Coupling of Internal and Space-Time Symmetries*

O. W. GREENBERG
Department of Physics and Astronomy,
University of Maryland, College Park, Maryland
(Received 13 May 1964)

When Lorentz invariance is ignored, internal symmetries which are violated in a given way, and which are independent of the space-time symmetries, can be rewritten in terms of a larger group which contains the internal group and the time-translation group in a coupled (noncommuting) way. This rewriting can be chosen so that the noncommutativity of the time-translation and internal groups splits the mass degeneracy of internal multiplets, i.e., accounts for what was previously called "violation" of the internal group. This procedure is illustrated explicitly for the SU(3) baryon octet with octet symmetry violation. When Lorentz invariance is required, the coupling of internal and space-time symmetries becomes more difficult. A global and more general proof is given of McGlinn's result that in a larger group $\mathcal{G}$ whose generators are those of the Poincaré (i.e., inhomogeneous Lorentz) group and an internal group, for certain internal groups the commutativity of the (homogeneous) Lorentz and internal subgroups implies the commutativity of the space-time translation and internal subgroups. In addition, it is shown that in such a group $\mathcal{G}$, which has an internal group for which the commutativity of the Lorentz and internal subgroups does not imply the commutativity of the translation and internal subgroups, the internal multiplets still remain degenerate in mass.

1. INTRODUCTION

THE recent history of particle physics is to a large extent the story of suggested internal symmetries, i.e., symmetries not related to space-time, which must be considered to be violated by small or large amounts in order to yield predictions which agree reasonably well with experimental data. Symmetries which are not exact also occur in other fields of physics; for example, the external electric and magnetic fields of the Stark and Zeeman effects in atomic physics violate rotation symmetry. However, the amount of violation of symmetries of the kind just cited has a clear experimental origin, and the degree of violation can be varied, and even reduced to zero, in contrast to the situation in particle physics. If the internal symmetries now considered in particle physics have a fundamental significance, one might expect that they would have some connection with the space-time symmetries, and that there would be some sense in which they would be exact. It seems worthwhile to consider some possibilities and some difficulties of achieving these two desiderata, even if one is not certain that the presently known internal symmetries are the true ones.

If we ignore the requirements of Lorentz invariance, we can strightforwardly translate an internal symmetry and a given way of violating this symmetry into a description in which there is a larger group of symmetries, including both the internal group and the time translation group, which is exactly represented. What was previously called "violation" of the symmetry, as evidenced by splitting of the mass degeneracy of the particles in an internal multiplet, now appears as noncommutativity of the internal and time translation subgroups. As an illustration of this possibility, in Sec. 2 we translate SU(3) symmetry with octet violation into an irreducible representation of a larger group in the baryon octet. We obtain the Gell-Mann mass formula as an *exact* relation in this representation.

In a recent paper,[1] McGlinn has made the important observation that because the Poincaré group $\mathcal{P}$ (i.e., the inhomogeneous Lorentz group) is a semidirect product in which the (homogeneous) Lorentz group $\mathcal{L}$ acts on the translations, noncommutativity of the translation group $\mathcal{T}$ with the internal group $\mathcal{I}$ may require that the Lorentz and internal groups also fail to commute. Under the condition $[\mathcal{L},\mathcal{I}]=0$, and with some further restrictive assumptions, McGlinn proved that $[\mathcal{T},\mathcal{I}]=0$, so that the mass degeneracy of internal multiplets could not be split. McGlinn's method of proof used the Jacobi identity for the generators of the larger group $\mathcal{G}$, whose generators are the generators of $\mathcal{P}$ and $\mathcal{I}$, and required that the internal group be a finite dimensional semisimple Lie group. We give an alternative proof of McGlinn's result in Sec. 3. Our proof is valid for quite general internal groups. Our analysis uses global group properties only, and brings forward the way in which the action of $\mathcal{L}$ on $\mathcal{T}$ enters the problem. We also consider the possibility of noncommutativity of the representatives of $\mathcal{P}$ and $\mathcal{I}$ in the ray representations of $\mathcal{G}$. Finally we point out that even in those cases in which $[\mathcal{T},\mathcal{I}]\neq 0$ is allowed the type of noncommutativity which occurs does not lead to splitting of the mass degeneracy of internal multiplets.

The preceding paragraph indicates that, under certain specific conditions, space-time and internal symmetries cannot be coupled in a nontrivial way. Nonetheless, we believe that such a linking of space-time and internal symmetries can be carried out if some of these conditions are relaxed.[2] Among the conditions to be relaxed is the

* Work supported in part by National Science Foundation Grant GP 1193 and U. S. Air Force Office of Scientific Research, Grant 500-74.

[1] W. D. McGlinn, Phys. Rev. Letters **12**, 467 (1964).

[2] F. Coester, M. Hammermesh, and W. D. McGlinn, Phys. Rev. **135**, B451 (1964), have suggested that the requirement

restricition that $\mathcal{G}$ have as generators only the generators of $\mathcal{P}$ and of $\mathcal{I}$, or in global terms, that each element in $\mathcal{G}$ have the unique decomposition $g=pi$, p in $\mathcal{P}$, and i in $\mathcal{I}$. We hope to discuss the situation when this restriction is relaxed in a later article.

2. MODEL OF OCTET VIOLATION OF SU(3) IN THE BARYON OCTET

In this section we give an *exact* eightfold representation of a large group, which contains time translations and SU(3) as noncommutative subgroups, and in which the "violation" of SU(3) is due to this noncommutativity. We ignore Lorentz invariance in this section. In this discussion our aim is to translate the usual approximate SU(3) symmetry into an exact symmetry involving a larger group with coupled time translation and SU(3) symmetry. We do not derive either the SU(3) symmetry or the octet violation of it; analogous models could be constructed for other groups and other types of violations.

We present our model in terms of the Hamiltonian (generator of time translations) and the SU(3) generators. We assume that our Hamiltonian H has the eight baryons as eigenvectors with their observed masses (neglecting electromagnetic mass differences).

$$H|N\rangle=M_N|N\rangle, \qquad H|\Lambda\rangle=M_\Lambda|\Lambda\rangle,$$
$$H|\Sigma\rangle=M_\Sigma|\Sigma\rangle, \qquad H|\Xi\rangle=M_\Xi|\Xi\rangle.$$

We assume that the SU(3) generators are represented in the space of the eight baryons, and that these generators change the mass as well as the particle species. For example, using the notation of Behrends, Dreitlein, Fronsdal, and Lee,[3]

$$6^{\frac{1}{2}}E_1|n\rangle=|p\rangle, \qquad 6^{\frac{1}{2}}E_1|\Sigma^0\rangle=-\sqrt{2}|\Sigma^+\rangle,$$
$$2\sqrt{3}E_2|\Xi^0\rangle=-\sqrt{2}|\Sigma^+\rangle, \qquad 2\sqrt{3}E_2|\Sigma^0\rangle=|p\rangle,$$
$$2\sqrt{3}E_3|\Xi^0\rangle=|\Sigma^0\rangle-\sqrt{3}|\Lambda\rangle, \qquad 2\sqrt{3}E_3|\Lambda\rangle=\sqrt{3}|n\rangle,$$

etc. The complete table of the action of the generators on the baryon octet is given in Table VII of the cited paper.[3] To introduce octet violation of SU(3), we require the existence of an SU(3) scalar operator L, which is related to H by octet violation terms:

$$L=H+\alpha Y+\beta[\mathbf{I}^2-(Y^2/4)],$$

where Y is the hypercharge, $\mathbf{I}$ is the isospin vector, and α and β are numbers to be determined by requiring that L rommute with all generators of SU(3) in the eightfold cepresentation. This calculation is straightforward.

From

$$[L,E_2]|\Xi^0\rangle=0$$

and

$$[L,E_3]|\Lambda\rangle=0,$$

we find

$$\alpha=\tfrac{1}{2}(M_\Sigma-M_\Xi)+\tfrac{3}{2}(M_\Lambda-M_N),$$
$$\beta=M_\Xi-M_\Sigma+M_N-M_\Lambda.$$

Then from

$$[L,E_2]|\Sigma^0\rangle=0,$$

the Gell-Mann mass formula,

$$\tfrac{1}{2}(M_N+M_\Xi)=\tfrac{1}{4}(3M_\Lambda+M_\Sigma)$$

results, and L is an SU(3) scalar.

The octet of physical baryons with their observed masses is the basis of the above irreducible representation of the direct product $\{e^{iLt}\}\otimes$SU(3). The mass formula is an *exact* relation for this representation.

We would like to point out an analogy between the Hamiltonian and Langrangian in relativistic theories and the operators H and L in our model of an SU(3)-invariant theory. In a relativistic theory, the Hamiltonian density is not a scalar under the Lorentz group, but the Lagrangian density, which is closely related to the Hamiltonian, is. Analogously, in the SU(3)-invariant model, the operator H (the Hamiltonian) is not an SU(3) scalar, but L, which is closely related to it, is. Thus the operator L plays a role analogous to that of the Lagrangian.

Concerning the structure of the Lie algebra associated with our model, we point out that although L together with the eight generators of SU(3) form a nine-dimensional Lie algebra, the operator H is not in this algebra. If H is added to this set of nine generators, the commuterors of H with the SU(3) generators give new elements which belong to neither the group $\{e^{iLt}\}$, nor the group SU(3).

3. CONDITIONS UNDER WHICH COMMUTATIVITY OF THE LORENTZ AND INTERNAL GROUPS, $[\mathcal{L},\mathcal{I}]=0$, IMPLIES COMMUTATIVITY OF THE TRANSLATION AND INTERNAL GROUPS, $[\mathcal{T},\mathcal{I}]=0$

We consider an abstract group $\mathcal{G}$ which is composed of the restricted[4] Poincaré group $\mathcal{P}=\mathcal{L}\times\mathcal{T}$ (i.e., the inhomogeneous restricted Lorentz group) and a group of internal transformtaions $\mathcal{I}$ in a way which we now make precise. We assume that an arbitrary element of $\mathcal{G}$ has the unique form

$$g=(a,\Lambda,i)=(a)[\Lambda]\{i\}, \qquad (1)$$

where $(a)=(a,1,e)$, $[\Lambda]=(0,\Lambda,e)$, and $\{i\}=(0,1,i)$ are elements in $\mathcal{T}$, $\mathcal{L}$, and $\mathcal{I}$, respectively, and (0), [1], $\{e\}$ are the respective unit elements. We will omit the

$[\mathcal{L},\mathcal{I}]=0$ should be replaced by $[\mathcal{P},H_i]=0$, where H_i are the commuting generators of $\mathcal{I}$ whose eigenvalues in a multiplet are the internal quantum numbers of the multiplet. We thank Dr. Hamermesh for an instructive conversation, and for making this paper available prior to publication. We hope to apply our global approach to analyze the requirement $[\mathcal{P},H_i]=0$.

[3] R. E. Behrends, J. Dreitlein, C. Fronsdal, and W. Lee, Rev. Mod. Phys. 34, 1 (1962), Table VII, p. 33.

[4] By restricted, we mean the group without the inversions.

brackets around these elements when no confusion will result. We assume the usual group multiplication laws for $\mathcal{P}$

$$(a_1)(a_2)=(a_1+a_2)\,, \tag{2a}$$

$$[\Lambda_1][\Lambda_2]=[\Lambda_1\Lambda_2]\,, \tag{2b}$$

$$[\Lambda](a)=(\Lambda a)[\Lambda]. \tag{2c}$$

We assume that $\mathcal{I}$ is a subgroup, so that multiplication in $\mathcal{I}$ is closed:

$$\{i_1\}\{i_2\}=\{i_1i_2\}. \tag{3}$$

We assume that the elements of the internal group commute with the elements of the Lorentz group,

$$\{i\}[\Lambda]=[\Lambda]\{i\}. \tag{4}$$

Finally, we assume that the product of an element of the internal group times a translation is an arbitrary element in the group[5]:

$$\{i\}(a)=(t(i)a,\lambda(a,i)1,T(a)i)\,, \tag{5}$$

and that $t(i)a$ is measurable in a. The correspondence $t(i)$ takes a translation (a) into a new translation $a'=t(i)a$ and the correspondences $\lambda(a,i)$ and $T(a)$ act in similar ways. Aside from the measurability property just mentioned, no special properties of these correspondences, such as linearity, are assumed. In the sequel we will suppress the transformation 1 on which $\lambda(a,i)$ acts. If in Eq. (5) either the element i is the identity of the group $\mathcal{I}$ or the element a is the identity of the group $\mathcal{T}$ then the equation should give a trivial result, which leads to two sets of three boundary conditions on our correspondences. These conditions are

$$t(i)0=0\,,\quad \lambda(0,i)=1,\quad T(0)i=i\,, \tag{6}$$

from the zero space-time translation, and from the identity of the internal group,

$$t(e)a=a\,,\quad \lambda(a,e)=1\,,\quad T(a)e=e. \tag{7}$$

We now apply the requirement of associativity to our group, and derive nine equations which our correspondences t, λ, and T must satisfy. The derivation of these requirements is straightforward; therefore, we will give only the derivation for the first three equations and then list the remaining ones.

Consider the effect of the associativity equation

$$(\{i\}[\Lambda])(a)=\{i\}([\Lambda](a)).$$

The left-hand side is

$$\begin{aligned}(\{i\}[\Lambda])(a)&=[\Lambda](\{i\}(a))\\&=[\Lambda](t(i)a,\lambda(a,i),T(a)i)\\&=(\Lambda t(i)a,\Lambda\lambda(a,i),T(a)i).\end{aligned}$$

The right-hand side is

$$\begin{aligned}\{i\}([\Lambda](a))&=(\{i\}(\Lambda a))[\Lambda]\\&=(t(i)\Lambda a,\lambda(\Lambda a,i),T(\Lambda a)i)[\Lambda]\\&=(t(i)\Lambda a,\lambda(\Lambda a,i)\Lambda,T(\Lambda a)i).\end{aligned}$$

Because of the uniqueness of a group element expressed in this form, we get three equations for our correspondences:

$$t(i)\Lambda a=\Lambda t(i)a\,, \tag{8}$$

$$\lambda(\Lambda a,i)=\Lambda\lambda(a,i)\Lambda^{-1}\,, \tag{9}$$

$$T(\Lambda a)i=T(a)i. \tag{10}$$

Our next set of three equations is derived from the equation

$$(\{i_1\}\{i_2\})(a)=\{i_1\}(\{i_2\}(a))$$

in a similar manner. These equations are

$$t(i_1i_2)a=t(i_1)t(i_2)a\,, \tag{11}$$

$$\lambda(a,i_1i_2)=\lambda(t(i_2)a,i_1)\lambda(a,i_2)\,, \tag{12}$$

$$T(a)(i_1i_2)=(T(t(i_2)a)i_1)(T(a)i_2). \tag{13}$$

Finally our last three equations are derived from the equality

$$(\{i\}(a_1))(a_2)=\{i\}((a_1)(a_2))\,,$$

and these equations are

$$t(i)(a_1+a_2)=t(i)a_1+\lambda(a_1,i)t(T(a_1)i)a_2\,, \tag{14}$$

$$\lambda(a_1+a_2,i)=\lambda(a_1,i)\lambda(a_2,T(a_1)i), \tag{15}$$

$$T(a_1+a_2)i=T(a_2)T(a_1)i. \tag{16}$$

This exhausts the set of independent equations.

To study the correspondence T, we consider Eqs. (10), (16), and (6). For any translation a, these equations imply

$$\begin{aligned}i=T(0)i=T(a-a)i&=T(a)T(-a)i\\&=T(a)T(-\Lambda a)i=T(a-\Lambda a)i.\end{aligned} \tag{17}$$

For a time-like, we can choose Λ in such a way that $a-\Lambda a$ is any space-like vector. Similarly, for a space-like, we can make $a-\Lambda a$ any time-like or light-like vector. Thus Eq. (17) implies that T is trivial:

$$T(a)i=i\,, \tag{18}$$

for all a and i. The analysis leading to Eq. (18) can be summarized by saying that there are no proper invariant linear subspaces in the group $\mathcal{T}$ under the action of the group $\mathcal{L}$.

Next, we study the correspondence λ, using Eqs. (9) and (15). We rewrite Eq. (15) in a new form using the fact just proved that T is trivial:

$$\lambda(a_1+a_2,i)=\lambda(a_1,i)\lambda(a_2,i). \tag{15'}$$

Equation (15′) states that the $\lambda(a,i)$ are an abelian subgroup of the Lorentz group, and Eq. (9) states

---

[5] Our Eq. (5) makes no assumption that either $\mathcal{P}$ or $\mathcal{I}$ is normal in $\mathcal{G}$, i.e., that $\mathcal{G}$ is a semidirect product of either of $\mathcal{P}$ or $\mathcal{I}$ by the other. The requirements that $\mathcal{P}$ be normal in $\mathcal{G}$ are that $T(a)i=i$, and $t(i)$ maps $\mathcal{T}$ onto $\mathcal{T}$. The requirements that $\mathcal{I}$ be normal in $\mathcal{G}$ are that $t(i)a=a$, $\lambda(a,i)=1$, and $\mathcal{T}(a)$ maps $\mathcal{I}$ onto $\mathcal{I}$.

that this subgroup is invariant. However, the Lorentz group is simple,[6] i.e., has no nontrivial invariant subgroups, and therefore the λ's which enter in Eq. (15′) are either the whole group $\mathcal{L}$, or just the identity. Since $\mathcal{L}$ is not abelian, it cannot satisfy Eq. (15′). Therefore the only possiblity is

$$\lambda(a,i)=1\,, \tag{19}$$

for all a and i, and thus the correspondence λ must be trivial.

The only equations that remain to be analyzed are Eqs. (8), (11), (14), and (7). We rewrite Eq. (14) in order to take account of the fact that the correspondences T and λ must be trivial:

$$t(i)(a_1+a_2)=t(i)a_1+t(i)a_2\,. \tag{14′}$$

With the assumption [stated below Eq. (5)] that $t(i)a$ is measurable in a, Eq. (14′) states that the correspondence t is linear, and Eq. (8) states that this linear transformation commutes with all Lorentz transformations. Therefore the linear transformation t must be a multiple of the identity which can depend only on the element i of the internal group;

$$t(i)a=c(i)a\,, \tag{20}$$

where $c(i)$ must be a real number to preserve the reality of the elements of $\mathcal{T}$. Equations (11) and (7) now state that the numbers $c(i)$ form a one-dimensional representation of the internal group $\mathcal{I}$:

$$c(i_1i_2)=c(i_1)c(i_2),\, c(e)=1\,. \tag{21}$$

Therefore, in order to have a nontrivial (i.e., noncommutative) combination of the Poincaré and internal groups, there must exist a nontrivial real one-dimensional representation of the internal group $\mathcal{I}$. For this to occur there must be a homomorphism which maps the group $\mathcal{I}$ onto the real numbers, which are an abelian group. Thus, the group $\mathcal{I}$ must have a nontrivial abelian factor group. In the event that this occurs, the internal group element multiplied by the translation element will have the new form shown in Eq. (5′):

$$\{i\}(a)=(c(i)a,1,i)=(c(i)a)\{i\}\,. \tag{5′}$$

We summarize this group theory result as a theorem:

Theorem. Let $\mathcal{G}$ be any group which is composed of the restricted Poincaré group and an internal group $\mathcal{I}$ in a way made precise by Eqs. (1) through (3). If the internal group commutes with the (homogeneous) Lorentz group $\mathcal{L}$, $[\mathcal{I},\mathcal{L}]=0$, then (a) if the internal group has no nontrivial real one-dimensional representation, the internal group also commutes with the space-time translation group, $[\mathcal{I},\mathcal{T}]=0$, so that $\mathcal{G}$ is the direct product $\mathcal{G}=\mathcal{I}\otimes\mathcal{P}$, or (b) if the internal group has a nontrivial real one-dimensional representation, the lack of commutativity with $\mathcal{T}$ is given by

$$\{i\}(a)=(c(i)a)\{i\}\,, \tag{22}$$

where the real numbers $c(i)$ are the nontrivial one-dimensional representation of $\mathcal{I}$, and $\mathcal{G}$ is the semidirect product $\mathcal{I}\times\mathcal{P}$.

Since the finite dimensional semisimple Lie groups have no nontrivial one-dimensional representations,[7] our case (a) includes McGlinn's result as a special case.

The type of noncommutativity of $\mathcal{T}$ and $\mathcal{I}$ which can occur in case (b), Eq. (22), cannot lead to splitting of the mass degeneracy of the members of a multiplet of $\mathcal{I}$, because the numbers $c(i)$ are representatives of an abelian factor group of $\mathcal{I}$. Thus, the $c(i)$ will be associated with abelian gauge transformations and cannot split the mass degeneracy of a multiplet. (It is the nonabelian elements in $\mathcal{I}$ which, in a representation, change particle species.)

Except for the phase factors which occur in the ray representations which must be admitted in quantum mechanics, our results so far show that if Eqs. (1) through (3) are assumed, and if $[\mathcal{L},\mathcal{I}]=0$, then the mass degeneracy of multiplets of $\mathcal{I}$ remains valid for an irreducible representation of the group $\mathcal{G}$. However, a theorem of G. W. Mackey[8] assures us that no nontrivial phase factors can occur because the Poincaré group has no nontrivial one-dimensional representations and therefore we draw the physical conclusion:

An irreducible representation of a group $\mathcal{G}$ composed of $\mathcal{P}$ and $\mathcal{I}$ according to Eqs. (1) through (3), and in which $[\mathcal{I},\mathcal{L}]=0$, has exact mass degeneracy in each multiplet of $\mathcal{I}$.

ACKNOWLEDGMENTS

We thank Professor C. Kacser, Professor A. Kleppner, and Professor J. C. Pati of the University of Maryland, and Dr. S. Meshkov of the National Bureau of Standards for helpful discussions. We also thank the referee for a careful reading of the paper, and for several helpful remarks.

[6] E. P. Wigner, Ann. Math. **40**, 149 (1939), especially pp. 167–168.

[7] Hermann Weyl, *The Classical Groups* (Princeton University Press, Princeton, New Jersey, 1946), 2nd ed., p. 263.

[8] G. W. Mackey, Acta Mathematica **99**, 265 (1958), Theorem 9.4, p. 303. The theorem is valid when $\mathcal{I}$ is a separable locally compact group. We thank Professor Adam Kleppner for calling our attention to this theorem.