# Normalization of Bethe-Salpeter Wave Functions and Bootstrap Equations*

R. E. CUTKOSKY AND M. LEON
*Carnegie Institute of Technology, Pittsburgh, Pennsylvania*
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The normalization condition for Bethe-Salpeter wave functions is derived directly from the integral equation for the Green's function. The result agrees with, but is more general than the usual one which requires the existance of a conserved current. An application is given to the derivation of bootstrap equations.

## 1. INTRODUCTION

THE usual way to normalize a Bethe-Salpeter wave function makes use of a conserved current: Integrating the time component of such a current over all space gives a quantity which is known *a priori*, e.g., nucleon number or electric charge. In this paper we show that the normalization condition can, in fact, be derived quite simply without reference to any conserved current. As a result, we can, for example, handle bound states of neutral mesons just like any other bound states. In Sec. 2 we derive and compare the two methods of normalization, while in Sec. 3 we show how the normalization condition plus the Bethe-Salpeter equation can be employed to derive bootstrap equations.

## 2. NORMALIZATION

Consider the integral equation satisfied by the two-particle Green's function, and the accompanying Bethe-Salpeter equation

$$G=G_1G_2+G_1G_2IG; \tag{1}$$

near a bound-state pole

$$G=i\phi_a\bar{\phi}_a/(P^2-m_a^2) \qquad +\text{terms regular at} \quad P^2=(P_1+P_2)^2=m_a^2. \tag{2}$$

Writing the product of one-particle Green's functions as

$$G_1G_2=K^{-1},$$

this gives the Bethe-Salpeter equation

$$\phi_a=K^{-1}I\phi_a. \tag{3}$$

(We suppress all irrelevant indices and integration variables, and usually drop the over-all energy-momentum $\delta$ function without change of notation.) It is useful to consider the Bethe-Salpeter equation off the bound-state mass shell, viz.,

$$\phi_a=\lambda_aK^{-1}I\phi_a \tag{3'}$$

for arbitrary $P$; the eigenvalue $\lambda_a(P^2)$ satisfies

$$\lambda_a(m_a^2)=1.$$

For a given $P$ we have, in addition to Eq. (3′),

$$\bar{\phi}_a=\lambda_a\bar{\phi}_aIK^{-1};$$

there follows the useful orthogonality relation

$$\bar{\phi}_aK\phi_b\left[\equiv\int d^4q\bar{\phi}_a(P,q)K(P,q)\phi_b(P,q)\right] =0 \quad \text{for} \quad \lambda_a(P)\neq\lambda_b(P). \tag{4}$$

Writing $G$ as an expansion on the solutions of Eq. (3′),

$$G=\sum_n \phi_n\psi_n\,,$$

and using Eqs. (1), (3′), and (4), we obtain

$$G=\sum_n[\phi_n\bar{\phi}_n/(1-\lambda_n^{-1})\bar{\phi}_nK\phi_n]. \tag{5}$$

Comparison of this result with Eq. (2) in the neighborhood of $P^2=m_a^2$ fixes the normalization

$$\bar{\phi}_aK\phi_a[\partial\lambda_a^{-1}(m_a^2)/\partial P_\mu]=2iP_\mu.$$

Now,

$$\lambda_a^{-1}=\bar{\phi}_aI\phi_a/\bar{\phi}_aK\phi_a$$

is an extremum for variations of $\phi_a$, so that

$$\bar{\phi}_aK\phi_a[\partial\lambda_a^{-1}(m_a^2)/\partial P_\mu] =\bar{\phi}_a(\partial I/\partial P_\mu)\phi_a-\bar{\phi}_a(\partial K/\partial P_\mu)\phi_a\,,$$

and the normalization condition can finally be written in the form

$$2P_\mu=i\bar{\phi}_a[\partial(K-I)/\partial P_\mu]\phi_a. \tag{6}$$

We will now show that this general expression, Eq. (6), agrees with the usual result which assumes existence of a conserved charge. Following Klein and Zemach[1] (see also Mandelstam[2]), we observe that introduction of an interaction with an external electromagnetic field $A_\mu$,

$$\mathfrak{L}'=j_\mu A_\mu\,,$$

leaves the form of Eq. (1) unchanged, while all its terms become functionals of $A_\mu$. Since

$$G^{-1}=K-I\,, \tag{7}$$

$$\delta G/\delta A_\mu=-G\Gamma_\mu G, \quad \text{with} \quad \Gamma_\mu=\delta(K-I)/\delta A_\mu.$$

However, we can also consider $\delta G/\delta A_\mu$ in terms of the constituent particle (renormalized Heisenberg) field operators[1]:

$$\delta G/\delta A_\mu={}_i\langle 0|T\{\psi(1)\psi(2)j_\mu\bar{\psi}(3)\bar{\psi}(4)\}|0\rangle.$$

In the standard fashion, we let times 1, 2 $\rightarrow+\infty$, times

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[1] A. Klein and C. Zemach, Phys. Rev. **108**, 126 (1957).
[2] S. Mandelstam, Proc. Roy. Soc. (London) **A233**, 248 (1955).

3, 4 → −∞, to find the bound-state poles and wave functions:

$$\delta G/\delta A_\mu = i\sum_{a,a'}\langle 0|T\{\psi(1)\psi(2)\}|P,a\rangle 2\pi\delta_+(P^2-m_a^2)d^4P\langle P,a|j_\mu|P',a'\rangle \times 2\pi\delta_+(P'^2-m_{a'}^2)d^4P'\langle P',a'|T\{\bar\psi(3)\bar\psi(4)\}|0\rangle$$
$$= i^3\sum_{a,a'}[\phi_a\langle P,a|j_\mu|P',a'\rangle\bar\phi_{a'}/(P^2-m_a^2)(P'^2-m_{a'}^2)] + \text{terms regular at } P^2=m_a^2,\quad P'^2=m_{a'}^2. \tag{8}$$

Factoring the $\delta$ function out of the right-hand side of Eq. (8) and comparing the results with Eqs. (2) and (7) gives

$$\langle P,a|j_\mu(P-P')|P',a'\rangle = i\bar\phi\Gamma_\mu(P,P')\phi_{a'}.$$

But for a state of unit charge,

$$\langle P,a|j_\mu(0)|P,a\rangle = 2P_\mu,$$

so that we have

$$2P_\mu = -i\bar\phi_a\Gamma_\mu\phi_a. \tag{9}$$

This is the normalization condition as it appears in Ref. 1; for comparison with Eq. (6), we must evaluate the right-hand side of Eq. (9) more explicitly. For coupling with a zero-momentum electromagnetic field, any operator $O$ which can be written as a sum of graphs satisfies

$$\delta O/\delta A_\mu = \sum e_i(\partial O/\partial q_\mu{}^i),$$

where $e_i$ is the charge and $q_\mu{}^i$ the momentum of each line, and the summation is over all lines and all graphs. In our case, integration by parts and the relation

$$\delta(\bar\phi_a(K-I)\phi_a) = 0$$

then gives

$$\bar\phi_a[\delta(K-I)/\delta A_\mu]\phi_a = \bar\phi_a(\partial/\partial P_\mu)(K-I)\phi_a.$$

Agreement with Eq. (6) is thus established.

### 3. APPLICATION

We now derive from the Bethe-Salpeter equation and the normalization condition equations for a class of bootstrap models. These equations express the coupling constants and masses of the particles (bound states) as functions of the coupling constants and masses. We restore the hitherto suppressed indices which characterize the constituent particles forming the bound state:

$$\phi_a \equiv \phi_a(P,q) \equiv \phi_a{}^{bc}(P_a,p_b,p_c).$$

Using any convenient momentum factor $U_{abc}$ which could appear at an $a \to b+c$ vertex to contract the (suppressed) spinor, vector, etc., indices of $\phi_a{}^{bc}$, we define the coupling constants $g_a{}^{bc}$ from the residue of $\phi_a{}^{bc}$ at

$$p_b^2 = m_b^2,\quad p_c^2 = m_c^2: \tag{10a}$$
$$\phi_a{}^{bc} = g_a{}^{bc}\phi_a{}'^{bc}(P_a,p_b,p_c),$$

with

$$[U_{abc}K\phi_a{}'^{bc}]_{p_b^2=m_b^2,\ p_c^2=m_c^2} = Y_{abc}, \tag{10b}$$

$Y_{abc}$ being an angular factor normalized in any convenient way. Employment of the variation

$$\delta\phi = \delta g\cdot\phi'$$

in our variational principle

$$\delta[\bar\phi(K-I)\phi] = 0$$

leads to the coupling constant equation

$$g_a{}^{bc} = \textstyle\sum_{ef} g_a{}^{ef}\bar\phi_a{}'^{bc}I_{bc,ef}{}^a\phi_a{}'^{ef}/\bar\phi_a{}'^{bc}K_{bc}{}^a\phi_a{}'^{bc}.$$

A generalized ladder approximation,

$$I_{bc,ef}{}^a = \textstyle\sum_i g_{bie}g_{cif}I'_{bc,ef}{}^{ai},$$

is introduced in which the vertices are assumed to be so calculated that we obtain exactly the same equations if particle $b$ or $c$ is considered as the bound state, and can identify

$$g_a{}^{bc} = g_{abc}.$$

This yields the relation

$$g_{abc} = \sum_{e,f,i} D_{abc}{}^{efi}g_{aef}g_{bie}g_{cif}. \tag{11}$$

Here we have defined

$$D_{abc}{}^{efi} = \tilde I_{bc,ef}{}^{ai}/\tilde K_{bc}{}^a$$

and

$$\tilde I = \bar\phi' I'\phi'$$

and

$$\tilde K = \bar\phi' K\phi';$$

also,

$$D_{abc}{}^{efi}Y_{abc} = [U_{abc}I'_{bc,ef}{}^{ai}\phi_a{}'^{ef}]_{p_b^2=m_b^2,\ p_c^2=m_c^2}.$$

The dependence of $D_{abc}{}^{efi}$ on $a$ comes only from the dependence on $P_a$ (which in the preceding section was called $P$).

The mass equation follows directly from the normalization condition. In the c.m. system, we have simply

$$2m_a = i(\partial/\partial P_{a0})[\bar\phi_a(K-I)\phi_a]$$
$$= i\sum_{bc}(g_{abc})^2(\partial/\partial P_{a0})\tilde K_{bc}{}^a - i\sum_{bc,e,f,i} g_{abc}g_{bie} \times g_{cif}g_{aef}(\partial/\partial P_{a0})\tilde I_{bc,ef}{}^{ai}, \tag{12}$$

employing the same approximations as above. Equations (11) and (12) form a convenient starting point for many bootstrap models, especially in discussions of the bootstrap origin of symmetries.[3]

[3] R. E. Cutkosky, Bull. Am. Phys. Soc. 8, 591 (1963); Carnegie Institute of Technology Report NYO-10565, 1963 (unpublished).