{x_0}(x)$ is a harmonic function inside the domain (a). We shall now minimize $V_{x_0}(x)$ with respect to x_0 to obtain the optimum bound. It is easily seen that, for any given x, the minimum occurs at $x_0=2/\pi$. There (V.7) becomes

$$V(x)\equiv V_{2/\pi}(x)=-\mathrm{Re}[(2/\pi)\cos^{-1}(1-4/\pi^2x^2)^{1/2}-x+(x^2-4/\pi^2)^{1/2}]. \tag{V.8}$$

We notice that, although we did not use condition (V.5) for $2/\pi<x<1$, this requirement is automatically satisfied by (V.8). As was noted already, $V(x)$ is a *harmonic* function in the domain (a). In fact $V(x)$ is a *subharmonic* function in the half-plane $\mathrm{Re}x>0$ *including the cut.* To show this it is enough to notice that the Laplacian of $V(x)$ is non-negative in $\mathrm{Re}x>0$:

$$\Delta V(x)=(2/x)(4/\pi^2-x^2)^{1/2}\theta(2/\pi-\mathrm{Re}x)\delta(\mathrm{Im}x). \tag{V.9}$$

It can be shown that $V(x)$ is the largest of all subharmonic functions defined in $\mathrm{Re}x>0$ satisfying the conditions (V.4) and (V.5).

Coming back to $a(x,s)$ we obtain

$$|a(x,s)|<s^{1-\mathrm{Re}(x^2-4/\pi^2)^{1/2}-(2/\pi)\,\mathrm{Re}\cos^{-1}(1-4/\pi^2x^2)^{1/2}}, \tag{V.10}$$

which may also be written as

$$|a(x,s)|<\left|\exp\left(-\ln s\int_{2/\pi}^{\infty}\left(1-\frac{4}{\pi^2x'^2}\right)^{1/2}dx'\right)\right|. \tag{V.11}$$

[25] The following arguments are not affected by this simplifying assumption.

[26] R. P. Boas, *Entire Functions* (Academic Press Inc., New York, 1954), p. 92.

This result is certainly an improvement compared with the original information we have put in. At first we knew only that $|a(x,s)|<1$ for $2/\pi<x<1$ but now find that, for any $x>2/\pi$, $|a(x,s)|$ decreases like some negative power of s. However, this is not enough to improve the s dependence of the Froissart forward bound because partial wave amplitudes may still take large values for angular momenta up to $l\sim(2/\pi)\sqrt{s}\ln s$.

On the other hand, it can shown that the bound (V.11) is not the best possible one compatible with conditions (i), (ii), and (iii) stated at the beginning of this section. In fact, it can be shown that the least upper bound $M(x,s)$ of the moduli of all functions ϕ satisfying (V.4) and (V.5) is a continuous subharmonic function of x, the logarithm of which is also a subharmonic function. Furthermore, if $\ln M(x,s)$ is harmonic in a connected subdomain G of $\text{Re}x>0$, it can be shown that there exists a function $\phi_0(x,s)$ satisfying (V.4) and (V.5) for which $|\phi_0(x,s)|=M(x,s)$ in G. These results are quite general and applicable to other analogous situations. Now assume for a moment that $|B_{2/\pi}(x,s)|$ is equal to $M(x,s)$. Then, since $\ln|B_{2/\pi}(x,s)|$ is harmonic in $\text{Re}x>0$, $\text{Im}x>0$, we would conclude that $B_{2/\pi}(x,s)$ is itself a function ϕ. This is clearly impossible since $B_{2/\pi}(x,s)$ is singular at $x=2/\pi$. Thus, $|B_{2/\pi}(x,s)|$ is not equal to $M(x,s)$.

So, at this stage, we cannot draw any conclusion, neither that the Froissart bound *can* be improved, nor that it *cannot* be improved. We have nevertheless some suspicion now that it is the latter that is true. We prove it in the following by constructing explicitly an analytic function $a(l,s)$ which satisfies conditions (i), (ii), and (iii) and has the property $\liminf_{s\to\infty}(\sigma_{\text{tot}}/(\ln s)^2)>0$. In what follows, we shall use (V.11) merely as a guide, and try to approach it as closely as possible for large energies, especially on the segment $0<x<2/\pi$.

The first method is based on the observation that the right-hand side of (V.11) can be regarded as the numerator of the W. K. B. solution of the following differential equation:

$$(d^2/dx^2+4(\ln s)^2/\pi^2x^2-(\ln s)^2)\psi(x)=0. \quad \text{(V.12)}$$

It will be interesting to study the solutions of this equation because they are analytic in $\text{Re}x>0$ and yet they will have a behavior which is not too different from that of the right-hand side of (V.11). The solutions of (V.12) are Hankel functions

$$x^{1/2}H_{i\lambda}^{(1)}(ix\ln s)\,,\quad x^{1/2}H_{i\lambda}^{(2)}(ix\ln s)$$

with

$$i\lambda(i\lambda+1)=-(4/\pi^2)(\ln s)^2\,,\quad \text{i.e.,}\quad \lambda\simeq(2/\pi)\ln s\,.$$

We discard $H_{i\lambda}^{(2)}$ because it increases exponentially for $x\to+\infty$.

To construct an analytic example of $a(x,s)$ starting from $H_{i\lambda}^{(1)}$, we need the following properties of $H_{i\lambda}^{(1)}$ for real and positive λ:

(1) $H_{i\lambda}^{(1)}(iy)$ is purely imaginary for real and positive y,

(2) $|H_{i\lambda}^{(1)}(iy)|<C_0\lambda^{-1/3}$ for all real and positive y, (V.13)

(3) $|H_{i\lambda}^{(1)}(iy)|<C_0(\lambda^2-y^2)^{-1/4}$ for $0<y<\lambda$, or more precisely,

$$H_{i\lambda}^{(1)}(iy)\simeq-i(2\pi)^{1/2}\times\frac{\sin(\pi/4+\lambda\cosh^{-1}(\lambda/y)-(\lambda^2-y^2)^{1/2})}{(\lambda^2-y^2)^{1/4}}, \quad \text{(V.14)}$$

for $\lambda\epsilon<y<\lambda(1-\epsilon)$, $\lambda\to\infty$, where ϵ is a small positive number,

(4)

$$|H_{i\lambda}^{(1)}(iy)|<C_0\frac{\exp((\lambda\pi/2)-\text{Re}y)}{(|\lambda+y|)^{1/2}} \text{ for } \text{Re}y>0. \quad \text{(V.15)}$$

For a derivation of these inequalities, see Appendix E.

From these inequalities one can easily deduce that the analytic function

$$F(x,s)=C(\ln s)^{1/2}\frac{(2/\pi)-x}{((2/\pi)+x)^{1/2}}H_{(2i/\pi)\ln s}^{(1)}(ix\ln s) \quad \text{(V.16)}$$

satisfies

$$|F(x,s)|<|s^{1-x}| \quad \text{for} \quad \text{Re}x>0$$

as is seen from (V.15), and

$$|F(x,s)|\le 1 \quad \text{for} \quad 0<x<2/\pi\,,$$

as is seen from (V.13) and (V.14), provided that C is taken to be small enough. These properties are the only ones necessary to get the upper bound (V.11). Hence the "unitarity" requirement

$$|F(x,s)|\le 1$$

is satisfied automatically for all real positive x. In addition, from the asymptotic estimate (V.14), we see that in the interval $\epsilon<x<2/\pi-\epsilon$, $F(x,s)$ is of order unity except when

$$\frac{\pi}{4}+\frac{2}{\pi}(\ln s)\cosh^{-1}\left(\frac{2}{\pi x}\right)-(\ln s)\left(\frac{4}{\pi^2}-x^2\right)^{1/2}=n\pi\,,$$

$n=$ integer.

This equation shows that the spacing of the zeros of $F(x,s)$ is of the order of $\Delta x=1/\ln s$, or, returning to the original variable l, $\Delta l=\sqrt{s}$.

Thus, for almost all integer values of l less than $(2/\pi)\sqrt{s}\ \ln s$, F is of order unity. However, F may take both positive and negative imaginary values. To construct an example of the absorptive amplitude which satisfies the unitarity condition

$$1\ge\text{Im}a_l(s)\ge 0 \quad \text{for integer } l\,,$$

we may therefore choose

$$\operatorname{Im}a_l(s)=-[F(l/(\sqrt{s}\ln s),s)]^2. \quad (V.17)$$

This quantity has the proper analyticity domain as a function of l and satisfies

$$|\operatorname{Im}a_l(s)|<s^2\exp(-2l/\sqrt{s}),$$

which is a particular case of (III.3). Finally, since $\operatorname{Im}a_l(s)$ is of order unity for almost all l less than $(2/\pi)\sqrt{s}\ln s$, the sum $\sum_0^\infty(2l+1)\operatorname{Im}a_l(s)$ is of the order of $s\ln^2 s$.

As was noticed above the density of zeros of $F(x,s)$ for real x less than $2/\pi$ increases with increasing s. This suggests to us an alternative approach to our problem of finding an analytic function $\phi(x,s)$ which is as close to $\exp[(\ln s)V(x)]$ as possible. Namely, we may try to simulate the discontinuity of $\exp[(\ln s)V(x)]$ across the cut $0<x<2/\pi$ by a distribution of zeros of $\phi(x,s)$. For this purpose, it is appropriate to write $(\ln s)V(x)$ in the following form:

$$(\ln s)V(x)=(\ln s)\int_0^{2/\pi}\rho(u)\ln\left|\frac{x-u}{x+u}\right|du, \quad (V.18)$$

where

$$\rho(u)=((4/\pi^2)-u^2)^{1/2}/\pi u.$$

This is obtained by integrating (V.9) with the help of the Green's function $(2\pi)^{-1}\ln|(x-u)/(x+u)|$. Introducing the function

$$y(u)=\int_u^{2/\pi}\rho(v)dv,$$

we can rewrite (V.18) as

$$(\ln s)V(x)=(\ln s)\int_0^\infty\ln\left|\frac{x-u(y)}{x+u(y)}\right|dy, \quad (V.19)$$

where $u(y)$ is the inverse function of $y(u)$.

In order to find $\phi(x,s)$, holomorphic in $\operatorname{Re}x>0$, such that $(\ln s)V(x)$ is an asymptotic (loosely speaking) expression for $\ln|\phi|$ when $s\to+\infty$, we shall assume that $\phi(x,s)$ is of the form

$$\prod_i\left(\frac{x-x_i}{x+x_i}\right)^r, \quad (V.20)$$

where x_i's are real numbers satisfying $0<x_i<(2/\pi)$ and r is a positive integer. We have chosen this form because it is holomorphic in $\operatorname{Re}x>0$ and also satisfies the condition (V.4). From (V.20) we obtain

$$\ln|\phi|=r\sum_i\ln\left|\frac{x-x_i}{x+x_i}\right|. \quad (V.21)$$

We want to approximate the right-hand side of (V.19) by a suitable choice of x_i's. To each x_i we can associate $y_i=y(x_i)$. Then, Eq. (V.19) suggests that the best choice for x_i's is obtained by requiring that y_i's are equally spaced:

$$y_i=(2i-1)r/2\ln s,\quad i=1,2,\cdots.$$

With this choice of x_i's, it is possible to show that there is a constant C such that

$$\phi(x,s)=C\prod_{i=1}^\infty\left(\frac{x-x_i}{x+x_i}\right)^r \quad (V.22)$$

satisfies the conditions (V.4) and (V.5). To guarantee the positiveness of $\phi(x,s)$ on the positive real axis, r should be restricted to a positive even integer. Furthermore, we find that (V.22) saturates the Froissart bound.

VI. REMARKS

In this paper we have tried to obtain upper bounds for the high-energy scattering amplitude assuming that it satisfies the Mandelstam representation and the unitarity condition but without making any further assumptions. More specifically we have only used the fixed energy analytic properties of the scattering amplitude and unitarity property in the inequality sense. Considering the small amount of information we have used, the results seem to be rather strong. However, from a practical point of view, they are so weak that they do not allow a crucial test of the theory by comparison with the experiment.

As far as the fixed angle upper bound is concerned, we do not know whether the $(\ln s)^{3/2}$ behavior can be improved further by our method or not. Even if some improvement were possible, we feel certain that the upper bound cannot be better than a positive constant since we do not take into account in our approach the fact that most (or even all) subtraction constants are not arbitrary.[1,27] On the other hand, one can show by other means that under the same assumptions the scattering amplitude can be as small as $\exp(-c\sqrt{s}\ln s)$ for fixed angles.[17] Furthermore, this seems to be rather close to the actual behavior of the scattering amplitude.[28]

We now want to comment on the forward scattering amplitude. First we want to emphasize that our result may not be the best possible upper bound because we did not take into account full analyticity and unitarity. In particular we disregarded analyticity with respect to energy and also unitarity in crossed channels.

Next, we note that the examples (V.17) and (V.22) which saturate the Froissart forward bound have been given in terms of the analytically continued partial-wave amplitude. Therefore, the angular dependence of the corresponding scattering amplitude $f(s,z)$ is not easy to recognize. It would be interesting if one could construct an explicit example of $f(s,z)$ which saturates the Froissart bound at $z=1$. On the other hand, it is not

[27] A. Martin, Phys. Rev. Letters **9**, 410 (1962).

[28] J. Orear, Phys. Rev. Letters **12**, 112 (1964); T. Kinoshita, *ibid.* **12**, 257 (1964).

difficult to find an example in which $f(s,1)\sim s\ln s$. One such example is the following[17]:

$$f(s,\,1+t/2k^2)=iC(\ln s)s^{1+t_0^{1/2}-(t_0-t)^{1/2}}, \quad \text{(VI.1)}$$

where t_0 is a positive number. This expression has obviously the correct analyticity domain in which it is bounded by $C(\ln s)s^{(1+\sqrt{t_0})}$ and for a suitable choice of C it satisfies[17]

$$\mathrm{Re}a_l(s)=0\,,\quad 0<\mathrm{Im}a_l(s)<1\,.$$

A characteristic feature of this example is that all derivatives of $\mathrm{Im}f$ with respect to t are positive for $-\infty<t<t_0$.

More generally one can show that, if the absorptive part of a scattering amplitude $A_s(s,t)$ satisfies the inequalities

$$A_s(s,t_1)\geq 0\,,\quad [(d^n/dt^n)A_s(s,t)]_{t=t_1}\geq 0\,, \quad \text{(VI.2)}$$

for all n and some fixed negative t_1, $A_s(s,0)$ has an upper bound of the form

$$A_s(s,0)<Cs(\ln s)(\ln\ln s)\,. \quad \text{(VI.3)}$$

This may be proved in the following manner.

From (VI.2) we find that $A_s(s,t)$ is positive and increasing in the interval $t_1<t<t_0$, where t_0 is positive. Thus we get

$$|t|(A_s(s,t))^2<\int_t^0(A_s(s,t'))^2dt'<s^2\sigma_{el}<sA_s(s,0) \quad \text{(VI.4)}$$

for $t_1<t<0$. Using the Froissart bound for $A_s(s,0)$ and the positiveness condition (VI.2), we obtain

$$|A_s(s,t)|<s(\ln s)^2 \quad \text{(VI.5)}$$

for t in the circle $|t-t_1|\leq|t_1|$. Combining this with the temperedness requirement

$$|A_s(s,t)|<s^N$$

for $|t-t_1|<|t_0-t_1|$, we obtain

$$|A_s(s,t)|<s\ln^2 s \quad \text{(VI.6)}$$

in the circle $|t-t_1|<|t_1|+\mathrm{const}/\ln s$, with the help of the Hadamard's three circle theorem.[14] On the other hand, the positiveness requirement together with inequality (VI.4) show that

$$|A_s(s,t)|<(sA_s(s,0)/\tau)^{1/2}, \quad \text{(VI.7)}$$

for $|t-t_1|<|t_1|-\tau$. Then, applying again the Hadamard's theorem to the three circles

$$|t-t_1|<|t_1|-\tau\,,$$
$$|t-t_1|<|t_1|\,,$$

and

$$|t-t_1|<|t_1|+\mathrm{const}/\ln s\,,$$

and adjusting τ in an appropriate way, we obtain the desired result (VI.3). Although assumption (VI.2) is rather strong, the actual physical amplitude might well possess such a property.

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APPENDIX A: GENERATING FUNCTION $L(z,u)$ OF LEGENDRE FUNCTIONS OF THE SECOND KIND

1. *Definitions.* We shall first recall some well-known formulas.[29,30]

For u in G (complement of the closed straight line segment $I\equiv[-1,1]$) we have

$$Q_n(u)=\frac{1}{2}\int_{-1}^{1}\frac{P_n(y)}{u-y}dy\,. \quad \text{(A1)}$$

In particular

$$Q_0(u)=\tfrac{1}{2}\ln((u+1)/(u-1))\,, \quad \text{(A2)}$$

(one-valued in G, $\to 0$ as $|u|\to\infty$). On the other hand, for y on I and $|z|<1$, we have

$$(1-2yz+z^2)^{-1/2}=\sum_{n=0}^{\infty}z^nP_n(y) \quad \text{(A3)}$$

(the square root being equal to 1 at $z=0$). For every z with $|z|<1$, the series is uniformly convergent for y on I. For $|z|<1$ and u in G we define

$$L(z,u)=\frac{1}{2}\int_{-1}^{1}(u-y)^{-1}(1-2yz+z^2)^{-1/2}dy\,, \quad \text{(A4)}$$

where the square root is chosen in the same way as above. Obviously L is holomorphic for $|z|<1$ and u in G. We have

$$L(0,u)=Q_0(u)\,. \quad \text{(A5)}$$

For every fixed z with $|z|<1$, we find that

$$L(z,u)\to 0 \quad \text{as} \quad |u|\to\infty\,. \quad \text{(A6)}$$

[29] E. T. Whittaker and G. N. Watson, Ref. 10, pp. 320–321.

[30] *Higher Transcendental Functions*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York, 1953), Vol. 1, p. 154.

Inserting (A3) in (A4), and using (A1), we obtain

$$L(z,u)=\sum_{n=0}^{\infty} z^n Q_n(u) \quad \text{for} \quad |z|<1 \text{ and } u \text{ in } G. \tag{A7}$$

2. *Analytic Continuation of L.* We shall now consider analytic continuation of L from the domain $|z|<1$, $u\in G$, where it is given by (A4), to a larger domain. In particular we are interested in the convergence domain of (A7).

An obvious way to make analytic continuation of (A4) is to carry out the integration. We thus obtain

$$L(z,u)=\frac{1}{t}\ln\frac{u-z+t}{(u^2-1)^{1/2}}, \tag{A8}$$

where $t=(1-2uz+z^2)^{1/2}$ and $u(u^2-1)^{-1/2}\to 1$ for $|u|\to\infty$. We still have to choose proper determination of the square root and the logarithm.

At first we fix u in G. Put

$$v=(u-z+t)(u^2-1)^{-1/2}. \tag{A9}$$

The mapping $(z,t)\to v$ is a one-to-one mapping of the Riemann surface of the points (z,t) satisfying $t^2=1-2uz+z^2$ onto the complex v plane. The inverse mapping is given by

$$z=u-\tfrac{1}{2}(u^2-1)^{1/2}(v+v^{-1}),$$
$$t=\tfrac{1}{2}(u^2-1)^{1/2}(v-v^{-1}). \tag{A10}$$

In terms of v the right-hand side of (A8) can be written as $2(u^2-1)^{-1/2}(v-v^{-1})^{-1}\ln v$. We now fix $\ln v$ by the condition

$$|\arg v|<\pi,$$

and put

$$F(v,u)=2(u^2-1)^{-1/2}(v-v^{-1})^{-1}\ln v, \tag{A11}$$

which is holomorphic and one-valued in the v plane cut along the negative real axis from 0 to $-\infty$. It follows from (A11) that $F(v^{-1},u)=F(v,u)$ which means that F is independent of the determination of t [see (A10)]. Thus $F(v,u)$ is equal to a holomorphic and one-valued function $L'(z,u)$ in the z plane with the cut

$$\Gamma(u)=\{z=u+(u^2-1)^{1/2}r,\ r \text{ real and } \geq 1\}.$$

L' is actually some determination of the right-hand side of (A8) so that $L-L'$ is of the form $2k\pi i/t$, k being some integer. Since $L-L'$ at $z=0$, however, k must be equal to zero. This means that

$$L(z,u)=F(v,u). \tag{A12}$$

We have thus obtained the result: *For u fixed in G, $L(z,u)$ is holomorphic in the z plane cut along $\Gamma(u)$.* At the point z on $\Gamma(u)$, one finds from (A10), (A11), and (A12) that

$$L(z(1+i0),u)-L(z(1-i0),u) = 2\pi i(1-2uz+z^2)^{-1/2}, \tag{A13}$$

where $(u^2-1)^{1/2}(1-2uz+z^2)^{-1/2}$ is real and positive. Thus, *L has a branch point at $z=u+(u^2-1)^{-1/2}=a(u)$* [see (II.7)]. Furthermore, for every large positive R there is some positive constant $M=M(u)$ such that

$$|L(z,u)|<M|z|^{-1}\ln|z| \tag{A14}$$

holds in the domain of the cut z plane satisfying $|z|>R$. Thus

$$L(z,u)=\int_{\Gamma(u)}(x-z)^{-1}(1-2ux+x^2)^{-1/2}dx \tag{A15}$$

[for the determination of square root, see (A13)].

Consequences: (a) The series (A7) converges for $|z|<|u+(u^2-1)^{1/2}|$. (b) $\lim\sup_{n\to\infty}|Q_n(u)|^{1/n}=|u-(u^2-1)^{1/2}|$. (c) For a fixed z, the series (A7) converges uniformly for u in every closed domain in $|u+(u^2-1)^{1/2}|>|z|$. If $|z|\leq 1$, this last domain is G. If $|z|>1$, this domain is the exterior of the ellipse with foci at $+1$ and -1, going through the point $(z+z^{-1})/2$.

3. *Analytic Property of L in u for Fixed Values of z.* If $|z|<1$, L is a holomorphic function of u in G by definition. In order to get an insight for other values of z, we make the following construction. Let γ be any smooth arc (without double points) connecting -1 and $+1$ in the u plane. Let $\Delta(\gamma)$ be the (open) complement of γ, including the point ∞. If we construct a set $\{z|(z+z^{-1})/2\in\Delta(\gamma)\}$, we find that it consists of two disconnected parts. Let $D(\gamma)$ be the part that contains $z=0$. For u in $\Delta(\gamma)$ and z in $D(\gamma)$, define

$$L_\gamma(z,u)=\frac{1}{2}\int_\gamma (u-y)^{-1}(1-2yz+z^2)^{-1/2}dy \tag{A16}$$

(square root$=1$ at $z=0$). Then L_γ is holomorphic in the simply connected domain $\{(z,u)|z\in D(\gamma),u\in\Delta(\gamma)\}$. Furthermore, for z in a certain neighborhood of 0 and u in a certain neighborhood of ∞, L_γ coincides with L, as is seen by continuously deforming γ into I. Thus, L_γ is an analytic continuation of L. Now, for fixed z with $|z|\geq 1$, choose γ so that the union of γ and I forms a closed curve without double points, with $(z+z^{-1})/2$ in its interior. Then L is holomorphic in the u plane cut along γ. The discontinuity of L at the cut can be read out of (A16).

APPENDIX B: UPPER BOUND FOR THE MODULUS OF THE GENERATING FUNCTION $L(z,u)$ IN CERTAIN DOMAINS

We start from the integral representation (A15) for $L(z,u)$. For our purpose, it is convenient to deform the path $\Gamma(u)$ to $\Gamma_1(u)=\{x=(u+(u^2-1)^{1/2})r',\ r' \text{ real}, \leq 1\}$. Then we obtain

$$L(z,u)=\int_1^\infty (wr'-z)^{-1}[(r'-1)(r'-w^{-2})]^{-1/2}dr', \tag{B1}$$

for u in G and z in the plane cut along $\Gamma_1(u)$, where w stands for $a(u)=u+(u^2-1)^{1/2}$ [see (II.7)]. Remember that a is a one-to-one mapping of G onto $|w|>1$. The right-hand side of (B1) is thus a holomorphic function of w and z in $|w|>1$ and in the z plane cut along $\Gamma_1(u)$.

Let us first consider the domain D_0 in the (z,w) space which consists of w on $\Gamma_1(u_0)$, $u_0\in G$, and z in the domain A (complement of the shaded domain shown in Fig. 2). For (z,w) in this domain and for $r'\geq 1$, we find that

$$|r'-w^{-2}|\geq r'-|w|^{-2}\geq r'-R^{-2}$$

and

$$|wr'-z|=|w|\,|r'|\,|1-z/wr'|\geq r'R\mu\,,$$

where $\mu=\text{Min}\{(R-r)/R,\sin\varphi\}$. Using these inequalities, we get

$$|L(z,u)|\leq\frac{1}{\mu}\int_1^\infty (Rr')^{-1}[(r'-1)(r'-R^{-2})]^{-1/2}dr'\,,$$

which may also be written as

$$|L(z,u)|\leq\frac{1}{\mu}Q_0(\rho)=\frac{1}{2\mu}\ln\frac{\rho+1}{\rho-1}\,, \tag{B2}$$

where $\rho=b(R)=(R+R^{-1})/2$ [see (II.8)]. Notice that, when w is on $\Gamma_1(u_0)$, u is a hyperbolic arc with foci at $+1$ and -1 which goes through u_0.

Next we consider the domain D_1 of the (z,w) space defined by $|w|\geq r_1>1$ and $|z|\leq r_2$ where $0<r_2<r_1$. We obtain in a similar manner

$$|L(z,u)|\leq\frac{r_1}{2(r_1-r_2)}\ln\frac{\rho+1}{\rho-1}\,, \tag{B3}$$

where ρ now stands for $b(r_1)$. Notice that when $|w|\geq r_1$, u is in the complement of the elliptical disk E_{r_1}.

As an application of these inequalities, let us consider the following problem. Let Δ be a closed curve without double points, situated outside the unit circle $|z|=1$, and made up of circular arcs centered at the origin and of segments of rays through the origin (see Fig. 3). For w on or outside of Δ, and for z in the domain $\Delta(\mu)$ shown in Fig. 3, inequalities (B2) and (B3) gives

$$|L(z,u)|\leq(1/2\mu)\ln(\rho+1)/(\rho-1)\,, \tag{B4}$$

where $\rho=\frac{1}{2}(R+R^{-1})$. The meaning of μ is explained in Fig. 3.

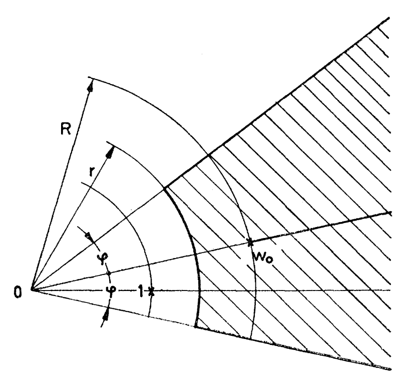

FIG. 2. $w_0=a(u_0)$, $|w_0|=R>1$, $0<r<R$. A is the closed complement of the shaded area.

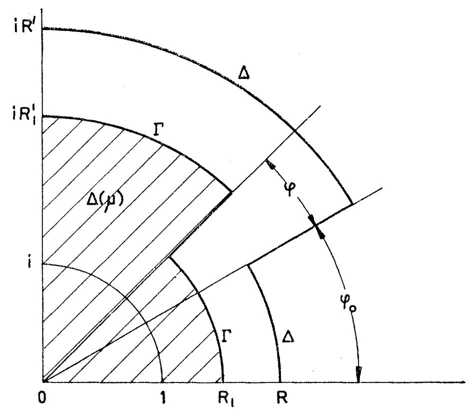

FIG. 3. The portion of $\Delta(\mu)$ in the first quadrant is represented by the shaded area. The closed curve Δ is characterized by R, R' and φ_0. R_1, R_1', φ and μ are defined by $0<\mu<1$, $\mu=(R-R_1)/R=(R'-R_1')/R'=\sin\varphi$.

APPENDIX C: ILLUSTRATION TO SEC. IIA

We shall illustrate the formulas obtained in Sec. IIA by applying them to functions of the type frequently encountered in dispersion theory.

Let us consider

$$g(z)=\frac{1}{\pi}\int_{x_0}^\infty\frac{\rho(x)}{x-z}dx\,, \tag{C1}$$

with $x_0>1$. For definiteness, assume that ρ is a complex valued function of the real variable x and satisfies the condition

$$\int_{x_0}^\infty x^{-1}|\rho(x)|dx<\infty \tag{C2}$$

(in Lebesgue's sense). Thus g is holomorphic in the z plane cut along the real axis from x_0 to ∞.

We now consider a function f which is constructed from the function g of (C1) according to the prescription of Theorem 1 (Sec. IIA). Let V be a domain of the type described there. Inserting (C1) in (II.9) we get

$$f(z)=\frac{1}{2\pi i}\int_{\partial V}duK(z,u)\left[\frac{1}{\pi}\int_{x_0}^\infty\frac{\rho(x)}{x-u}dx\right].$$

Since we can find a positive number M such that

$$|K(z,u)(x-u)^{-1}\rho(x)|<Mx^{-1}|\rho(x)|$$

for u on ∂V and $x\geq x_0$, and since

$$\int_{\partial V}|du|\int_{x_0}^\infty x^{-1}|\rho(x)|dx<\infty\,,$$

we can write[31]

$$f(z)=\frac{1}{\pi}\int_{x_0}^{\infty}dx\rho(x)\left[\frac{1}{2\pi i}\int_{\partial V}(x-u)^{-1}K(z,u)du\right].$$

The integral with respect to u is readily evaluated by applying the residue calculus to the exterior of the closed curve ∂V; it gives $K(z,x)$. Thus

$$f(z)=\frac{1}{\pi}\int_{x_0}^{\infty}dx\rho(x)K(z,x). \quad \text{(C3)}$$

For every fixed $x\geq x_0$, $K(z,x)$ can be written as

$$K(z,x)=\frac{1}{\pi(2x)^{1/2}}\int_{y}^{\infty}(\eta-y)^{-1/2}(\eta-z)^{-1}d\eta$$

in the z plane cut from $y=(x+x^{-1})/2=b(x)$ to ∞ (the square roots are non-negative). Inserting this in (C3), we obtain

$$f(z)=\frac{1}{\pi}\int_{x_0}^{\infty}dx(x^2+1)^{-1/2}\rho(x) \times\left[\frac{1}{\pi}\int_{1}^{\infty}(v-1)^{-1/2}(v-y^{-1}z)^{-1}dv\right]$$

after a change of variable. Since

$$|(x^2+1)^{-1/2}\rho(x)|<x^{-1}|\rho(x)|$$

and

$$|(v-1)^{-1/2}(v-y^{-1}z)^{-1}|<C(v-1)^{-1/2}v^{-1},$$

for x and v in thier integration ranges, we can interchange the order of integration and obtain

$$f(z)=\frac{1}{\pi}\int_{y_0}^{\infty}\frac{\sigma(\eta)}{\eta-z}d\eta \quad \text{(C4)}$$

for all z in the plane cut from $y_0=b(x_0)$ to ∞, where, for (almost all) $\eta\geq y_0$,

$$\sigma(\eta)=\frac{1}{\pi}\int_{x_0}^{a(\eta)}(2\eta x-1-x^2)^{-1/2}\rho(x)dx \quad \text{(C5)}$$

(integral transform of Abel type). $a(\eta)$ is defined by (II.7).

Conversely, if $\sigma(\eta)$ is defined for $\eta\geq y_0$, and if

$$\int_{y_0}^{\infty}\eta^{-1}|\sigma(\eta)|d\eta<\infty,$$

we get, using (II.15), (II.16), and (A15), the expression

$$g(z)=\frac{1}{\pi}(2z\partial_z+1)\int_{x_0}^{\infty}(x-z)^{-1}\rho_0(x)dx \quad \text{(C6)}$$

for the g function corresponding to the f function (C4), where

$$\rho_0(x)=\int_{y_0}^{b(x)}(1-2\eta x+x^2)^{-1/2}\sigma(\eta)d\eta \quad \text{(C7)}$$

for (almost all) $x\geq x_0$. If $\rho_0(x)$ is continuous and continuously differentiable for $x\geq x_0$, and if $\rho_0(x)\to 0$ as $x\to x_0$ and $x\to\infty$, an easy manipulation of (C6) gives

$$g(z)=\frac{1}{\pi}\int_{x_0}^{\infty}\frac{\rho(x)}{x-z}dx, \quad \text{(C8)}$$

where

$$\rho(x)=(2x\partial_x+1)\rho_0(x). \quad \text{(C9)}$$

We have derived the formula (C5) and the converse formulas (C7) and (C9) under rather restrictive assumptions. There is no doubt, however, that they can be extended to more general cases where σ and ρ are tempered distributions with suitable behavior at infinity.

As an application of such generalized formulas, one could derive the well-known fact that the Regge interpolation of the coefficients of the Legendre series for f is a Mellin transform of some tempered distribution (here ρ).

APPENDIX D: BOUNDS ON $P_l(\cos\theta)$ AND $\sin\theta P_l'(\cos\theta)$ FOR $0\leq\theta\leq\pi$ AND COMPLEX l

For real l the following inequality is well known[32]:

$$|P_l(\cos\theta)|<\frac{C_0}{((2l+1)\sin\theta)^{1/2}}. \quad \text{(D1)}$$

On the other hand, for $l=-\frac{1}{2}+i\lambda$, we have[32]

$$P_{-\frac{1}{2}+i\lambda}(\cos\theta)=\frac{2}{\pi}\int_0^{\theta}\frac{\cosh(\lambda u)du}{(2(\cos u-\cos\theta))^{1/2}}. \quad \text{(D2)}$$

It is not difficult to majorize this:

$$|P_{-\frac{1}{2}+i\lambda}(\cos\theta)|<2(2/\pi)^{1/2}e^{|\lambda|\theta}/(|\lambda|\sin\theta)^{1/2}. \quad \text{(D3)}$$

More generally, one can prove that $P_l(\cos\theta)$ as a function of complex l does not increase faster than $\exp(|\mathrm{Im}l|\theta)$ as l goes to infinity. Therefore the function

$$((2l+1)\sin\theta)^{1/2}e^{il\theta}P_l(\cos\theta)$$

satisfies the Phragmén-Lindelöf conditions in the angles between $\arg(l+\frac{1}{2})=0$ and $\arg(l+\frac{1}{2})=\frac{1}{2}\pi$, and is less than a fixed constant independent of θ on both lines.[21] Hence we have

$$|P_l(\cos\theta)|<Ce^{|\mathrm{Im}l|\theta}/(|2l+1|\sin\theta)^{1/2} \quad \text{(D4)}$$

for $\mathrm{Re}l>-\frac{1}{2}$.

Next we consider a bound for $\sin\theta P_l'(\cos\theta)$. Starting

[31] See, for example, N. Wiener, *The Fourier Integral and Certain of its Applications* (Cambridge University Press, New York, 1933), p. 18, Theorem X_{25}.

[32] W. Magnus and F. Oberhettinger, *Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), pp. 58, 74, 51, 52.

from[32]

$$\sin\theta P_l'(\cos\theta)=\frac{l(l+1)}{2l+1}\frac{P_{l-1}(\cos\theta)-P_{l+1}(\cos\theta)}{\sin\theta}, \quad \text{(D5)}$$

we can easily obtain

$$|\sin\theta P_l'(\cos\theta)|<C(|2l+1|)^{1/2}e^{|\mathrm{Im}l|\theta}/(\sin\theta)^{3/2}, \quad \text{(D6)}$$

for large $|l|$ making use of (D4). However, this bound is far from being the best possible. To find a better bound, let us express $P_{l-1}-P_{l+1}$ as follows[32]:

$$\begin{aligned}P_{l-1}(\cos\theta)-P_{l+1}(\cos\theta)&=\frac{2\sqrt{2}}{\pi}\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]\sin\phi d\phi}{(\cos\phi-\cos\theta)^{1/2}}\\&=\frac{2\sqrt{2}}{\pi}\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]\sin\theta d\phi}{(\cos\phi-\cos\theta)^{1/2}}\\&\quad+\frac{2\sqrt{2}}{\pi}\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi](\sin\phi-\sin\theta)d\phi}{(\cos\phi-\cos\theta)^{1/2}}. \quad \text{(D7)}\end{aligned}$$

Let us first consider the second term. Apart from a numerical factor, it may be written as

$$\int_0^\theta \sin[(l+\tfrac{1}{2})\phi]\cos\tfrac{1}{2}(\phi+\theta)\left(\frac{\sin(\frac{1}{2}(\theta-\phi))}{\sin(\frac{1}{2}(\theta+\phi))}\right)^{1/2}d\phi.$$

Integrating by part, we see easily that it is less than

$$(C/|l+\tfrac{1}{2}|)e^{|\mathrm{Im}l|\theta}. \quad \text{(D8)}$$

To estimate the first term of (D7), we have to study

$$\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]d\phi}{(\cos\phi-\cos\theta)^{1/2}}.$$

Majorizing it by

$$\int_0^\theta\frac{e^{|\mathrm{Im}l|\phi}d\phi}{(\cos\phi-\cos\theta)^{1/2}},$$

it is not difficult to prove that it is less than

$$Ce^{|\mathrm{Im}l|\theta}/(|\mathrm{Im}l|\sin\theta)^{1/2}.$$

Unfortunately this is not enough. However, for real l we may use[33]

$$\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]d\phi}{(\cos\phi-\cos\theta)^{1/2}}=-\sqrt{2}Q_l(\cos\theta)+\int_0^\infty\frac{e^{-(l+\frac{1}{2})t}dt}{(\cosh t-\cos\theta)^{1/2}}. \quad \text{(D9)}$$

This may be easily majorized since

$$|Q_l(\cos\theta)|<\frac{C}{(l\sin\theta)^{1/2}} \quad \text{for real } l, \quad \text{(D10)}$$

and

$$\left|\int_0^\infty\frac{e^{-(l+\frac{1}{2})t}dt}{(\cosh t-\cos\theta)^{1/2}}\right|<\frac{1}{(l+\frac{1}{2})(1-\cos\theta)^{1/2}}. \quad \text{(D11)}$$

Thus we get

$$\left|\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]d\phi}{(\cos\phi-\cos\theta)^{1/2}}\right|<\begin{cases}\dfrac{Ce^{|\mathrm{Im}l|\theta}}{(|\mathrm{Im}l|\sin\theta)^{1/2}} & \text{for complex } l,\\ \dfrac{C}{(l\sin\theta)^{1/2}} & \text{for real } l.\end{cases} \quad \text{(D12)}$$

Application of Phragmén-Lindelöf techniques yields

$$\left|\int_0^\theta\frac{\sin[(l+\frac{1}{2})\phi]d\phi}{(\cos\phi-\cos\theta)^{1/2}}\right|<\frac{Ce^{|\mathrm{Im}l|\theta}}{(|l|\sin\theta)^{1/2}}, \quad \mathrm{Re}l>0. \quad \text{(D13)}$$

Hence we obtain

$$|P_{l-1}(\cos\theta)-P_{l+1}(\cos\theta)|<Ce^{|\mathrm{Im}l|\theta}\left[\frac{1}{|l|}+\left(\frac{\sin\theta}{|l|}\right)^{1/2}\right], \quad \text{(D14)}$$

and finally

$$|\sin\theta P_l'(\cos\theta)|<Ce^{|\mathrm{Im}l|\theta}\left[\frac{1}{\sin\theta}+\left(\frac{|l|}{\sin\theta}\right)^{1/2}\right], \quad \mathrm{Re}l>0. \quad \text{(D15)}$$

APPENDIX E: BOUNDS AND ASYMPTOTIC EXPRESSIONS OF $H_{i\lambda}^{(1)}(iz)$ FOR LARGE REAL POSITIVE λ AND $\mathrm{Re}z>0$

We start from the Sommerfeld representation[34]

$$H_{i\lambda}^{(1)}(iz)=\frac{1}{\pi}\int_{+i\infty}^{-i\infty}e^{-z\cos t+\lambda(\pi/2-t)}dt. \quad \text{(E1)}$$

Let us first treat the case where z is real and positive. Since the integrand decreases extremely rapidly as $\mathrm{Im}t\to\pm\infty$ ($-\frac{1}{2}\pi<\mathrm{Re}t<\frac{1}{2}\pi$), we can displace the path of integration to the line $\mathrm{Re}t=t_1$, $0\le t_1<\frac{1}{2}\pi$. Then we obtain

$$|H_{i\lambda}^{(1)}(iz)|<\frac{1}{\pi}\int e^{-z\cos t_1\cosh t_2+\lambda(\pi/2-t_1)}dt_2. \quad \text{(E2)}$$

Since $\cosh t_2>1+\frac{1}{2}(t_2)^2$, we can reduce this to a Gaussian integral and obtain

$$|H_{i\lambda}^{(1)}(iz)|<\frac{Ce^{-z\cos t_1+\lambda(\pi/2-t_1)}}{(z\cos t_1)^{1/2}}. \quad \text{(E3)}$$

[33] L. Robin, *Fonctions Spheriques de Legendre et Fonctions Spheroïdales* (Gauthier-Villars, Paris, 1957), Vol. 2, p. 155 [Eq. (281)].

[34] E. Jahnke and F. Emde, *Table of Functions* (Dover Publications, New York, 1945), p. 148.

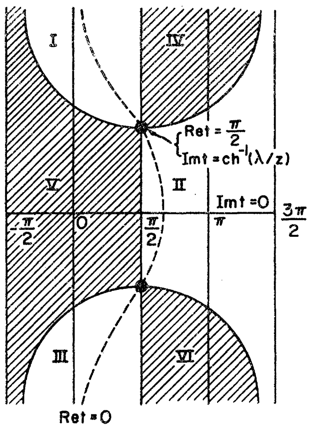

FIG. 4. The t plane for real z satisfying $0<z<\lambda$. The integrand is less than unity in the unshaded regions I, II, and III. The dashed line represents an integration path convenient for calculating $H_{i\lambda}^{(1)}$.

If we restrict ourselves to $z\geq\lambda-\lambda^{1/3}$, we can choose $\pi/2-t_1=\lambda^{2/3}/z$. Noticing that

$$\cos t_1>(\tfrac{1}{2}\pi-t_1)-\tfrac{1}{6}(\tfrac{1}{2}\pi-t_1)^3\,,$$

we then find

$$|H_{i\lambda}^{(1)}(iz)|<(C/\lambda^{1/3})\exp[\tfrac{1}{6}\lambda^2/z^2-\lambda^{2/3}(1-(\lambda/z))].$$

Hence

$$|H_{i\lambda}^{(1)}(iz)|<(C'/\lambda^{1/3})\exp[-\lambda^{2/3}(1-(\lambda/z))]<C''/\lambda^{1/3}\quad\text{for}\quad z\geq\lambda-\lambda^{1/3}.\qquad\text{(E4)}$$

If we choose $\frac{1}{2}\pi-t_1=1/\lambda$, this simple treatment gives also the bound

$$|H_{i\lambda}^{(1)}(iz)|<C(\lambda/z)^{1/2}\,,$$

for $z<\lambda-\lambda^{1/3}$. However, as we shall see, this can be considerably improved.

For $z<\lambda$ we have to study more carefully the integrand of the Sommerfeld representation (E1). In this case we find that the integrand is less than unity in regions I, II, III, and greater than unity in regions IV, V, VI (see Fig. 4). To optimize the bound on $H_{i\lambda}^{(1)}$ we have to integrate along the dashed line which goes through the two saddle points $\mathrm{Re}t=\frac{1}{2}\pi$, $\mathrm{Im}t=\pm\cosh^{-1}(\lambda/z)$. Then, applying the method of steepest descent, we can find a bound on $H_{i\lambda}^{(1)}$ and also an asymptotic expression for $H_{i\lambda}^{(1)}$.

The method of steepest descent gives an asymptotic expression of $H_{i\lambda}^{(1)}$ only for $\epsilon<(z/\lambda)<1-\epsilon$, this condition being necessary to guarantee that the saddle points are at finite distance and do not coincide. Then we obtain easily

$$H_{i\lambda}^{(1)}(iz)\simeq-i(2\pi)^{1/2}\times\frac{\sin(\frac{1}{4}\pi+\lambda\cosh^{-1}(\lambda/z)-(\lambda^2-z^2)^{1/2})}{(\lambda^2-z^2)^{1/4}}\,,\qquad\text{(E5)}$$

$$\epsilon<z/\lambda<1-\epsilon\,,\quad\lambda\to\infty\,.$$

In the region $1-\epsilon<(z/\lambda)<1$, one can find an upper bound of $H_{i\lambda}^{(1)}(iz)$ by using the contour

$$\mathrm{Re}t=\pi/2-\lambda^{-1/3}\quad\text{and}\quad\cosh^{-1}(\lambda/z)<\mathrm{Im}t<+\infty\,,$$
$$\mathrm{Im}t=\cosh^{-1}(\lambda/z)\quad\text{and}\quad\pi/2-\lambda^{-1/3}<\mathrm{Re}t<\pi\,,$$
$$\mathrm{Re}t=\pi\quad\text{and}\quad 0\leq\mathrm{Im}t<\cosh^{-1}(\lambda/z)\,,$$

the rest being symmetrical with respect to the real axis. This gives the inequality

$$|H_{i\lambda}^{(1)}(iz)|<C/\lambda^{1/3}\quad\text{for}\quad 1-\epsilon<z/\lambda<1.\qquad\text{(E6)}$$

For z close to zero, one can easily convince oneself that the saddle-point method still gives a reliable bound except that the phase might be in error. Thus we have

$$|H_{i\lambda}^{(1)}(iz)|<C/\sqrt{\lambda}\quad\text{for}\quad z/\lambda<\epsilon.\qquad\text{(E7)}$$

This checks with the upper bound obtained by the power-series expansion at $z=0$.

Let us now investigate the complex z region with $\mathrm{Re}z\geq0$. Integrating the Sommerfeld integral along $\mathrm{Re}t=0$, we obtain

$$|H_{i\lambda}^{(1)}(iz)|<Ce^{\lambda\pi/2}e^{-\mathrm{Re}z}/(\mathrm{Re}z)^{1/2}.\qquad\text{(E8)}$$

For $\mathrm{Re}z=0$, we can still use the Sommerfeld representation if the limits of integration are taken to be $-\pi/2+i\infty$ and $\pi/2-i\infty$. Then, if we put $z=-ix$, we find a unique saddle point at $\sin t=i(\lambda/x)$, namely $\mathrm{Re}t=0$ and $\mathrm{Im}t=\sinh^{-1}(\lambda/x)$, which leads us to

$$|H_{i\lambda}^{(1)}(x)|<Ce^{\lambda\pi/2}/(\lambda^2+x^2)^{1/4}.\qquad\text{(E9)}$$

This bound is valid for $x>0$ as well as for $x<0$ because, $H_{i\lambda}^{(1)}(iz)$ being purely imaginary for real and positive z, we can apply the Schwartz reflection principle along the line $\mathrm{Im}z=0$, $\mathrm{Re}z>0$. Then we find that the function

$$H_{i\lambda}^{(1)}(iz)(\lambda+z)^{1/2}e^{-\lambda\pi/2}e^{z}$$

satisfies the conditions of the Phragmén-Lindelöf theorems in the range $-\frac{1}{2}\pi<\arg z<\frac{1}{2}\pi$. Hence we obtain

$$|H_{i\lambda}^{(1)}(iz)|<Ce^{\lambda\pi/2}e^{-\mathrm{Re}z}/(|\lambda+z|)^{1/2}\,,\qquad\text{(E10)}$$

$$\mathrm{Re}z>0\,,\quad\lambda\text{ real and }\gg0.$$

More details on these bounds will be found in Ref. 35.

[35] A. Martin, Nuovo Cimento (to be published).