Renormalizable Electrodynamics of Scalar and Vector Mesons. II

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(Received 19 March 1964)

The "gauge technique" for solving field theories introduced in an earlier paper is applied to scalar and vector electrodynamics. It is shown that for scalar electrodynamics there is no $\lambda\varphi^{*2}\varphi^2$ infinity in the theory, while with conventional subtractions vector electrodynamics is completely finite. The essential ideas of the guage technique are explained in Part II, Sec. 1 of the paper, and a preliminary set of rules for finite computation in vector electrodynamics is set out in Part III, Sec. 2.

INTRODUCTION

THE phenomenal success of renormalized perturbation theory for electron-photon interactions has on balance probably been a disaster for the development of quantum field theory as such. The disaster lay in the somewhat fortuitous circumstance that the magnitudes which the theory was powerless to compute happened to be "unobservable" quantities like self-mass and self-charge. Even when for renormalized meson interactions it became apparent that such magnitudes included measurable quantities like the $\pi^+-\pi^0$ (self-) mass differences, theoretical interest unfortunately did not shift back to the central problem of trying to discover if there might exist nonperturbative solutions of field-theory integral equations for which all measurable S-matrix elements can be computed in finite terms.

Physically, of course, the most attractive possibility would be if at least for some theories—and this may include those that are currently considered nonrenormalizable—finite solutions did exist but only for special values[1] of the constants of field theory. In an earlier paper[2] it was suggested that this last possibility might be the one realized for electrodynamics of spin-one charged mesons. The suggestion was based on the use of a new (nonperturbative) approximation procedure which made consistent use of Ward's identity. One purpose of the present paper is to exploit this approximation technique not only for the electrodynamics of spin-one, but also of spin-zero mesons. Even though in the latter case one is dealing with a theory with only a few infinities, it is instructive to see how at least some of these disappear at the crudest attempt to improve the conventional perturbation approximation. For the spin-one case, this technique makes the theory finite, though the conjecture about the constants of the theory is not realized in the strong form it was stated in in Ref. 2.

The paper is divided into four parts: Part I gives the main theoretical ideas and the general approximational scheme; Part II sets out the equations for the general-two- and three-particle Green's functions, in the two particle unitarity approximation, in scalar and vector-electrodynamics; in Part III these equations are solved consistently with the requirements of analyticity and unitarity of the theory; and in Part IV we use these solutions to state rules for computations of general S-matrix elements. These rules take the place of the Feynman rules and even in the preliminary version of this paper provide finite integrals (including those for some of the renormalization constants).

The paper unfortunately is long. The impatient reader may perhaps find it easier to go straight on to Part II Sec. 1 which briefly sets out the main ideas of the calculational technique before returning to Part I.

Part I

This part is an amplification of the ideas of paper I. In Sec. 1A we derive a boundary condition for the high-energy behavior of (a product of) the basic two- and three-particle Green's functions. If this boundary condition is satisfied, integrals involved in *all other Green's functions* would exhibit an approximational stability for high energy behavior and will be essentially finite. In Sec. 1B we turn to the basic Green's function and show that for gauge theories an approximation technique exists which makes the vertex-function equation essentially redundant, so that the general boundary condition is equivalent in such theories to a high-energy limitation

* Work supported in part by the Air Force Office of Scientific Research OAR through the European Office, Aerospace Research, U. S. Air Force.
† Work supported in part by the U. S. Atomic Energy Commission, the National Science Foundation, and the Research Committee of the Graduate School of Wisconsin from special funds voted by the State Legislature.

[1] In this context the following remarks of A. Einstein [Phys. Rev. **89**, 329 (1953)] are possibly relevant: "If there exist elementary solutions of the equations which depend upon a continuous parameter, then the field equations must prevent the coexistence within one system of such elementary solutions pertaining to arbitrary values of their parameters · · · . If a theory does not possess these features then the theory is inadmissible." We are indebted to Dr. J. Bronowski for pointing out this reference.

[2] A. Salam, Phys. Rev. **130**, 1287 (1963). This paper will be referred to as I.

on the behavior of just the two-particle propagators. This allows us finally in Sec. 1C to connect the finiteness (of all integrals) in a gauge theory with the information on high-energy behavior contained in the well-known spectral representation of the propagator.

1. THE INTEGRAL EQUATIONS OF FIELD THEORY

A field theory is defined by a set of integral relations among its Green's functions. As a rule all such relations involve an infinite number of terms; thus, of their nature the integral equations need some type of approximation procedure for their solution. Depending on the approximation procedure to be followed, one can write down a wide variety of equivalent sets for one and the same theory. One such set is due to Dyson[3] and Schwinger[4]; another is the unitarity set defined in Sec. 1C; still another set is due to Symanzik.[5]

A. Dyson-Schwinger Set; (the fundamental criterion for the stability of an approximation scheme and high-energy boundary conditions on Γ*, D, and S)*

For a typical 3-field (e.g., electron-photon) interaction the well-known Dyson equations are

$$S^{-1}=Z_2S_0^{-1}+Z_1e^2\int \Gamma S\Gamma_0 D \tag{I.1}$$

$$D^{-1}=Z_3D_0^{-1}+Z_1e^2\int \Gamma S\Gamma_0 S \tag{I.2}$$

$$\Gamma=Z_1\Gamma_0+e^2\int \Gamma S\Gamma S\Gamma D+e^4\cdots \tag{I.3}$$

$$M=M[\Gamma,D,S]. \tag{I.4}$$

S and D are the (renormalized) electron and photon Green's functions; Γ is the (renormalized) vertex function and e is the physical charge.[6] The three functions Γ, D, and S we shall refer to as the *basic* Green's functions. M represents any other Green's function; the important remark of the Dyson formalism is that all M's are functionals of the three *basic* Green's functions Γ, S, and D.[7]

The Feynman solution of field theory is recovered as a power series iteration of (I.1)–(I.3), the iteration starting with

$$S^{-1}=\gamma\cdot p-m\,,\quad D^{-1}=p^2-\mu^2\,,\quad \text{and}\quad Z_1=Z_2=Z_3=1\,.$$

This assumption of course immediately precludes the possibility that the Z's might be zero. {It is worth remarking that these zeroth approximations (e.g., $S^{-1}=\gamma\cdot p-m$) do *not* coincide with the inhomogeneous terms of the corresponding integral equations [e.g., $Z_2(\gamma\cdot p-m_0)$].}

In the sequel, we wish to set up a different approximation procedure for solving a field theory. Basically, the idea is to find nonperturbative solutions of the basic set (I.1)–(I.3) and then to substitute for D, S, and Γ in (I.4). To estimate the high-energy behavior of the integrals involved in M, one knows from Dyson-Schwinger formalism that $M=\sum M^{(N)}$, where $M^{(N)}$ has the following structure:

$$M^{(N)}=\int (S\Gamma D^{1/2})^N S^{-E_e/2}D^{-E_\gamma/2}\times(d^4k)^{[(N-E_e-E_\gamma)/2+1]}. \tag{I.5}$$

Here E_γ and E_e are the number of external photon and electron lines and N is the number of irreducible vertices. Now it is clear from (I.5) that the high-energy behavior of the integrand (and therefore of the integral as a function of external momenta) is independent of the order of iteration[8] N if and only if[9]

$$S\Gamma D^{1/2}=O(1/k^2) \tag{I.6}$$

for large k. Equation (I.6) is the *fundamental criterion for the stability* of any approximation procedure for computing M which bases itself on the Dyson formalism. It serves as the boundary condition to be satisfied by the (product of the) three basic Green's functions.

This stability criterion is satisfied by the basic Green's functions in all "renormalizable" theories. A straight-

[3] F. J. Dyson, Phys. Rev. **75**, 1736 (1949).

[4] The Schwinger formulation of Green's function theory [J. Schwinger, Proc. Natl. Acad. Sci. (U.S.) **37**, 452 (1951)] is parallel to the Dyson (Ref. 3) formulation above; it however offers some advantages for gauge theories (see Part III, Sec. 3).

[5] K. Symanzik, J. Math. Phys. **1**, 249 (1960); an extensive study of this set has recently been completed by J. G. Taylor, Nuovo Cimento (to be published).

[6] The constants Z_1, Z_2, Z_3 occur in the Lagrangian and are themselves defined as boundary values of S, D, and Γ. For electrodynamics $S_0^{-1}=\gamma\cdot p-m_0$, $D_0^{-1}=p^2-\mu_0^2$. $\Gamma_0=\gamma$ and m_0^2 and μ_0^2 (which always appear multiplied by Z_2 and Z_3, respectively) are the unrenormalized mass constants. Graphically, the Dyson-Schwinger set corresponds to the drawing of Dyson's irreducible diagrams for any Green's function and then making vertex and self-energy corrections to these. For all 3-field interactions the structure of the Eqs. (I.1)–(I.4) is the same, the distinctive differences of one Lagrangian from another appear only in the specification of the inhomogeneous terms S_0, D_0, and Γ_0. Also if 4-field interactions occur in the Lagrangian, this only increases the number of what we have called the *basic* Green's functions.

[7] The lack of symmetry between Γ and Γ_0 in Eqs. (I.1) and (I.2) has always been an embarrassment (the well-known problem of "*b* divergences"). One recent suggestion to deal with this problem is due to Symanzik (Ref. 5). Other alternatives are due to A. Salam [Phys. Rev. **82**, 217 (1951)] and J. C. Ward [Phys. Rev. **84**, 897 (1951)]. All these proposed solutions convert the single term on the right-hand side of (I.1) or (I.2) into a series in e^2 and show that the integral behaves effectively like $\int\Gamma S\Gamma D$. To see this at its simplest, eliminate $Z_1\Gamma_0$ in (I.1) and (I.2), by using (I.3), i.e., set $Z_1\Gamma_0=-e^2\int\Gamma S\Gamma S\Gamma D+\cdots$. The treatment of this problem in the text (see the unitarity set Sec. 1C automatically restores the "*a*" and "*b*" vertex symmetry).

[8] If this is the case the only integrals of type $M^{(N)}$ which may still be nonfinite and divergent belong to the class satisfying $[2-s/2]E_e+E_\gamma<4$ where s is given by $S_0=0[1/k^s]$. The relation of these infinities to the subtraction constants in renormalization theory is well known and will be discussed as the occasion arises.

[9] More precisely, it is stated by the stability criterion that $S_{\mu\mu'}^{1/2}(p)\Gamma_{\mu'\nu'b'}(p,p',t)S_{\nu'\nu}^{1/2}(t)D_{ba}^{1/2}(t)$ should decrease as fast or faster than $1/(\text{length})^2$ along any direction in the (p,p') plane.

forward iteration of (I.1)–(I.3) shows that $S\approx S_0=O(1/k)$, $D\approx D_0=O(1/k^2)$, $\Gamma\approx\Gamma_0=O(1)$ for spinor electrodynamics and $S\approx S_0=O(1/k^2)$, $\Gamma\approx\Gamma_0=O(k)$ for electrodynamics of spin-zero particles.[10] Our contention in paper I was that the boundary condition (I.6) can be satisfied by the solutions of Eqs. (I.1)–(I.3) even for some nonrenormalizable theories, provided these equations are solved nonperturbatively. The present paper is concerned with a detailed verification of this statement for electrodynamics of spin-one particles. We have chosen electrodynamics[4] as the prime example because a proper use of Ward's identity makes the problem of finding nonperturbative solutions much easier, and because for such theories, condition (I.6) can be reduced (as will be shown in Sec. 1B) to a boundary condition on the charged particle propagator alone. Just to illustrate how the new technique in any case improves the convergence properties of the integrals in field theory, we consider also the conventionally renormalizable theory of electrodynamics of spin-zero particles.

B. The Gauge Approximation and a Reformulation of the Stability Criterion

In a gauge invariant theory as a consequence of current conservation, the vertex function Γ and the propagator S satisfy the Ward-Takahashi identity

$$t_a\Gamma_a(p,p')=S^{-1}(p)-S^{-1}(p');\quad p=p'+t. \tag{I.7}$$

This identity makes Eq. (I.1) of the Dyson-Schwinger set redundant. In fact one may define S from the relation

$$S^{-1}(p)=t_a\Gamma_a(p,p')\,|_{\gamma\cdot p'=m}. \tag{I.8}$$

Also, $Z_1=Z_2$ $(=Z)$. Conversely, of course, the identity states that if S is known a part of Γ is determined and is no longer arbitrary. We shall call this part $\Gamma^A[S]$; thus

$$\Gamma=\Gamma^A[S]+\Gamma^B,$$

where

$$t_a\Gamma_a{}^A[S]=S^{-1}-S'^{-1},\quad t_a\Gamma_a{}^B\equiv 0. \tag{I.9}$$

Now these identities do not in any way uniquely define the split of Γ, but whatever the precise definition of $\Gamma^A[S]$, since ZS_0^{-1} is part of S [see Eq. (I.1)], $Z\Gamma_{0a}$, the inhomogeneous part of the vertex function equation which is (usually) defined as $Z(S_0^{-1}-S_0'^{-1})(p+p')_a/(p^2-p'^2)$, must form part of $\Gamma^A[S]$. Therefore, quite generally, $Z\Gamma_0$ can be eliminated in Eq. (I.3) in terms of Γ^A in the following manner: Write Eq. (I.3) in the form

$$\Gamma_a=Z\Gamma_{0a}+K_a[\Gamma,S]. \tag{I.10}$$

Define

$$\Gamma_a{}^A=Z\Gamma_{0a}+F_a, \tag{I.11}$$

where F_a is a *linear functional* of $(S^{-1}-ZS_0^{-1})$ with the property

$$t_aF_a=(S^{-1}-ZS_0^{-1})-(S'^{-1}-ZS_0'^{-1})=t_aK_a[\Gamma,S]. \tag{I.12}$$

Using (I.11),

$$\Gamma_a=\Gamma_a{}^A+K_a-F_a=\Gamma_a{}^A+X_{ab}K_b[\Gamma,S]. \tag{I.13}$$

Equation (I.13) differs[11] from Eq. (I.3) in having as its "inhomogeneous" term Γ^A in place of $Z\Gamma_{0a}$. Equation (I.13) together with Eq. (I.8), viz.,

$$S^{-1}=Z(\gamma\cdot p-m_0)+t_aK_a[\Gamma,S]\,|_{\gamma\cdot p'=m} \tag{I.14}$$

now replace Eqs. (I.1) and (I.3) of the Dyson-Schwinger set.[12]

To solve (I.13) and (I.14) we use the simple iteration scheme suggested in paper I. This scheme is based on taking the inhomogeneous term $\Gamma^A[S]$ of Eq. (I.13) as the first approximation $\Gamma^{(0)}$ to Γ. This will be called the "*gauge approximation*" in subsequent work.

The "gauge approximation" has the merit of decoupling (I.13) and (I.14). Explicitly, define

$$[S^{(n)}]^{-1}=Z^{(n)}(\gamma\cdot p-m)+t_aK_a[\Gamma^{(n)},S^{(n)}]\,|_{\gamma\cdot p'=m} \tag{I.15}$$

and[13]

$$\Gamma_a{}^{(n+1)}=\Gamma_a{}^A[S^{(n)}]+X_{ab}K_b[\Gamma^{(n)},S^{(n)}], \tag{I.16}$$

where

$$\Gamma^{(0)}=\Gamma^A[S^{(0)}] \tag{I.17}$$

and $S^{(0)}$ is a solution of the equation

$$[S^{(0)}]^{-1}=Z^{(0)}(\gamma\cdot p-m)+t_aK_a[\Gamma^{(0)},S^{(0)}]\,|_{\gamma\cdot p'=m}. \tag{I.18}$$

Clearly S, $\Gamma=\lim_{n\to\infty}S^{(n)}$, $\Gamma^{(n)}$, provided the sequences $S^{(n)}$ and $\Gamma^{(n)}$ converge to a limit.

To see the decoupling of (I.13) and (I.14), it is sufficient to remark that $\Gamma^A[S]$ is a functional of S alone, *so that Eq. (I.18) is an equation for just one unknown* $S^{(0)}$. Once $S^{(0)}$, and therefore $Z^{(0)}$ and $\Gamma^{(0)}$, have been determined, one simply writes down $\Gamma^{(1)}$ from (I.16). At each

[10] It is important to emphasize that the stability criterion guarantees a uniformity of high-energy behavior in each order of iteration, only to the extent of a power count of external momenta. The extra powers of logarithms which arise in each order $M^{(N)}$ can lead to a different behavior for the sum of the series $M^{(N)}$; our problem in this paper is *not* the determination of the high energy behavior of this sum $\Sigma M^{(N)}$, our concern is with each term $M^{(N)}$.

[11] To take a concrete example, for scalar electrodynamics one may choose

$$F_a=\frac{(p+p')_a}{p^2-p'^2}[S^{-1}(p)-S^{-1}(p')]-Z(p+p').$$

Clearly $X_{ab}=g_{ab}-(p+p')_at_b/(p^2-p'^2)$; note $t_aX_{ab}\equiv 0$.

[12] If K_a has the form $\int\cdots S\Gamma_aS'\cdots$, the corresponding expression for t_aK_a would contain $\int\cdots(S'-S)\cdots$ in the equation for S^{-1}. In other words, in writing down the expression for t_aK_a in Eq. (I.14) one makes consistent use of the Ward-Takahashi identity.

[13] One may perhaps stress once again the close analogy of the above approximation procedure to that followed by perturbation theory. Perturbation theory starts with the first approximation $\Gamma^{(0)}=\Gamma_0$ where Γ_0 by definition equals $\Gamma^A[S_0]$. Since $t_aX_{ab}K_b=0$ and $t_a\Gamma_a{}^A[S^{(n)}]=S^{(n)-1}-S'^{(n)-1}$, Eq. (I.16) satisfies the Ward-Takahashi identity to each order in e^2 of the iteration.

subsequent iteration stage there is again just one equation [Eq. (I.15)] to be solved for $S^{(n)}$, $\Gamma^{(n+1)}$ being determined by substitutions in Eq. (I.16). This makes the method fairly practical.[14]

In terms of Γ^A *and the gauge approximation*, we can now restate the stability criterion. It is easy to see from the structure of the kernel[15] K in Eq. (I.13) that if $\Gamma^A[S]$ satisfies

$$S\Gamma^A[S]D^{1/2}=O(1/k^2), \tag{I.19}$$

then

$$\Gamma^{(n+1)}=O(\Gamma^A[S^{(n)}]). \tag{I.20}$$

To see this, we remark that K, in general, has the form [see (I.5) with $E_\gamma=1$, $E_e=2$] $\sum_r \int \Gamma(S\Gamma D^{1/2})^r(d^4k)^{r/2}$ so that any iteration solution of (I.16) which starts with $\Gamma\approx\Gamma^A$ and with $(S\Gamma^A D^{1/2})\approx 1/k^2$ will always reproduce a Γ satisfying $\Gamma\approx O(\Gamma^A)$.

Equation (I.20) therefore not only gives us a justification for the iteration scheme set up in (I.16), it allows us also to replace the stability criterion (I.6) by the much simpler relation (I.19). Assuming for the moment $D\approx D_0=O(1/k^2)$, (I.19) reduces still further to read

$$S\Gamma^A[S]=O(1/k). \tag{I.21}$$

In this final form, the stability criterion is providing a high-energy boundary condition for just one Green's function, i.e., the meson propagator S alone. The crux of the whole problem of solving in finite terms an entire (gauge-invariant) field theory in our new approximation scheme is therefore reduced to the following: Is there a choice of $\Gamma^A[S]$ for which one single equation $S^{-1}=ZS_0^{-1}+t_aK_a[\Gamma^A,S]$ possesses a solution satisfying (I.21)?

For gauge theories one may even anticipate the answer. Since Ward's identity "roughly" states that $\Gamma^A\approx S^{-1}/k$ it would seem that (I.21) is always satisfied and electrodynamics of charged particles (of any spin whatever) is intrinsically divergence free. The mistake in the past has been taking as the starting approximation $\Gamma^A=\Gamma_0$ and $\Delta=\Delta_0$ which do not satisfy (I.21). All we shall do in this paper is to try to choose a different $\Gamma^A[S]$ for which (I.21) is automatically true.

The discussion above may appear highly complicated As we shall see in Part II, Sec. 1, in practice, the procedures are rather simple. Summarizing the contents of this section, we have used Ward's identity to provide a first approximation $\Gamma^A[S]$ to the full vertex function Γ which depends only on the meson propagator. In specifying Γ^A there is a degree of arbitrariness; we exploit this to choose $\Gamma^A[S]$ in such a manner that (if at all possible) $S\Gamma^A[S]=O(1/k)$, *for all directions in* (p,p') *plane* where S is the solution of $S^{-1}(p)=ZS_0^{-1}+t_aK_a\times[\Gamma^A[S],S]$. If such S and $\Gamma^A[S]$ do exist, the structure of the equation for the full-vertex function Γ already guarantees that for high energies the full Γ and Γ^A behave similarly. In this sense Γ^A is a good approximation to Γ. The entire question of the finiteness of a theory is thus made to depend, in this approximation scheme, on the possibility of solving just one equation —the equation for the two-particle Green's function— with the boundary condition stated above.

C. The Unitarity Set and the Equations for S and D

The end result of the discussion above is to make the high-energy behavior of the charged particle propagator S—and possibly also of the photon propagator D in case Γ^A is chosen to be a functional of both[16] S and D—*the pivotal questions for a discussion of the finiteness of a theory*. Now the Dyson-Schwinger equations for S and D even for the simplest choice of Γ^A, i.e.,

$$\Gamma^A=\frac{(p+p')_a}{p^2-p'^2}(S^{-1}-S'^{-1}), \tag{I.22}$$

and even for the simplest approximation to the kernel K still present a horribly nonlinear aspect (see Part III, Sec. 3 where the equation for S is written out in full). To expect that one can solve these equations using the mathematical theory of integral equations as well as guarantee that at the same time one can preserve the physical properties of the theory—like unitarity and causality—is to ask for miracles.

That these physical properties are of crucial importance for a correct estimation of high-energy behavior cannot be stressed too strongly. It is perfectly possible to find approximate solutions to the Dyson-Schwinger equations for S and D which show highly convergent behavior[17] but which were obtained for example by

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[14] A still more practical iteration procedure can be set up as follows: Define

$$S^{(n+1)-1}=Z^{(n)}S_0^{-1}+t_aK_a[\Gamma^{(n)},S^{(n)}],$$
$$\Gamma_a^{(n+1)}=\Gamma_a^A[S^{(n)}]+X_{ab}K_b[\Gamma^{(n)},S^{(n)}],$$

where $\Gamma^{(0)}=\Gamma^A[S^{(0)}]$, $S^{(0)}\equiv S^{(1)}$. To start off the iteration, $S^{(0)}$ is the solution of the equation

$$S^{(0)}=Z^{(0)}S_0^{-1}+t_aK_a[\Gamma^{(0)},S^{(0)}]|_{\gamma\cdot p'=m}.$$

This iteration has the additional merit that the above equation is the only one which needs to be solved. All higher orders are given by substitutions in the orders below.

[15] The integrals which arise when (I.14) is iterated with Γ^A as the zeroth approximation have the general form given by (I.5) with $E_e=2$, $E_\gamma=1$, so that $\Gamma=0[\Gamma^A]$. The situation here is similar to the case of renormalizable theories where a straightforward iteration of (I.1)–(I.3) shows that if $S_0\Gamma_0D_0^{1/2}=0[1/k^2]$, then $S\Gamma D^{1/2}$ is also $0[1/k^2]$ and Γ is $0[\Gamma_0]$.

[16] Equation (I.2) for the photon propagator can be included in the iteration scheme thus: $D^{(n)-1}=Z_3^{(n)}D_0^{-1}+(\int\Gamma^{(n)},D^{(n)},S^{(n)})$ with $D=\lim_{n\to\infty}D^{(n)}$. This means that Γ^A should be chosen (see Part III, Sec. 1B) to depend explicitly not only on S, but also on D. At each iteration stage one then solves two equations, one for S and one for D. In practice, using the procedure of footnote 14, it will always suffice to solve altogether two equations and no more.

[17] See for example Ning Hu, Phys. Rev. 80, 1109 (1950), where the expression for S obtained as an approximate solution to Dyson's Eq. (I.1) contradicts Lehmann's result. The only hope of renormalizing an unrenormalizable theory (with positive definite metric) is through an improved high-energy behavior of Γ, and not S.

sacrificing the positive definiteness which comes by considering unitarity properly.

Now field theory has unfortunately not progressed to the extent that one can write down for any Green's function a (spectral) expression exhibiting the consequences of causality, unitarity and crossing. But just for the case of the two-point functions a complete spectral representation does exist and we would like to exploit this representation to estimate the crucial high-energy behavior of S and D in conjunction with the *gauge-technique* described above.

The following is a summary of the known results. By considering the momentum transform of $\langle 0|\psi\bar{\psi}|0\rangle$ (for simplicity of writing the formulas assume a scalar electron)

$$\mathrm{Im}S=-S\left[\int \Gamma\delta_+\delta_+\Gamma^* + \int M_4\delta_+\delta_+\delta_+M_4^*+\cdots\right]S^*, \quad \text{(I.23)}$$

where (1) $\delta_\pm$ stands for mass shell wave functions like $\theta(\pm p_0)\delta(p^2-m^2)$, (2) M_4, M_5, etc. are the contributions from $\langle 0|\psi|3\rangle$, $\langle 0|\psi|4\rangle$, $\cdots$ and (3) $\mathrm{Re}S(p^2)$ is recovered from $\mathrm{Im}S(p^2)$ by the standard Lehmann-Kallen dispersion relation. It is this dispersion relation which sets a powerful (minimal) limit to the high-energy behavior of S; for a theory with positive definite metric it states that S must be at least as divergent at infinity as S_0.

The manner in which we propose to use (I.23) is as follows: The right-hand side of (I.23) contains Γ on the mass shell for two particles as well as the four and higher point Green's functions M_4, M_5, $\cdots$. For Γ (on the mass shell of two of the external momenta), one can likewise write the following unitarity relation:

$$\mathrm{Im}\,F(s)=\mathrm{Re}\left[\int F\delta_+\delta_+M_4^* + \int SM_4\delta_+\delta_+M_5^*+\cdots\right]\Bigg|_{p^2=q^2=m^2}, \quad \text{(I.24)}$$

$$0=\mathrm{Im}\left[\int F\delta_+\delta_+M_4^*+\cdots\right]\Bigg|_{p^2=q^2=m^2}.$$

Here

$$F(s)=F[(p+q)^2] =\langle 0|\,j(x)\,|p,q\rangle_{p^2=q^2=m^2}=S(s)\Gamma(s) \quad \text{(I.25)}$$

and $\mathrm{Re}\,F(s)$ is related to $\mathrm{Im}\,F(s)$ by a dispersion relation. Using (I.24), (I.23) can be reduced to read

$$S=S[M_4,M_5,\cdots]. \quad \text{(I.26)}$$

At this stage, not having dispersion relations like (I.23), (I.24) for M_4, M_5, $\cdots$[18] we must use the Dyson-Schwinger formalism to express the M's as functionals of S, D, and Γ. Our final procedure will then be the following:

(1) Use the Ward-Takahashi identity to define $\Gamma^A[S,D]$.

(2) Write (I.26) with M_4, M_5, $\cdots$ expressed as known functionals of S, D and Γ. Approximate $\Gamma=\Gamma^A[S]$ in the integral relation (I.26) for S.

(3) If a solution to (I.26) exists satisfying the self-consistent boundary condition $D^{1/2}S\Gamma^A[S]=O(1/k^2)$, the theory is finite and renormalizable.

(4) The full Γ (on two-particle mass shell), computed from (I.24) by a straightforward iteration, will behave similarly to Γ^A at high energies. To see the force of this last remark, note that the arguments of Sec. I. 1B regarding the behavior of the integrals concerned apply equally to (I.24). This is because the dependence on external momenta (p) of unitarity integrals like $\int F(p,k)\cdots\delta_+(k^2-m^2)d^4k$ is (in general) similar to that of integrals like $\int F(p,k)\cdots d^4k/(k^2-m^2+i\epsilon)$.

Summarizing: Whereas the stability criterion derived by considering Green's functions other than S, D, and Γ gives a general boundary condition on a product of S, D, and Γ, the spectral representations of S and D give additional information about the minimal high-energy behavior of S and D themselves. This information combined with the stability criterion sharpens the requirements on the initial choice of $\Gamma^A[S,D]$.

In Part II we shall write down in detail Eqs. (I.23), (I.24), and (I.26) for scalar and vector electro-

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[18] To appreciate the problems involved in writing a general unitary set, consider the Wightman (W) and the time-ordered (τ) products:

$$W_n=\langle 0|\,\varphi(X_1)\cdots\varphi(X_n)\,|0\rangle, \quad \tau_n=\langle 0|\,T(\varphi(X_1)\cdots\varphi(X_n))\,|0\rangle,$$

with the definition

$$\tau_n=\sum_{\text{perm}}\theta(X_{i_1}-X_{i_2})\theta(X_{i_2}-X_{i_3})\cdots W(X_{i_1},X_{i_2},\cdots).$$

Using the completeness relation

$$W_n=\langle 0|\,\varphi(X_1)\,|\mathrm{in}\rangle\langle \mathrm{in}|\,\varphi(X_2)\,|\mathrm{out}\rangle,$$

and the reduction formula

$$\langle \mathrm{out}|\,\varphi(x)\,|\mathrm{in}\rangle=K_1K_2\cdots\langle 0|\,T(\varphi(x)\varphi(x_1)\varphi(x_2)\cdots)\,|0\rangle,$$

where $K=(\partial^2+m^2)$, we may write

$$\tau_n=\Sigma\theta(X_{i_1}-X_{i_2})\cdots\prod_{n\text{ blocks}}(K)^r\tau_{r+1}(X_{i_1},\eta)\Delta^+ \times(\eta-\xi)(K)^s\tau_{s+1}(X_{i_2},\xi).$$

This last set will be called the *Unitarity Set*. One could, in principle, completely replace the Dyson-Schwinger set by this unitarity set, but to make any use of it one must learn to approximate to it, consistently with the general principles of field theory. The simplest suggestion (analogous to perturbation theory) of approximating to τ_n, is to retain on the right only two (or three, or four $\cdots$) particle intermediate states (i.e. keep only τ_2, τ_3, or τ_2, τ_3, and τ_4, etc. on the right). This idea however comes to grief on account of the presence of the $\theta(x)$ factors. In general it is not clear that even the Lorentz invariance of the T products would be preserved with this type of unitarity approximation and it is at this stage that use of something analogous to local commutativity becomes necessary.

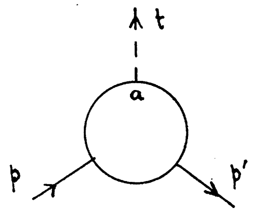

FIG. 1. The vertex function.

dynamics,[19] in a two-particle unitarity approximation. In Part III of the paper these equations will be solved and the general statements (3) and (4) above will be explicitly verified. In Part IV we shall indicate how one might set up a practical calculational scheme, consider its gauge covariance and indicate how the scheme may, in principle, be improved in successive stages to include higher particle states and also to go beyond the approximation of $\Gamma=\Gamma^A$.

Part II

This part forms the calculational heart of the paper. In Sec. 1, we rapidly illustrate the main ideas of the new approximation technique which incorporates Ward's identity. In Sec. 2 the general spectral representation of Δ and D, and the form factor decomposition of Γ, are written down and the two-particle contributions to $\mathrm{Im}\Delta$ and $\mathrm{Im}D$ are evaluated. (Δ is the charged meson propagator). In Sec. 3 the same is done for $\mathrm{Im}\Gamma$ both for scalar and vector electrodynamics. We also consider in Sec. 3 representations for C parts (two-meson, two-photon graphs).

1. THE APPROXIMATION SCHEME

In this section we illustrate the approximation scheme in its *barest essentials* by computing Δ, the charged meson propagator for spin-zero and spin-one electrodynamics. We wish to show in particular how the use of Ward's identity improves the convergence of the integrals in the theory and start by solving for Δ and Γ by using the following two exact equations for the basic Green's functions (see Fig. 1):

$$\mathrm{Im}\Delta(p)=-\sum_n|\langle 0|\varphi(0)|n\rangle|^2 \tag{II.1}$$

$$t_a\Gamma_a(p,p')=\Delta^{-1}(p)-\Delta^{-1}(p');\; p'+t=p. \tag{II.2}$$

(Ward-Takahashi identity).

Equation (II.2) determines the Δ-dependent part Γ^A of Γ. In this lowest approximation other Green's functions are computed by drawing "irreducible diagrams" for these and then by inserting Δ and Γ^A for the meson lines and the vertices. The result is a theory which is more convergent than conventional perturbation theory.

[19] It may be noted that for any gauge except the radiation (i.e., Coulomb) gauge, electrodynamics uses an indefinite metric. Thus though Lehmann's theorem applies directly to the gauge-independent part of D, its use for the charged particle propagator needs care. See Part III for a fuller discussion.

A. Scalar Electrodynamics

If m is the meson mass, one can introduce a spectral representation for $\Delta^{-1}(p)$ in the form

$$\Delta^{-1}(p)=(p^2-m^2)Z(p^2) \tag{II.3}$$

$$Z(p^2)=1-\int_{m^2}^{\infty}\frac{(p^2-m^2)G(x)}{p^2-x+i\epsilon}dx. \tag{II.4}$$

Define

$$Z=\lim_{p^2\to\infty}Z(p^2)=1-\int G(x)dx. \tag{II.5}$$

Provided that all integrals converge,

$$Z(p^2)=Z-\int\frac{(x-m^2)G(x)dx}{p^2-x+i\epsilon}. \tag{II.6}$$

Since

$$\Delta^{-1}(p)-\Delta^{-1}(p')=(p^2-p'^2)\left[Z+\int\frac{(x-m^2)^2G(x)dx}{(x-p^2)(x-p'^2)}\right], \tag{II.7}$$

Ward's identity gives the exact relation

$$t_a[\Gamma_a-\Gamma_a{}^A]\equiv 0, \tag{II.8}$$

where

$$\Gamma_a{}^A=(p+p')_a\left[Z+\int\frac{(x-m^2)^2G(x)dx}{(x-p^2)(x-p'^2)}\right]. \tag{II.9}$$

Note that

$$\Gamma_a{}^A\delta_+(p'^2-m^2)=(p+p')_aZ(p^2)\delta_+(p'^2-m^2). \tag{II.10}$$

Quite generally Γ has the form[20]:

$$\Gamma_a=\Gamma_a{}^A-d_{ab}(t)(p+p')_bB(p^2,p'^2,t^2). \tag{II.11}$$

From PT invariance, $B(p^2,p'^2,t^2)$ is symmetric in p^2 and p'^2 but is otherwise arbitrary.

Now assuming two-particle unitarity, one can write the following equation for $\mathrm{Im}\Delta$:

$$\mathrm{Im}\Delta=-\left[\delta_+(p^2-m^2)+\Delta\left(\int\Gamma\delta_+\delta_+\Gamma^*\right)\Delta^*\right]$$

or more precisely

$$\frac{1}{\pi}\mathrm{Im}\Delta^{-1}(p)=\frac{e^2}{(2\pi)^3}\int\Gamma_a(p,p')\delta_+(p'^2-m^2)\times[d_{ab}(t)-ae_{ab}(t)]\delta_+(t^2)\Gamma_b{}^*(p,p')d^4p'. \tag{II.12}$$

Here $[d_{ab}(t)-ae_{ab}(t)]\delta_+(t^2)$ is the absorptive part of the free photon propagator in an arbitrary gauge specified

[20] The transverse projection operator $d_{\mu\nu}(t)=-g_{\mu\nu}+t_\mu t_\nu/t^2$ and the longitudinal projection $e_{\mu\nu}(t)=t_\mu t_\nu/t^2$ were introduced in Paper I. Writing $\mathbf{d}$ and $\mathbf{e}$ for these, respectively, note $\mathbf{e}\cdot\mathbf{e}=\mathbf{e}$, $\mathbf{d}\cdot\mathbf{d}=-\mathbf{d}$, $\mathbf{e}-\mathbf{d}=1$, $\mathbf{e}\cdot\mathbf{d}=0$. Also if $\mathbf{\Delta}=\lambda_1\mathbf{d}+\lambda_2\mathbf{e}$, then $\mathbf{\Delta}^{-1}=\lambda_1{}^{-1}\mathbf{d}+\lambda_2{}^{-1}\mathbf{e}$.

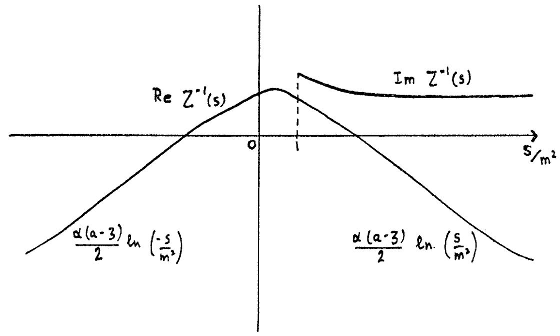

FIG. 2. Graph of $Z^{-1}(s)$ versus s.

by the constant a. Let us now make the first approximation of our theory and take $\Gamma=\Gamma^A$ [Eq. (II.9)] on the right side of (II.12). Using (II.3) and (II.6), and evaluating the integral, we get:

$$\frac{1}{\pi}\,\mathrm{Im}Z(p^2)=-\frac{\alpha(a-3)}{2}\frac{(p^2+m^2)}{p^2}\times|Z(p^2)|^2\theta(p^2-m^2) \quad \text{(II.13)}$$

with $\alpha=e^2/8\pi^2$. Dividing by $|Z(p^2)|^2$, we obtain equivalently

$$(1/\pi)\,\mathrm{Im}Z^{-1}(x)=\alpha(3-a)[(s+m^2)/2s]\times\theta(s-m^2);\; s=p^2. \quad \text{(II.14)}$$

From the well-known analyticity[21] properties of the propagator and using $Z^{-1}(m^2)=1$, we can write the dispersion integral[22] (Fig. 2)

$$Z^{-1}(s)=1+\frac{\alpha(3-a)}{2}(s-m^2)\int_{m^2}^{\infty}\frac{dx}{x(x-s+i\epsilon)} \quad \text{(II.15)}$$

$$=1+\tfrac{1}{2}\alpha(a-3)(1-m^2/s)\times[\ln|(s/m^2)-1|-i\pi\theta(s/m^2-1)]. \quad \text{(II.16)}$$

Clearly,

$$Z=\lim_{s\to\infty}Z(s)=\lim_{s\to\infty}[\ln(s/m^2)]^{-1}=0 \quad \text{(II.17)}$$

for all $\alpha>0$ and $a\neq 3$. Asymptotically,

$$\Delta(s)\approx\frac{\alpha(a-3)\ln(s/m^2)}{2s},$$
$$G(s)=\frac{\mathrm{Im}Z(s)}{\pi(s-m^2)}\approx\left[\frac{\alpha(a-3)s}{2}\ln^2\!\left(\frac{s}{m^2}\right)\right]^{-1}. \quad \text{(II.18)}$$

To obtain Γ^A, one may substitute (II.18) in (II.9). Now note that the factor $[\ln(s/m^2)]^{-2}$ in the expression for $G(s)$ acts as a built-in convergence factor for the theory. To see this, consider for example the lowest order irreducible diagram for meson-meson scattering as shown in Fig. 3.

Let the mesons p_1, p_2, p_3, p_4 lie on their mass shells. If meson self-energy insertions corresponding to Δ and vertex insertion Γ^A are made in this diagram, the contribution to the Dyson integrand from the (p_1,p_2) line in Fig. 3, equals $\Gamma(p_1,u)\Delta(u)\Gamma(u,p_2)$. Since [Eq. (II.10)] $\Gamma_a(p_1,u)=(p_1+u)Z(u^2)|_{p_1^2=m^2}$, the net contribution from the (p_1,p_2) line is $(p_1+u)_a(p_2+u)_bZ(u^2)/(u^2-m^2)$. Thus, Fig. 3 with all its insertions gives

$$I=\int d^4u\,X(u)[Z(u^2)]^2,$$

where $X(u)$ is the normal perturbation theory expression which behaves like $1/u^4$ for large u. Since $[Z(u)]^2=0[\ln^{-2}(u^2/m^2)]$, integral I is no longer divergent. Even with this simplest of modifications, apparently the need for a new subtraction constant to cancel the meson-meson scattering infinity[23] has disappeared and no $\varphi^{+2}\varphi^2$ type of counter term is necessary.

It is perhaps instructive to compare our expressions for Δ and Γ with those obtained from perturbation theory. The perturbation Δ may be obtained by substituting $\Gamma_a=(p+p')_a$ and $\Delta=(p^2-m^2)^{-1}$ on the right-hand side of (II.12). The resulting expression happens to coincide exactly with the one obtained in (II.16). This, however, is not the case for Γ; whereas, in our expression for Γ^A the coupling constant α occurs in the combinations $[1+\alpha\ln(s/m^2)]^{-1}$ and $[1+\alpha\ln(s'/m^2)]^{-1}$, standard perturbation theory expands these same com-

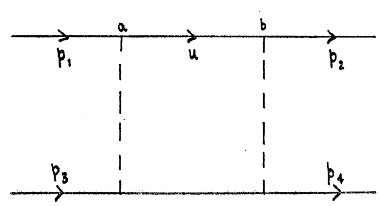

FIG. 3. Meson-meson scattering.

[21] In writing (II.15) we have ignored the part of the integrand which gives rise to the usual infrared divergence $\lim_{\mu^2\to 0}\ln(m^2/\mu^2)$. This has no bearing on the high-energy limits of $Z^{-1}(s)$.

[22] $Z^{-1}(s)$ may possess real zeroes (especially when $a<3$) for $S_0/m^2<1$ where

$$S_0/m^2=\tfrac{1}{2}\alpha(3-a)(S_0/m^2-1)\ln(1-S_0/m^2).$$

If α is small, the real zero occurs at $S_0/m^2\approx -e^{2/\alpha(3-a)}$ and the representation of $Z(s)$ must be modified to

$$Z(s)=1-\frac{2}{\alpha(3-a)}\cdot\frac{(s-m^2)}{(s+e^{2/\alpha(3-a)})}-(s-m^2)\int_{m^2}^{\infty}\frac{G(x)dx}{s-x+i\epsilon}$$

for small α and $a<3$. This CDD pole in the inverse propagator (and therefore in Γ^A) seems innocuous since the combination $\Gamma^A\Delta$ does not contain a pole. (A complete discussion of this point should include also the C part CDD poles. We wish to stress here only one thing: the appearance of a CDD pole in Γ^A has nothing to do with the approximation procedure above. *Whenever* Δ has a zero, this pole *will* turn up in Γ^A through the Ward-Takahashi identity). It can easily be checked that the pole is innocuous for the high-energy behavior of the theory but of course precludes expansion around $\alpha=0$.

[23] P. T. Matthews, Phil. Mag. **16**, 185 (1950).

binations in the form $1-\alpha\ln(s/m^2)$ and $1-\alpha\ln(s'/m^2)$ thereby visibly sacrificing the high-energy convergence properties of Γ in order to achieve consistency in the power series sense.[24]

B. Vector Electrodynamics

The procedure is completely analogous to the spin-zero case. Here we set down the barest essentials leaving to Part III, Sec. 2 the fuller details. With the notation of Ref. 2 the general propagator $\Delta_{\mu\nu}$ depends on two spectral functions (associated with the transverse and longitudinal projections $d_{\mu\nu}$ and $e_{\mu\nu}$). Write

$$\mathbf{\Delta}^{-1}=\mathbf{d}(p^2-m^2)Z_1(p^2)+\mathbf{e}m^2Z_2(p^2)\,, \qquad \text{(II.19)}$$

where

$$Z_1(p^2)=1-(p^2-m^2)\int\frac{G_1(x)dx}{p^2-x+i\epsilon}, \qquad \text{(II.20)}$$

$$Z_2(p^2)=1-m^2\int\frac{G_1(x)dx}{x}-p^2\int\frac{G_2(x)dx}{p^2-x+i\epsilon}. \qquad \text{(II.21)}$$

The wave function renormalization constant Z and the bare mass constant are given by the relations

$$Z=\lim_{p^2\to\infty}Z_1(p^2)=1-\int G_1(x)dx\,, \qquad \text{(II.22)}$$

$$\frac{m_0^2Z}{m^2}=\lim_{p^2\to\infty}Z_2(p^2)=1-m^2\int\frac{G_1(x)dx}{x}-\int G_2(x)dx. \qquad \text{(II.23)}$$

Defining

$$\mathcal{G}(x)=(x-m^2)^2G_1(x)/x+m^2G_2(x) \qquad \text{(II.24)}$$

and

$$\mathfrak{z}(p^2)=Z-\int\frac{\mathcal{G}(x)dx}{p^2-x+i\epsilon}, \qquad \text{(II.25)}$$

one may rewrite (II.19) as

$$\Delta_{\mu\nu}^{-1}(p)=-g_{\mu\nu}(p^2-m^2)Z_1(p^2)+p_\mu p_\nu\mathfrak{z}(p^2). \qquad \text{(II.26)}$$

Since

$$\begin{aligned}\Delta_{\mu\nu}^{-1}(p)-\Delta_{\mu\nu}^{-1}(p')&=-g_{\mu\nu}[(p^2-m^2)Z_1(p^2)-(p'^2-m^2)Z_1(p'^2)]\\&\quad+p_\mu p_\nu'[\mathfrak{z}(p^2)-\mathfrak{z}(p'^2)]\\&\quad+p_\mu t_\nu\mathfrak{z}(p^2)+t_\mu p_\nu'\mathfrak{z}(p'^2)\,,\end{aligned} \qquad \text{(II.27)}$$

one can satisfy Ward's identity in the form

$$t_\alpha[\Gamma_{\alpha\mu\nu}-\Gamma_{\alpha\mu\nu}{}^A]=0\,, \qquad \text{(II.8)}$$

where

$$\begin{aligned}\Gamma_{\alpha\mu\nu}{}^A=-(p+p')_\alpha\Bigg[g_{\mu\nu}\bigg\{Z+\int\frac{(x-m^2)^2G_1(x)dx}{(x-p^2)(x-p'^2)}\bigg\}\\-p_\mu p_\nu'\int\frac{\mathcal{G}(x)dx}{(x-p^2)(x-p'^2)}\Bigg]\\+p_\mu g_{\alpha\nu}\mathfrak{z}(p^2)+p_\nu'g_{\alpha\mu}\mathfrak{z}(p'^2).\end{aligned} \qquad \text{(II.28)}$$

To evaluate Δ^{-1} use as before the equation

$$\text{Im}\Delta^{-1}=e^2\int\Gamma^A\delta_+\delta_+\Gamma^{*A}. \qquad \text{(II.12)}$$

Noting that

$$(p^2-m^2)^2G_1=-(1/3\pi)\,\text{Tr}(\mathbf{d}\,\text{Im}\mathbf{\Delta}^{-1})\,,\quad m^2p^2G_2=(1/\pi)\,\text{Tr}(\mathbf{e}\,\text{Im}\mathbf{\Delta}^{-1})\,, \qquad \text{(II.29)}$$

we obtain

$$\begin{aligned}(1/\pi)\,\text{Im}Z_1^{-1}(s)&=[\alpha(s+m^2)/24m^2s^2]\theta(s-m^2)[3a(s+m^2)^2\\&\quad-2(s-m^2)^2-3(s^2+10m^2s+m^4)]\end{aligned} \qquad \text{(II.30)}$$

and

$$\begin{aligned}(1/\pi)\,\text{Im}Z_2(s)&=[\alpha(s-m^2)\theta(s-m^2)/8m^4s^2]\\&\quad\times[3am^4(s+m^2)|Z_2(s)|^2-3(s-3m^2)|F_1(s)|^2\\&\quad-2(s-m^2)(2s+m^2)\,\text{Re}\,F_1^*(s)F_2(s)\\&\quad-(5/4)(s-m^2)^2(s+m^2)|F_2(s)|^2]\,,\end{aligned} \qquad \text{(II.31)}$$

where

$$F_1(s)=-(s-m^2)Z_1(s)-m^2Z_2(s)\,,\quad F_2(s)=2Z_1(s). \qquad \text{(II.32)}$$

The equation for Z_1 is immediately soluble. For large s, $\text{Im}Z_1^{-1}(s)=O(s)$. Thus,

$$Z_1(s)=O[(s\ln s)^{-1}]\ (\text{and } Z=0,\ \text{all } \alpha>0). \qquad \text{(II.33)}$$

From (II.31) and (II.32)

$$\text{Im}Z_2^{-1}(s)=O(1+\text{Re}[Z_2^{-1}(s)(\ln s)^{-1}])$$

and the solution has the form

$$Z_2(s)=O((\ln s)^{-1})\quad(\text{and } Zm_0^2=0). \qquad \text{(II.34)}$$

We shall return to the full discussion of (II.33) and (II.34) in Part III, Sec. 2. Here we simply remark that high energy behavior of type (II.33) and (II.34) is precisely what was stipulated for $Z_1(p^2)$ and $Z_2(p^2)$ in paper I, in order that a "stable" approximation scheme for a finite vector electrodynamics can be set up.

We have not considered in this section the computation of the full Γ. In Part III we make this computation and explicitly verify the assertion of Sec. I. 1B and I. 1C that Γ and Γ^A behave similarly for large energies.

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[24] That the perturbation Γ must contain terms of the form $(1-\alpha\ln s/m^2)$ and $(1-\alpha\ln s'/m^2)$ can be verified directly by writing the perturbation expression for $\Delta^{-1}(s)-\Delta^{-1}(s')$ and checking through explicitly for Ward's identity.

2. FORM FACTOR DECOMPOSITION OF Γ AND TWO PARTICLE CONTRIBUTIONS TO Δ AND *D*

In Sec. II. 1 we set down the spectral representations for Δ and those parts of Γ which depend directly on D through Ward's identity. In Sec. II. 2, we write down the general spectral representation of D, the photon propagator, decompose Γ into form factors and compute the two-particle contributions to $\mathrm{Im}\Delta$ and $\mathrm{Im}D$. In the next section Sec. II. 3 are written the two-particle contribution to $\mathrm{Im}\Gamma$. We also rectify in Sec. II. 3 the omission so far in not considering C parts (2-meson, 2-photon graphs) which for spin-zero and spin-one electrodynamics play as crucial a role as the vertex function Γ itself and are in any case necessary for ensuring gauge invariance.

(i) Neutral Vector Meson Propagator *D* (Massive Photon in an Arbitrary Gauge)

Following Feldman and Matthews[25] we write the photon propagator in the form

$$\mathbf{D}^{-1}(t)=\mathbf{d}(t)(t^2-\mu^2)Z_3(t^2)-\mathbf{e}(t)Z_3\mu_0{}^2(t^2-\lambda^2)/\lambda^2\,, \quad \text{(II.35)}$$

where

$$Z_3(t^2)=1-\int\frac{(t^2-\mu^2)G_3(x)dx}{t^2-x+i\epsilon}\,, \quad \text{(II.36)}$$

$$Z_3=1-\int G_3(x)dx\,, \quad \text{(II.37)}$$

$$\frac{\mu_0{}^2Z_3}{\mu^2}=1-\int\frac{G_3(x)dx}{x}\,. \quad \text{(II.38)}$$

Note that in this formalism the absorptive part of the free photon propagator equals

$$(1/\pi)\,\mathrm{Im}\mathbf{D}_0{}^{-1}(t)=\mathbf{d}(t)\delta_+(t^2-\mu^2)-\lambda^2\mu^{-2}\mathbf{e}(t)\delta_+(t^2-\lambda^2)\,.$$

(ii) The Vertex Function

(1) From C and P invariance

$$\Gamma_a(p,p')=\tilde{\Gamma}_a(p',p)=-\tilde{\Gamma}_a(-p',-p)\,, \quad \text{(II.39)}$$

where transposition (~) refers to charged meson indices.

(2) From the Ward-Takahashi identity Γ_a must have the form (true for any arbitrary gauge)

$$\Gamma_a=\Gamma_a{}^A+\Gamma_a{}^B=\Gamma_a{}^A-d_{ab}(t)B_b(t)\,,$$

where

$$t_a\Gamma_a{}^A=\Delta^{-1}(p)-\Delta^{-1}(p') \quad\text{and}\quad t_a\Gamma_a{}^B=0\,. \quad \text{(II.40)}$$

Possible forms for $\Gamma_a{}^A$ for scalar and vector mesons were displayed in equations (II.9) and (II.28). The properties of $\Gamma_a{}^B$ are listed below:

A. Scalar Case

(1) From symmetry considerations

$$B_b=(p+p')_bB(p^2,p'^2,t^2) \quad \text{(II.41)}$$

with

$$B(p^2,p'^2,t^2)=B(p'^2,p^2,t^2)\,. \quad \text{(II.42)}$$

(2) The physical requirement that Γ_a does not exhibit a pole at $t^2=0$ implies that

$$\lim_{t^2\to 0}\frac{B(p^2,p'^2,t^2)}{t^2}=\text{finite}\,. \quad \text{(II.43)}$$

(3) The requirement that the mesons carry unit charge imposes the restriction

$$\Gamma_a(p,p')\,|_{p^2=p'^2=m^2}=2p_a \quad\text{as}\quad t\to 0\,.$$

Thus $B(m^2,m^2,t^2)\to 0$ for $t\to 0$. This condition is in fact part of (II.43).

(4) On the meson-photon mass shell

$$\Gamma_a(p,p')=(p+p')_aE(p^2)-\frac{t_a(p^2-m^2)}{\mu^2}[E(p^2)-Z(p^2)]\,,$$

where

$$E(p^2)=Z(p^2)+B(p^2,m^2,\mu^2)\,. \quad \text{(II.44)}$$

For $\mu^2\to 0$ write

$$\lim_{\mu^2\to 0}\frac{E(p^2)-Z(p^2)}{\mu^2}=\lim_{\mu^2\to 0}\frac{B(p^2,m^2,\mu^2)}{\mu^2}=E'(p^2)\,. \quad \text{(II.45)}$$

Thus,

$$\Gamma_a(p,p')\,|_{p'^2=m^2,\mu^2=0}=(p+p')_aZ(p^2)-t_a(p^2-m^2)E'(p^2)\,. \quad \text{(II.46)}$$

On the two-meson mass shell

$$\Gamma_a(p,p')\,|_{p^2=p'^2=m^2}=(p+p')_a[1+B(m^2,m^2,t^2)]=(p+p')_a\mathcal{E}(t^2)\,. \quad \text{(II.47)}$$

This last equation defines electric form factor $\mathcal{E}(t^2)$.

B. Vector Case

The general expression for $\Gamma_{a\mu\nu}{}^B$ is $d_{ab}(t)B_{b\mu\nu}(p,p',t)$, with

$$B_{b\mu\nu}=(p+p')_b[B_1g_{\mu\nu}+B_2p_\mu t_\nu+B_3t_\mu p_\nu'+B_4p_\mu p_{\nu'}+B_5t_\mu t_\nu]+(g_{b\nu}t_\mu-g_{b\mu}t_\nu)B_6+(g_{b\nu}t_\mu+g_{b\mu}t_\nu)B_7+g_{b\nu}p_\mu B_8+g_{b\mu}p_\nu'B_9\,. \quad \text{(II.48)}$$

[25] G. Feldman and P. T. Matthews, Phys. Rev. **130**, 1623 (1963). In this formulation of electrodynamics, λ^2 is introduced with the significance of the mass of a "time-like" photon whose polarization is always along the propagation direction t_μ. Current conservation guarantees that the mass shell S-matrix elements (though of course not the Green's functions) are independent of λ^2. The constant λ^2 specifies a particular covariant gauge: $a=\lambda^2/\mu^2=0$ defines the Landau gauge, and $a=\lambda^2/\mu^2=1$ defines the Fermi-Stuckelberg gauge. For $\lambda^2/\mu^2\to\infty$ one recovers conventional theory of massive neutral vector mesons.

Thus,

$$\Gamma_{a\mu\nu}=-(p+p')_a g_{\mu\nu}\left[Z+\int\frac{(x-m^2)^2G_1(x)dx}{(x-p^2)(x-p'^2)}\right]$$
$$+\int\frac{\mathcal{G}(x)dx}{(x-p^2)(x-p'^2)}[(p+p')_a p_\mu p_\nu'+(x-p^2)g_{a\mu}p_\nu'$$
$$+(x-p'^2)g_{a\nu}p_\mu]+d_{ab}(t)B_{b\mu\nu}(p,p'). \quad \text{(II.49)}$$

(1) The symmetry properties are

$$\begin{aligned}B_{1,4,5,6}(p^2,p'^2,t^2)&=B_{1,4,5,6}(p'^2,p^2,t^2),\\ B_7(p^2,p'^2,t^2)&=-B_7(p'^2,p^2,t^2),\\ B_{8,9}(p^2,p'^2,t^2)&=B_{9,8}(p'^2,p^2,t^2),\\ B_{2,3}(p^2,p'^2,t^2)&=-B_{3,2}(p'^2,p^2,t^2).\end{aligned} \quad \text{(II.50)}$$

Clearly only seven of the nine B_i are independent.

(2) In order that $\Gamma_{a\mu\nu}$ have no pole at $t^2=0$,

$$\begin{aligned}\lim_{t^2\to0}B_{1,4}&=(p^2-p'^2)B_{2,3}+B_{8,9}\\ &=(p^2-p'^2)B_5+2B_7=0.\end{aligned} \quad \text{(II.51)}$$

(3) Contracting out on the polarization vectors of one meson [i.e., $\Gamma_{a\mu\nu}(p,p')d_{\nu\nu'}(p')$], we get

$$B_{b\mu\nu}(p,p')\to(p+p')_b$$
$$\times[B_1g_{\mu\nu}+B_2p_\mu t_\nu+B_5t_\mu t_\nu]+(g_{b\nu}t_\mu-g_{b\mu}t_\nu)B_6$$
$$+(g_{b\nu}t_\mu+g_{b\mu}t_\nu)B_7+g_{b\nu}p_\mu B_8, \quad \text{(II.52)}$$

so that just six form factors remain. Contracted out on the second meson polarization vector, only three form factors survive:

$$d_{\mu\rho}(p)\Gamma_{a\rho\sigma}(p,p')d_{\sigma\nu}(p')=d_{\mu\rho}(p)[(p+p')_a(-g_{\rho\sigma}\mathcal{E}+t_\rho t_\sigma\mathcal{Q})$$
$$+(g_{a\rho}t_\sigma-g_{a\sigma}t_\rho)\mathfrak{M}]d_{\sigma\nu}(p'), \quad \text{(II.53)}$$

with

$$\begin{aligned}\mathcal{E}(t^2)&=1+B_1(m^2,m^2,t^2),\\ \mathcal{Q}(t^2)&=-B_5(m^2,m^2,t^2),\\ \mathfrak{M}(t^2)&=B_6(m^2,m^2,t^2).\end{aligned} \quad \text{(II.54)}$$

We shall call $\mathcal{E}$, $\mathfrak{M}$, and $\mathcal{Q}$ the electric, magnetic and quadrupole form factors (though the designation is not strictly correct in the last case). These must be *gauge invariant* just like $\mathcal{E}$ in the scalar case. In the static limit, we obtain

$\mathcal{E}(0)=1$ (unit charge),

$\mathfrak{M}(0)=K$ (magnetic moment in units of $e/2m$),

$\mathcal{Q}(0)=(2/m^2)(q+K-1)$ (q is the quadrupole moment in units of e/m^2). (II.55)

(iii) The 2-Particle Contributions to D and Δ; General Expressions

In our calculations, the 2-particle contributions will play a crucial testing role. These are computed in this section with completely general spectral functions of Part II, Sec. 1 for scalar and vector electrodynamics.

A. Scalar Electrodynamics

(1) *Photon self-energy.*

$$\frac{1}{\pi}\,\mathrm{Im}D_{ab}^{-1}(t)=\frac{e^2}{(2\pi)^3}\int d^4p\Gamma_a(p,p')$$
$$\times\delta_+(p^2-m^2)\delta_+(p'^2-m^2)\Gamma_b^*(p,p').$$

In terms of the spectral function for D^{-1} and the form factor of Γ this reduces to

$$G_3(t^2)=[\alpha t^2/6(t^2-\mu^2)^2]$$
$$\times(1-4m^2/t^2)^{3/2}|\mathcal{E}(t^2)|^2\theta(t^2-4m^2) \quad \text{(II.56)}$$

with $\mathcal{E}(t^2)$ defined in (II.47).

(2) *Meson self-energy.* This has been computed before in Sec. 1A for $\mu=0$ and $\Gamma=\Gamma^A$. For the general case it is convenient to separate the photon mass-shell contributions into two parts; one part coming from the Fermi gauge $-g_{ab}\delta_+(t^2-\mu^2)$, and the second coming from the remaining terms $(t_at_b/\mu^2)[\delta_+(t^2-\mu^2)-\delta_+(t^2-\lambda^2)]$, in the photon propagator. Thus,

$$(1/\pi)\,\mathrm{Im}\Delta(p)=\Delta(p)[X(p^2)+Y(p^2)]\Delta^*(p^2),$$

where the gauge-dependent contribution $Y(p^2)$ equals

$$Y(p^2)=\frac{e^2}{(2\pi)^3}\int(t_a\Gamma_a)\delta_+(p'^2-m^2)(t_b\Gamma_b^*)$$
$$\times\frac{1}{\mu^2}[\delta_+(t^2-\mu^2)-\delta_+(t^2-a\mu^2)]d^4t$$
$$=[\alpha(p^2-m^2)^2/2p^2\mu^2]|Z(p^2)|^2$$
$$\times[\varphi(\mu^2)-\varphi(a\mu^2)]. \quad \text{(II.57)}$$

Here $\varphi(\mu^2)$ is the quantity which arises naturally in evaluating the phase space integral

$$\int d^4t\delta_+(t^2-\mu^2)\delta_+[(p-t)^2-m^2]$$
$$=\frac{\pi\varphi(\mu^2)}{2p^2}=\frac{\pi}{2p^2}\theta[p^2-(m+\mu)^2]$$
$$\times[(p^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}. \quad \text{(II.58)}$$

Indeed the result with $\mu=m$ has already been used in (III.56).] The Fermi gauge contribution, $X(p^2)$, to $\mathrm{Im}\Delta(p)$ equals

$$X(p^2)=\frac{\alpha\varphi(\mu^2)}{2p^2}\left[(2p^2+2m^2-\mu^2)|E(p^2)|^2\right.$$
$$\left.-\frac{(p^2-m^2)^2}{\mu^2}\{|E(p^2)|^2-|Z(p^2)|^2\}\right] \quad \text{(II.59)}$$

with $E(p^2)$ defined in (II.44). All in all then,

$$G(p^2)=\frac{\alpha|Z(p^2)|^2}{2p^2\mu^2}[\varphi(\mu^2)-\varphi(a\mu^2)]+\frac{X(p^2)}{(p^2-m^2)^2}. \quad \text{(II.60)}$$

It is worth remarking that for $a\rightarrow\infty$ (i.e., no time-like photons in intermediate states) $\varphi(a\mu^2)=0$ and G is positive definite as indeed it should be. In the limit as $\mu^2\rightarrow 0$ (II.60) gives

$$G(p^2)=\alpha\theta(p^2-m^2)[(a-3)(p^2+m^2)/(p^2-m^2)\times|Z(p^2)|^2+2(p^2-m^2)\,\mathrm{Re}Z^*(p^2)E'(p^2)]/2p^2 \quad \text{(II.61)}$$

with $E'(p^2)$ defined in (II.45).

The total number of unknowns appearing in the equations for Δ and D equals four $[G(p^2), G_3(t^2), E(p^2)$, and $\mathcal{E}(t^2)]$. Thus besides the two equations for G and G_3 we need two more. These are provided by writing the two-particle approximation to Γ using (I.24) and are set out in Sec. 3.

B. Vector Electrodynamics

(1) *Photon self-energy*. From the equation

$$(t^2-\mu^2)^2G_3(t^2)=-\frac{e^2}{3(2\pi)^3}d_{ab}(t)\int d^4p\,\Gamma_{a\mu\rho}(p,p')d_{\mu\nu}(p)d_{\rho\sigma}(p')\Gamma_{b\nu\sigma}{}^*(p,p')\delta_+(p^2-m^2)\delta_+(p'^2-m^2)$$

we arrive at[26]

$$\begin{aligned}G_3(t^2)=&\frac{\alpha t^2}{24m^4(t^2-\mu^2)^2}\left(1-\frac{4m^2}{t^2}\right)^{3/2}\theta(t^2-4m^2)\\&\times[(t^4-4m^2t^2+12m^4)|\mathcal{E}(t^2)|^2-2t^2(t^2-2m^2)\,\mathrm{Re}\mathcal{E}^*(t^2)\mathfrak{M}(t^2)-t^2(t^2-2m^2)(t^2-4m^2)\,\mathrm{Re}\mathcal{E}^*(t^2)\mathcal{Q}(t^2)\\&+t^2(t^2+4m^2)|\mathfrak{M}(t^2)|^2+t^4(t^2-4m^2)\,\mathrm{Re}\mathfrak{M}^*(t^2)\mathcal{Q}(t^2)+t^4(t^2-4m^2)^2|\mathcal{Q}(t^2)|^2]\end{aligned}\quad \text{(II.62)}$$

with the form factors $\mathfrak{M}$, $\mathcal{E}$, and $\mathcal{Q}$ defined in (II.54).

(2) *Meson self energy*. A rather lengthy evaluation gives the following general expressions:

$$\begin{aligned}G_1(p^2)=&\frac{\alpha}{2p^2\mu^2}\Bigg\{-\left(3+\frac{\varphi^2(a\mu^2)}{4m^2p^2}\right)\varphi(a\mu^2)|Z_1(p^2)|^2+\frac{\varphi^3(\mu^2)}{(p^2-m^2)^2}\Bigg[\left(3+\frac{\varphi^2(\mu^2)}{4m^2p^2}\right)|E(p^2)|^2\\&+\frac{\mu^2(p^2+m^2-\mu^2)}{2m^2p^2}\,\mathrm{Re}E^*(p^2)M(p^2)+\frac{(p^2+m^2-\mu^2)\varphi^2(\mu^2)}{4m^2p^2}\,\mathrm{Re}E^*(p^2)Q(p^2)+\frac{\mu^2\varphi^2(\mu^2)}{4m^2p^2}\,\mathrm{Re}Q^*(p^2)M(p^2)\\&+\frac{(p^2-m^2)(\mu^2-p^2-m^2)}{2m^2p^2}\,\mathrm{Re}E^*(p^2)N(p^2)+\frac{3\mu^2(p^2-m^2)}{2m^2p^2}\,\mathrm{Re}M^*(p^2)N(p^2)+\frac{\mu^2(2p^2+2m^2-\mu^2)}{4m^2p^2}|M(p^2)|^2\\&-\frac{(p^2-m^2)^2\varphi^2(\mu^2)}{4m^2p^2}\,\mathrm{Re}Q^*(p^2)N(p^2)+\frac{\varphi^4(\mu^2)}{16m^2p^2}|Q(p^2)|^2+\frac{(p^2-m^2)^2+2\mu^2(p^2+m^2)}{4m^2p^2}|N(p^2)|^2\Bigg]\Bigg\}\,,\end{aligned}\quad \text{(II.63)}$$

with

$$E(p^2)=Z_1(p^2)+B_1(p^2,m^2,\mu^2)\,, \quad \text{(II.64)}$$

$$Q(p^2)=-B_5(p^2,m^2,\mu^2)\,, \quad \text{(II.65)}$$

$$M(p^2)=B_6(p^2,m^2,\mu^2)\,, \quad \text{(II.66)}$$

$$N(p^2)=B_7(p^2,m^2,\mu^2)\,. \quad \text{(II.67)}$$

Equation (II.64) is analogous to the scalar counterpart (II.65). Further,

$$\begin{aligned}G_2(p^2)=&\frac{\alpha}{2m^2p^6}\Bigg\{-\frac{m^2\varphi^3(a\mu^2)}{4\mu^2}|Z_2(p^2)|^2+\varphi(\mu^2)\\&\times\Bigg[\left(3+\frac{\varphi^2(\mu^2)}{4m^2\mu^2}\right)|F_1(p^2)|^2+\frac{\varphi^4(\mu^2)}{16m^2\mu^2}|F_2(p^2)|^2+\frac{(p^2-\mu^2-m^2)\varphi^2(\mu^2)}{4m^2\mu^2}\,\mathrm{Re}F_1{}^*(p^2)F_2(p^2)\Bigg]\Bigg\}\,,\end{aligned}\quad \text{(II.68)}$$

[26] That G_3 in (II.62) is indeed positive definite can be seen by defining in place of $\mathcal{Q}$ the combination

$$\chi(t^2)=-\left(1-\frac{t^2}{2m^2}\right)\mathcal{E}(t^2)-\frac{t^2}{2m^2}\mathfrak{M}(t^2)+t^2\left(1-\frac{t^2}{4m^2}\right)\mathcal{Q}(t^2),$$

in which case (II.62) simplifies to

$$G_3(t^2)=\frac{\alpha t^2}{3(t^2-\mu^2)^2}\left(1-\frac{4m^2}{t^2}\right)^{2}\left[|\mathcal{E}(t^2)|^2+\frac{t^2}{2m^2}|\mathfrak{M}(t^2)|^2+\tfrac{1}{2}|\chi(t^2)|^2\right].$$

where

$$F_1(p^2)=-(p^2-m^2)Z_1(p^2)-m^2Z_2(p^2)+\tfrac{1}{2}(p^2-m^2+\mu^2)(M(p^2)+N(p^2))+p^2B_8(p^2,m^2,\mu^2)\,,$$
$$F_2(p^2)=2E(p^2)+2p^2B_2(p^2,m^2,\mu^2)-(p^2-m^2+\mu^2)Q(p^2)+N(p^2)-M(p^2)\,. \tag{II.69}$$

In the limit $\mu^2\to 0$ as noted in (II.51), E, Q, M are nonzero. Other form factors B_1, $B_4\to 0$. Also in this limit, $E(p^2)\to Z_1(p^2)$, $F_1(p^2)\to-(p^2-m^2)[Z_1(p^2)-\frac{1}{2}M(p^2)]-m^2Z_2(p^2)$, $F_2(p^2)\to 2Z_1(p^2)-M(p^2)$. Writing

$$\lim_{\mu^2\to 0} Q(\mu^2)/\mu^2=Q' \text{ etc.}\quad\text{and}\quad E'(p^2)=\partial E(p^2,\mu^2)/\partial\mu^2|_{\mu^2=0}\,, \tag{II.70}$$

We have adopted here the stronger form of (II.51) in which every $B_i\to 0$ as $\mu^2\to 0$, except $i=6$. This makes no essential difference to (II.71) and (II.72) but simplifies the expressions considerably.

$$G_1(p^2)=-\frac{\alpha\theta(p^2-m^2)}{24m^2p^4}\Bigg(\frac{3a(p^2+m^2)^3}{p^2-m^2}|Z_1(p^2)|^2-\frac{(p^2+m^2)}{p^2-m^2}[5(p^2-m^2)^2+36m^2p^2]|Z_1(p^2)|^2$$
$$+2(p^2-m^2)[(p^2-m^2)^2+12m^2p^2]\,\mathrm{Re}Z_1{}^*(p^2)E'(p^2)+(p^2-m^2)^3(p^2+m^2)\,\mathrm{Re}Z_1{}^*(p^2)Q'(p^2)$$
$$-2(p^2-m^2)^2(p^2+m^2)\,\mathrm{Re}Z_1{}^*(p^2)N'(p^2)+2(p^2-m^2)(p^2+m^2)[\mathrm{Re}Z_1{}^*(p^2)M(p^2)+|M(p^2)|^2]\Bigg)\,, \tag{II.71}$$

$$G_2(p^2)=\frac{\alpha(p^2-m^2)\theta(p^2-m^2)}{8m^2p^6}\{3am^4(p^2+m^2)|Z_2(p^2)|^2-3(p^2-3m^2)|F_1(p^2)|^2+2(p^2-m^2)^2\,\mathrm{Re}F_1{}^*(p^2)F_1{}'(p^2)$$
$$-2(p^2-m^2)(2p^2+m^2)\,\mathrm{Re}F_1{}^*(p^2)F_2(p^2)+(p^2-m^2)^3[\mathrm{Re}F_1{}^*(p^2)F_2{}'(p^2)+\mathrm{Re}F_1{}'(p^2)F_2{}^*(p^2)]$$
$$-(5/4)(p^2-m^2)^2(p^2+m^2)|F_2(p^2)|^2+\tfrac{1}{2}(p^2-m^2)^4\,\mathrm{Re}F_2{}^*(p^2)F_2{}'(p^2)\}\,. \tag{II.72}$$

The total number of unknowns for vector electrodynamics in the limit $\mu^2\to 0$ is 12,

$$G_1,\ G_2,\ G_3,\ E(p^2),\ \mathcal{E}(t^2),\ M(p^2),\ \mathfrak{M}(t^2),$$
$$Q(p^2),\ \mathcal{Q}(t^2),\ N(p^2),\ B_2(p^2),\ B_8(p^2)\,.$$

Besides the three equations for Δ and D we need nine more equations (from the two-particle approximations to the vertex function) to make a complete set.

3. VERTEX FUNCTION IN THE TWO-PARTICLE APPROXIMATION AND C-PARTS (TWO-MESON, TWO-PHOTON PROCESSES)

The Basic Equation

By considering

$$\langle 0|j(0)|p,q\rangle_{\text{in}}=\langle 0|j(0)|2\rangle_{\text{out}}\,{}_{\text{out}}\langle 2|p,q\rangle_{\text{in}}$$

and using PT invariance, we get

$$\mathrm{Im}(\Delta(s)\Gamma(s))=\mathrm{Re}\int\Delta(s)\Gamma(s)\delta_+\delta_+M^*(s)\,, \tag{II.73}$$

$$0=\mathrm{Im}\int\Delta(s)\Gamma(s)\delta_+\delta_+M^*(s);\ s=(p+q)^2. \tag{II.74}$$

One may extract from M the one-particle reducible parts in the s-channel, thus writing (Fig. 4)

$$M=M_{(1)}+\Gamma(s)\Delta(s)\Gamma(s)\,. \tag{II.75}$$

Using $\mathrm{Im}\Delta=\int|\Delta\Gamma|^2\delta_+\delta_+$, Eqs. (II.73) and (II.74) reduce as follows:

$$\mathrm{Im}\Gamma=\mathrm{Re}\int\Gamma\delta_+\delta_+M_{(1)}{}^*\,, \tag{II.76}$$

$$0=\mathrm{Im}\int\Gamma\delta_+\delta_+M_{(1)}{}^*. \tag{II.77}$$

To see the character of these functions, assume all amplitudes are scalar functions. Equations (II.76) and (II.77) are equivalent to the (two-particle) unitarity condition on $M_{(1)}$:

$$\mathrm{Im}\int\delta_+\delta_+M_{(1)}{}^*+\left|\int\delta_+\delta_+M_{(1)}{}^*\right|^2=0\,, \tag{II.78}$$

and the equation

$$\tan\theta=\frac{\mathrm{Im}\Gamma}{\mathrm{Re}\Gamma}=\frac{\mathrm{Im}\int\delta_+\delta_+M_{(1)}{}^*}{\mathrm{Re}\int\delta_+\delta_+M_{(1)}{}^*}\,. \tag{II.79}$$

This is the homogeneous Riemann-Hilbert equation

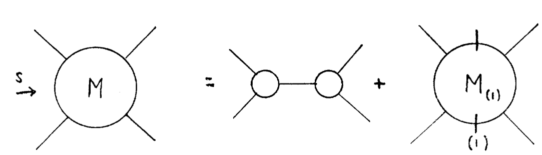

FIG. 4. The one-particle irreducible scattering.

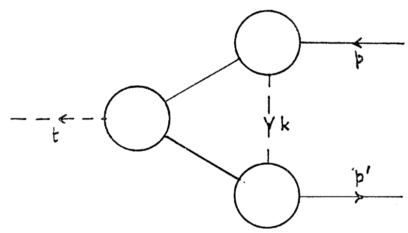

FIG. 5. Basic triangle graph for the vertex equation.

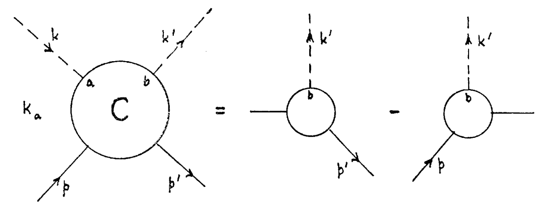

FIG. 6. Diagrammatic interpretation of the gauge identity for proper Compton scattering.

which has been extensively studied by Muskhelishvili[27] and Omnes[27] and has the solution

$$\Gamma(s)=cX(s)\,, \qquad \text{(II.80)}$$

where

$$X(s)=\exp\left[\frac{1}{\pi}\int_{-\infty}^{\infty}\frac{\theta(x)dx}{x-s}\right]. \qquad \text{(II.81)}$$

The inclusion of three- and higher particle states converts (II.79) essentially to the inhomogeneous form

$$\text{Im}\Gamma=\tan\theta\;\text{Re}\Gamma+U(s)\,, \qquad \text{(II.82)}$$

which has the solution

$$\Gamma=\left[c+\frac{1}{\pi}\int\frac{U(x)dx}{(x-s)X(x)}\right]X(s). \qquad \text{(II.83)}$$

(i) Scalar Electrodynamics

We will now consider the two cases:

A. Γ for the unphysical photon ($t^2\neq0$, $p^2=p'^2=m^2$),

B. Γ for the unphysical meson ($p^2\neq m^2$, $p'^2=m^2$, $t^2=0$).

A. *Unphysical Photon*

To make the problem tractable we must approximate to $M_{(1)}$. From crossing symmetry for M, $M_{(1)}$ must include the one-photon contribution $\Gamma(k)\Delta(k)\Gamma(k)$. (See Fig. 5.) This contribution, by itself, however, does not satisfy unitarity; in fact for the relevant values of k^2 (i.e., $k^2<0$), $M_{(1)}$ is purely real and thus needs to be supplemented by further terms. These are given by (II.78). Neglecting $|\text{Im}M_{(1)}|^2$ compared to $\text{Im}M_{(1)}$ unitarity gives $\text{Im}M_{(1)}=-[\text{Re}M_{(1)}]^2$. With $\text{Re}M_{(1)}=\Gamma\Delta\Gamma$ in this approximation we finally obtain[28]

$$\tan\theta=\int\delta_+\delta_+\Gamma\Delta\Gamma\,. \qquad \text{(II.84)}$$

Using the phase-space integrals listed in the Appendix, this works out as

$$\tan\theta(t^2)=\frac{\alpha\theta(t^2-4m^2)}{2(t^4-4m^2t^2)^{1/2}}\int_{4m^2-t^2}^{0}dk^2\frac{(4m^2-t^2-2k^2)}{4m^2-t^2}\cdot\frac{(2t^2+k^2-4m^2)}{k^2}\mathcal{E}^2(k^2)Z_3^{-1}(k^2)\,, \qquad \text{(II.85)}$$

where $\mathcal{E}(k^2)$ and $Z_3^{-1}(k^2)$ are real for the integration range of k^2.

Collecting all the relevant equations in the two particle approximation with the photon unphysical, the propagator and vertex function equations are

$$\frac{1}{\pi}\,\text{Im}Z_3(t^2)=\frac{\alpha}{6}\left(1-\frac{4m^2}{t^2}\right)^{3/2}\theta(t^2-4m^2)\,|\mathcal{E}(t^2)|^2\,, \qquad \text{(II.86)}$$

$$\frac{1}{\pi}\,\text{Im}\,\mathcal{E}(t^2)=\frac{\alpha\theta(t^2-4m^2)\,\text{Re}\,\mathcal{E}(t^2)}{2(t^4-4m^2t^2)^{1/2}}\int_{4m^2-t^2}^{0}dk^2\frac{(4m^2-t^2-2k^2)}{4m^2-t^2}\cdot\frac{(2t^2+k^2-4m^2)}{k^2}\mathcal{E}^2(k^2)Z_3^{-1}(k^2)\,, \qquad \text{(II.87)}$$

with the boundary conditions $Z_3(0)=\mathcal{E}(0)=1$.

B. *The Unphysical Meson and Consideration of C Parts*

The calculation is similar to the above. It is more complicated insofar as gauge invariance demands a proper treatment of two-meson two-photon (C-part) contributions which are involved in the intermediate states. Some remarks about the C parts are therefore needed:

[27] N. Muskhelishvili, *Singular Integral Equations* (P. Noordhof Ltd., Gröningen, The Netherlands, 1946), p. 111. The solution in the text is the so-called "fundamental solution"; it is the solution which applies when the change in the argument equals zero when x goes from $-\infty$ to $+\infty$. When this argument change is nonzero, the constant C in (III.80) is replaced by a polynomial. The high-energy behavior of Γ is thus determined by the phase shift $[\theta]_{-\infty}^{\infty}$ of the scattering amplitude and therefore (through Levinson's theorem) by the number of possible bound states and CDD poles. A detailed discussion of this will be published elsewhere. See also, R. Omnes, Nuovo Cimento 8, 316 (1958).

[28] In writing this we have the familiar dilemma of S-matrix theory; how to reconcile within one (*approximate*) expression, the demands of *causality, crossing, and unitarity.* Thus absolutely strict (2-particle) unitarity has been sacrificed in $M_{(1)}$ by not including terms of the form $\int(\Gamma\Delta\Gamma)\Delta\Delta(\Gamma\Delta\Gamma)$.

If $C_{ab}(p,p';k,k')$ stands for a proper Compton scattering graph (i.e., one-meson irreducible graph) it is easy to show that[29]

$$k_a C_{ab}(p,p';k,k')=\Gamma_b(p+k,\,p')-\Gamma_b(p,\,p'-k)\,, \tag{II.88}$$

$$k_b' C_{ab}(p,p';k,k')=\Gamma_a(p,\,p+k)-\Gamma_a(p'-k,\,p')\,. \tag{II.89}$$

These identities can be represented diagrammatically as shown in Fig. 6.

The symmetries obtained by C and P invariance are

$$C_{ab}(p,p';k,k')=\tilde{C}_{ab}(-p',\,-p;\,k,k')=\tilde{C}_{ab}(p',p;\,-k,\,-k')=\tilde{C}_{ba}(p',p;\,k',k)\,. \tag{II.90}$$

Similarly to the vertex function, we may define a "gauge covariant" separation

$$C_{ab}=C_{ab}{}^A+C_{ab}{}^B \quad \text{where} \quad k^a C_{ab}{}^B=k'^b C_{ab}{}^B=0\,. \tag{II.91}$$

Now Γ itself consists of two parts, Γ^A and the purely transverse part Γ^B [see (II.40)]. Thus C^A $(kC^A=\Gamma-\Gamma)$ itself consists of C^{AA} terms $(kC^{AA}=\Gamma^A-\Gamma^A)$ and C^{AB} terms $(kC^{AB}=\Gamma^B-\Gamma^B)$. Now,

$$\Gamma_b{}^A(p+k,\,p')-\Gamma_b{}^A(p,\,p'-k)=2k_bZ+\int dx(x-m^2)^2G(x)\left[\frac{(p+k+p')_b}{(x-p'^2)[x-(p+k)^2]}-\frac{(p+p'-k)_b}{(x-p^2)[x-(p'-k)^2]}\right],$$

so that we may write

$$C_{ab}{}^{AA}(p,p';k,k')=2g_{ab}Z+\int\frac{dx(x-m^2)^2G(x)N_{ab}(p,p';k,k';x)}{[x-(p+k)^2](x-p^2)(x-p'^2)[x-(p'-k)^2]}, \tag{II.92}$$

where

$$N_{ab}(p,p';k,k';x)=2g_{ab}(x-p^2)(x-p'^2)+[(p+p')_bk_a'-(p+p')_ak_b](p^2-p'^2)$$
$$-[(p+p')_a(p+p')_b-k_bk_a'](p^2+p'^2-2x)\,. \tag{II.93}$$

Similarly, from $\Gamma^B-\Gamma^B$ we may write

$$C_{ab}{}^{AB}(p,p';k,k')=-d_{ac}(k)\bigg[(p+p')_c\frac{k_b'}{k'^2}\{B[p^2,(p'+k')^2,k^2]-B[(p-k')^2,p'^2,k^2]\}$$
$$+g_{cb}\{B[p^2,(p'+k')^2,k^2]+B[(p-k')^2,p'^2,k^2]\}\bigg]-\bigg[\frac{k_a}{k^2}(p+p')_c\{B[(p+k)^2,p'^2,k'^2]$$
$$-B[p^2,(p'-k)^2,k'^2]\}+g_{ac}\{B[(p+k)^2,p'^2,k'^2]+B[p^2,(p'-k)^2,k'^2]\}\bigg]d_{cb}(k')\,. \tag{II.94}$$

The expressions (II.92)–(II.94) for C possess all the requisite symmetry properties[30] and together give the Γ-dependent part of C. It is this part whose inclusion in any calculation is necessary to preserve gauge invariance.

We can now return to the discussion of the vertex function with the meson unphysical. At the outset we set the photon mass zero so that

$$\Gamma_a(p,p')\,|_{p'^2=m^2,t^2=0}=(p+p')_aZ(p^2)-(p^2-m^2)(p-p')_aE'(p^2)\,.$$

Clearly,

$$(p-p')_a\Gamma_a(p,p')=(p^2-m^2)Z(p^2)=\Delta^{-1}(p) \quad\text{and}\quad (p+p')_a\Gamma_a(p,p')=2(p^2+m^2)Z(p^2)-(p^2-m^2)^2E'(p^2)\,.$$

Approximate to $M_{(1)}$ in (II.76) by the gauge covariant combination[31]

$$M_{(1)}=\Gamma^A\Delta\Gamma^A+C^{AA}\,,$$

[29] K. Nishijima, Phys. Rev. **119**, 485 (1960); T. D. Lee, Phys. Rev. **128**, 899 (1962).

[30] It is perhaps instructive to write the spectral form for C^{AA} in Mandelstam variables $s=(p+k)^2$, $u=(p'-k)^2$.

$$C_{ab}{}^{AA}(p,p',k,k')\,|_{\text{mass shells}}=2g_{ab}Z+\int dx\frac{G(x)N_{ab}(s,u;x)}{(x-s)(x-u)}$$

with

$$N_{ab}(s,u;\,x)=2g_{ab}(x-m^2)^2+2[(p+p')_a(p+p')_b-k_bk_a'](x-m^2).$$

Using the identities,

$$\int\frac{(x-m^2)^2G(x)dx}{(x-s)(x-u)}=\frac{\Delta^{-1}(s)-\Delta^{-1}(u)}{s-u}-Z,\qquad \int\frac{(x-m^2)G(x)dx}{(x-s)(x-u)}=\frac{Z(s)-Z(u)}{s-u},$$

this may be more simply expressed as

$$C_{ab}{}^{AA}(s,u)=2g_{ab}\left[\frac{\Delta^{-1}(s)-\Delta^{-1}(u)}{s-u}\right]+2[(p+p')_a(p+p')_b-k_bk_a']\cdot\left[\frac{Z(s)-Z(u)}{s-u}\right].$$

[31] $M_{(1)}$ is approximate in the sense that Γ^B and the intrinsic spectral functions of C are not taken into account in writing $M_{(1)}$.

we obtain (after a lengthy calculation) for the vertex function,

$$\frac{1}{\pi}\operatorname{Im}E'(p^2)=\frac{\alpha\theta(p^2-m^2)}{2(p^2-m^2)^3}\int_{2m^2-p^2}^{m^4/p^2}dq^2\bigg[|Z(p^2)|^2\bigg(\frac{4(p^2+m^2)^2}{p^2-m^2}-\frac{9p^4+5p^2q^2+3m^2q^2+9m^2p^2+6m^4}{p^2-q^2}\bigg)$$
$$+\operatorname{Re}Z(p^2)Z^*(q^2)\bigg(\frac{(p^2+q^2+2m^2)^2}{q^2-m^2}-\frac{4p^4+9p^2q^2+q^4+7m^2p^2+5m^2q^2+6m^4}{p^2-q^2}\bigg)\bigg]. \quad \text{(II.95)}$$

Equations (II.86), (II.87), (II.95), and (II.61) together with the boundary condition $Z(m^2)=1$ complete the set of two- and three-point function unitarity equations. Before closing this section, it is as well to be reminded that $\mathcal{E}(t^2)$ and $E'(p^2)$ are themselves boundary values of the same function $B(p^2,p'^2,t^2)$,

$$\mathcal{E}(t^2)=1+B(m^2,m^2,t^2)\,,\quad\text{and}\quad E'(p^2)=\lim_{\mu^2\to 0}\frac{B(p^2,m^2,\mu^2)}{\mu^2}\,,$$

and that the full stability criterion [relation (I.6)] specifies for these the boundary conditions

$$\lim_{p^2\to\infty}p^2E'(p^2)\quad\text{and}\quad\lim_{t^2\to\infty}\mathcal{E}(t^2)=0(1)\,.$$

(ii) Vector Electrodynamics

The exact unitarity equations for Γ are far too long and complicated to write down in full generality, particularly with the meson unphysical. This is on account of the large number of form factors involved and the need to include the C parts properly for preserving gauge covariance. The problem is slightly more amenable with the photon unphysical because here one deals only with three (gauge-independent) form factors $\mathcal{E}$, $\mathfrak{M}$, and $\mathcal{Q}$ as defined in Eqs. (II.53) and (II.54). Nevertheless the general expressions for $\operatorname{Im}\mathcal{E}$, $\operatorname{Im}\mathfrak{M}$, and $\operatorname{Im}\mathcal{Q}$ in terms of $\mathcal{E}$, $\mathfrak{M}$, and $\mathcal{Q}$ (given by the one-photon exchange approximation to $M_{(1)}$ as in the scalar case) are still very complicated. We shall content ourselves here by stating the unitarity equation for $\operatorname{Im}\mathcal{E}$ (the simplest):

$$\frac{1}{\pi}8m^2t^4\bigg(1-\frac{t^2}{4m^2}\bigg)^3\operatorname{Im}\mathcal{E}(t^2)=\frac{\alpha\theta(t^2-4m^2)}{2(t^4-4m^2t^2)^{1/2}}\int_{4m^2-t^2}^{0}\frac{dk^2}{k^2}Z_3^{-1}(k^2)\cdot(t^2+2k^2-4m^2)$$
$$\times\{\operatorname{Re}\mathcal{E}(t^2)\{(2t^2+k^2-4m^2)[(I_2^2I_{18}-I_9I_{19})\mathcal{E}^2(k^2)+(I_2^2I_6-I_1^2I_9)I_6\mathcal{Q}^2(k^2)-2\mathcal{E}(k^2)\mathcal{E}(k^2)(I_2^2I_{20}-I_1I_9I_{15})]$$
$$+4\mathfrak{M}(k^2)\mathcal{E}(k^2)[I_2^2(I_{22}-I_{21})-I_9(I_2I_{15}-I_1I_{16})]+4\mathfrak{M}(k^2)\mathcal{Q}(k^2)[I_2^2I_6(I_8-I_7)-I_1I_9(I_1I_8-I_2I_6)]$$
$$+\mathfrak{M}^2(k^2)[2I_2^2(I_{24}-I_5I_6)-I_9(2I_1I_{17}-I_6I_9-I_1^2I_5)]\}$$
$$-\operatorname{Re}\mathcal{Q}(t^2)\{(2t^2+k^2-4m^2)[(I_2^2I_{20}-I_9I_{10}^2)\mathcal{E}^2(k^2)+I_3^2(I_2^2I_6-I_1^2I_9)\mathcal{Q}^2(k^2)-2\mathcal{E}(k^2)\mathcal{Q}(k^2)I_3(I_2^2I_{12}-I_1I_9I_{10})]$$
$$+4\mathfrak{M}(k^2)\mathcal{E}(k^2)[I_2^2(I_3I_{13}-I_4I_{12})-I_9I_{10}(I_2I_3-I_1I_4)]+4\mathfrak{M}(k^2)\mathcal{Q}(k^2)[I_2^2(I_4I_6-I_3I_7)-I_9I_1(I_1I_4-I_2I_3)]$$
$$+\mathfrak{M}^2(k^2)[I_2^2(2I_3I_{14}-I_6^2-I_3^2I_5)-I_9(2I_1I_3I_{11}-I_1^2I_6-I_3^2I_9)]\}\}. \quad \text{(II.96)}$$

The I's are simple traces of $\mathbf{d}(p)$, $\mathbf{d}(p')$, $\mathbf{d}(p-k)$, and $\mathbf{d}(p'-k)$, and are listed in Appendix II. In the region of large t^2 we get

$$\frac{1}{\pi}\operatorname{Im}\mathcal{E}(t^2)\approx\frac{\alpha}{32m^4t^8}\int_{-t^2}^{0}dk^2Z_3^{-1}(k^2)(t^2+2k^2)(t^2+k^2)$$
$$\times\{\operatorname{Re}\mathcal{E}(t^2)\{t^4(k^2+2t^2)[-4\mathcal{E}^2(k^2)+4k^2\mathcal{Q}(k^2)\mathcal{E}(k^2)+k^4\mathcal{Q}^2(k^2)]$$
$$+8k^4(t^2+k^2)\mathfrak{M}(k^2)\mathcal{E}(k^2)-16k^2t^4m^2\mathfrak{M}(k^2)\mathcal{Q}(k^2)+2t^6\mathfrak{M}^2(k^2)\}$$
$$-\operatorname{Re}\mathcal{Q}(t^2)\{t^2k^4(k^2+2t^2)[\mathcal{E}^2(k^2)-k^2\mathcal{Q}(k^2)\mathcal{E}(k^2)+\tfrac{1}{2}k^4\mathcal{Q}^2(k^2)]$$
$$-4k^4t^2(t^2+k^2)\mathfrak{M}(k^2)\mathcal{E}(k^2)+2t^2k^6(k^2+t^2)\mathfrak{M}(k^2)\mathcal{Q}(k^2)+k^4t^4\mathfrak{M}^2(k^2)\}\}. \quad \text{(II.97)}$$

This equation together with two similar ones for $\operatorname{Im}\mathfrak{M}$ and $\operatorname{Im}\mathcal{Q}$ (which have been worked out) and Eq. (II.62) for $\operatorname{Im}Z_3^{-1}$ completes the two-particle unitarity set for $D(t)$ and $\Gamma(t)$.

Part III

The two- and three-particle Green's function equations of Part II are solved (in the gauge approximation) to obtain explicit expressions for D, Δ and Γ^A in Sec. 1 for scalar electrodynamics and in Sec. 2 for vector electrodynamics. Also verified by explicit computation is the dimensional statement of Part I. 1B that the full Γ (in the two-particle unitarity approximation) behaves similarly to Γ^A. In Sec. 3 we go back to the Dyson-Schwinger equation for Δ and show by actual substitution that the unitarity solutions of Secs. 1 and 2 satisfy this so far as the high-energy behavior is concerned. The solutions obtained are stable so that with these as starting approximations, the full theory is divergence free.

1. THE SOLUTION OF UNITARITY EQUATIONS FOR Γ, D AND Δ FOR SCALAR ELECTRODYNAMICS

For scalar electrodynamics we are dealing with a renormalizable theory. One solution (the perturbation solution) of the equations is well known; it involves (for Δ, Γ, and D) a total of two subtraction constants [$Z(m^2)=1$, $Z_3(0)=1$]. In this section we attempt nonperturbative solutions; these will serve as guides for the more complicated case of vector electrodynamics. We also find that the photon subtraction $Z_3(0)=1$ is not really necessary.

A. Meson Equations

Rewrite the meson propagator equation (II.61) in the form

$$\operatorname{Im}Z^{-1}(p^2)=\tan\gamma\,\operatorname{Re}Z^{-1}(p^2)+U(p^2)\,,\qquad\text{(III.1)}$$

where

$$\tan\gamma=-\frac{\operatorname{Re}[2\alpha(p^2-m^2)^2/p^2]\theta(p^2-m^2)E'(p^2)}{1+\operatorname{Im}[2\alpha(p^2-m^2)^2/p^2]E'(p^2)}\,,\qquad\text{(III.2)}$$

$$U(p^2)=-\frac{\alpha(a-3)(p^2+m^2)\theta(p^2-m^2)}{2p^2[1+\operatorname{Im}(2\alpha(p^2-m^2)^2/p^2)E'(p^2)]}\,.\qquad\text{(III.3)}$$

The equation for $E'(p^2)$ has the form

$$\operatorname{Im}(p^2-m^2)E'(p^2)=|Z(p^2)|^2f(p^2)\,,\qquad\text{(III.4)}$$

where for large p^2

$$f(p^2)\approx\int_{-1}^{0}dx[A(x)+\{\operatorname{Re}Z^{-1}(p^2)Z^*(p^2x)\}B(x)]\qquad\text{(III.5)}$$

with

$$A(x)=(9x+5)/(x-1)$$

and

$$B(x)=(2x^3+10x^2-3x+1)/x(x-1)\,.$$

Note $U(p^2)=0$ for the special gauge $a=3$.

Using Muskhelishvili's result (III.1) has the formal solution [subject to one subtraction $Z^{-1}(m^2)=1$],

$$Z^{-1}(p^2)=X(p^2)\left[1+\frac{(p^2-m^2)}{\pi}\times\int\frac{dxU(x)}{(x-m^2)X(x)(x-p^2)}\right],\qquad\text{(III.6)}$$

where

$$X(p^2)=\exp\left[\frac{(p^2-m^2)}{\pi}\int\frac{\gamma(x)dx}{(x-p^2)(x-m^2)}\right].\qquad\text{(III.7)}$$

Also from (III.4)

$$(p^2-m^2)E'(p^2)=\frac{1}{\pi}\int\frac{|Z(k^2)|^2f(k^2)dk^2}{p^2-k^2}\,.\qquad\text{(III.8)}$$

In writing (III.8) we have assumed that the vertex function for scalar electrodynamics needs no extra subtraction besides the one at $Z(m^2)=1$. Since $f(p^2)\to$ const for $p^2\to\infty$, this assumption is equivalent to $Z(p^2)\to 0$ at least as fast as $1/\ln p^2$. We must now show that the expression (III.6) for $Z(p^2)$ indeed does confirm this.

To see this, note that if (III.8) holds,

$$(p^2-m^2)E'(p^2)\to 0\,,$$

so that $U(p^2)\to$ constant and $\tan\gamma(p^2)\to 0$. Therefore, $X(p^2)$ in (III.7) $\to$ constant apart from a possible logarithmic factor. But (whatever this factor) for $a\neq 3$, it is easy to see that $X(p^2)$ times the integral within brackets in (III.6) must increase at least as fast as $(\ln p^2)$.[32] This is precisely what we set out to show for $Z^{-1}(p^2)$ [compare Part II. 1, Eq. (II.16)].

So much for the formal solutions of (III.1) and (III.4). To obtain explicit expressions for $Z(p^2)$ and E' one may now set up a detailed iteration scheme to solve (III.6) and (III.8). A wide variety of schemes are possible depending on what we take as the effective coupling constant; a particularly convenient scheme is to go back to Eqs. (II.61) and (II.95) for E' and Z and to start with the first iteration $\Gamma=\Gamma^A[\Delta]$, (Γ^A defined in Part II. 1) i.e., $E'^{(0)}=0$[33] and writing the higher iterations

[32] We are indebted to Dr. J. G. Taylor for a discussion of this point.

[33] Considering the manner in which $E'(p^2)$ arises in (III.13) one can easily see that $E'(p^2)$ came from the following combination of terms in the integral in (II.59):

$$\lim_{\mu^2\to 0}\int\frac{\delta(t^2-\mu^2)}{t^2}B(p^2,m^2,t^2)dt^2.$$

Now from (III.31)

$$B(p^2,m^2,t^2)=[Z_3(t^2)-1]Z(p^2)=(t^2-\mu^2)Z(p^2)\int\frac{G_3(x)dx}{x-t^2}\,.$$

We thus get $E'(p^2)=0$ for this choice of Γ^A.

It is perhaps worth stressing that the choice of Γ^A in this paper is not sacrosanct. For calculational case other choices satisfying the requisite boundary conditions may be equally acceptable.

as follows:

$$\frac{1}{\pi}\,\mathrm{Im}Z^{(n+1)}(p^2)=\frac{\alpha\theta(p^2-m^2)}{2p^2}[(a-3)(p^2+m^2)\,|Z^{(n)}(p^2)|^2+2(p^2-m^2)^2\,\mathrm{Re}Z^{(n)*}(p^2)E^{(n)\prime}(p^2)],\tag{III.9}$$

$$\frac{1}{\pi}\,\mathrm{Im}E^{(n+1)\prime}(p^2)=\frac{\alpha\theta(p^2-m^2)}{2(p^2-m^2)^3}\int_{2m^2-p^2}^{m^4/p^2}dq^2\Bigg[\,|Z^{(n)}(p^2)|^2\bigg(\frac{4(p^2+m^2)^2}{p^2-m^2}-\frac{9p^4+5p^2q^2+3m^2q^2+9m^2p^2+6m^4}{p^2-q^2}\bigg)$$
$$+\mathrm{Re}Z^{(n)}(p^2)Z^{(n)*}(q^2)\bigg(\frac{(p^2+q^2+2m^2)^2}{q^2-m^2}-\frac{4p^4+9p^2q^2+q^4+7m^2p^2+5m^2q^2+6m^4}{p^2-q^2}\bigg)\Bigg].\tag{III.10}$$

At each iteration stage Eq. (III.9) improves $Z(p^2)$ and therefore $\Gamma^A[\Delta]$, while Eq. (III.10) gives the corresponding Γ^B ($\Gamma=\Gamma^A+\Gamma^B$). The starting expressions from (III.9) are

$$(1/\pi)\,\mathrm{Im}[Z^{(0)}(p^2)]^{-1}=-[\alpha\theta(p^2-m^2)/2p^2](a-3)(p^2+m^2)\tag{III.11}$$

giving

$$[Z^{(0)}(p^2)]^{-1}=1+\tfrac{1}{2}\alpha(a-3)(1-m^2/p^2)\ln(1-p^2/m^2),\tag{III.12}$$

i.e., precisely the expressions written down in Part II. 1. In the next order of the iteration, substitute (III.12) on the right-hand side (III.10). This gives $(1/\pi)\,\mathrm{Im}E'^{(1)}$ which in the limit of large p^2 equals[34]

$$\frac{1}{\pi}\,\mathrm{Im}E^{(1)\prime}(p^2)\approx\frac{\alpha}{2p^4}\int_{-p^2}^{0}dq^2\bigg(\frac{2}{\alpha(3-a)}\bigg)^2\bigg[\frac{1}{p^2\ln^2p^2}\bigg(4-\frac{9p^2+5q^2}{p^2-q^2}\bigg)+\frac{1}{p^2q^2\ln p^2\ln q^2}\bigg(\frac{(p^2+q^2)^2}{q^2}+\frac{4p^4+9p^2q^2}{p^2-q^2}\bigg)\bigg]$$
$$\approx\frac{2\alpha}{p^2[\alpha(3-a)\ln(p^2/m^2)]^2},\quad a\neq 3.\tag{III.13}$$

Using the dispersion relation (III.8), we get for $E^{(1)\prime}(p^2)$

$$(p^2-m^2)E^{(1)\prime}(p^2)\approx\frac{2\alpha}{[\alpha(3-a)]^2\ln(p^2/m^2)}.\tag{III.14}$$

Collecting all terms, this means that the "first-order" correction to the full Γ equals

$$\Gamma_a{}^{(1)}(p,p')\,|_{p'^2=m^2,t^2=0}$$
$$\approx\frac{(p+p')_a}{\frac{1}{2}\alpha(3-a)\ln(p^2/m^2)}-\frac{(p-p')_a}{\frac{1}{2}\alpha(3-a)^2\ln(p^2/m^2)}.\tag{III.15}$$

As expected from the earlier discussion, the correction term (for the full Γ) $t_a/\ln p^2$, behaves asymptotically in the same way as the initial term $\Gamma^A\approx(p+p')_a/\ln p^2$. If we insert $E^{(1)\prime}(p^2)$ (i.e., the full Γ) of Eq. (III.15) into (III.9), we obtain $Z(p^2)$ (and therefore $\Gamma^A[\Delta]$) to the next order:

$$\frac{1}{\pi}\,\mathrm{Im}Z^{(2)}(p^2)\approx\frac{2}{\alpha(a-3)\ln^2(p^2/m^2)}\bigg[1+\frac{2}{(a-3)^2}\bigg],$$

i.e.,

$$[Z^{(2)}]^{-1}(p^2)$$
$$\approx 1+\frac{\alpha(a-3)}{2}\ln\bigg(\frac{p^2}{m^2}\bigg)\bigg[1+\frac{2}{(a-3)^2}\bigg]^{-1},\tag{III.16}$$

which has exactly the same form as (III.15) apart from an effective change in coupling[35] and this behavior will persist in all higher orders.

B. The Photon Equations

Like the meson case, we first set down formal Hilbert-Muskhelishvili solutions for the photon propagator and vertex equations and then construct explicit expressions for D and Γ by an iteration procedure where (as in Part I. 1) the iteration starts by assuming $\Gamma=\Gamma^A[D]$. Rewrite Eqs. (II.56) and (II.85) in the form:

$$\frac{1}{\pi}\,\mathrm{Im}Z_3{}^{-1}(t^2)=-\frac{\alpha t^2}{(t^2-\mu^2)}\bigg(1-\frac{4m^2}{t^2}\bigg)^{3/2}$$
$$\times\theta(t^2-4m^2)\,|Z_3{}^{-1}(t^2)\mathcal{E}(t^2)|^2,\tag{III.17}$$

$$\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{E}(t^2)=\tan\theta(t^2)\,\mathrm{Re}\mathcal{E}(t^2),\tag{III.18}$$

where from (II.85),

$$\tan\theta\underset{t^2\to\infty}{\approx}\alpha\int_{-\infty}^{0}\frac{\mathcal{E}^2(k^2)Z_3{}^{-1}(k^2)dk^2}{k^2-\mu^2}.\tag{III.19}$$

[34] The part of the integrand which gives rise to infrared divergence difficulties has been discarded and the asymptotic behavior of the logarithmic integral $\int_0^{t^2}dk^2/\ln k^2\approx t^2/\ln t^2$ has been used.

[35] The fact that the change in coupling seems appreciable is not to be taken too seriously. Thus our neglect of many-particle contributions to (III.9) and (III.10) must be borne in mind as well as the approximation used to derive (III.10) which is only roughly unitary. The only important point is that the high-energy behavior remains stable.

The boundary conditions are $Z_3(0)=\mathcal{E}(0)=1$ while the stability criterion specifies that (apart from logarithmic factors) at worst $\mathcal{E}(t^2)\approx 1$ for large t^2.

Now the Muskhelishvili solution of (III.18) [incorporating $\mathcal{E}(0)=1$] gives

$$\mathcal{E}(t^2)=\exp\left[\frac{t^2}{\pi}\int\frac{\theta(x)dx}{(x-t^2)x}\right] \tag{III.20}$$

and (apart from logarithmic factors) this behaves at infinity like[36]

$$\mathcal{E}(t^2)\approx\left[\frac{4m^2-t^2}{4m^2}\right]^{-1/\pi\theta(\infty)}. \tag{III.21}$$

Since from (III.15) $\theta(\infty)\neq 0$, it is clear from a substitution of (III.21) into (II.37) that $Z_3=(1-\int G_3)<\infty$. Also since $\mathcal{E}(t^2)$ falls rapidly at infinity it is perfectly possible to write an unsubtracted dispersion relation for $Z_3(t^2)$,

$$Z_3(t^2)=\frac{\alpha}{6}\int\left(1-\frac{4m^2}{x}\right)^{3/2}\frac{|\mathcal{E}(x)|^2}{(x-t^2)}dx, \tag{III.22}$$

provided α satisfies[37] the boundary equation

$$1=Z_3(0)=\frac{\alpha}{6}\int\left(1-\frac{4m^2}{x}\right)^{3/2}\frac{|\mathcal{E}(x)|^2}{x}dx. \tag{III.23}$$

To see how the unsubtracted integral (III.22) behaves, consider the case $\mu^2=0$ discussed in footnote 36. If $\mathcal{E}\approx 1/(t^2)^{1/2}$, (III.22) gives[38] (apart from logarithmic factors) $Z_3\approx 1/t^2$, i.e., the photon propagator $D\approx$ constant for large t^2. Intrinsically there is nothing in the discussion so far to preclude such behavior (the stability criterion is still satisfied since $Z_3^{-1/2}\mathcal{E}\approx 1$); however (in spite of the theoretical attractiveness of the eigenvalue Eq. (III.23) and the possibility of determining the fine structure constant $2\pi\alpha$ from it), we do not feel warranted to entertain an unsubtracted dispersion relation just from a discussion based on the two-particle unitarity approximation with one-photon exchange. Taking the conservative attitude, i.e., that the dispersion integral for $Z_3(t^2)$ needs one subtraction, we write

$$Z_3^{-1}(t^2)=1-\frac{\alpha}{6}t^2\int\left(1-\frac{4m^2}{x}\right)^{3/2}\times\frac{|Z_3^{-1}(x)\mathcal{E}(x)|^2}{x(x-t^2)}dx, \tag{III.24}$$

so that, apart from logarithmic factors, $Z_3^{-1}(t^2)\approx 1$ just like perturbation theory.

Like the case of the meson equations, one may now set up an iterative scheme to solve (III.20) and (III.24). One attractive scheme which is fairly close to perturbation solution can be obtained by writing (III.18) in the form

$$\frac{1}{\pi}\operatorname{Im}Z_3^{-1}(t^2)\mathcal{E}(t^2)=\tan(\theta+\beta)\operatorname{Re}[Z_3^{-1}(t^2)\mathcal{E}(t^2)], \tag{III.25}$$

where

$$\tan\beta=\frac{\operatorname{Im}Z_3^{-1}(t^2)}{\operatorname{Re}Z_3^{-1}(t^2)}.$$

Assuming that the fine structure constant $(2\pi\alpha)$ is small so that $\tan(\theta+\beta)\approx 0$, a first approximation to (III.20) which incorporates $\mathcal{E}(0)=Z_3(0)=1$ is given by[39]

$$\mathcal{E}^{(0)^{-1}}(t^2)Z_3^{(0)}(t^2)=1.$$

Substituting this on the right in (III.24) we get

$$(1/\pi)\operatorname{Im}[Z_3^{(0)}(t^2)]^{-1}=-[\alpha t^2/6(t^2-\mu^2)]\times(1-4m^2/t^2)^{3/2}\theta(t^2-4m^2),$$

so that

$$[Z_3^{(0)}(t^2)]^{-1}=1+\frac{\alpha t^2}{6}\int_{4m^2}^{\infty}\frac{dx}{x(t^2-x)}\left(1-\frac{4m^2}{x}\right)^{3/2}=[\mathcal{E}^{(0)}(t^2)]^{-1}. \tag{III.26}$$

The $Z_3(t^2)$ thus obtained coincides with the usual perturbation expression. The higher iterations (defined below), however, differ. Thus if we write

$$\frac{1}{\pi}\operatorname{Im}Z_3^{(n+1)}(t^2)=\frac{\alpha}{6}\left(1-\frac{4m^2}{t^2}\right)^{3/2}\times\theta(t^2-4m^2)|\mathcal{E}^{(n)}(t^2)|^2, \tag{III.27}$$

$$\frac{1}{\pi}\operatorname{Im}\mathcal{E}^{(n+1)}(t^2)=\frac{\alpha\theta(t^2-4m^2)}{2(t^4-4m^2t^2)^{1/2}}\times\int_{4m^2-t^2}^{0}dk^2\operatorname{Re}\mathcal{E}^{(n)}(t^2)\mathcal{E}^{(n)2}(k^2)Z_3^{(n)-1}(k^2)\times\frac{(4m^2-t^2-2k^2)(2t^2+k^2-4m^2)}{(4m^2-t^2)k^2}, \tag{III.28}$$

[36] From (III.19) clearly $\theta(\infty)\neq 0$. In order, however, that $\mathcal{E}(t^2)$ does not increase faster than a constant (the stability criterion), the fine structure constant α must be restricted so that

$$\tan^{-1}\left[\alpha\int_{-\infty}^{0}\frac{\mathcal{E}^2(k^2)Z_3^{-1}(k^2)dk^2}{k^2-\mu^2}\right]\geqslant 0.$$

For $\mu^2=0$, $\theta(\infty)=\pi/2$ so that this condition holds for all $\alpha>0$ and

$$\mathcal{E}(t^2)\approx(1/t^2)^{1/2}.$$

[37] In order that $Z_3(t^2)$ has no CDD zeroes, α has to lie within a special range of values (see footnote 22). Just to avoid introducing the unwanted CDD ambiguities, in (III.24) we have written a dispersion relation for $Z_3^{-1}(t^2)$.

[38] Because of the gauge invariance of $\mathcal{E}(t^2)$, $\mathcal{E}$ cannot depend on the gauge varying quantity G but only on the gauge-independent function G_3.

[39] In a sense this is equivalent to an expansion of the exponent in

$$Z_3^{-1}(t^2)\,\mathcal{E}(t^2)=\exp\left[\frac{t^2}{\pi}\int\frac{\tan^{-1}(\theta+\beta)}{x(t^2-x)}dx\right]$$

assuming that $\tan^{-1}(\theta+\beta)$ is proportional to α.

with $Z_3^{(0)}=Z_3^{(1)}$, it is easy to see that explicitly in the "next order," for example,

$$\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{E}^{(1)}(t^2)\approx\frac{\alpha\theta(t^2-4m^2)}{2t^2(\alpha/6)\ln(t^2/m^2)}\int_{-t^2}^{0}dk^2\times\frac{(t^2+2k^2)(2t^2+k^2)}{t^2k^2}\frac{6}{\alpha\ln(k^2/m^2)}\approx\frac{72\theta(t^2-4m^2)}{\alpha\ln^2(t^2/m^2)},\quad\text{(III.29)}$$

i.e.,

$$\mathcal{E}^{(1)}(t^2)\approx\int_{4m^2}^{\infty}\frac{72dx}{\alpha(x-t^2)\ln^2(x/m^2)}\approx\frac{72}{\alpha\ln(t^2/m^2)}.\quad\text{(III.30)}$$

This expression for $\mathcal{E}^{(1)}(t^2)$ [i.e. Γ^B] is to be compared with $\mathcal{E}^{(0)}(t^2)$ (i.e. Γ^A) $\approx 6/\alpha\ln(t^2/m^2)$. The magnitude of the "correction" gives an idea of the "effective expansion parameter." So far as the behavior at infinity is concerned this is

$$\lim_{t^2\to\infty}\frac{\mathcal{E}^{(1)^{-1}}-\mathcal{E}^{(0)^{-1}}}{\mathcal{E}^{(0)^{-1}}}=\frac{(1+(\alpha/72)\ln t^2)-(1+(\alpha/6)\ln t^2)}{(1+(\alpha/6)\ln t^2)}\approx-\frac{11}{12}$$

with $\mathcal{E}^{(0)}$ and $\mathcal{E}^{(1)}$ given by (III.26) and (III.30), respectively. The important remark is not the size of this "expansion parameter"; what we wish to stress repeatedly is that our major concern in this paper is with the high-energy behavior, and a consistent and stable behavior is being provided by our iteration procedure which starts with $\Gamma=\Gamma^A[\Delta,D]$ (i.e., $\mathcal{E}^{(0)}=Z_3^{(0)}$).

C. Summary

To summarize the work of this section:

(1) We have shown that $Z=\lim_{p^2\to\infty}Z(p^2)=0$ for all $\alpha>0$, $a\neq 3$ in the two-particle unitarity approximation.

(2) We have obtained two solutions for the photon equations; one of these corresponds to the usual perturbation solution with one subtraction, the other is a new solution and exists only for special values of α, provided a no subtraction dispersion relation is assumed to hold. In this paper we do not wish to choose between the alternative solutions.

(3) For either case a good first approximation to Γ is provided by

$$\Gamma^A=[\Delta^{-1}(p)-\Delta^{-1}(p')]/(p^2-p'^2)\times[(p+p')_a-d_{ab}(t)(p+p')_b(\mathcal{E}(t^2)-1)].\quad\text{(III.31)}$$

This expression has the merit that the two-particle *photon* (propagator as well as vertex) *equations are exactly and identically solved for* $\Gamma=\Gamma^A$. This is not true for the corresponding meson equations, but a simple iteration scheme can be set up which starting with Γ^A, computes the full Γ out in successive stages. A convenient lowest iterate for $\Delta^{(0)}$ is given by

$$\Delta^{(0)}(p)=(p^2-m^2)^{-1}\times[1+\tfrac{1}{2}\alpha(a-3)(1-m^2/p^2)\ln(1-p^2/m^2)]\quad\text{(III.12)}$$

and the corresponding iterate for $\mathcal{E}^{(0)}(t^2)$ (for the case when we allow one subtraction constant) is given by

$$[\mathcal{E}^{(0)}(t^2)]^{-1}=[Z_3^{(0)}(t^2)]^{-1}=1+\frac{\alpha t^2}{6}\int_{4m^2}^{\infty}\frac{dx}{x(t^2-x)}\left(1-\frac{4m^2}{x}\right)^{3/2}.\quad\text{(III.26)}$$

This Γ^A has the property that when any pair of particles is placed on the mass shell its asymptotic behavior in the unphysical momentum $k(=p,\,p'$ or $t)$ is

$$\Gamma[\Delta^{(0)},D^{(0)}]\approx(p+p')_a/\gamma\ln k^2-(p-p')_a/\beta\ln k^2\quad\text{(III.32)}$$

with constants γ and β proportional to the fine structure constant $2\pi\alpha$. This characteristic behavior *is always retained in every subsequent iteration* where $\Delta^{(0)}$ and $D^{(0)}$ are replaced by $\Delta^{(n)}$ and $D^{(n)}$. This same behavior is exhibited by any iteration to the full Γ (on the two-particle mass shell)—i.e., Γ^B behaves in the same manner as Γ^A for large values of the momenta.

2. VECTOR ELECTRODYNAMICS

A. Alternative Solutions

The power of the stability criterion in specifying acceptable high energy behavior of Δ, Γ and D is first really exhibited in the conventionally unrenormalizable theory of vector electrodynamics. A full discussion was given in I; we summarize the conclusions.

The Dyson equations for $\mathbf{\Delta}$ are

$$Z_1(p^2)=Z\left(\frac{p^2-m_0^2}{p^2-m^2}\right)+\tfrac{1}{3}\,\mathrm{Tr}\left(\frac{\mathbf{d}\cdot(t_a\mathbf{K}_a)}{p^2-m^2}\right),\quad\text{(III.33)}$$

$$Z_2(p^2)=Zm_0^2+\mathrm{Tr}(\mathbf{e}\cdot t_a\mathbf{K}_a).\quad\text{(III.34)}$$

Since by definition[40]

$$Z=\lim_{p^2\to\infty}Z_1(p^2)\quad\text{and}\quad Zm_0^2=\lim_{p^2\to\infty}Z_2(p^2),\quad\text{(III.35)}$$

the second terms on the right of (III.33) and (III.34) must approach zero. Now a *sufficient* condition for $\Gamma^A\Delta\approx 1/k$ to hold [with Γ^A for example defined in

[40] Dyson defined Z and Zm_0^2 differently from (III.35), viz., as boundary values of Δ (and its derivative) at $p^2=m^2$. The equivalence of the two definitions when these constants are finite was shown in Paper I (footnote 6). In so far as Z is the boundary value of Δ^{-1}, the renormalized Dyson equation, rewritten as

$$\Delta^{-1}=(\lim\Delta^{-1})\Delta_0^{-1}+\int\Gamma\Delta\Gamma D+\cdots$$

is clearly for from being just a simple integral equation of a conventional type.

(II.28)] is given by [see paper I, (2.21)]

$$\lim_{p^2\to\infty} p^2 Z_1(p^2) \approx Z_2(p^2). \qquad \text{(III.36)}$$

There are two distinct possibilities:—[A] *Either* Zm_0^2 is finite and (III.33) solves with the *boundary behavior* $Z_1(p^2)\approx 1/p^2$ (i.e., $Z=0$ automatically), *or* [B] $Z\neq 0$, but $Z_2(p^2)\approx p^2$. For this case the equations (III.33) and (III.34) must be carefully interpreted and m_0^2 *must be intrinsically quadratically infinite*. For case [A], we expect therefore $\Delta\approx\Delta_0\approx 1$, $\Gamma^A\approx 1/p$. For case [B], $\Delta\approx 1/p^2$, $\Gamma^A\approx p$.

These two will be called *the vector and the scalar alternatives, respectively*.[41] The two alternatives are differentiated by the number and character of subtraction constants.

In Secs. B–E we assume $m_0^2<\infty$ and investigate in detail the "vector alternative" turning briefly to the "scalar alternative" in Appendix III. It appears that the "vector alternative" gives results similar to lowest order perturbation theory for the propagator Δ though not for Γ. The "scalar alternative" ($m_0^2=\infty$) *if it exists* has no correspondence with the perturbation solution for Δ or Γ.

B. Construction of Γ^A

Using Muskhelishvili methods we may (as for the case of scalar-electrodynamics) seek to find (formal) solutions for the set of equations for Δ, Γ, and D, and thereby verify that

$$\Gamma\approx 1/(ap+bp'), \quad D\approx 1/t^2, \quad Z_1(p^2)\approx 1/p^2, \quad Z_2(p^2)\approx 1. \qquad \text{(III.37)}$$

Alternatively, we may use the simpler procedure of Part I. 1B; i.e., choose Γ^A which satisfies $\Gamma^A\sim 1/(ap+bp')$ and check from Eqs. (II.62–II.64) that Z_1, Z_2, and D as well as Γ^B (on two-particle mass shell) do exhibit the behavior (III.37).

On account of the complexity of the equations for vector electrodynamics we use here the simpler procedure of Part I. 1B. The required acceptable form for $\Gamma^A[\Delta,D]$ will be constructed in stages, first by considering the photon equations and then the meson equations. For the photon case we explicitly solve both the propagator and the vertex-function equations, and directly verify $\Gamma^B/\Gamma^A\approx 1$ on the two-meson mass shell. For the meson case, only the propagator equation is solved explicitly, reliance being placed on a dimensional argument for the assertion $\Gamma^B/\Gamma^A\approx 1$ (on the meson-photon mass shell).

C. The Photon Equations

Since the spectral function is gauge-independent and positive definite,[26] one can use Lehmann's theorem directly. This states

$$Z_3(t^2)\underset{t^2\to\infty}{=} D^{-1}(t^2)/t^2 = O(1), \qquad \text{(III.38)}$$

implying from (II.62) that $t^2\mathcal{E}(t^2)$, $t^2\mathfrak{M}(t^2)$ and $t^4\mathcal{Q}(t^2)$ behave in the like manner at infinity.[42]

The stability criterion (on the mass shell $p^2=p'^2=m^2$) however gives relations like $\mathcal{E}(t^2)Z_3^{-1/2}(t^2)\approx 1/t^2$. Thus for consistency one needs

$$G_3(t^2)\approx\mathcal{E}(t^2)\approx\mathfrak{M}(t^2)\approx 1/t^2, \quad \mathcal{Q}(t^2)\approx 1/t^4 \qquad \text{(III.39)}$$

for large t^2.

Rewrite now Eqs. (II.62) and (II.97) in the form

$$\mathrm{Im}Z_3^{-1}\approx -\alpha[t^4|Z_3^{-1}\mathcal{E}|^2 - 2t^4\,\mathrm{Re}(Z_3^{-1}\mathcal{E}) \times(Z_3^{-1}\mathfrak{M})^* + t^4|Z_3^{-1}\mathfrak{M}|^2+\cdots] \qquad \text{(III.40)}$$

$$\mathrm{Im}(Z_3^{-1}\mathcal{E}) = \tan\theta_1\,\mathrm{Re}(Z_3^{-1}\mathcal{E}) + \tan\theta_2\,\mathrm{Re}(Z_3^{-1}\mathfrak{M})+\cdots \qquad \text{(III.41)}$$

$$\mathrm{Im}(Z_3^{-1}\mathfrak{M}) = \tan\theta_3\,\mathrm{Re}(Z_3^{-1}\mathcal{E}) + \tan\theta_4\,\mathrm{Re}(Z_3^{-1}\mathfrak{M})+\cdots \qquad \text{(III.42)}$$

with a similar equation for $Z_3^{-1}\mathcal{Q}$. Here $\tan\theta$'s are complicated functionals of Z_3, $\mathcal{E}$, $\mathcal{Q}$ and $\mathfrak{M}$. Just to examine the structure of these equations, consider (III.38) and (III.39) only. If $\mathfrak{M}$ and $\mathcal{Q}$ are treated as unknown functions, Eq. (III.40) has the inhomogeneous Muskhelishvili form

$$\mathrm{Im}(Z_3^{-1}\mathcal{E}) = \tan\delta\,\mathrm{Re}(Z_3^{-1}\mathcal{E}) + U(t^2).$$

According to Muskhelishvili results, the (homogeneous) equation possesses a solution vanishing at infinity like $1/t^2$ if and only if the phase change $[\delta_{\mathcal{E}}]_{-\infty}^{\infty}=\pi$. This (and the analogous conditions for $\delta_{\mathfrak{M}}$, etc.) then are the (gauge-independent) restrictions on the possible values of the constants of the theory like α and the observed magnetic moment K, and are the restrictions mentioned in the introduction of this paper.

This condition and its implications in terms of Castillejo, Dalitz, and Dyson (CDD) poles and bound-states (Levinson's theorem) will be discussed in a separate paper. Assuming, however, the behavior (III.39) we may approximate $\Gamma=\Gamma^A$ or equivalently $\mathcal{E}=\mathcal{E}^A[Z_3]$ as follows: Define[43]

$$\mathcal{E}^A[Z_3] = AZ_3(t^2)\int\frac{G_3(x)dx}{t^2-x}, \qquad \text{(III.43)}$$

[41] The "*scalar*" alternative is the one conjectured for the full propagator Δ by T. D. Lee and C. N. Yang. [Phys. Rev. **128**, 885 (1962)] and arises after a summation of perturbation graphs.

[42] More precisely, from footnote (26) one infers that it is $\mathcal{E}(t^2)$, $(t^2)^{1/2}\mathfrak{M}(t^2)$ and $\chi(t^2)$ which must possess the stipulated behavior. An examination of the form of $\chi(t^2)$, however, shows that unless there are cancellations between $t^2\mathcal{Q}$, $\mathfrak{M}$ and $\mathcal{E}$, in general the restriction must obtain as stated in the text.

[43] One can rewrite (III.43) in the form

$$\mathcal{E}^A[Z_3] = \frac{Z_3(t^2)[1-Z_3(t^2)]}{t^2 Z_3'(0)}.$$

and take as first approximation:

$$\mathcal{E}=\mathcal{E}^A[Z_3],\quad \mathfrak{M}=\mathfrak{M}^A[Z_3]=K\mathcal{E}^A(t^2),\quad \mathcal{Q}=\mathcal{Q}^A=0,\qquad \text{(III.44)}$$

where A is the normalization constant $[-\int G_3/xdx]$ specified to ensure $\mathcal{E}(0)=1$. Using $\Gamma=\Gamma^A$ from (III.44), Eq. (II.62) reduces to the form

$$\frac{1}{\pi}\,\mathrm{Im}Z_3^{-1}(t^2)=-\frac{\alpha}{24m^4t^4}\left(1-\frac{4m^2}{t^2}\right)^{3/2}$$
$$\times[(t^4-4m^2t^2+12m^4)-2Kt^2(t^2-2m^2)+K^2t^2(t^2+4m^2)]$$
$$\times|1-Z_3(t^2)|^2\theta(t^2-4m^2)(Z_3'(0))^2.\qquad \text{(III.45)}$$

If $K=1$ (normal perturbation theory has such a magnetic moment) we notice that the bracketed expression in (III.40) has a less singular behavior at large t^2 than for any other K. For reasons stressed further when we consider the meson equation, we wish to allow for a general value of K. In that case, a solution of (III.40) (exact in the asymptotic limit) can be constructed by first neglecting Z_3 on the right side and solving for a constant $Z_3'(0)$. This gives [allowing for one subtraction constant $Z_3(0)=1$],

$$Z_3^{-1}(t^2)=1-\frac{\alpha t^2}{24m^4}\int_{4m^2}^{\infty}\frac{dx}{x^3}\left(1-\frac{4m^2}{x}\right)^{3/2}$$
$$\times\frac{[(x^2-4m^2x+12m^4)-2Kx(x-2m^2)+K^2x(x+4m^2)]}{(Z_3'(0))^2(x-t^2)}$$
$$\approx 1-\frac{\alpha}{24m^4(Z_3'(0))^2}\ln\frac{t^2}{4m^2}.\qquad \text{(III.46)}$$

To obtain $Z_3'(0)$ in (III.46) all we need to do is solve the equation which it provides for itself, viz.,

$$Z_3^{-1\prime}(0)=Z_3'(0)=\frac{\alpha}{24m^4(Z_3'(0))^2}\int_{4m^2}^{\infty}\frac{dx}{x^4}\left(1-\frac{4m^2}{x}\right)^{3/2}$$
$$\times[(x^2-4m^2x+12m^4)-2kx(x-2m^2)+k^2x(x+4m^2)].\qquad \text{(III.47)}$$

Clearly, $\mathcal{E}(t^2)\approx m^2/\alpha^{2/3}t^2\ln t^2$. The eletcric and magnetic form factors for spin-one electrodynamics for the case $K\neq 1$ (more precisely $\lim_{t^2\to\infty}[\mathcal{E}(t^2)/\mathfrak{M}(t^2)]\neq 1$) are therefore highly convergent quantities. The solution (III.46) was obtained by neglecting Z_3 on the right of (III.45). The form of (III.46) shows that this neglect was perfectly justified if we want to get the asymptotic behavior. To solve (III.45) exactly, substitute (III.46) for Z_3 on the right and iterate.[44]

[44] To all intents and purposes we can regard Eq. (III.46) as the exact initial Z_3, since all boundary conditions are incorporated in its integral representation. The point is that it will be subject to corrections from further iterations specified in (III.48) in any case.

So far we have not considered (II.97) for the full vertex function. We first verify that if the approximations $\mathcal{E}=\mathcal{E}^A[Z_3]$, $\mathfrak{M}=\mathfrak{M}^A[Z_3]$ with Z_3 given by (III.46) are substituted on the right side of (II.97), the full $\mathcal{E}$ behaves similarly to $\mathcal{E}^A[Z_3]$. Explicitly, a lengthy calculation gives

$$\frac{1}{\pi}\,\mathrm{Im}\,\mathcal{E}(t^2)\approx\frac{\alpha}{32m^4t^8}\int_{-t^2}^{0}dk^2Z_3^{-1}(k^2)\,\mathcal{E}^2(k^2)$$
$$\times\mathrm{Re}\,\mathcal{E}(t^2)\cdot(t^2+2k^2)(t^2+k^2)$$
$$\times[-4t^4(k^2+2t^2)+8k^4(t^2+k^2)K+2t^6K^2]$$
$$\approx\frac{m^4}{\alpha^{2/3}t^{10}}\int_{-t^2}^{0}\frac{k^6dk^2}{\ln t^2\ln k^2}\approx\frac{m^2}{\alpha^{2/3}t^2(\ln t^2)^2}.\qquad \text{(III.48)}$$

This has exactly the same behavior as $\mathrm{Im}\,\mathcal{E}^A(t^2)$. Likewise we have computed $\mathrm{Im}\,\mathfrak{M}(t^2)$ and $\mathrm{Im}\,\mathcal{Q}(t^2)$ and have shown that the form factors tend to zero in the expected manner, i.e., $1/t^2(\ln t^2)^2$ and $1/t^4(\ln t^2)^2$. These were some of the most lengthy and arduous calculations of this paper and the verification of the stated results was a consequence of a number of cancellations which could not have been foreseen when the calculations were first set up.

To obtain $\mathrm{Re}\,\mathcal{E}$, $\mathrm{Re}\,\mathfrak{M}$, and $\mathrm{Re}\,\mathcal{Q}$, one can write down dispersion relations. On account of the rapid convergence of $\mathcal{E}$, $\mathfrak{M}$, and $\mathcal{Q}$, dispersion relations for these quantities need no subtractions. If this is the case, the three conditions

$$\mathcal{E}(0)=1=-\frac{1}{\pi}\int\frac{\mathrm{Im}\,\mathcal{E}(x)}{x}dx,\qquad \text{(III.49)}$$

$$\mathfrak{M}(0)=K=-\frac{1}{\pi}\int\frac{\mathrm{Im}\,\mathfrak{M}(x)}{x}dx,\qquad \text{(III.50)}$$

$$\int\mathrm{Im}\,\mathcal{Q}(x)dx=0\qquad \text{(III.51)}$$

[the last necessary in order that $\mathcal{Q}(t^2)$ satisfies the stability criterion and exhibits at infinity the behavior $1/t^4$], determine three equations for the three unknowns α, K and the (physical) quadrupole moment q, introduced through the definition [see (II.55)]

$$\frac{2q}{m^2}=-\frac{1}{\pi}\int\frac{\mathrm{Im}\,\mathcal{Q}(x)dx}{x}+1-K.$$

Conditions (III.44)–(III.46) are the analogs (for our iteration solution) of the exact Muskhelishvili restrictions mentioned earlier on the phase change like $[\delta(\mathcal{E})]_{-\infty}{}^{\infty}=\pi$.

D. The Meson Propagator and the Final Form of Γ^A

We follow the procedure of Part III. 2C and examine first Eqs. (II.63) and (II.65) for indications of high-energy behavior of Δ and the full Γ. Unlike the photon case the spectral functions G_1 and G_2 are not positive definite except in the radiation gauge. Thus we cannot use the stronger form of Lehmann's theorem. However, the equations have the form

$$\operatorname{Im} Z_1^{-1} \approx \alpha[p^2 + \text{terms involving } Z_1^{-1}E,\ Z_1^{-1}M, \text{ etc.}],$$

$$\operatorname{Im} Z_2^{-1} \approx \alpha[1 + \text{terms involving } Z_2^{-1}Z_1,\ Z_2^{-1}E, \text{ etc.}]$$

so that barring a cancellation of the leading terms within the brackets [ ], one may expect $Z_1 \approx 1/p^2$, $Z_2 \approx 1$. In more detail if we consider (II.63) and (II.64) as they stand and impose the requirements *of the vector alternative* (see Part III. 2A), $\int G_1 dx < \infty$ and $\int G_2 dx < \infty$ (*Z and Zm_0^2 finite*), we would need

$$Z_1 \approx E \approx M \approx p^2 E' \approx p^2 N' \approx p^4 Q' \approx 1/p^2, \quad Z_2 \approx 1,$$

or equivalently $\Delta \approx 1$ and $\Gamma \approx 1/(\alpha p + \beta p')$. With these boundary conditions, we are in a position to specify a form for $\Gamma^A[\Delta, D]$. The tentative form suggested in Eq. (II.28) satisfies Ward's identity but behaves unacceptably like $a/p + b/p'$. It also has no functional dependence on the photon propagator D. Both these shortcomings are simultaneously removed provided we replace in (II.28) the factor

$$(p+p')_a g_{\mu\nu} \text{ by } [g_{\mu\nu}(p+p')_b - K(g_{b\mu}t_\nu - g_{b\nu}t_\mu)] \times [g_{ab} + d_{ab}(t)(1 - \mathcal{E}^A(t^2))]$$

$$g_{a\mu} \text{ by } [g_{a\mu} + d_{a\mu}(t)(1 - \mathcal{E}^A(t^2))]$$

$$g_{a\nu} \text{ by } [g_{a\nu} + d_{a\nu}(t)(1 - \mathcal{E}^A(t^2))]$$

and

$$p_\mu p_\nu'(p+p')_a \text{ by } p_\mu p_\nu'(p+p')_b [g_{ab} + d_{ab}(t)(1 - \mathcal{E}^A(t^2))].$$

Writing out in full, we propose to choose[45]

$$\begin{aligned}\Gamma_{a\mu\nu}{}^A[\Delta, D] = {} & [-g_{\mu\nu}(p+p')_b + K(g_{b\mu}t_\nu - g_{b\nu}t_\mu)] \\ & \times[-g_{ab} + d_{ab}(t)(1 - \mathcal{E}^A(t^2))]\left[Z + \int \frac{(x-m^2)^2 G_1(x)dx}{(x-p^2)(x-p'^2)}\right] \\ & + p_\mu[g_{a\nu} + d_{a\nu}(t)(1 - \mathcal{E}^A(t^2))]\int \frac{\mathcal{G}(x)dx}{x-p^2} + p_\nu'[g_{a\mu} + d_{a\mu}(t)(1 - \mathcal{E}^A(t^2))]\int \frac{\mathcal{G}(x)dx}{x-p'^2} \\ & + p_\mu p_\nu'(p+p')_b[g_{ab} + d_{ab}(t)(1 - \mathcal{E}^A(t^2))]\int \frac{\mathcal{G}(x)dx}{(x-p^2)(x-p'^2)}. \qquad \text{(III.52)}\end{aligned}$$

With this Γ^A, the meson equations reduce exactly to the form

$$\frac{1}{\pi}\operatorname{Im} Z_1^{-1}(s) = \frac{\alpha\theta(s-m^2)}{24m^2 s^2}(s+m^2)[3a(s+m^2)^2 - 2(s-m^2)^2 - 3(s^2 + 10m^2 s + m^4) + 2(s-m^2)^2(K+K^2)], \qquad \text{(III.53)}$$

$$\begin{aligned}\frac{1}{\pi}\operatorname{Im} Z_2(s) = {} & \frac{\alpha(s-m^2)\theta(s-m^2)}{8m^4 s^2}[3am^4(s+m^2)|Z_2(s)|^2 - 3(s-3m^2)|F_1(s)|^2 \\ & - (5/4)(s-m^2)^2(s+m^2)|F_2(s)|^2 - 2(s-m^2)(2s+m^2)\operatorname{Re} F_1^*(s)F_2(s)], \qquad \text{(III.54)}\end{aligned}$$

where

$$F_1(s) = -\tfrac{1}{2}(s-m^2)(2-K)Z_1(s) - m^2 Z_2(s), \quad F_2(s) = (2-K)Z_1(s). \qquad \text{(III.55)}$$

Whereas Eq. (III.53) is immediately soluble, the same is not true for (III.54) except in the special circumstance $K=2$, corresponding to the fully symmetric Salam-Ward-Glashow electrodynamics.[46] For simplicity of solution we might fix on $K=2$ to determine Γ^A, other possible values being treated as a perturbation. For $K=2$, Eqs. (III.53) and (III.54) reduce to the form

$$\frac{1}{\pi}\operatorname{Im} Z_1^{-1}(s) = \frac{\alpha(s+m^2)}{24m^2 s^2}\theta(s-m^2)[3(a-1)(s+m^2)^2 - 2(s^2 + 10m^2 s + m^4) + 12(s-m^2)^2], \qquad \text{(III.56)}$$

$$\frac{1}{\pi}\operatorname{Im} Z_2^{-1}(s) = -\frac{3\alpha(s-m^2)}{8s^2}\theta(s-m^2)[a(s+m^2) - (s-3m^2)]. \qquad \text{(III.57)}$$

Since $\operatorname{Im} Z^{-1}(s) \approx s$, the dispersion representations for Δ *will involve one extra subtraction constant other than*

[45] Here we define more properly

$$\mathcal{E}^A(t^2) = \lim_{\mu^2 \to 0} A Z_3(t^2) \frac{(t^2 - \mu^2)}{t^2} \int \frac{G_3(x)}{t^2 - x} dx,$$

where the limit $\mu^2 \to 0$ is taken at the end of a calculation. This ensures that $E' = N' = Q' = 0$ in (II.71) and (II.72) for the starting approximation $\Gamma = \Gamma^A$.

[46] A. Salam and J. C. Ward. Nuovo Cimento **11**, 568 (1959); S. Glashow, Nuc. Phys. **10**, 107 (1959).

$Z_1(m^2)=1$. This will be taken as $Z(0)=Z_1(0)=Z_2(0)$.[47] The fact that this subtraction is needed is another statement of the boundary condition $Z_1(p^2)\sim 1/p^2$, i.e., $Z\equiv 0$ for all $\alpha>0$. The equations (III.56) and (III.57) possess the following solutions[48]:

$$Z_1^{-1}(s)=1+(s-m^2)\left[\frac{1-Z^{-1}(0)}{m^2}-\frac{s}{\pi}\int_{m^2}^{\infty}\frac{\operatorname{Im}Z_1^{-1}(x)dx}{x(x-m^2)(s-x)}\right]$$

$$=1+(s-m^2)\frac{[1-Z^{-1}(0)]}{m^2}-\frac{s(s-m^2)\alpha}{24m^2}\int\frac{dx(x+m^2)}{x^3(s-x)}[3(a-1)(x+3m^2)+(10x-34m^2)]$$

$$\approx\frac{s}{m^2}[1-Z^{-1}(0)]-\frac{\alpha(3a+7)s}{24m^2}\ln\left(\frac{s}{m^2}\right), \tag{III.58}$$

and

$$Z_2^{-1}(s)=Z^{-1}(0)-\frac{s}{\pi}\int_{m^2}^{\infty}\frac{\operatorname{Im}Z_2^{-1}(x)dx}{x(s-x)}$$

$$=Z^{-1}(0)+\frac{3\alpha s}{8}\int_{m^2}^{\infty}\frac{dx(x-m^2)}{x^3(s-x)}[a(x+m^2)-(x-3m^2)]$$

$$\approx Z^{-1}(0)+\frac{3\alpha(a-1)}{8}\ln\left(\frac{s}{m^2}\right). \tag{III.59}$$

Surprisingly, $Zm_0^2=\lim_{s\to\infty}Z_2(s)$ is also zero for this particular approximation ($Z\equiv 0$, $\alpha>0$).

To treat the case of arbitrary K is not difficult. With the solutions (III.58) and (III.59) as the basic solutions a simple subsidiary iteration of (III.53) and (III.54) can readily be set up. We shall not write down the resulting expressions. With $Z_1(p^2)$ and $Z_2(p^2)$ known, Γ^A can now be explicitly written down as below.

E. Summary

To summarize, vector electrodynamics is a finite theory, provided the following insertions are made in vertices and lines of irreducible diagrams for all Green's functions other than Γ, Δ, and D. This is because Γ, Δ, and D satisfy (III.37).

$$\Gamma=\Gamma^A=[-g_{\mu\nu}(p+p')_b+2(g_{b\mu}t_\nu-g_{b\nu}t_\mu)][g_{ab}+d_{ab}(t)(1-\mathcal{E}(t^2))]$$
$$\times\frac{(p^2-m^2)Z_1(p^2)-(p'^2-m^2)Z_1(p'^2)}{p^2-p'^2}+p_\mu p_\nu'(p+p')_b[g_{ab}+d_{ab}(t)(1-\mathcal{E}(t^2))]\frac{\mathfrak{z}(p^2)-\mathfrak{z}(p'^2)}{p^2-p'^2}$$
$$+p_\mu[g_{a\nu}+d_{a\nu}(t)(1-\mathcal{E}(t^2))]\mathfrak{z}(p^2)+p_\nu'[g_{a\mu}+d_{a\mu}(t)(1-\mathcal{E}(t^2))]\mathfrak{z}(p'^2) \tag{III.60}$$

$$\Delta_{\mu\nu}^{-1}(p)=-g_{\mu\nu}Z_1(p^2)(p^2-m^2)+p_\mu p_\nu\mathfrak{z}(p^2) \tag{III.61}$$

$$D_{ab}^{-1}(t)=d_{ab}(t)t^2Z_3(t^2)-t_at_b/a. \tag{III.62}$$

A convenient first choice for Z_1, Z_2, Z_3, and $\mathcal{E}$ is as follows:

$$Z_3^{-1}(t^2)=1-\frac{\alpha t^2}{24m^4}\int_{4m^2}^{\infty}\frac{dx}{x^3}\left(1-\frac{4m^2}{x}\right)^{3/2}\frac{(x^2+20m^2x+12m^4)}{(Z_3'(0))^2(x-t^2)}, \tag{III.63}$$

$$\mathcal{E}(t^2)=\frac{Z_3(t^2)[1-Z_3(t^2)]}{t^2Z_3'(0)}, \tag{III.64}$$

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[47] If the renormalization constants possess a correspondence with the constants in a Lagrangian, a term $\xi((\partial A_\mu^+/\partial x_\mu)(\partial A_\mu^-/\partial x_\mu))$ in L_0 would get renormalized to $\xi'((\partial A_\mu^+/\partial x_\mu)(\partial A_\mu^-/\partial x_\mu))$ where $\xi'=\xi+Z_2'(0)$. Terms of this type however are not gauge-independent. In conventional formulation of the theory, L_0 contains only terms $F_{\mu\nu}^+F_{\mu\nu}^-+M_0^2A_\mu^+A_\mu^-$ (i.e., $\xi=0$) so that one may require $\xi'=0$. i.e, $Z_2'(0)=\int[G_2(x)/x]dx=0$. This requirement might be used to compute $Z_2(0)$.

[48] In writing (III.58) we have neglected the part of the integrand, proportional to $(a-3)$ characteristically, which gives rise to an infrared divergence.

$$Z_1^{-1}(p^2)=1+\frac{(p^2-m^2)[1-Z^{-1}(0)]}{m^2}-\frac{\alpha p^2(p^2-m^2)}{24m^2}\int_{m^2}^{\infty}dx\frac{(x+m^2)}{x^3(p^2-x)}[3(a-1)(x+3m^2)+10x-34m^2], \quad \text{(III.65)}$$

$$p^2\mathfrak{z}(p^2)=(p^2-m^2)Z_1(p^2)+m^2\left[Z^{-1}(0)+\frac{3\alpha p^2}{8}\int_{m^2}^{\infty}dx\frac{(x-m^2)}{x^3(p^2-x)}[(a-1)x+(a+3)m^2]\right]^{-1}. \quad \text{(III.66)}$$

To complete this set we also write down C^{AA} (computed from Γ^A of Eq. (III.60) as in the scalar case),

$$C^{AA}(p,p';k,k')$$
$$=Z[g_{a\mu}g_{b\nu}+g_{b\mu}g_{a\nu}-2g_{ab}g_{\mu\nu}]+\int\frac{[-g_{\mu\nu}(x-m^2)^2G_1(x)+p_\mu p_\nu'\mathcal{G}(x)]dx}{[x-p^2][x-(p+k)^2][x-p'^2][x-(p'-k)^2]}N_{ab}(p,p';k,k';x)$$
$$+K\int\frac{\nu_{ab\mu\nu}(p,p';k,k';x)(x-m^2)^2G_1(x)dx}{[x-p^2][x-(p+k)^2][x-p'^2][x-(p'-k)^2]}+\int\frac{\mathfrak{N}_{ab\mu\nu}(p,p';k,k';x)\mathcal{G}(x)dx}{[x-p^2][x-(p+k)^2][x-p'^2][x-(p'-k)^2]}, \quad \text{(III.67)}$$

where K is the total magnetic moment, N_{ab} is defined by Eq. (II.93), and G_1, $\mathcal{G}$ are the spectral functions of the meson propagator introduced in Eqs. (II.20) and (II.24),

$$\nu_{ab\mu\nu}(p,p';k,k';x)=[g_{a\nu}k_\mu-g_{a\mu}k_\nu][(2x-p^2-p'^2)(p+p')_b+k_b(p'^2-p^2)]$$
$$+[g_{b\mu}k_\nu'-g_{b\nu}k_\mu'][(2x-p^2-p'^2)(p+p')_a+k_a'(p^2-p'^2)], \quad \text{(III.68)}$$

and

$$\mathfrak{N}_{ab\mu\nu}(p,p';k,k';x)=(x-p^2)(x-p'^2)\{g_{a\mu}g_{b\nu}[x-(p'-k)^2]+g_{a\nu}g_{b\mu}[x-(p+k)^2]\}$$
$$+[(p+p'+k')_a p_\mu g_{b\nu}(x-p'^2)+(p+p'+k)_b p_\nu' g_{a\mu}(x-p^2)][x-(p'-k)^2]$$
$$+[(p+p'-k')_a p_\nu'(x-p^2)+(p+p'-k)_b p_\mu g_{a\nu}(x-p'^2)][x-(p+k)^2]. \quad \text{(III.69)}$$

These expressions provide a first (stable) approximation and essentially solve in a good approximation the two-particle unitarity equations. These can be improved upon, within the two-particle unitarity system of equations and also by incorporating higher particle states. Any improvement means a recomputation of Δ, Γ, and D by the methods described in Part III. 2 and is a major undertaking. It is, however, a perfectly feasible undertaking. *The "stability" of the starting approximation, however, will guarantee that any improvements will not alter high-energy behavior.* This is an important point. The general procedure for computing these improvements has been described already in Parts I. 1, III. 1, and III.2. It will be summarized in a practical form again in Part IV, where also we discuss how good an approximation to the full theory is obtained by just the ansatz of using (III.60)–(III.66).

3. THE DYSON-SCHWINGER EQUATIONS AND THE UNITARITY SOLUTIONS

Just to make doubly sure, we go finally back to the Dyson set for the basic Green's functions. We saw that the unitarity equations for Δ and Γ in Parts III. 1 and III. 2 were relatively easy to solve. The corresponding Dyson equations are much more complicated; also, as mentioned earlier, any truncation of these results in equations which do not possess even approximate unitarity. Rather than solve these equations *ab initio,* we show in this section that the high-energy behavior exhibited by the unitarity solutions is consistent with the corresponding Dyson set. This will ease our conscience.

In the equation

$$\Gamma=Z\Gamma_0+K[\Gamma,\Delta,D] \quad \text{(I.10)}$$

approximate to K by the gauge-covariant expression

$$K=[\Gamma\Delta\Gamma\Delta\Gamma+\Gamma\Delta C+C\Delta\Gamma]D, \quad \text{(III.70)}$$

where C is the proper Compton graph. The choice of K is the closest in Dyson-Schwinger terms to two-particle unitarity. Since

$$t_aK_a=-\frac{ie^2}{(2\pi)^4}\int d^4k[\Gamma(p,\,p-k)\Delta(p-k)\Gamma(p-k,\,p)D(k)$$
$$-\Gamma(p',\,p'-k)\Delta(p'-k)\Gamma(p'-k,\,p')D(k)], \quad \text{(III.71)}$$

we obtain the following equation for Δ^{-1}:

$$\Delta^{-1}(p)=Z\Delta_0^{-1}(p)-\frac{ie^2}{(2\pi)^4}$$
$$\times\int d^4k\Gamma(p,\,p-k)\Delta(p-k)\Gamma(p-k,\,p)D(k). \quad \text{(III.72)}$$

A. Scalar Electrodynamics

Writing

$$\Gamma_a(p,p')=(p+p')_a \times[g_{ab}A(p^2,p'^2)-d_{ab}(t)B(p^2,p'^2,t^2)], \quad \text{(III.73)}$$

where

$$A(p^2,p'^2)=\frac{\Delta^{-1}(p)-\Delta^{-1}(p')}{p^2-p'^2}, \quad \text{(III.74)}$$

Eq. (III.72) reduces, when p is time-like, to the form

$$\Delta^{-1}(x)=Z\Delta_0^{-1}(x)-\frac{ie^2}{4x(2\pi)^3}\int dydz\varphi(x,y,z)\Delta(y) \times\left(\{A(x,y)+B(x,y,z)\}^2\frac{Z_3^{-1}(z)}{z-\mu^2}\frac{\varphi^2(x,y,z)}{z} -\frac{a\mu^2}{Z_3\mu_0^2}\frac{A^2(x,y)}{z-a\mu^2}\frac{(x-y)^2}{z}\right). \quad \text{(III.75)}$$

Here

$$\varphi^2(x,y,z)=x^2+y^2+z^2-2xy-2yz-2zx, \quad \text{(III.76)}$$

with

$$x=p^2, \quad y=(p-k)^2 \quad \text{and} \quad z=k^2. \quad \text{(III.77)}$$

μ^2 is the photon mass and a specifies the gauge.

Using now the iteration procedure described in Part III. 1 starting the approximation as in Eq. (III.9) with

$$B(x,y,z)=A(x,y)[Z_3(z)-1]$$

Eq. (III.75) reduces in the Fermi gauge ($a=1$) to

$$\Delta^{-1}(x)=Z\Delta_0^{-1}(x)-\frac{ie^2}{4x(2\pi)^3}\int dydz\varphi(x,y,z)\Delta(y) \times\left[\frac{\Delta^{-1}(x)-\Delta^{-1}(y)}{x-y}\right]^2\frac{1}{z-\mu^2+i\epsilon} \times\left[Z_3^{-1}(z)\frac{\varphi^2(x,y,z)}{z}-\frac{\mu^2}{Z_3\mu_0^2}\frac{(x-y)^2}{z}\right]. \quad \text{(III.78)}$$

To recover second-order perturbation theory, make the usual approximations on the right-hand side

$$Z_3=1, \quad \mu^2=\mu_0^2, \quad \frac{\Delta^{-1}(x)-\Delta^{-1}(y)}{x-y}=1.$$

This gives

$$\Delta^{-1}(x)=Z\Delta_0^{-1}(x)-\frac{ie^2}{4x(2\pi)^3}\int dydz \times\frac{\varphi(x,y,z)(z-2x-2y)}{(z-\mu^2+i\epsilon)(y-m^2+i\epsilon)}. \quad \text{(III.79)}$$

For $\mu^2=0$ this verifies the well-known result

$$\frac{1}{\pi}\,\mathrm{Im}\Delta^{-1}(x)=-\frac{\alpha}{x}(x^2-m^4). \quad \text{(III.80)}$$

More generally, neglecting the radiative corrections to the photon line in (III.79) (these give rise to inessential complications), the equation reduces to the form

$$x\Delta^{-1}(x)=xZ\Delta_0^{-1}(x) +\int dyK(x,y)\Delta(y)[\Delta^{-1}(x)-\Delta^{-1}(y)]^2, \quad \text{(III.81)}$$

where[49]

$$K(x,y)\approx(\alpha x^2+\beta xy+\alpha y^2)/(x-y)^2 \quad \text{(III.82)}$$

for large x and y. As in the conventional Dyson treatment, we shall make two subtractions before considering the convergence of (III.81). This is achieved by using the boundary conditions

$$\Delta^{-1}(m^2)=0, \quad \left.\frac{\partial\Delta^{-1}(x)}{\partial x}\right|_{x=m^2}=1, \quad \text{(III.83)}$$

which finally give for Δ^{-1}

$$\Delta^{-1}(x)=Z[\Delta_0^{-1}(x)-\Delta_0^{-1}(m^2)-(x-m^2)(\partial\Delta_0^{-1}/\partial x)|_{m^2}] +(x-m^2)^2\int dy\chi(x,y)\Delta(y) \times[\Delta^{-1}(x)-\Delta^{-1}(y)]^2, \quad \text{(III.84)}$$

with

$$\chi(x,y)\approx\frac{1}{x}\frac{\partial^2K(x,y)}{\partial x^2}\approx\frac{1}{x(x-y)^2}. \quad \text{(III.85)}$$

Clearly,

$$\frac{1}{\pi}\,\mathrm{Im}\Delta^{-1}(x)\approx x\int\frac{dy}{(x-y)^2} \times\{\mathrm{Im}\Delta(y)\{\mathrm{Re}^2[\Delta^{-1}(x)-\Delta^{-1}(y)] -\mathrm{Im}^2[\Delta^{-1}(x)-\Delta^{-1}(y)]\} +2\,\mathrm{Re}\Delta(y)\,\mathrm{Re}[\Delta^{-1}(x)-\Delta^{-1}(y)] \times\mathrm{Im}[\Delta^{-1}(x)-\Delta^{-1}(y)]\}. \quad \text{(III.86)}$$

Inserting on the right of (III.86) the unitarity asymptotic values

$$-(1/\pi)\,\mathrm{Im}\Delta(x)\approx\delta(x-m^2)+(A/x)$$
$$\mathrm{Im}\Delta^{-1}(x)\approx x/A\ln^2x \quad \text{(III.87)}$$
$$\Delta(x)\approx(A\ln x/x)\approx\mathrm{Re}\Delta(x),$$

[49] This result remains the same even if the complete photon propagator is included in (III.81), assuming for $Z_3^{-1}(z)$ the unitarity behavior $\ln(z/m^2)$.

and using the following results[50]

$$\int_a^\infty \frac{dy}{(x-y)\ln^n y} \approx \frac{C(n)}{(\ln x)^{n-1}}, \quad n>1 \qquad \text{(III.88)}$$

and

$$\int_a^\infty \frac{dy}{(x-y)^2 \ln^n y} \approx \frac{C(n)(n-1)}{x(\ln x)^n}, \quad \text{all } n, \qquad \text{(III.89)}$$

we check that integrals appearing in (III.86), integrals like

$$\int \frac{dy}{y(x-y)^2}\left[\frac{x}{\ln x}-\frac{y}{\ln y}\right]^2,$$

are precisely of order $1/\ln^2 x$. This verifies that for large x we do indeed recover back the unitarity solution $\mathrm{Im}\Delta^{-1}(x) \approx x/A' \ln^2 x$. This argument does no more than verify asymptotic self-consistency; it is impossible of course to make any statement about matching of A' with A in (III.87) without fully solving (III.78).[51]

B. Vector Electrodynamics

The Dyson equations for the longitudinal and transverse parts of the meson propagator are much more complicated than in the scalar case but can still be written in the form

$$x\boldsymbol{\Delta}^{-1}(x) \approx \int dydzK(x,yz)[\boldsymbol{\Delta}^{-1}(x)-\boldsymbol{\Delta}^{-1}(y)] \times \boldsymbol{\Delta}(y)[\boldsymbol{\Delta}^{-1}(x)-\boldsymbol{\Delta}^{-1}(y)]+Zx\boldsymbol{\Delta}_0{}^{-1}(x), \qquad \text{(III.90)}$$

$$x=p^2, \quad y=(p-k)^2, \quad z=k^2.$$

To show that the asymptotic solutions of (III.90) are correctly described by the unitarity solutions of Part III. 2 we shall concentrate on the transverse part of $\boldsymbol{\Delta}$. Equation (III.90) may then be reduced to the form

$$x(x-m^2)Z_1(x) \approx \int dyK(x,y)[(x-m^2)Z_1(x)-(y-m^2)Z_1(y)]^2 \times (y-m^2)Z_1(y). \qquad \text{(III.91)}$$

The kernel $K(x,y)$ arise from quantities like

$$\mathrm{Tr}[\mathbf{d}(p)\mathbf{e}(p')]$$

and it has the form

$$(\alpha x^2+\beta xy+\gamma y^2)(\alpha' x^2+\beta' xy+\gamma' y^2)/xy(x-y)^2$$

for large values of x and y.

If we substitute the unitarity solutions of Part II,I.2 viz.,

$$-\frac{1}{\pi}\mathrm{Im}Z_1^{-1}(x) \approx [\delta(x-m^2)+A](x-m^2)$$

$$\frac{1}{\pi}\mathrm{Im}Z_1(x)(x-m^2) \approx (1/A)\ln^2 x \qquad \text{(III.92)}$$

$$(x-m^2)Z_1(x) \approx \mathrm{Re}(x-m^2)Z_1(x) \approx A \ln x$$

in the right-hand side of (III.91) and make just two subtractions: $Z_1(m^2)=0$, $Z_1'(m^2)=1$, we obtain an integral of the type

$$\frac{x}{\pi}\mathrm{Im}Z_1(x) \approx \int \frac{y^2dy}{(x-y)^2}\mathrm{Im}[Z_1(y)(xZ_1(x)-yZ_1(y))^2]$$

$$\approx \int \frac{y^2dy}{(x-y)^2}\mathrm{Im}Z_1(y)\,\mathrm{Re}^2(xZ_1(x)-yZ_1(y))^2, \quad \text{typically}$$

$$\approx \int \frac{ydy}{(x-y)^2}\left[\frac{1}{\ln x}-\frac{1}{\ln y}\right]^2$$

which does not converge. This means that an extra subtraction constant $Z(0)$ is needed, exactly as we encountered in the unitarity equations. Making use of (III.91) and (III.92) we recover, as expected:

$$\frac{1}{\pi}\mathrm{Im}Z_1(x) \approx \int \frac{dy}{(x-y)^2}\left(\frac{1}{\ln x}-\frac{1}{\ln y}\right)^2 \approx \frac{1}{x\ln^2 x}.$$

Part IV

SUMMARY AND CONCLUSIONS

In Part III. 2 we have set down the general form for $\Gamma^A[\Delta,D]$. (The actual expressions were not merely functionals of Δ and D but also of the form factors $\mathcal{E}$; this was done in order to let Γ^A approximate as closely as possible to the full Γ.) We have also written down a convenient first choice for Δ and D consistent with and solving the two-particle unitarity approximation. The claim is that if we draw irreducible diagrams for any other S-matrix element and insert Γ^A, Δ, and D from (III.60)–(III.66) for the vertices and the lines, no infinities will ever appear in the theory. Our discussion, however, is still inadequate in two vital directions: (A) How to improve the approximation scheme; (B) the inclusion of C parts and the gauge covariance of resulting S-matrix elements.

[50] For proof of (III.88) see Appendix D, G. Frye and R. L. Warnock, Phys. Rev. **130**, 478 (1963). We are indebted to Dr. V. Barger for pointing out this reference. The relation (III.89) may be obtained by differentiating both sides of (III.88). Note that $C(2)=1$.

[51] It has not been shown here but we have checked that iterations of (III.78) (or indeed of the unitarity equations using the unitarity solutions), do lead to complicated transcendental functions of the coupling constant with an essential singularity at $\alpha=0$.

A. Improvement of the Approximation Scheme

In principle this problem is completely solved once the general *form of* $\Gamma^A[\Delta,D]$ is fixed. Given this form, one determines Δ, D, *and the full* Γ *by solving the Dyson equations to any desired approximation as fully described in* Part I. 1B.

In practice, we did not solve these equations; we found it more convenient to use the two-particle unitarity equations to construct explicit expressions for Δ and D. The practical problem of improving the approximation scheme is thus different from the problem *in principle.* First let us take the practical problem and indicate how to improve the unitarity approximation.

(1) Fix the form of $\Gamma^A[\Delta,D]$ (satisfying the Ward-Takahashi identity with the requisite boundary properties and approximating as closely as possible to the full two-particle unitarity Γ). Compute, as in the text, Δ^A and D^A from the relevant two-particle unitarity equations [this involves only $(0|\varphi|1)=\delta_+$ and $(0|\varphi|2)=\Gamma^A$].

(2) Write down the Dyson expressions for all other Green's functions $M[\Delta,\Gamma,D]$ and substitute $\Gamma=\Gamma^A$, $\Delta=\Delta^A$, $D=D^A$ to get $M^A[\Delta^A,\Gamma^A,D^A]$ and therefore $(0|\varphi|3)$, $(0|\varphi|4)$.

(3) Use $(0|\varphi|3)$, etc., to recompute Δ^A, D^A, with three-particle and higher contributions taken into account. This gives the new $\Gamma^A[\Delta^A,D^A]$ and the corresponding improved expressions for M^A.

The scheme above is a consistent approximation to the *full* field theory; *it is approximate to the extent that* $\Gamma=\Gamma^A$. The next problem is to compute a better approximation to Γ and then carry through the steps (1)–(3) once again. If we possessed a spectral representation for Γ^B (for all three particles off the mass shell) analogous to the general spectral representation for Δ, one need only modify rules (1)–(3) above in an obvious manner. Unfortunately, no such representation exists.[52] We are forced therefore to fall back on the Dyson equations for Δ, Γ, and D. Here then is a practical prescription for using Δ^A, D^A, Γ^A computed above to provide a starting point for solving the Dyson equations.

(4) Suppose step 1 (i.e., two-particle unitarity) has been carried through and Δ^A, D^A, Γ^A are known. The corresponding approximate Dyson equations (see Sec. III. 3) are

$$\mathrm{Im}(\Delta^{-1})=\mathrm{Im}\left[Z\Delta_0{}^{-1}+\int\Gamma\Delta\Gamma D\right]=\mathrm{Im}\int\Gamma\Delta\Gamma D \quad \text{(IV.1)}$$

$$\Gamma=\Gamma^A+X\left[\int\Gamma\Delta\Gamma\Delta\Gamma D+\int C\Delta\Gamma D+\int\Gamma\Delta C D\right] \quad \text{(IV.2)}$$

provided Γ^A depends only on Δ (and not D).

Rewrite (1) as

$$\mathrm{Im}\Delta^{-1}=\mathrm{Im}(\Delta^{-1})^A+\mathrm{Im}\int(\Gamma\Delta\Gamma D-\Gamma^A\delta_+\Gamma^{A*}\delta_+) \quad \text{(IV.3)}$$

and solve (2) and (3) by iterations which start with $\Gamma=\Gamma^A$, $\Delta=\Delta^A$. We now possess the full Δ (and similarly D). Further terms can obviously be introduced on the right-hand side of (1) and (2) and procedures similar to the above carried through to any desired stage.

In practice, it is hardly likely that step (4) will ever be carried out and the real possibility of improvement in computing Γ beyond the approximation Γ^A will come when a proper dispersion formula for Γ is discovered.

One merit of our formalism is perhaps worth stressing; the causality of this formalism in a Feynman sense is fully manifest. Every S-matrix element can be written in the form

$$\int\cdots dxdydz\cdots G(x)G(y)\int\cdots\frac{d^4k}{(k^2-x)[(p-k)^2-y]}\cdots.$$

This is true whether we use $\Gamma=\Gamma^A$ or the full Γ obtained by iterating (2).

B. C Parts

The C-part contribution has a dual role; first, what we called the C^{AA} and C^{AB} parts in Part II. 3 are necessary to preserve gauge covariance; second the intrinsic C^{BB} parts are basic insertions, in fact as basic as the vertex or self-energy insertions. So far as the problem of C^{BB} contributions is concerned, we need to write a fourth Dyson-Schwinger equation. Gauge covariance of the theory, however, needs only the inclusion of C^{AA} and C^{AB} contributions. In this sense the shortcoming of the present paper in not considering C^{BB} terms is not terribly serious. The problem of the computation of C^{AA} and C^{AB} contributions was solved in Part II. 3 for scalar electrodynamics and for C^{AA} terms in vector electrodynamics in Part III. 2. In a separate paper we propose to return to the computation of the C^{BB} contributions. The procedure of choice is to use β formalism instead of the wave-formalism of this paper. In this case there are no C-part insertions at all though of course the number of form-factors are Γ (as well as Δ) is considerably increased. To anticipate the results, no difficulties appear so far as the problem of convergence of the integrals in the theory is concerned. In this subsequent paper will also be treated more fully the gauge covariance of the formalism which demands a definition of an (n-photon) $C^{AA\cdots A}$ part analogous to the C^{AA} part above and its addition to higher matrix elements.

The methods of this paper apply for all theories where a gauge transformation (partial or exact) exists. By

[52] In the authors' opinion there is no problem in field theory more pressing than an integral representation for the full three-point function such that (like the representation for the two-point function) the connection of the kernel through unitarity with higher Green's functions is manifest.

using these methods the range of "renormalizable" theories appears capable of extension—it seems to include almost all spin-one gauge theories. Discussions of Lie group gauges on the lines presented in this paper will be published separately, as also the discussion of the subtraction constants and their significance in terms of bound states and CDD zeroes in the theory.

ACKNOWLEDGMENTS

We are indebted to Professor P. T. Matthews for his careful reading of Part I of the manuscript and to J. Strathdee for discussions. One of the authors (A. S.) would like to thank the hospitality extended by Professor R. G. Sachs at the University of Wisconsin where part of the work was carried out.

APPENDIX I: PHASE SPACE INTEGRALS

We will present here a collection of relevant phase-space integrals for easy reference. Some overlap with equations in the text is unavoidable. The self-energy integrals are

$$\int \delta_+(t^2-\mu^2)\delta_+[(p-t)^2-m^2]d^4t \equiv \frac{\pi}{2p^2}\varphi(p^2,m^2,\mu^2)=\frac{\pi}{2p^2}\theta[p^2-(m+\mu)^2][(p^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}. \tag{A1}$$

In the special case when $\mu \to m$

$$\int \delta_+(p^2-m^2)\delta_+[(p-t)^2-m^2]d^4p=\frac{\pi}{2}\left(1-\frac{4m^2}{t^2}\right)^{1/2}\theta(t^2-4m^2). \tag{A2}$$

As for the vertex integrals,

$$\int d^4q f(p^2,q^2)\delta_+[(p'-q)^2-\mu^2]\delta_+[(t-q)^2-m^2]=\frac{\pi\theta[p^2-(m+\mu)^2]}{2[(p^2-m^2-\mu^2)^2-4m^2\mu^2]^{1/2}}\int_{2m^2+2\mu^2-p^2}^{(m^2-\mu^2)^2/p^2} dq^2 f(p^2,q^2) \tag{A3}$$

and

$$\int d^4k\delta_+[(k-p)^2-m^2]\delta_+[(p'-k)^2-m^2]f(t^2,k^2)=\frac{\pi\theta(t^2-4m^2)}{2(t^4-4m^2t^2)^{1/2}}\int_{4m^2-t^2}^{0} dk^2 f(t^2,k^2) \tag{A4}$$

in the limit as $\mu \to m$.

APPENDIX II: LIST OF TRACES USED IN EQ. (II.96)

In the evaluation of the spectral function of the photon propagator and the vertex function with the photon unphysical, we encounter a host of terms which involve traces of momenta and **d** and **e** projections. I_1–I_{24} give definitions of symbols used in (II.96).

$I_1=k\mathbf{d}(p)p'$ $=k\mathbf{d}(p')p$ $=-k^2t^2/4m^2$

$I_2=p'\mathbf{d}(p)p'$ $=p\mathbf{d}(p')p$ $=-t^2(1-t^2/4m^2)$

$I_3=k\mathbf{d}(p-k)k$ $=k\mathbf{d}(p'-k)k$ $=-k^2(1-k^2/4m^2)$

$I_4=k\mathbf{d}(p-k)p'$ $=k\mathbf{d}(p'-k)p$ $=-k^2(1-k^2/4m^2-t^2/4m^2)$

$I_5=\mathrm{Tr}\mathbf{d}(p)\mathbf{d}(p')$ $=\mathrm{Tr}\mathbf{d}(p-k)\mathbf{d}(p'-k)$ $=(t^4/4m^4-t^2/m^2+3)$

$I_6=k\mathbf{d}(p)\mathbf{d}(p')k$ $=k\mathbf{d}(p-k)\mathbf{d}(p'-k)k$

$I_7=k\mathbf{d}(p)\mathbf{d}(p')p$ $=k\mathbf{d}(p')\mathbf{d}(p)p'$

$I_8=k\mathbf{d}(p-k)\mathbf{d}(p'-k)p$ $=k\mathbf{d}(p'-k)\mathbf{d}(p-k)p'$

$I_9=p'\mathbf{d}(p)\mathbf{d}(p')p$

$I_{10}=p'\mathbf{d}(p)\mathbf{d}(p-k)k$ $=p\mathbf{d}(p')\mathbf{d}(p'-k)k$

$I_{11}=p'\mathbf{d}(p)\mathbf{d}(p'-k)k$ $=p\mathbf{d}(p')\mathbf{d}(p-k)k$

$I_{12}=k\mathbf{d}(p-k)\mathbf{d}(p)\mathbf{d}(p')k$ $=k\mathbf{d}(p'-k)\mathbf{d}(p')\mathbf{d}(p)k$

$$I_{13}=k\mathbf{d}(p-k)\mathbf{d}(p)\mathbf{d}(p')p \qquad =k\mathbf{d}(p'-k)\mathbf{d}(p')\mathbf{d}(p)p'$$

$$I_{14}=k\mathbf{d}(p'-k)\mathbf{d}(p)\mathbf{d}(p')k \qquad =k\mathbf{d}(p-k)\mathbf{d}(p')\mathbf{d}(p)k$$

$$I_{15}=k\mathbf{d}(p-k)\mathbf{d}(p'-k)\mathbf{d}(p')p \qquad =k\mathbf{d}(p'-k)\mathbf{d}(p-k)\mathbf{d}(p)p'$$

$$I_{16}=p'\mathbf{d}(p-k)\mathbf{d}(p'-k)\mathbf{d}(p')p \qquad =p\mathbf{d}(p'-k)\mathbf{d}(p-k)\mathbf{d}(p)p'$$

$$I_{17}=k\mathbf{d}(p-k)\mathbf{d}(p'-k)\mathbf{d}(p)p' \qquad =k\mathbf{d}(p'-k)\mathbf{d}(p-k)\mathbf{d}(p')p$$

$$I_{18}=\mathrm{Tr}\mathbf{d}(p)\mathbf{d}(p-k)\mathbf{d}(p'-k)\mathbf{d}(p')$$

$$I_{19}=p'\mathbf{d}(p)\mathbf{d}(p-k)\mathbf{d}(p'-k)\mathbf{d}(p')p$$

$$I_{20}=k\mathbf{d}(p-k)\mathbf{d}(p)\mathbf{d}(p')\mathbf{d}(p'-k)k$$

$$I_{21}=k\mathbf{d}(p')\mathbf{d}(p)\mathbf{d}(p-k)\mathbf{d}(p'-k)k \qquad =k\mathbf{d}(p)\mathbf{d}(p')\mathbf{d}(p'-k)\mathbf{d}(p-k)k$$

$$I_{22}=k\mathbf{d}(p')\mathbf{d}(p)\mathbf{d}(p-k)\mathbf{d}(p'-k)p \qquad =k\mathbf{d}(p)\mathbf{d}(p')\mathbf{d}(p'-k)\mathbf{d}(p-k)p'$$

$$I_{23}=p\mathbf{d}(p')\mathbf{d}(p)\mathbf{d}(p-k)\mathbf{d}(p'-k)k \qquad =p'\mathbf{d}(p)\mathbf{d}(p')\mathbf{d}(p'-k)\mathbf{d}(p-k)k$$

$$I_{24}=k\mathbf{d}(p')\mathbf{d}(p)\mathbf{d}(p'-k)\mathbf{d}(p-k)p' \qquad =k\mathbf{d}(p)\mathbf{d}(p')\mathbf{d}(p-k)\mathbf{d}(p'-k)p.$$

For the case of the meson self-energy with φ defined as in (A.1) the following relations are useful:

$$p\mathbf{d}(t)p = \varphi^2/4\mu^2$$

$$t\mathbf{d}(p)t = \varphi^2/4p^2$$

$$t\mathbf{d}(p-t)t = \varphi^2/4m^2$$

$$\mathrm{Tr}\mathbf{d}(p)\mathbf{d}(p-t) = 3+\varphi^2/4m^2p^2$$

$$\mathrm{Tr}\mathbf{d}(p-t)\mathbf{d}(t) = 3+\varphi^2/4m^2\mu^2$$

$$\mathrm{Tr}\mathbf{d}(p)\mathbf{d}(t) = 3+\varphi^2/4p^2\mu^2$$

$$t\mathbf{d}(p)\mathbf{d}(p-t)t = (\mu^2-p^2-m^2)\varphi^2/8m^2p^2$$

$$t\mathbf{d}(p-t)\mathbf{d}(t)p = (p^2-m^2-\mu^2)\varphi^2/8m^2\mu^2$$

$$p\mathbf{d}(t)\mathbf{d}(p)t = (p^2+\mu^2-m^2)\varphi^2/8p^2\mu^2$$

$$t\mathbf{d}(p-t)\mathbf{d}(t)\mathbf{d}(p)t = [(p^2-m^2)^2-\mu^4]\varphi^2/16m^2p^2\mu^2$$

$$p\mathbf{d}(t)\mathbf{d}(p)\mathbf{d}(p-t)t = [(m^2-\mu^2)^2-p^4]\varphi^2/16m^2p^2\mu^2$$

$$t\mathbf{d}(p)\mathbf{d}(p-t)\mathbf{d}(t)p = [m^4-(p^2-\mu^2)^2]\varphi^2/16m^2p^2\mu^2.$$

APPENDIX III

As remarked in Part III. 2A, if $Zm_0^2=\infty$, it is possible that alternative solutions to the Dyson equations may exist, with the boundary behavior

$$\Delta\approx 1/p^2, \quad \Gamma\approx p, \quad D\approx 1/t^2.$$

Very crudely one may see the effect of $m_0^2=\infty$ ($Z<\infty$) in the following manner. The so-called "free unrenormalized" propagator has the form

$$-\frac{g_{\mu\nu}}{p^2-m_0^2}+\frac{p_\mu p_\nu}{m_0^2}\frac{1}{p^2-m_0^2}.$$

One may expect that as a result of self-energy corrections the first term changes to the form $-g_{\mu\nu}/(p^2-m^2)$ while the second drops off in the limit $m_0^2=\infty$ giving for Δ the scalar behavior $\Delta\approx 1/p^2$.

To examine if the equations (II.71) and (II.72) for $Z_1(p^2)$ and $Z_2(p^2)$ can possibly admit of solutions behaving like $Z_1(p^2)\approx 1$ and $Z_2(p^2)\approx p^2$, rewrite these equations in the form,

$$\mathrm{Im}Z_1(p^2)[1+\alpha p^2\,\mathrm{Im}M+\cdots] =\alpha p^2[|Z_1|^2+\mathrm{Re}Z_1{}^*M+\cdots] \quad \text{(A5)}$$

$$\mathrm{Im}Z_2(p^2)[1+\alpha p^2\,\mathrm{Im}Z_1+\alpha p^2\,\mathrm{Im}M+\cdots] =\alpha[|Z_2|^2+(p^2)^2|Z_1|^2+\cdots], \quad \text{(A6)}$$

or alternatively

$$\mathrm{Im}Z_1^{-1}(p^2)=U(p^2)+\tan\theta\,\mathrm{Re}Z_1^{-1}(p^2) \quad \text{(A7)}$$

where

$$\tan\theta\approx\alpha p^2\,\mathrm{Re}M/(1+\alpha p^2\,\mathrm{Im}M), \quad \text{(A8)}$$

$$U(p^2)\approx(\alpha p^2/(1+\alpha p^2\,\mathrm{Im}M))[1+|Z^{-1}M|^2], \quad \text{(A9)}$$

with a similar equation for $Z_2^{-1}(p^2)$.

Now if $M(p^2)\approx 1$ (i.e. $\Gamma\approx p$), the term $\alpha p^2\,\mathrm{Im}M$ dominates over 1 in the denominator of $\tan\theta$ and $U(p^2)$. Provided therefore that $[\theta]_{m^2}{}^{\infty}=0$, solutions may exist with the characteristic behavior $Z_1(p^2)\approx 1$, $Z_2(p^2)\approx p^2$, $m_0^2=\infty$. In this case, however, since $M(p^2)\approx 1$, one may expect that (unlike the "vector" alternative) a new subtraction constant would be needed for the magnetic form factor.

In a sense the "scalar alternative" would exist if in the original Eqs. (A5) and (A6)

$$\mathrm{Im}Z_1-\alpha p^2(|Z_1|^2+\mathrm{Re}MZ_1^*+|M_1|^2+\cdots)=0.$$

One can neglect completely $\mathrm{Im}Z_1$ for large p^2 in comparison with the rest of the expression. Note that if $M(p^2)\approx 1$, $U(p^2)$ and $\tan\theta$ are essentially independent of the fine structure constant $2\pi\alpha$.

It is a puzzle to the authors how to construct the "scalar" solutions, to check their existence, and to check that no extra subtraction constants are introduced in the vertex function Γ (besides possibly a subtraction for magnetic moment) and in its relation to the vector alternative. It may be that a *summation* of all contributions in the "vector" alternative produces cancellations which reduce $\Delta(p^2)\approx 1$ to $\approx 1/p^2$. Such cancellations are not impossible to conceive of for the meson propagator on account of the indefinite nature of G_1 and G_2 but for the photon propagator they seem unlikely to occur except the special and exceptional case of the total magnetic moment $K=1$ for the charged meson, a case where (as shown in the text) the high-energy behavior of the photon propagator has a very different character than for any other K. Note that for both the "vector" and "scalar" alternatives $\Delta^{1/2}\Gamma\Delta^{1/2}\sim 1/p$.

If the scalar alternative indeed exists, we have the interesting demonstration that the (renormalized) unitarity equations (which essentially state relations between Re and Im parts) need supplementation by a specification of the number and character of subtraction constants to distinguish between possible alternative solutions. The number and nature of allowed subtractions in turn determines if the solution investigated exists for all values of the constants of the theory or only for special values.

It has repeatedly been emphasized in the text that our stable solution to vector (and scalar) electrodynamics is not a solution where convergence has been obtained by summing certain subsets of graphs. Even though the final two-particle unitarity expressions for Δ resemble second-order perturbation expressions, the convergence of the theory comes not from Δ but from $\Gamma^A[\Delta,D]$ which displays scant resemblance to its perturbation counterpart. Further, in principle, we have a procedure which builds up the *full* theory—i.e., *we can include every* Dyson-Schwinger graph and without redundance at any stage. We do not claim that the precise form of Γ^A chosen in the paper gives the best approximation from the point of view getting closest to physical answers, nor that after three-, four-, and higher particle contributions have been *summed* up, the behavior of Δ, Γ, etc., may not change. All we claim is that *each* unitarity contribution individually behaves in the manner indicated in the text and that the stability criterion continues to be satisfied at each stage of computation.