# Limited Applicability of the Theory of Nucleon Tunneling*

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The assumptions made in the theory of the tunneling process of single-nucleon transfer are reviewed and their shortcomings discussed. A schematic form of a black-box treatment is systematized and assumptions made are pointed out. The reason for the agreement between results obtainable from this form and the earlier way of ascertaining quantum-mechanical corrections is discussed. Among the assumptions made is the legitimacy of neglecting terms that vanish when the ratio of the nucleon mass to the reduced mass is set equal to zero. The character of effects arising from these terms is illustrated by means of a one-dimensional model, and the associated inadequacy of dealing only with the relative motion of the heavy aggregates without including effects of motion with respect to the inertial system is pointed out. Further discussion concerns itself with the effects of the angular momentum of the transferred nucleon in the emitting nucleus and in the receiving one on the space dependence of the transfer function; the symmetrized forms of the transfer cross section; the effect of the relative velocity of the emitter and receiver nuclei on the matching of the angular momentum of the nucleon; the transition from the isotropic-transfer quantum treatment to the corresponding semiclassical one, including the double-limit situation involved in making the scattering angle and the space-decay parameter approach zero; the questions involved in the consideration of the exterior region, including the possible effect of deuteron, triton, alpha particle and other types of tentacles in configuration space; and the distinction between the nucleon configuration and wave-function assignments of shell-structure theory and the nucleon configurations and wave functions that matter more directly for the treatment of the exterior region and of single-nucleon transfer. The bearing of virtual Coulomb-excitation processes on applications of usual potential-barrier penetrability estimates for reaction-yield estimates made in astrophysics is mentioned, and it is pointed out that as the kinetic energy is decreased, the nuclear radii in ordinary estimates must be increased.

## I. INTRODUCTION

APPLICATIONS[1] of a form of nucleon-transfer theory[2] have been made employing primarily the semiclassical (SC) approximation. In view of the large amount of experimental material which has been compared with theoretical expectation and of the variable character of the agreement, it appears desirable to point out and to discuss some of the limitations on the applicability of the equations used. The considerations in the earlier theoretical papers[2–5] do not take into account some essential aspects of the process. Disagreements between calculation and measurement may thus be due, at least partly, to the incompleteness of the theory originally intended for qualitative purposes and the examination of the earlier data[6] on $N^{14}(N^{14}, N^{13})N^{15}$. The limitations on the applicability of the equations in the earlier work[2–5] were partly discussed in papers read at conferences[7–9] which were sufficiently concerned, however, also with other matters to make the presentation of the limitations unsystematic and not sufficiently clear.

The possibility of exploring the density of nucleons at the nuclear surface[10] which in the case of reactions with small $Q$ values amounts to the determination of reduced widths for nucleons in their bound states has been discussed concretely[2] for the case of $N^{14}(N^{14}, N^{13})N^{15}$ in the SC approximation which is frequently used in applications.[1] This possibility still remains one of the attractions of the study of nucleon transfer since the reduced widths are valuable parameters for nuclear structure models.

A systematic presentation of a black box treatment is presented in Sec. II. The "black boxes" are the nuclear interiors of the transmitter and receiver nuclei. The problem is specialized to the case of the isotropic transfer function and it is brought out that the quantum-mechanical (QM) equations can be transformed in a manner similar to that used in BE-I for the treatment of adiabatic wave functions and the reduction to reduced widths in the case of reactions with small $Q$ values. This process leads to Eq. (2.16) and its partner

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[1] D. E. Fischer, A. Zucker, and A. Gropp, Phys. Rev. **113**, 542 1959); J. A. McIntyre, T. L. Watts, and F. C. Jobes, *ibid.* **119**, 1331 (1960); A. Zucker, in *Proceedings of the Conference on Direct Interactions and Nuclear Reaction Mechanisms, Padua, 1962* (Gordon and Breach Publishers, Inc., New York, 1963), p. 857; K. S. Toth and E. Newman, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei*, edited by A. Ghiorso, R. M. Diamond, and H. E. Conzett (University of California Press, Berkeley, 1963), p. 114; L. C. Becker, F. C. Jobes, and J. A. McIntyre, *ibid.*, p. 106; a complete bibliography is not attempted.

[2] G. Breit and M. E. Ebel, Phys. Rev. **103**, 679 (1956). This paper will be occasionally referred to as BE–I.

[3] M. E. Ebel, Phys. Rev. **103**, 958 (1956).

[4] G. Breit and M. E. Ebel, Phys. Rev. **104**, 1030 (1956). This paper will be occasionally referred to as BE–II.

[5] G. Breit, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1959), Vol. 41, Part 1, especially Sec. 48.

[6] H. L. Reynolds and A. Zucker, Phys. Rev. **101**, 166 (1956).

[7] G. Breit, in *Proceedings of the Second Conference on Reactions Between Complex Nuclei, Gatlinburg, 1960* (John Wiley & Sons, Inc., New York, 1960), p. 1.

[8] G. Breit, in *Proceedings of the Conference on Direct Interactions and Nuclear Reaction Mechanisms, Padua, 1962* (Gordon and Breach Publishers, Inc., New York, 1963), p. 480.

[9] G. Breit, in *Proceedings of the Third Conference on Reactions Between Complex Nuclei, Asilomar, 1963* (University of California Press, Berkeley, 1963), p. 97.

[10] G. Breit, M. H. Hull, Jr., and R. L. Gluckstern, Phys. Rev. **87**, 74 (1952).

obtained by interchange of the receiver and emitter. A further simple reduction yields the coupled equations (2.17), (2.18). Neglect of the last two terms in curly braces involving the ratio of the nucleon mass to the reduced mass of the collision process, $M/\mu$, gives Eq. (2.20) which is the QM extension of the SC equations in BE-I. The relationship of consequences of these equations to an earlier derivation of the QM correction factor to the SC formulas is then discussed and the reason for the agreement of the two approaches is mentioned. The insufficiency of reduced widths for the general treatment of the tunneling process in cases of the reaction $Q$ value being $\neq 0$ is brought out in the course of the presentation and the nature of the quantities entering instead is briefly described. In Sec. III the effect of the omitted terms in $M/\mu$ is examined employing a one-dimensional example of two equal $\delta$-function potentials. For the case of complete symmetry of masses and fields these effects become small but in the case of $N^{14}(N^{14},N^{13})N^{15}$ for which many comparisons between theory and experiment have been made there is no assurance of symmetry of the fields. The modified space dependence of addition to transfer function $\beta$ contained in $1-\alpha_0 x$ of Eq. (3.20) and the presence of momentum dependence appear noteworthy since analogous terms in a more complete treatment may affect comparison with experiment. Both terms in (3.20) are needed to verify reciprocity, i.e., to obtain Hermiticity.

The angular momentum of the transferred nucleon affects the space dependence of the transfer function as discussed in Sec. IV. Since the wave function of the transferred nucleon may be a linear superposition of functions with different orbital angular momenta, and since, for anisotropic transfer, the space dependence is affected by the orientation of the projection of the orbital angular momentum, these effects can modify the angular and energy dependence of the transfer cross section. In Sec. V, the effect on the angular momentum of nucleon $c$ about nucleus $b$ of the relative motion of nucleus $a$ containing $c$ is discussed in a simple special case. The presence of such effects is well known in the theory of deuteron stripping, but since it has not been taken into account explicitly for nucleon transfer from heavier nuclei, its discussion appeared desirable.

In Sec. VI, the limit $M/\mu=0$ for the dynamic correction term in the case of the isotropic-transfer function is returned to and the connection between the QM and SC results is discussed. The relationship of the cross section formulas to the Rutherford scattering formula at small angles $\theta$, and the double-limit situation that arises when $\theta$ and the range parameter of the neutron usually denoted by $\alpha$ approach zero, is discussed. This matter is of interest in connection with the comparison of theoretical formulas with experiment for the $N^{14}(N^{14},N^{13})N^{15}$ reaction. In this section the possibility of enhancement of effects of competition with other reactions through virtual Coulomb excitation at the larger internuclear distances is briefly discussed. It may result in an effective increase of the range of spatial extension of the imaginary part of the effective optical potential commonly called $W$, and to the presence of wave-absorption effects at lower bombarding energies than otherwise expected. In the same connection, the possible presence of resonance effects like those in the Heitler-London theory of molecular binding may be useful to recall especially in connection with the collision of $N^{14}$ with $N^{14}$. The possible bearing of virtual Coulomb-excitation processes on reaction yield estimates made in astrophysics is mentioned at the end of this section.

The limitations on quantitative applicability of single-nucleon transfer theory caused by tentacles of deuterons, tritons, etc., in configuration space are briefly mentioned in Sec. VII. This section is also concerned with the indirectness of the connection between the wave function of the transferred nucleon and shell-model configurations.

Some of the main symbols reoccurring in the paper which it might be difficult to identify are as follows: $]\!]$ indicates termination of influence of operator in $(\ ]\!]$; $\hbar l^a$, $\hbar l^b$ are orbital angular momenta of $c$ while attached to $a$ and $b$, respectively; $m^a$, $m^b$ are the magnetic quantum numbers of $l^a$ and $l^b$. In subscripts and superscripts $l^a$, $m^a$, $l^b$, $m^b$ are printed as $l(a)$, $m(a)$, $l(b)$, $m(b)$; $\mathbf{s}=\sin(\theta/2)$ where $\theta$ is the angle made by the velocity of the reaction product with that of the incident nucleus.

## II. THE BLACK-BOX TREATMENTS

As carried out,[2] the SC approximation makes use of the following assumptions: (a) The motion of the heavy aggregates, such as the two colliding nuclei may be approximated by classical mechanics; (b) the forces between nucleons have a short range; consequently, the effects of Coulomb excitation, real or virtual, are supposed to have a negligible effect; (c) transitions between levels of the same nucleus caused by acceleration effects are not taken into account; (d) reactions other than the transfer reaction under consideration have a negligible probability at any point on the classical orbit.

The limitations on assumption (a) have been discussed in Ref. 5 from two viewpoints. The first was concerned with qualitative estimates of the possibility of satisfying the requirements of sufficiently good wave-packet localization. These turned out to be higher than could be satisfied during the collision time which is long enough to allow appreciable spreading of the wave packet. The reason for this is the large separation energy of the neutron in $N^{14}$ and in $N^{13}$ which makes it necessary to localize the relative distance of the nuclei better than within the limits of $1/(2\alpha)$ where

$$\alpha=(2\mu_{ac}|E_{ca}|/\hbar^2)^{1/2} \tag{2.1}$$

is the reciprocal of the range constant of the neutron

wave function in the outer regions of the emitter nucleus $a$. Here $\mu_{ac}$ is the reduced mass of relative motion of the neutron $c$ and nucleus $a$ while $E_{ac}$ is the binding energy of $c$ in $a$. The reason for this high degree of localization is that the chance of the neutron penetrating from $a$ to $c$ varies approximately as $\exp(-2\alpha r_{ab})$. The relative largeness of $E_{ac}$ which is $\approx 10$ MeV in the case of $N^{14}$ makes it difficult therefore to localize the wave packet sufficiently.

Secondly, a perturbation calculation, Eqs. (48.27)–(48.36), employing a $\delta$-function type of interaction energy between the neutron and the receiver nucleus $b$ has been made and the results compared with those of the SC approximation. This calculation made use of the evaluation of an integral in the work of Ter-Martirosyan[11] on $(d,p)$ reactions which is closely related to independent work on $(d,p)$ reactions by Biedenharn, Boyer, and Goldstein.[12] Although the treatment of the motion of the transferred neutron is considered in this quantum-mechanical (QM) treatment with much less generality than in the corresponding SC considerations made in Ref. 2, it is nevertheless to be expected that the ratio of the QM to the SC quantities in this special case should have a close relationship to the value of the same ratio under the more general conditions used in Ref. 2. A reason for this expectation is that the errors of the SC treatment have to do primarily with the impossibility of the exact localization on the classical orbit of the heavy aggregates so that qualitatively the replacement of a $\delta$-function interaction by one with space extension gives rise to an averaging which is similar for the $\delta$ function and other interactions. This supposition is supported by a more detailed argument presented at the Padua conference.[8] The result of the calculation is that within the accuracy of the approximation used the cross section is proportional to $|I_0(\theta)|^2$, where

$$I_0(\theta)=\frac{1}{(4\pi)^{1/2}}\int\psi^{(-)*}(\eta_f,\mathbf{k}_f;\mathbf{r})\psi^{(+)}(\eta_i,\mathbf{k}_i;\mathbf{r})\times(e^{-\alpha r}/r)d\mathbf{r} \quad (2.2)$$

is the Ter-Martirosyan integral for scattering angle $\theta$ connected with initial and final relative momenta $\mathbf{k}_i$, $\mathbf{k}_f$ of the heavy aggregates by

$$\cos\theta=(\hat{k}_i\cdot\hat{k}_f).$$

Employing the steepest descents approximation to the hypergeometric function that occurs in $I_0(\theta)$ as worked out by Ter-Martirosyan[11] the ratio of QM to SC values of the collision cross section $\sigma$ is obtainable from (48.33) of Ref. 5. In the special case of $k_i=k_f$, $\eta_i=\eta_f$, approaching the limit of $\eta\to\infty$, $k\to\infty$, keeping $\eta/k=a'=\frac{1}{2}$ times the classical distance of closest approach, there results Eq. (48.34) of Ref. 5 which reproduces the SC dependence of $\sigma$ on $\alpha_n$ and $\theta$ as in Eq. (23.1) of Ref. 2. Employing the most critically varying factor in the ratio of QM to SC values of $\sigma$ this ratio is approximately that of the corresponding values of

$$\exp\{-4\eta\tan^{-1}[\alpha/2k\mathbf{s}]\}=\exp\{-2a'\alpha[1-\tfrac{1}{3}(\alpha/2k\mathbf{s})^2+\cdots]/\mathbf{s}\}, \quad (2.3)$$

where

$$\mathbf{s}=\sin(\theta/2). \quad (2.4)$$

For the $(N^{14},N^{13})$ reaction at bombarding energy of 14 MeV on $N^{14}$, this effect amounts to[5] a $\sim 13\%$ effect at 180° and $\sim 40\%$ effect at 90°. In these numbers only the contribution caused by direct scattering is considered as though $N^{14}$ [13] and $N^{14}$ were not identical particles, the main purpose being to illustrate the order of magnitude of effects. According to Eqs. (48.35) and (48.36) of Ref. 5 and the discussion following them the effect on the total cross section at the same energy is about 25%. The difference between QM and the SC approximation has thus been known to be non-negligible for some time. The characteristic parameter which enters the effects calculated in Ref. 5, such as that in Eq. (2.3) above, is

$$\alpha/(2k\mathbf{s}). \quad (2.5)$$

Aside from the factor $1/(2\mathbf{s})$, this parameter is proportional to the ratio of the wavelength at an infinite distance to the characteristic length $1/\alpha$, in qualitative agreement with the consideration of the possibility of wave-packet localization.

An extension of the QM treatment in Ref. 5 including the effects of the capturing nucleus without the employment of the $\delta$-function potential has been briefly described in Ref. 8. The arrangement of the calculation is similar to that of the SC calculation in Ref. 2 regarding the employment of the adiabatic functions $u$, $v$ representing the condition of the nucleons for fixed positions of the centers of mass of the aggregates $a$ and $b$. Instead of being multiplied by time-dependent coefficients as in Ref. 2, the adiabatic functions are multiplied, however, by functions of the vector displacement $\mathbf{r}$ from $a$ to $b$. The presentation in Ref. 8 points out some of the main assumptions that have been made in arriving at the final forms but the enumeration of the various omissions is involved and incomplete. The generality of the treatment regarding the inclusion of many-body features of the problem may also not have been apparent from Ref. 8. The derivation will now be restated in an improved form.

The wave function of the whole system is approximated by

$$\Psi\approx\hat{u}^a(\mathbf{r}_c,\mathbf{r},q)\psi^a(\mathbf{r})+\hat{u}^b(\mathbf{r}_c,\mathbf{r},q)\psi^b(\mathbf{r}). \quad (2.6)$$

[11] K. A. Ter-Martirosyan, Zh. Eksperim. i Teor. Fiz. **29**, 713 (1955) [English transl. Soviet Phys.—JETP **2**, 620 (1956)].

[12] L. C. Biedenharn, K. Boyer, and M. Goldstein, Phys. Rev. **104**, 383 (1956).

[13] The writer is grateful to Dr. K. W. Chun for communicating his very useful result before publication.

All coordinates are in the center-of-mass system, the transferred neutron is $c$ and its displacement vector is $\mathbf{r}_c$, $q$ denotes collectively the relative coordinates of the system other than those contained in $\mathbf{r}$ and $\mathbf{r}_c$. In this treatment the masses $m_a$, $m_b$ of $a$ and $b$ are supposed large in comparison with the mass $M$ of $c$ and the other nucleons, an assumption the faults of which will be discussed later. It is therefore possible to group the many $q$ according to whether they refer to particles in $a$ or $b$ as in Eqs. (7.1) and (7.2) of BE-I. The identity of nucleons is not explicitly considered at this point but can be taken care of, so far as the tunneling process goes in Sec. IV of BE-I.

The possibility of expressing the function in this manner implies not only that the probability of other reactions of other types is small but also that the probability of other single-nucleon transfer reactions, including those involving other states of $a$ and of $b$ is sufficiently small to contribute negligibly to wave absorption and also to the transfer to one state occurring as a result of virtual transfer to another state followed by a transition by long-range forces or other tunneling transfers. The Coulomb field is one of the possible long-range forces. The dominant process for low-incident energies or distant collisions generally consists of the succession of virtual Coulomb excitation by field of $b$ on $c$ in $a$ followed by Coulomb de-excitation of $c$ at $b$ by field of $a$. The potential energies

$$V_{ac}(\mathbf{r}_c,q_1,q_2,\cdots,q_n), \quad V_{bc}(\mathbf{r}_c,q_{n+1},q_{n+2},\cdots,q_{n+m}), \tag{2.7}$$

between $c$ and $a$ as well as $c$ and $b$ are meant as above, without restriction regarding form but it is supposed that the range of these interactions is short so that between the surfaces of $a$ and $b$ there is a gap within which $c$ is force free. The functions $u^a(\mathbf{r}_c,\mathbf{r},q)$, $u^b(\mathbf{r}_c,\mathbf{r},q)$ occurring in (2.6) are the adiabatic functions of BE-I which correspond to the solution of the Schrödinger equation for fixed positions of the centers of mass of the nuclei but whose definition otherwise implies no specialization of the general many-body problem. In general, the equations satisfied by $\hat{u}^a$, $\hat{u}^b$ are

$$\begin{gathered}(H_{\rm ad}-E^a)\hat{u}^a=0, \quad (H_{\rm ad}-E^b)\hat{u}^b=0,\\ H_{\rm ad}=-(\hbar^2/2m_c)\Delta_c+H_a+H_b+V_{ac}+V_{bc}.\end{gathered} \tag{2.8}$$

The subscript "ad" to $H$ indicates that the Hamiltonian is that for adiabatic functions. It is supposed at this point that the consideration of two functions in the expansion of $\Psi$ suffices. This assumption involves an approximation which implies for example that tunneling from the ground state of $a$ to level $E_{2b}$ of $b$ followed by tunneling from $E_{2b}$ to level $E_{2a}$ of $a$ and then to level $E_{1b}$ of $b$ is neglected. Such processes, while doubtless negligible at the larger distances, especially in the absence of virtual Coulomb excitation (VCE), may become more pronounced at the shorter distances. Although the equations in Ref. 8 as well as in the present presentation are written for two states, one of which is mainly associated with $a$ and the other mainly with $b$ there is no inherent difficulty in extending the treatment to groups of degenerate sublevels of two energy levels having energies $E^a$ and $E^b$. In the limit of $r=\infty$, these correspond to magnetic sublevels of the two-space degenerate levels. For simplicity the sums over the sublevels will not be dealt with here. This simplification is legitimate in special cases such as the $j$–$j$ coupling configurations considered in BE-I for $N^{14}(N^{14},N^{13})N^{15}$ for which the transfer is essentially isotropic. This idealization is useful also because it makes it easier to bring out other shortcomings of the theory.

In the space outside $a$ and $b$, for the case of $a$ and $b$ definitely separated the functions $\hat{u}^a$, $\hat{u}^b$ have the form

$$\begin{gathered}\hat{u}^a=u(\mathbf{r}_c,\mathbf{r})\varphi^a(q_1,\cdots,q_n)\varphi^b(q_{n+1},\cdots,q_{n+m}),\\ (c>a,\ c>b)\\ \hat{u}^b=v(\mathbf{r}_c,\mathbf{r})\varphi^a(q_1,\cdots,q_n)\varphi^b(q_{n+1},\cdots,q_{n+m}),\end{gathered} \tag{2.9}$$

and on account of assumed isotropy of transfer, the argument $\mathbf{r}$ of $u$ and $v$ may be replaced by $r$. As shown in BE-I, the adiabatic functions may lead to a slow convergence of the perturbation calculation and it is for this reason desirable to introduce related but different functions as in Eq. (9.3) of BE-I,

$$\bar{u}=u\mathbf{C}-v\mathbf{S}, \quad \bar{v}=u\mathbf{S}+v\mathbf{C}, \tag{2.10}$$

$$\mathbf{S}=\sin\gamma, \qquad \mathbf{C}=\cos\gamma, \tag{2.11}$$

where $\gamma$ is an angle which is defined in BE-I and is called $\theta$ there. The transformation has the qualitative significance of making the functions $\bar{u}$, $\bar{v}$ resemble the unperturbed functions in the case of large internuclear separations.

The same transformation applied to $\hat{u}^a$, $\hat{u}^b$ gives functions $\bar{u}_g$, $\bar{v}_g$ which are applicable also if $c$ is inside either $a$ or $b$. The subscript $g$ is meant to indicate the more general applicability of the functions. Since the angle $\gamma$ is a function of $r$ only it is not affected by the operators of the Hamiltonians $H_a$, $H_b$ which refer to the internal motions of particles within $a$ and $b$, respectively, and commutes therefore with $H_a+H_b$. Making use of this property as well as of $[\Delta_c,\gamma]=0$, it is then found by a short calculation that

$$\begin{gathered}(H_{\rm ad}-\bar{E}^a)\bar{u}_g+(\hbar^2\beta/2M)\bar{v}_g=0,\\ (\hbar^2\beta/2M)\bar{u}_g+(H_{\rm ad}-\bar{E}^b)\bar{v}_g=0,\end{gathered} \tag{2.12}$$

where as in BE-I the energies $\bar{E}^a$, $\bar{E}^b$ are the values of $E^a$, $E^b$ for $r=\infty$. The quantity $\beta$ as introduced in BE-I through Eq. (4.9), depends on the barrier penetration factors and the reduced widths of the neutron in the channel in which it leaves $a$ and arrives at $b$. Within certain approximations described in BE-I an explicit expression for the $N^{14}(N^{14},N^{13})N^{15}$ reaction is given in Eq. (24.1) of BE-I. The precise form of $\beta$ does not matter for the present discussion, but the fact that it depends on $r$ as $[\exp(-\alpha r)]/r$ will be used. The reduction of $\beta$ to a form characterizing the interior regions of the two nuclei entirely by means of reduced widths is possible only if the $Q$ value of the reaction is

close to zero. If this is not the case, Eqs. (4.1) and (4.2) of BE-I, which introduce the reduced widths through essentially their reciprocals $\lambda_1$, $\lambda_2$ do not suffice for expressing the differences in logarithmic derivatives occurring on the left-hand side of BE-I. In place of $\lambda_1$, $\lambda_2$ there enter quotients of finite differences in the logarithmic derivatives to energy differences. Such quotients are not directly expressible in terms of densities at the nuclear surface but related expressions involving averages over energy intervals exist. Introducing functions $\bar{\psi}_u$, $\bar{\psi}_v$ by means of

$$\hat{u}^a\psi^a+\hat{u}^b\psi^b=\bar{u}_g\bar{\psi}_u+\bar{v}_g\bar{\psi}_v \tag{2.13}$$

and substituting the approximate $\Psi$ of (2.6) in the Schrödinger equation

$$(H-E)\Psi=0, \tag{2.14}$$

where

$$H=H_{\rm ad}+V_{ab}(r)-(\hbar^2/2\mu)\Delta_{\rm r}, \tag{2.15}$$

with $\mu$ standing for the reduced mass of relative motion of $a$ and $b$ and with $V_{ab}(r)$ representing the central potential acting between $a$ and $b$, such as the Coulomb potential, there results

$$\left(-\frac{\hbar^2}{2\mu}\Delta_{\rm r}+V_{ab}+\bar{E}^a-E\right)\bar{\psi}_u-\frac{\hbar^2}{2M}\beta\bar{\psi}_v-\frac{\hbar^2}{2\mu}$$
$$\times\{(\bar{u}_g,[\Delta_{\rm r},\bar{u}_g])\bar{\psi}_u+(\bar{u}_g,[\Delta_{\rm r},\bar{v}_g])\bar{\psi}_v\}=0, \tag{2.16}$$

together with the result of interchanging $a$ with $b$ and $u$ with $v$. Here and below the inner products indicated by the symbol $(X,Y)$ are taken over all coordinates except the three components of $\mathbf{r}$. Thus,

$$(X,Y)=\int X^*Yd\mathbf{r}_c dq_1\cdots dq_{n+m}. \tag{2.16'}$$

Evaluation of the commutators gives

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\rm r}+V_{ab}-\bar{E}_a+\frac{\hbar^2}{2\mu}\int(\nabla_{\rm r}\bar{u}_g]\!]^2dqd\mathbf{r}_c\right]\bar{\psi}_u$$
$$+\frac{\hbar^2}{2M}\left\{-\beta-\frac{M}{\mu}(\bar{u}_g,(\Delta_{\rm r}\bar{v}_g]\!])\right.$$
$$\left.-\frac{2M}{\mu}(\bar{u}_g,(\nabla_{\rm r}\bar{v}_g]\!])\cdot\nabla_{\rm r}\right\}\bar{\psi}_v=0, \tag{2.17}$$

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\rm r}+V_{ab}-\bar{E}_b+\frac{\hbar^2}{2\mu}\int(\nabla_{\rm r}\bar{v}_g]\!]^2dqd\mathbf{r}_c\right]\bar{\psi}_v$$
$$+\frac{\hbar^2}{2M}\left\{-\beta-\frac{M}{\mu}(\bar{v}_g,(\Delta_{\rm r}\bar{u}_g]\!])\right.$$
$$\left.+\frac{2M}{\mu}(\bar{u}_g,(\nabla_{\rm r}\bar{v}_g]\!])\cdot\nabla_{\rm r}\right\}\bar{\psi}_u=0,$$
$$\bar{E}_a=E-\bar{E}^a,\quad \bar{E}_b=E-\bar{E}^b. \tag{2.18}$$

These equations are somewhat similar in content to the first equation in slide 2 of Ref. 8 and the equation obtainable from it by interchanging $a$ and $b$ throughout. Equations essentially identical in form with those of (2.17) have in fact been first obtained by Dr. K. W. Chun employing the equation of slide 2 of Ref. 8 and transforming it to the barred variables.[13] There is actually a difference in the content of these results from that of (2.17) because the functions $\bar{u}_g$, $\bar{v}_g$ are defined not only for $c$ being outside both $a$ and $b$ but also for $c$ being inside one of these nuclei, i.e., $\bar{u}_g$, $\bar{v}_g$ are introduced here by the same linear transformations as $\bar{u}$, $\bar{v}$ but in terms of the complete adiabatic functions $\hat{u}^a$, $\hat{u}^b$ rather than the $u$, $v$ of (2.9). In obtaining (2.17) use is made of the orthonormality of $\bar{u}_g$ and $\bar{v}_g$ which leads to

$$(\bar{u}_g,(\nabla_{\rm r}\bar{v}_g]\!])+(\bar{v}_g,(\nabla_{\rm r}u_g]\!])=0, \tag{2.19}$$

the functions $\bar{u}_g$, $\bar{v}_g$ being chosen to be real. Here as well as in (2.17) the ( ]\! indicate that the differential operator in ( ]\! is applied only to the quantity on its right in ( ]\!. Equation (2.19) leads to opposite signs of the last terms in the two parts of (2.17). If $(\bar{u}_g,(\nabla_{\rm r}\bar{v}_g]\!])$ and $\nabla_r$ were commuting operators this relationship of signs would lead directly to reciprocity of a reaction and its inverse. In fact, neglecting the lack of commutativity the effective Hamiltonian for a two-component wave function with components $\bar{\psi}_u$, $\bar{\psi}_v$ has a Hermitian structure of the off-diagonal elements contributed by $\beta$ and the last terms in curly braces. However, since the factors in the latter terms do not commute the verification of reciprocity necessitates the consideration of both terms of order $M/\mu$ in each of the curly braces and is readily carried out. It follows more obviously, however, from Eq. (2.16) and the result of the interchanges $(a,b)$, $(u,v)$ in the latter. In this form it is a consequence of the identity

$$\int[\bar{\psi}_u^*(\bar{u}_g,\Delta_{\rm r}\bar{v}_g)\bar{\psi}_v-\bar{\psi}_v(\bar{v}_g,\Delta_{\rm r}\bar{u}_g)\bar{\psi}_u^*]d\mathbf{r}=0. \tag{2.19'}$$

If the mass of the nucleon $M$ is supposed small in comparison with the masses of the nuclei $a$ and $b$, an approximation to (2.17) is obtained by neglecting the terms in $M/\mu$ in the curly braces. This approximation is of the same type as in BE-I where the orbits were not supposed to be affected by the transfer process. In this approximation, and neglecting the integrals in the square brackets of (2.17) for the same reason

$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\rm r}+V_{ab}-\bar{E}_a\right]\bar{\psi}_u=\frac{\hbar^2}{2M}\beta\bar{\psi}_v,$$
$$\left[-\frac{\hbar^2}{2\mu}\Delta_{\rm r}+V_{ab}-\bar{E}_b\right]\bar{\psi}_v=\frac{\hbar^2}{2M}\beta\bar{\psi}_u, \tag{2.20}$$

as in slide 5 of Ref. 8. Assuming that the incident wave contains only $\bar{u}_g$ and that the transfer probability is

small, one may use for $\bar{\psi}_u$ the solution of the first of the two equations in (2.20) neglecting the right-hand side with appropriate boundary conditions which, in the case of a Coulombian $V_{ab}$, amount to the employment for $\bar{\psi}_u$ of

$$\bar{\psi}_u=\psi^{(+)}(\eta_i,\mathbf{k}_i,\mathbf{r}) \tag{2.21}$$

of Eq. (2.2). By standard methods the substitution of this approximate value in the second part of (2.20) gives the asymptotic form of $\bar{\psi}_v$ at large $r$ as

$$\bar{\psi}_v\sim\frac{\mu}{4\pi Mr}I_\beta^{(-)(+)}\exp\{i[k_fr-\eta_f\ln(2k_fr)]\}, \tag{2.22}$$

in the notation of Ref. 8. Here

$$I_\beta^{(-)(+)}=\int\psi^{(-)*}(\eta_f,\mathbf{k}_f,\mathbf{r})\beta(r)\psi^{(+)}(\eta_i,\mathbf{k}_i,\mathbf{r})d\mathbf{r}, \tag{2.23}$$

which within a constant of proportionality is $I_0(\theta)$ of Eq. (2.2). Since $\beta(r)$ does not contain the incident energy, both the angular and the energy dependence of the collision cross section are such as correspond to the $\delta$-function potential used in Ref. 5.

## III. THE DYNAMIC REACTIONS

The terms in (2.17) neglected in obtaining (2.20) are caused by the interaction of the relative motion of $a$ and $b$ with the motion described by the adiabatic functions and those described in terms of the coordinates $\mathbf{r}_c$, $q$. In this sense these terms are concerned with the dynamic reaction of the $\mathbf{r}$ motion on the motion of the transferred nucleon and on the motion of nucleons contained in $a$ and $b$. The dynamic reaction terms have been neglected in BE-I and in Ref. 8 but have been briefly considered in Ref. 9 in a semiclassical analogy.

The dynamic reaction terms occurring in the square brackets in (2.17) have an effect like that of a change in $V_{ab}$. Since there are other reasons such as nuclear polarizability for considering $V_{ab}$ as not being exactly the Coulomb potential and since such effects are difficult to estimate with certainty they will not be discussed further here, their consideration being inseparable from that of elastic scattering. The dynamic reaction terms in curly braces in (2.17) add themselves to $\beta$ and thus contribute directly to the nucleon transfer. They contain the coefficient $M/\mu$ which has the value $\frac{1}{7}$ in the case of $N^{14}(N^{14},N^{13})N^{15}$ and is not truly small. The neglect of the dynamic reaction terms on nucleon transfer probability cannot be justified therefore on the grounds of the smallness of $M/\mu$ alone.

The dynamic reactions depend on the ratio of the masses $m_a$ and $m_b$ of $a$ and $b$ to each other, quite apart from the unavoidable connection of $m_a$ and $m_b$ with the nuclear structure of $a$ and $b$. Thus, even if there were no important change in the adiabatic wave functions $u$, $v$, but if the masses of $a$ or $b$ which do not affect strongly the important parts of $u$ and $v$ were changed, an effect would result on the dynamic reaction terms as below. The adiabatic functions $u$, $v$ should have as their arguments any set of relative coordinates which correspond to the separation of the c.m. These coordinates may be introduced by first replacing the coordinates of individual particles in $a$ by those of the center of mass of $a$ and the relative coordinates of particles in $a$ by well-known procedures. Similarly, the relative coordinates of particles in $b$ and the center-of-mass coordinates of $b$ can be introduced. For the sake of simplicity the smallness of $M/m_a$ and $M/m_b$ will be made use of in presenting the particular point under discussion to the extent of neglecting the effect of the transferred neutron on the position of the center of mass of the whole system. The inclusion of this effect would result in a modification of the terms in curly braces in (2.18) containing the factor $M/\mu$ to the relative order $M/\mu$ and a qualitative idea of the effect of these terms can be obtained without a complete calculation. In this approximation the difference between the two coordinates intended for the initial and final states in BE-II need not be considered.

If $m_a=m_b$, the relative coordinates $\boldsymbol{\varrho}=(\xi,\eta,\zeta)$ containing $\mathbf{r}_c$ may be introduced by referring the position of $c$ to the center of mass of $a$ and $b$ so that

$$\boldsymbol{\varrho}=\mathbf{r}_c-(\mathbf{r}/2). \tag{3.1}$$

If $m_a\neq m_b$ the coordinates of $c$ referred to center of mass of $a$ and $b$ are

$$\boldsymbol{\varrho}'=(\xi',\eta',\zeta')=\mathbf{r}_c-\mathbf{R}, \tag{3.2}$$

where

$$\mathbf{R}=m_b\mathbf{r}/m, \quad m=m_a+m_b. \tag{3.3}$$

Referring to $(\mathbf{r},\boldsymbol{\varrho})$ as the old and to $(\mathbf{r},\boldsymbol{\varrho}')$ as the new coordinates and designating partial differentiations in these two systems by $o$ and $n$, respectively,

$$\left(\frac{\partial}{\partial x}\right)_n=\left(\frac{\partial}{\partial x}+\frac{m_b-m_a}{2m}\frac{\partial}{\partial\xi}\right)_o, \tag{3.4}$$

$$(\partial/\partial\xi')_n=(\partial/\partial\xi)_o. \tag{3.5}$$

According to (3.4) terms containing $(\nabla_r v_g]]$ and $(\Delta_r u_g]]$ in (2.17) become modified through the inclusion of terms involving $(\nabla_\rho v_g]]$ and $(\Delta_\rho u_g]]$. Since $\bar{\psi}_u$ is a function of $\mathbf{r}$ only, the $\nabla_r\bar{\psi}_u$ remains unchanged in the equation. Equation (2.17) is thus affected by $m_b-m_a$. Since the motions of nucleons within $a$ and $b$ are affected by the accelerations of $a$ and $b$ as they move about their center of mass this situation is a natural one. The dynamic effects under consideration are in this sense acceleration effects and have been referred to as such in Refs. 8 and 9. The terms corresponding to these terms in the SC calculation in BE-I have been neglected there. They are formally of the order of magnitude $M\beta/\mu$, not truly negligible in comparison with

the first term in curly braces in (2.17). The qualitative reason for the entrance of barrier penetration factors in $(\bar{v}_g,(\Delta_r\bar{u}_g]])$ and $(\bar{v}_g,(\nabla_r\bar{u}_g]])$ in much the same way as in $\beta$ is that even though $\bar{u}_g$ and $\bar{v}_g$ are orthogonal to each other according to (2.10), the change in $\bar{u}_g$ caused by a change in $\mathbf{r}$ is not orthogonal to $\bar{v}_g$ and that the important change occurs where $\bar{v}_g$ is large, i.e., in $b$ which is reached by the tail of the changed $\bar{u}_g$ by barrier penetration from $a$.

It is not immediately obvious from (2.17) and (2.18) that the dynamic reaction terms satisfy reciprocity. In fact neither of the two sets of terms in $M/\mu$ in the curly braces taken separately satisfies it. The different signs of the terms containing a free $\nabla_r$ produce at first sight the false impression of furnishing a Hermitian operator on the two-component wave function $\bar{\psi}_u$, $\bar{\psi}_v$. On the other hand, the two coupled equations on $\bar{\psi}_u$, $\bar{\psi}_v$ one of which is (2.16) allow an easy verification of reciprocity making use of two partial integrations. Since (2.17) is obtained from (2.16) they also satisfy reciprocity as may be verified by using the quantities in curly braces as a whole. By means of (2.16) it is seen that reciprocity is not affected by the dynamic reaction terms also if $m_b\neq m_a$.

The qualitative character of the dynamic reaction terms may be illustrated by means of the following one-dimensional problem. The particle $c$ is supposed to be free to move in a straight line. At points $x_c=0$ and $x_c=x$ it is acted on by $\delta$-function potentials of equal strength. In terms of the variable $\xi$ used earlier in this section the adiabatic functions satisfy the boundary conditions

$$\frac{dw}{wd\xi}\Big|_{(x/2)-0}^{(x/2)+0}=\frac{dw}{wd\xi}\Big|_{-(x/2)-0}^{-(x/2)+0}=-2\alpha_0,\quad w=(u\text{ or }v). \tag{3.6}$$

The symmetric and antisymmetric solutions are identified with $u$ and $v$, respectively. In the line segment between the two potentials

$$u=N_+\cosh(\alpha_+\xi),\quad v=N_-\sinh(\alpha_-\xi),\quad (-x/2<\xi<w/2), \tag{3.7}$$

where $N_+$, $N_-$ are normalization constants. Continuing these functions in the intervals $(-\infty,-x/2)$ and $(x/2,\infty)$ by means of (3.6) and requiring that $u$ and $v$ vanish at $\infty$ there result conditions on $\alpha_+$ and $\alpha_-$ as follows:

$$\alpha_+=\alpha_0[1+\exp(-\alpha_+x)],\qquad \alpha_-=\alpha_0[1-\exp(-\alpha_-x)]. \tag{3.8}$$

The values of the normalization constants are then also obtained as

$$N_+=\{2\alpha_+/[1+\alpha_+x+\exp(\alpha_+x]\}^{1/2},\qquad N_-=\{2\alpha_-/[-1-\alpha_-x+\exp(\alpha_-x)]\}^{1/2}. \tag{3.9}$$

If

$$\tau\equiv\exp(-\alpha_0x)\ll1, \tag{3.10}$$

then from (3.8), to within the first order of $\tau$,

$$\alpha_+^2=\alpha_0^2(1+2\tau),\quad \alpha_-^2=\alpha_0^2(1-2\tau). \tag{3.11}$$

Maintaining the same relation between $\beta$ and $\alpha_0$, $\alpha_+$, $\alpha_-$ as in BE-I,

$$\beta=2\alpha_0^2\tau. \tag{3.12}$$

The functions $u$, $v$ may also be expressed as

$$u=N_uI_u,\quad v=N_vI_v, \tag{3.13}$$

where

$$N_u=(\alpha_+/\pi)N_+\exp(\alpha_+x/2),\qquad N_v=(-i\alpha_-/\pi)N_-\exp(\alpha_-x/2) \tag{3.14}$$

and

$$I_u=\int_{-\infty}^{+\infty}\frac{e^{i\omega\xi}\cos(\omega x/2)}{\alpha_+^2+\omega^2}d\omega,\qquad I_v=\int_{-\infty}^{+\infty}\frac{e^{i\omega\xi}\sin(\omega x/2)}{\alpha_-^2+\omega^2}d\omega. \tag{3.15}$$

These forms of $u$ and $v$ are convenient for calculation, applying for all $\xi$ and not just to a finite interval as is the case for (3.7). The functions

$$\bar{u}=2^{-1/2}(u-v),\quad \bar{v}=2^{-1/2}(u+v) \tag{3.16}$$

have the special values

$$\bar{u}_{x=\infty}=\alpha_0^{1/2}\exp(-\alpha_0|\xi+x/2|),\qquad \bar{v}_{x=\infty}=\alpha_0^{1/2}\exp(-\alpha_0|\xi-x/2|), \tag{3.17}$$

so that for infinite separation of the potential wells, the states $\bar{u}$ and $\bar{v}$ are concentrated on the left and right potential wells, respectively. From (3.15), it follows that

$$(u,\partial v/\partial x)=0,\quad (v,\partial u/\partial x)=0, \tag{3.18}$$

the integrand of the integral over $\omega$ being odd after the integration over $\xi$ is performed. From (3.18) and (3.16),

$$(\bar{u},\partial\bar{v}/\partial x)=0,\quad (\bar{v},\partial\bar{u}/\partial x)=0. \tag{3.18'}$$

It is also found that

$$(\bar{u},\partial^2\bar{v}/\partial x^2)=\mathcal{O}(\tau^2) \tag{3.19}$$

and hence all terms of order $M/\mu$ in curly braces of the second equation in (2.17) are of higher order in $\tau$ than the term $-\beta$, provided $m_b-m_a$ is negligible. For non-negligible $(m_b-m_a)/m$, employing (3.4) and (3.5), the terms in curly braces in the one-dimensional analog of (2.17) combine to

$$-\frac{\hbar^2}{2M}\beta\{1-(M/\mu)[m_b(m_b-m_a)/2m^2](1-\alpha_0x)-(M/\mu)[(m_b-m_a)/2m]x\partial/\partial x\}, \tag{3.20}$$

where terms of higher order in $\tau$ than the first have been dropped. In the $N^{14}(N^{14},N^{13})N^{15}$ reaction $|(m_b-m_a)/m|=1/27$ while $M/\mu\approx1/7$. The product of the two factors is small and formally one may expect

the effects caused by the extra terms to be small in this case. But for $Mg^{26}(N^{14},N^{13})Mg^{27}$, $|(m_b-m_a)/m|=\frac{1}{3}$, $M/\mu\approx 1/9.1$ and without closer investigation it appears unjustifiable to claim that these effects are negligible. The presence of $x$ in the second term in curly braces in (3.20) and of $x\partial/\partial x$ in the third indicate a different energy dependence from that of the main term and are an additional reason for caution regarding neglecting the dynamic reaction terms. The one-dimensional example cannot be expected to be a quantitative guide for the three-dimensional situation but the relative order of magnitude of the three terms depends on the action of the tails of the wave functions in both cases.

The term containing $\nabla_r\overline{\psi_u}$ in (2.18) represents an effect of the momentum of relative motion of $a$ and $b$. If $Q=0$, the factor $e^{i\omega t}$ in Eq. (24.2) of BE-I, which furnishes the transition amplitude, is unity and for a head-on collision the contribution of the term in question is proportional to

$$\int \dot{r}(\bar{u},D\bar{v}/Dr)dt, \tag{3.21}$$

where $D/Dr$ is the derivative with respect to $r$ keeping $\gamma$ of BE-I constant. Taking the origin of $t$ at the turning point, $\dot{r}$ is antisymmetric in $t$ while the remaining factor of the integrand in (3.21) is symmetric. Therefore, the term vanishes. If $Q\neq 0$ the imaginary part of $e^{i\omega t}$ gives a contribution to the integrand which is symmetric in $t$ and a nonvanishing effect of the term in $\dot{r}$ results as mentioned at Asilomar.[9] This term is 90° out of phase with the main one and produces therefore insignificant effects when small compared to the principal term. It has not been shown that it is negligible in general. Since, as discussed shortly before Eq. (3.5), the validity of reciprocity depends on the combined effect of both terms involving $M/\mu$ in the curly braces the separate consideration of the two terms gives only a partial view of the problem.

## IV. INTERNAL ANGULAR MOMENTUM EFFECTS

The calculations in BE-I have been carried out in detail for special configurations of nucleons in $N^{14}$ and $N^{15}$. For these the spatial dependence of the tunneling effect on the cross section was expressible in terms of $(1/r)\exp(-\alpha r)$. In general, the $r$ dependence involves additional powers of $1/r$ combined with angular functions. The presence of additional powers of $1/r$ matters for quantitative comparisons of the energy dependence of the cross section and the angular distribution.

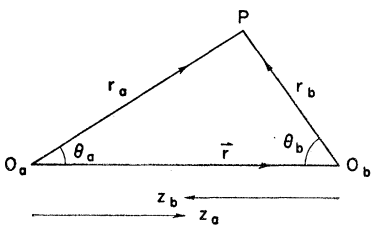


FIG. 1. Angles, distances, and quantization axes used in calculations of angular momentum effects.

The wave function of a nucleon originally in the right nucleus $b$ with angular momentum $l^b$ and magnetic quantum number $m^b$ is, within a constant factor,

$$r_b^{-1/2}K_{l(b)+\frac{1}{2}}(\alpha r_b)Y_{l(b)m(b)}(\theta_b,\varphi_b) = \sum_{l(a),m(a)} b_{l(a)m(a)}Y_{l(a)m(a)}(\theta_a,\varphi_a), \tag{4.1}$$

where $r_b$, $\theta_b$, $\varphi_b$ are polar coordinates of point $P$ in Fig. 1, referred to the polar axis $O_bz_b$ which is directed from the center of nucleus $b$ to the center of nucleus $a$. For convenience of printing the quantities $l^b$, $m^b$, $l^a$, $m^a$ are written as $l(b)$, $m(b)$, $l(a)$, $m(a)$ in subscripts and superscripts. Disregarding spin orientations the capture of a nucleon from the state of Eq. (4.1) into a state of nucleus $a$ of aximuthal quantum number $l_a$ and magnetic quantum number $m_a$ depends on the coefficients of the $Y_{l(a)m(a)}(\theta_a,\varphi_a)$ in the expansion of the wave function in (4.1). The expansion is complicated in the general case but a qualitative idea of the relative importance of different terms can be obtained if $\alpha r\gg 1$, $\alpha b\gg 1$ where $b$ is the nuclear radius. In this case[14] one may approximate at the surface of $a$

$$r_b^{-1/2}K_{l(b)+\frac{1}{2}}(\alpha r_b)\approx \mathfrak{A}(r)\exp(\alpha' r_a\cos\theta), \qquad \alpha'=\alpha r/(r-r_a), \tag{4.2}$$

and

$$\mathfrak{A}(r)=[(\pi/2\alpha)^{1/2}/(r-r_a)] \times\exp\{-\alpha[r+r_a^2/(r-r_a)]\}. \tag{4.3}$$

This approximation is obtained by means of the asymptotic form of $K_{l+\frac{1}{2}}(\alpha r_b)$ approximating $r_b$ in $\exp(-\alpha r_b)$ as

$$r_b=[(r-r_a)^2+2rr_a(1-\cos\theta)]^{1/2} \cong r-r_a+[rr_a/(r-r_a)](1-\cos\theta), \tag{4.4}$$

and employing the cruder approximation,

$$r_b=r-r_a, \tag{4.5}$$

in the denominator of the fraction $[\exp(-\alpha r_b)]/r_b$. The reason for making these approximations is that (4.2) is to be used in (4.1) for an analysis in the $Y_{l(a)m(a)}(\theta_a,\varphi_a)$ at fixed $r_a$ and that for fixed $r_a$ the values of $[\exp(-\alpha r_b)]/r_b$ on the sphere $r_a=$ const are largest close to $\cos\theta=1$. The main requirement in the calculation is therefore to represent the function well close to $\cos\theta=1$ and since the dependence on the exponent is the more critical the approximation used for the denominator as in (4.5) is the cruder of the two. The spherical harmonic factors on the left- and right-hand sides of Eq. (4.1) also vary and can spoil the approximation by making the integrand small in a sufficiently large range of values of $\theta$ around $\theta=0$.

[14] The notation for the Bessel functions of imaginary argument is as in G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, Cambridge, 1922).

Employment of (4.1) gives

$$r_b^{-1/2}K_{l(b)+\frac{1}{2}}(\alpha r_b)\approx(-)^{l(b)+m(b)}\pi\sum_{l_a}\frac{[(2l^a+1)(2l^b+1)]^{1/2}}{m^b!}\left[\frac{(l^b+m^b)!(l^a+m^b)!}{(l^b-m^b)!(l^a-m^b)!}\right]^{1/2}$$
$$\times\frac{e^{-\alpha r}}{(2\alpha r)^{m^b+1}}\frac{I_{l(a)+\frac{1}{2}}(\alpha r)}{r^{1/2}}Y_{l(a)m(b)}(\theta_a,\varphi_a). \quad (4.6)$$

Here $I_{l+\frac{1}{2}}$ is the Bessel function of imaginary argument of the first kind and is related to the regular density function $F_l$ by

$$F_l(i\rho)/(i^{l+1}\rho)=(\pi/2\rho)^{1/2}I_{l+\frac{1}{2}}(\rho). \quad (4.7)$$

These relations have been mentioned in Ref. 9 but with insufficient explanation concerning the meaning of the approximation used.

For $\alpha=0$ the Bessel functions of imaginary argument of the second kind are taken place of by inverse powers of $r$, the wave equation $(\Delta-k^2)\psi=0$ being replaced by the Laplace equation. In this case without approximations

$$(-)^{\lambda+\lambda'-m}\frac{Y_{\lambda'm}(\theta_b,\varphi_b)}{r_b^{\lambda'+1}}=\sum_{\lambda=m}^{\infty}\frac{(\lambda+\lambda')![(2\lambda'+1)/(2\lambda+1)]^{1/2}r_a^{\lambda}r^{-\lambda-\lambda'-1}Y_{\lambda m}(\theta_a,\varphi_a)}{[(\lambda-m)!(\lambda+m)!(\lambda'-m)!(\lambda'+m)!]^{1/2}}. \quad (4.8)$$

The function on the left-hand side is the irregular, i.e., the source type, solution of the Poisson equation emanating from $O_b$. It is represented on the right as a sum of solutions regular at $r_a=0$, with coefficients that are functions of the internuclear distance $r$. While exact, Eq. (4.8) applies only to nucleons having zero separation energy, a situation completely opposite to that of the approximation used in Eq. (4.6). In both cases the expansion coefficients are functions of the magnetic quantum number $m$ but in (4.8) the dependence on $r$ is $m$-independent while in (4.6) there is a marked $m$ dependence. For zero separation energy, according to (4.8) the ratios of coefficients of the regular functions for different $m$ are independent of $r$. In this sense the anisotropy is in this case independent of the internuclear distance. On the other hand, for high nucleon separation energies, according to (4.6) the anisotropy depends on $r$.

It is thus seen that there is no reason for expecting the $e^{-\alpha r}/r$ function which applies for the special nucleon configurations which appeared probable for $N^{14}(N^{14}, N^{13})N^{15}$ in BE-I to have a general significance. The transfer probability amplitude may be expected therefore to contain additional powers of the distance of closest approach in the SC approximation. As a result both the angular distribution of the reaction products and the energy dependence of the total cross section may be expected to differ from those derived by means of $e^{-\alpha r}/r$. This conclusion will be made use of in Sec. VI. It should be mentioned that the approximation of (4.2) cannot be expected to be universally good and that numerical examination shows it to be poor under many circumstances. Nevertheless, the general features of (4.6) and the presence of many powers of $1/r$ remain in the improvements.

Experimental data have been analyzed[7,15,16] in terms of $\sigma/\Lambda^2$ plots against

$$x=\alpha(b_1+b_2-2a')+\bar{\alpha}(\bar{b}_1+\bar{b}_2-2\bar{a}'), \quad (4.9)$$

where $\Lambda$ is the wavelength for the incident state divided by $2\pi$. Here $\alpha$ and $\bar{\alpha}$ are the space-decay constants of the neutron in the initial and final states, $b_1$, $b_2$, and $2a'$ are the radii of the two nuclei and the distance of closest approach in the initial state and the barred quantities have the corresponding meaning for the final state. Similarly, plots of $d\sigma/d(r_{\min}\Lambda^2)$ against $r_{\min}$ have been used in some data analyses.[15] Here $r_{\min}$ is the distance of closest approach for a Rutherford orbit corresponding to a given angle. In both cases the assumption of the applicability of the $e^{-\alpha r}/r$ type function is involved, and these ways of analyzing data have therefore only qualitative significance. There have been two reasons for carrying out this analysis. In the first place, it appeared desirable to see whether the bulk of the data is in agreement with the general tunneling picture, combining the latter with the supposition that for angles corresponding to the more distant collisions one may neglect forces additional to the Coulombian as well as the effects of competing reactions. In this application the variation of $e^{-\alpha r}/r$ contained in $e^{-\alpha r}$ is often more marked than that in the factor $1/r$. The presence of large effects other than tunneling may be detectable therefore in the approximation used. The possibility of virtual Coulomb excitation (VCE) was paid particular attention to in Ref. 7. Secondly, it appeared desirable to make some obvious improvements

[15] G. Herling, Y. Nishida, and G. Breit, Bull. Am. Phys. Soc. 5, 293 (1960).

[16] K. S. Toth and E. Newman, *Proceedings of the Third Conference on Reactions Between Complex Nuclei, Asilomar, 1963* (University of California Press, Berkeley, 1963), p. 114.

in the early attempts to provide a comparison of different transfer reactions by means of "universal curves" which were first made by Halbert *et al.*[17] A removal of the dissymmetry in the roles of the initial and final states was made[5] earlier but Eqs. (48.100), (48.101), and (48.102) of the last-mentioned reference contained unnecessary approximations in formulas for the total cross section which were avoided in the work reported on in Refs. 7 and 15. The starting point was however very similar as is seen from (48.79′) of Ref. 5 which has the same meaning as Eq. (4.9) of this paper.

Another reason for introducing the approximate expressions was the lack of proper symmetry between initial and final states which was present in the introduction of the $d\sigma/dr_{\min}$ plots by McIntyre, Jobes, and Watts.[18] The results obtained in BE-I employing $e^{-\alpha r}/r$ and $Q=0$ were therefore somewhat arbitrarily generalized as in Ref. 5. This generalization supposed that

$$d\sigma/d\Omega=(C/\mathbf{s}^3)\exp[-(\alpha a'+\bar{\alpha}\bar{a}')/\mathbf{s}], \quad (4.10)$$

where $\mathbf{s}$ is as in (2.4) and $C$ is angle-independent. Integration over solid angles gives then

$$\sigma=(8\pi C/(\alpha a'+\bar{\alpha}\bar{a}'))\exp(-\alpha a'-\bar{\alpha}\bar{a}'). \quad (4.11)$$

On the other hand, as in Ref. 5 and Eq. (4.9) above, it is expected that for the total cross section

$$\sigma=K\Lambda^2\exp\{-\alpha(2a'-b_1-b_2)-\bar{\alpha}(2\bar{a}'-\bar{b}_1-\bar{b}_2)\}, \quad (4.12)$$

with

$$K=\frac{\pi^2}{2\alpha^2\lambda_1\lambda_2}\left(\frac{\alpha b_1}{1+\alpha b_1}\right)^2\left(\frac{\bar{\alpha}b_2}{1+\bar{\alpha}b_2}\right)^2, \quad (4.12a)$$

the latter being taken from BE-I and having a direct meaning only for the particular nucleon configurations used there. Comparison of (4.11) with (4.12) gives $C$ in terms of $K$. Since, for both the initial and final orbit, the minimum distance of closest approach $r_{\min}$ is proportional to $1+1/\mathbf{s}$, there was introduced a mean $r_{\min}$.

$$\langle r_{\min}\rangle=(a'\bar{a}')^{1/2}(1+1/\mathbf{s}). \quad (4.13)$$

The relationship between $d\langle r_{\min}\rangle$ and the solid angle $d\Omega$ contained between the cones of angular openings $\theta$ and $\theta+d\theta$ is, according to (2.4),

$$d\langle r_{\min}\rangle=-[(a'\bar{a}')^{1/2}/8\pi\mathbf{s}^3]d\Omega, \quad (4.14)$$

and hence, introducing the values of $r_{\min}$ for the initial and final orbits

$$r_{\min}=\alpha'(1+1/\mathbf{s}), \quad \bar{r}_{\min}=\bar{a}'(1+1/\mathbf{s})$$

there was obtained

$$\frac{d\sigma}{d\langle r_{\min}\rangle}=\frac{K\Lambda^2}{(a'\bar{a}')^{1/2}}(\alpha a'+\bar{\alpha}\bar{a}')\times\exp[\alpha(b_1+b_2)+\bar{\alpha}(\bar{b}_1+\bar{b}_2)-\alpha r_{\min}-\bar{\alpha}\bar{r}_{\min}]. \quad (4.15)$$

The object in introducing these improvements and in employing them has been mainly that of taking into account approximately the effects of the $Q$ value of the reaction. These effects enter through the differences between $\alpha$ and $\bar{\alpha}$ and between $a'$ and $\bar{a}'$ as in (4.12) and (4.15). On the other hand, neither the quantum corrections for heavy particle motion nor the expected dependence of $K$ on the nucleon configuration have been taken into account in the applications of (4.12) and (4.15). Since the former of these approximations affects the energy and angular dependence and since the procedure for symmetrization between the initial and final states has not been uniquely established, it appeared justifiable to neglect in these comparisons the higher powers of $r$ which are expected to enter in the generalization of the transfer function $e^{-\alpha r}/r$.

On the other hand, attempts to decide which method of calculation is the better on the basis of comparison with experiment without taking into consideration the possibility of additional powers of $1/r$ entering the transfer function are obviously meaningless, especially in view of the lack of consideration of virtual Coulomb excitation and of $M/\mu$ type terms as will be discussed in more detail in Sec. VI.

## V. EXTERNAL ANGULAR MOMENTUM EFFECTS

In addition to the effects of anisotropy of transfer and of the related occurrence of additional powers of $1/r$ classical mechanics suggests that the transferred nucleon, while still attached to $a$, possesses an angular momentum around $b$ on account of the motion of $a+c$ as a whole. By analogy with the Franck-Condon principle of molecular physics a certain degree of independence between the relative motion of $a$ with respect to $b$ and the motion of $c$ around the combined system formed by $a$ and $b$ may be expected. Such an independence is in fact present in most of the discussion in Sec. II of the present paper but is violated by the dynamic reaction terms of Eq. (2.17). The effect under discussion in this section is concerned with another type of violation of this independence. If the angular momentum of $c$ around $b$, owing to the motion of $c$ along with $a$, matches the angular momentum of an available state of $c$ in $b$, the transfer to the state in $b$ may be expected to be favored. In order to be significant, the matching has to occur approximately at the distance $r_{\min}$ because the tunneling is most pronounced for this value of $r$. In this respect the consideration is similar to that used in applications of the Franck-Condon principle. The way in which this occurs in a quantum-

[17] M. L. Halbert, T. H. Handley, J. J. Pinajian, W. H. Webb, and A. Zucker, Phys. Rev. **106**, 251 (1957).

[18] J. A. McIntyre, F. C. Jobes, and T. L. Watts, *Proceedings of the Second Conference on Reactions Between Complex Nuclei, Gatlinburg, Tennessee, 1960* (John Wiley & Sons, Inc., New York, 1960), Paper A-2, p. 16.

mechanical treatment of angular momenta may be illustrated in a simple example.

As in BE-II, two sets of relative coordinates are appropriate: one for the condition of the system in which $c$ is attached to $a$, one for that in which it is attached to $b$. This complication is minimized in the special case of $m_b=\infty$. To simplify conditions still further $l_a$, the angular momentum of $c$ while in $a$, will be taken as $l_a=0$. If $\mathbf{L}$ the angular momentum of $a+c$ with respect to $b$, is oriented along the axis of quantization the wave function representing the motion of the center of mass of $c$ and $a$ with respect to $b$ may be written

$$\Psi_{ca,b}=[(x_{ca,b}+iy_{ca,b})/r_{ca,b}]^L R_{ca,b}(r_{ca,b}). \quad (5.1)$$

Here $x_{ca,b}$ is the difference in the $x$ coordinates of the center of mass of $c$ and $a$ and the coordinates of $b$ and the axis of quantization is perpendicular to $\mathbf{r}_{ca,b}$. Similarly for $y_{ca,b}$, and $r_{ca,b}$ is the absolute value of the vector $\mathbf{r}_{ca,b}$ having components $x_{ca,b}$, $y_{ca,b}$, $z_{ca,b}$. Since $m_b=\infty$, one may set $x_b=y_b=z_b=0$ so that

$$x_{ca,b}=(m_a x_a+m_c x_c)/m_{ac}, \quad m_{ac}=m_a+m_c. \quad (5.2)$$

Substituting into (5.1)

$$\Psi_{ca,b}=\sum_s \binom{L}{s}\left(\frac{m_a}{m_{ac}}\right)^{L-s}\left(\frac{m_c}{m_{ac}}\right)^s (x_a+iy_a)^{L-s} \times (x_c+iy_c)^s R_{ca,b}(r_{ca,b})/r_{ca,b}{}^L. \quad (5.3)$$

If $c$ is tightly bound to $a$ and if $r_{ca,b}$ is not too small, the square of the absolute value of the wave function is small unless

$$r_{ca}/r_{ca,b}\ll 1. \quad (5.4)$$

Under these conditions, the factor $(x_a+iy_a)^{L-s}(x_c+iy_c)^s$ is insensitive to $s$ and will be treated as a constant. If

$$m_c/(m_a+m_c)\ll 1, \quad (5.5)$$

the variation of successive terms in (5.3) with $s$ is contained in the factor

$$\frac{L!}{s!(L-s)!}\left(\frac{m_c}{m_{ac}}\right)^s. \quad (5.6)$$

Approximating logarithms of the factorials by means of Stirling's series with neglect of the term in $1/(12n)$ for $\log n$ the condition for a maximum on the supposition that $(s/L)\ll 1$, $L\gg 1$ becomes

$$s/L\approx m_c/m_{ac}. \quad (5.7)$$

Had one set

$$s\hbar=m_c v R, \quad L\hbar=m_{ac} v R \quad (5.8)$$

on the classical picture of the two angular momentum components under discussion corresponding either to mass $m_c$ or mass $m_a+m_c$ moving with the same velocity $v$ and at the same distance $R$ around the center of $b$, one would have obtained the same value of the ratio $s/L$ as in (5.7).

Equations (5.1) to (5.7) thus show that for $s$ states of relative motion of $c$ with respect to $a$, for $m_b=\infty$, and with the restrictions imposed in (5.4) and (5.5), the maximum probability of a state with angular momentum projection $s\hbar$ occurs at the classical mechanics value of the angular momentum projection. In calculating the quantum mechanical probability the variability of the two factors containing the combinations $x+iy$ was disregarded making use of the supposed spatially condensed nature of the wave function of relative motion of $c$ with respect to $a$. Had this assumption not been made, a diffuseness in the condition (5.7) would have been introduced. Some diffuseness in the correspondence of the probable quantum values to the classical picture, expressed by (5.8), is present besides because the expression listed in Eq. (5.6) does not have a sharp maximum. Under the assumptions made, the width of the square of the quantity in (5.6) is $s_m{}^{1/2}$, where $m$ indicates value at the maximum, the measure of the width used being a decrease to $1/e$ of value at maximum in going to either side of $s_m$ by the "width."

The choice of the factor used in (5.6) may appear arbitrary because the powers of $x_{ca,b}+iy_{ca,b}$ and of $x_c+iy_c$ in (5.1) and (5.3) do not occur with the same numerical coefficients as in the normalized spherical harmonics. If the calculation is made in terms of the latter the quantity replacing that in (5.6) is

$$\frac{2^s L!}{(L-s)![(2s+1)!]^{1/2}}\left(\frac{m_c}{m_{ac}}\right)^s, \quad (5.6')$$

the position of the maximum in the $m_c/m\ll 1$, $L\gg 1$ approximation is

$$(s_m+\tfrac{1}{2})/L\approx m_c/m_{ac}, \quad (5.7')$$

and the square of the quantity in (5.6′) is approximately proportional to

$$\exp\{-(s-s_m)^2/(s_m+\tfrac{1}{2})\},$$

the width of the transfer probability maximum as a function of transferred orbital angular momentum being now $(s_m+\frac{1}{2})^{1/2}$ in the same sense as it was $s_m{}^{1/2}$ before. The conclusions are seen to be rather insensitive to the change in convention regarding normalization. Some of the anomaly in the behavior of $Pb^{207}(N^{14}, N^{13})Pb^{208}$ and $Pb^{208}(N^{14},N^{13})Pb^{209}$ observed by Watts and McIntyre may be caused perhaps by the participation of the effect of selection of a band of angular momenta in a manner just discussed. If the neutron is not in an $s$ state initially, an additional diffuseness of the maximum may be expected.

## VI. TRANSITION TO SEMICLASSICAL THEORY AND SMALL-ANGLE LIMIT

At low energies the interactions between $a+c$ with $b$ initially and of $a$ with $b+c$ finally may be expected to be predominantly Coulombian. In this limit the integral $I_\beta^{(-+)}$ of Eq. (2.23) becomes a multiple of $I_0(\theta)$ of Eq. (2.2). In the special case of equal initial and final energies, on the basis of Ter-Martirosyan's results employing the collection of formulas in Ref. 5,

$$|I_0(\theta)|=\frac{4\pi^{3/2}\eta}{\alpha^2}\left\{\exp\left[2\eta\tan^{-1}\frac{2k}{\alpha}-2\pi\eta\right]\right\}\times\left|\frac{F(i\eta,i\eta;1;-\zeta)}{1+\zeta}\right|, \quad (6.1)$$

where

$$\zeta=4k^2\mathbf{s}^2/\alpha^2. \quad (6.2)$$

The quantity

$$(1+\zeta)^{i\eta}F(i\eta_2,i\eta_1;1;-\zeta)$$

may be evaluated by employing an analytic continuation of the hypergeometric series in powers of $1/(1+\zeta)$. Taking the limit of this series for $\xi=\eta_2-\eta_1=0$, there results the relation

$$(1+\zeta)^{i\eta}F(i\eta,i\eta;1;-\zeta)=\frac{2/\eta^2}{|\Gamma(i\eta)|^4}\sum_0^\infty\left[-\tfrac{1}{2}\ln(1+\zeta)+\frac{\Gamma'(s+i\eta)}{2\Gamma(s+i\eta)}+\frac{\Gamma'(1+s-i\eta)}{2\Gamma(1+s-i\eta)}-\frac{\Gamma'(1+s)}{\Gamma(1+s)}\right]\times\frac{\Gamma(s+i\eta)\Gamma(1+s-i\eta)}{[\Gamma(1+s)]^2}\left(\frac{1}{1+\zeta}\right)^s. \quad (6.3)$$

The passage to the classical limit employed in the semiclassical treatment (SCT) requires keeping

$$\eta/k=a' \quad (6.4)$$

finite and approaching a constant, while $\eta$ and $k$ approach $\infty$. In this way curves normal to wave fronts become eventually the rays of geometrical optics, i.e., the classical orbits. In this limit[19]

$$\lim(1+\zeta)^{i\eta}F(i\eta,i\eta;1;-\zeta)=\frac{2i/\eta}{|\Gamma(i\eta)|^2}K_0(x) \quad (\eta=\infty). \quad (6.5)$$

Here

$$x=2\eta(1+\zeta)^{-1/2}\longrightarrow a'\alpha\epsilon, \quad \epsilon=1/\mathbf{s}, \quad (6.6)$$

where the quantity to the right of the arrow is the result of taking the same limit as in (6.5). From Eqs. (6.1) and (6.5)

$$\lim|I_0(\theta)|=\pi^{1/2}\eta[\exp(-\alpha a')]\frac{K_0(\alpha a'\epsilon)}{k^2\mathbf{s}^2}. \quad (6.7)$$

This gives, therefore,

$$\frac{d\sigma}{d\Omega}=\frac{C}{v^6\mathbf{s}^4}[K_0(\alpha a'\epsilon)]^2\exp(-2\alpha a'), \quad (6.8)$$

in agreement with the exact evaluation of the angular dependence according to the SCT in Eq. (23) of BE-I. The dependence on $v$ also checks if the first of the two forms for the probability $|\bar{a}_v(+\infty)|^2$ listed in Eq. (25) of BE-I is employed. Similarly, the constant of proportionality $C$ is readily verified to be the same in the QM and the SCT. The agreements just mentioned are natural and follow from general principles discussed by means of Fig. 2 and Eqs. (18) to (19.4), pp. 694, 695 of BE-I.

For $\theta\ll 1$, the Bessel function of imaginary argument of the second kind $K_0$ approaches its asymptotic form for large values of the argument, and accordingly,

$$d\sigma/d\Omega\approx(C'/v^4\mathbf{s}^3)\exp[-2\alpha a'(1+\epsilon)], \quad (6.9)$$

where $C'$ is a constant. If, in this form, one makes $\alpha\to 0$, the angular dependence is as $1/\mathbf{s}^3$, and is not the same as that for Rutherford scattering. It is claimed on the other hand by Greider[20] that, in the limit of $\alpha=0$, the factor multiplying exponentials behaving with angle much like that in (6.9) should approach proportionality to $1/\mathbf{s}^4$ because agreement with the Rutherford-Coulomb scattering angular dependence would result for such a dependence. The expectation of agreement with the Coulomb-scattering law appears to be inherent in the approach of Ref. 20, which neglects the distortion operators in the evaluation of the formulas. The agreement of the QM and the SCT verified above[21] shows, however, that the distortion of the wave by the Coulomb field is very important, the main transfer occurring in the SCT near the point of closest approach along the orbit. According to Fig. 2 of BE-I, this point lies in the region in which the distortion of the plane wave by the Coulomb field is relatively large.

Equation (6.9) was obtained from the more exact (6.8) by going to small $\theta$. The comparison with the Rutherford law then took place by making $\alpha\to 0$ and neglecting $2\alpha a'(1+\epsilon)$ in the exponential function. Such a comparison contradicts, however, the assumption made in obtaining (6.9), viz., that $\alpha a'\epsilon\gg 1$. This assumption was used in replacing $K_0$ by its asymptotic value for large values of the argument. Thus, the derivation

[19] The writer is indebted to Dr. S. Ohnuma and Dr. C. R. Fischer for their check on Eqs. (6.3) and (6.4) a few years ago in another connection, and for having found an error in the first calculation of the limit in Eq. (6.5).

[20] K. R. Greider, Phys. Rev. **133**, B1483 (1964); Phys. Rev. Letters **9**, 392 (1962).

[21] This agreement and the agreement of both theories with the angular dependence expected for the Rutherford cross section was communicated to the author of Ref. 20 by the writer of the present paper before the publication of Ref. 20.

of Eq. (6.9) breaks down for sufficiently small $\alpha$ even in the semiclassical limit.

The employment of the SCT limit by means of (6.4) need not give the QM limit for $\theta \to 0$ for the actual situation, however, because it implies the passage to infinite values of the momentum $\mu v$, a limit which can be secured by making $\mu$ infinite but does not correspond to reality. The SCT limit makes use of a finite $v$ so that $\mu v$ cannot be made infinite by making $v = \infty$. For fixed $\mu$, $v$, and $\theta$ the behavior of $d\sigma/d\Omega$ is obtainable from (6.3). Since, for $\alpha \to 0$, the quantity $\zeta \to \infty$ and

$$|F(i\eta, i\eta; 1; -\zeta)| \to \frac{\ln(4k^2\mathbf{s}^2/\alpha^2)}{\eta|\Gamma(i\eta)|^2}, \quad (\alpha \to 0). \quad (6.10)$$

The limiting form of $|I_0|$ is, accordingly,

$$|I_0(\theta)| \to \frac{\pi^{1/2}\eta}{2k^2\mathbf{s}^2}(1-e^{-2\pi\eta}) \ln\left(\frac{4k^2\mathbf{s}^2}{\alpha^2}\right), \quad (\alpha \to 0). \quad (6.11)$$

If $\alpha$ assumes values such that $|\ln(1/\alpha^2)| \gg |\ln(4k^2\mathbf{s}^2)|$, the differential cross section which varies as $|I_0(\theta)|^2$ has the same angular dependence as that for Rutherford scattering. Agreement with the latter is thus obtained for $\alpha \to 0$ in the quantum-mechanical treatment. The passage to the limit $\alpha \to 0$, $\theta \to 0$ is seen to be not uniquely defined. In fact, if $\alpha$ is kept fixed and $\theta \to 0$, then $\zeta \to 0$ and $F(i\eta, i\eta; 1; -\zeta) \to 1$. In this limit

$$|I_0(\theta)| \to (4\pi^{3/2}/\alpha^2) \exp\left(2\eta \tan^{-1}\frac{2k}{\alpha} - 2\pi\eta\right), \quad (\theta \to 0). \quad (6.12)$$

The exponential function is in practice nearly $\exp(-\pi\eta - \alpha a')$ and the cross section is quite small on account of the factor $\exp(-2\pi\eta)$. Nevertheless, it is seen that the limit depends on the order in which $\alpha$ and $\theta$ are made to approach zero.

Passage to the limit $\alpha = 0$ as in (6.10) and (6.11) has a definite mathematical meaning but the physical assumptions justifying the use of perturbation theory are violated since the interaction is not weak when the nucleon separation energy is small. The infinite value of $|I_0(\theta)|$ which results for $\alpha = 0$ implies this as well. For $\alpha \to 0$, the SCT also gives a logarithmic term since

$$|K_0(\alpha a'\epsilon)|^2 \to [\ln(\alpha a'/2\mathbf{s}) + \gamma]^2, \quad (6.13)$$

and the velocity as well as $\mathbf{s}$ occur with the same powers in (6.8) and in (6.11). In this respect there is agreement between the SC and the QM approaches[21] and there is thus no reason for mistrusting the SC result on the alleged grounds of disagreement with the angular dependence expected from Rutherford's scattering formula. In summary, there is a logarithmic dependence on $\mathbf{s} = \sin(\theta/2)$, but as $\alpha \to 0$ in any fixed range of $\theta$ excluding $\theta = 0$, one can make $\alpha$ small enough to give the Rutherford dependence of the cross section on $\theta$. On the other hand for fixed $\alpha$ one can make $\theta$ small enough to make ln**s** interfere with the agreement. But if $\theta$ is small enough, the argument of $K_0$, viz., $\alpha a'\epsilon \gg 1$ and ln**s** has little bearing on the value of $K_0$ which is then approximated better employing the asymptotic exponential form as in (6.9). In this limit the critical variation of the impact parameter $\mathcal{P} \propto 1/\mathbf{s}$ with $\theta$ dominating the transfer process since the latter depends primarily on the value of $e^{-\alpha R}/R$ with $R$ corresponding to closest approach.

It is seen from the above discussion that the argument concerning the limit $\alpha \to 0$ in Sec. V of Ref. 20 is not valid. On the other hand, Ref. 20 lays stress on the improvement in the agreement with experiment that is obtained on account of the extra factor $\Delta$ in Eq. (42) of that reference. This emphasis on preference between different approximations to the solution of essentially the same equations on the basis of agreement with experiment is being made both in Sec. IV on p. B1492 and in Sec. V of Ref. 20. At the same time, Ref. 20 is critical of the work of Ter-Martirosyan[11] and of Biedenharn, Boyer, and Goldstein[12] regarding the limit which the results in these two references approach when $\alpha \to 0$, and also regarding the derivations used. By implication it is also critical of the applicability of the quantum corrections to the SCT which have been used by the writer.[5] The discussion in the present section shows, however, that from the point of view of the form of the answer all of these treatments behave properly at small $\theta$ and small $\alpha$ except for defects having their origin in the largeness of the transfer probability which cannot be corrected by inserting a factor $1/\sin(\theta/2)$ in the formula for $\sigma(\theta)$ as Ref. 20 recommends.

It will be noted, however, that the dissymmetry caused by the introduction of the zero-range interaction in Refs. 11 and 12 is not present in the quantum treatments presented at the Padua[8] and Asilomar[9] conferences and explained more systematically in the present paper. Even though the treatment of quantum corrections presented in Ref. 5 has made use of a zero-range interaction unsymmetrically and employed the integral evaluations of Ref. 11 this was done only to ascertain the ratio of QM and SCT results for the zero-range interaction. Otherwise, reliance was put on the treatment in BE-I which is free of the dissymmetry inherent in the use of the $\delta$-function potential. On the other hand, at energies much below the Coulomb barrier the same differences between a classical orbit (SCT) and a wave-mechanical (QM) treatment of the motion of the heavy nucleus enter for the symmetrical treatment of emitter and receiver nuclei and for the short-range unsymmetrical one. The application of the correction factor to the results in BE-I could be expected therefore to be adequate. This expectation is borne out by the work in Refs. 8, 9, and the present paper. On the other hand, the possible presence of virtual Coulomb excitation as well as the various other

effects discussed in Refs. 8 and 9 as well as here make a judgment, regarding correctness of calculations which do not take these effects into account on the basis of comparison with measurements, inadequate. It may also be mentioned that according to Ref. 22 the straightforward application of Eq. (6.1) employing the steepest descents approximation to the hypergeometric function according to Ref. 11 in the adaptation to the heavy ion case as in Ref. 5 gives a reasonable representation of experimental data on $N^{14}(N^{14},N^{13})N^{15}$ at 12.3-MeV bombarding energy. This treatment is more accurate in detail than that in Ref. 9 but is similar to the latter in the essential points. The claim made in Ref. 20 concerning the steepest descents treatment of Ter-Martirosyan being insufficient for the determination of the phase factor of the wave is correct only in the most literal and not an essential sense since the steepest descents integral over the saddle point furnishes the steepest descents approximation to the hypergeometric function including the phase factor. The result of this integration was used in Ref. 22.

The presentation in Ref. 20 makes use of an $s$-state wave function for the transferred nucleon without justification. The neutron actually transferred in the reaction discussed can hardly be an $s$ neutron in any ordinary sense. A partial justification for such a procedure is contained in BE-I. It is shown in that paper that for special coupling conditions of the $p$-shell neutrons, the net effect on energy and angular dependence of their transfer is such as though an $s$-state neutron were transferred. The $jj$ coupling assumed in BE-I is only an approximation to the actual one, however, and the additional powers of $1/r$ mentioned in connection with Eqs. (4.6)–(4.8) may be expected to have an effect on the energy dependence and the angular distribution. Unless it is shown that these effects are negligible there is an additional reason for considering agreement with experiment as insufficient evidence for the criticism[20] of standard formulas. The employment of the $s$ function through the nuclear interior defeats from the outset the possibility of discussing the nuclear interior with generality and realism since in the interior there are strong interactions between nucleons leading to admixtures of other configurations. It is also not known that wave absorption caused by competition with other reactions has a negligible effect on the angular distribution at 12.3-MeV bombarding energy. Such an effect does not appear unreasonable if virtual Coulomb excitation should enhance the transfer of larger groups of particles. Even the endothermic reactions $N^{14}(N^{14},B^{11})F^{17}$ and $N^{14}(N^{14},C^{11})O^{17}$ are energetically possible with 6.15 MeV available in the center-of-mass system, the first reaction being endothermic by 4.9 and the second by 4.1 MeV. The separation energies of $He^3$ in $N^{14}$ and $F^{17}$ are 27.7 and 22.8 MeV, and are large enough to inhibit ordinary tunneling very strongly, but it is not clear that a transfer of $He^3$ would take place with a probability insufficient to cause appreciable wave absorption. The transfer of $H^3$ corresponds to separation energies of 22.7 and 18.6 MeV initially and finally, and is also somewhat improbable. The exothermic deuteron transfer $N^{14}(N^{14},C^{12})O^{16}$ may be expected, however, to be more probable. The separation energies of the deuteron in the initial and final states are 10.3 and 20.7 MeV, respectively, the $Q$ value is 10.4 MeV. This, and perhaps other exothermic reactions, may perhaps introduce appreciable wave absorption with the participation of VCE. In this process, it will be remembered, the largeness of the separation energy does not count as heavily as for ordinary tunneling. These effects have not been definitely excluded as having negligible effects on the angular distribution of $N^{14}(N^{14},N^{13})N^{15}$ and similarly resonance force effects caused by the identity of the two colliding nuclei have not been shown to produce an influence on the relative motion of $N^{14}$ with respect to $N^{14}$ that is insufficient to affect the fits to the $E_l=12.3$-MeV data in Ref. 22 and in Ref. 20.

The possibility that at energies below the Coulomb-barrier, virtual Coulomb excitation may result in an enhancement of nuclear reactions, not necessarily of the single-nucleon transfer type, may be of interest in astrophysics. It is customary in considering nuclear reaction yields in this subject to make use of estimates of barrier penetrabilities in combination with the assumption that it is necessary for the colliding nuclei to come together within the range of nuclear forces in order that a reaction take place. The VCE processes should, on the other hand, be the dominant ones in low-energy collisions. Speaking qualitatively, the nuclear radii to be used neglecting VCE should become increasingly greater as the kinetic energy of colliding particles is decreased.

## VII. THE NUCLEON HALO REGION

The treatments in BE-I and the succeeding ones consider only one plausible coupling scheme and one configuration of nucleons. This was done for simplicity in the desire to obtain a quantitative comparison with experiment without too many complications. The underlying picture is appreciably more general and it appears appropriate to give a brief account of it. In the spirit of the black-box point of view employed in Wigner's $\mathcal{R}$-matrix theory of nuclear reactions a surface separating the configuration space into an interior and an exterior region will be used. For one nucleus the interior region will be defined as corresponding to positions of nucleons inside a three-dimensional sphere of radius $b$ having the center of mass of the nucleus for its center. In the case of deformed nuclei, it would of course be more rational for some purposes to use an ellipsoid rather than a sphere but, for simplicity, only a

[22] G. Breit, K. W. Chun, and H. G. Wahsweiler, Phys. Rev. 133, B403 (1964).

spherical separation surface will be discussed. Whenever all nucleons are inside the "sphere," the collection of nucleons will be said to be in the interior region. Otherwise, it will be said to be in the exterior region. The criterion for the radius of the sphere will be the possibility of neglecting the interactions between nucleons whenever they are outside the three-dimensional sphere. Approximately this is a density criterion. It cannot be formulated with complete exactness, but estimates indicate that it has an approximate meaning for the supposed short range of nucleon-nucleon forces. The chance of a collision in the exterior region decreases rapidly with $b$ to such an extent that it becomes reasonable to neglect it for sufficiently large $b$.

There are situations for which this is not the case. Thus, part of the exterior region is occupied by virtual deuterons, tritons, alpha particles, etc., forming tentacles in the many-dimensional configuration space[23] which are approximated by the clusters of Wilkinson.[24] However, a virtual alpha particle in a tentacle need not bear a strong resemblance to an ordinary alpha particle except at the end of the tentacle. At very low bombarding energies, the tentacles usually do not tunnel through the gap between two nuclei as well as a single nucleon. On the other hand, according to Ref. 23, some short-range tunneling effects are expected to be appreciable and in the quantitative analysis of nucleon transfer data single-nucleon tunneling via tentacles of larger mass may have to be considered. If the formulation of tentacles and their penetration through the gap is enhanced by VCE, a possibility which was left out of consideration in Ref. 23, their importance may become greater especially because the separation energy does not matter as critically at larger distances.

Returning to the simpler one-nucleon tunneling transfer without VCE and leaving the tentacles out of consideration, the wave function of a nucleus in the interior of the sphere $S$ may be expanded in terms of antisymmetrized products of central field single-nucleon functions. The goodness of the shell model is immaterial for the possibility of doing so, the only relevant consideration being the completeness of the set of expansion functions. The most important situation for the transfer is obtained if just one of the nucleons is outside $S$. In Wigner's terminology it is then in a closed channel and its "partner" is the residual nucleus. The convenient expansion for this part of the configuration space is in terms of products of wave functions of the residual nucleus and of the nucleon outside $S$. The boundary condition for the single-nucleon wave function is that of vanishing at $\infty$. Since the energy of the nucleon is determined by the separation energy, the set of channel functions is completely defined. The energy of the single-nucleon wave function, i.e., the energy of relative motion of the nucleon and the residual nucleus will not match in general the single-nucleon energies used in the Slater determinants of the interior region. The connection of the actual wave function in the exterior region with the shell-model function is thus not direct. If the nucleus is well described by a shell-model configuration, then one may expect the function in the exterior region to be approximately represented by products just described employing for the relative motion of nucleon and residual nucleus one of the shell theory functions and assigning the others to the residual nucleus. But the function for the external nucleon must then either be used with an energy different from the separation energy or else the shell of nucleons in the residual nucleus must be used with the separation energy of the transferred nucleon—an unrealistic and hardly warranted alternative—or else the nucleons in the shell become unequivalent. In the general case, it is thus necessary to join the functions in the exterior region to a function consisting of a sum of different shell-structure functions in the interior. There is no general reason, therefore, for the functions in the exterior region to contain only the $l$ values of the nucleons in the last shell of the nucleus. In particular, the neutron transferred from $N^{14}$ may be partly in a $p$ and partly in $f$ and $h$ states for the same state of the residual nucleus $N^{13}$ and a similar situation obtains for a nucleon captured by $N^{14}$ to form a state of $N^{15}$. The possibility considered in BE-I is thus only one of many although it appears that it is likely to be the more important one.

The fractional parentage considerations of shell structure theory which are said by some to have a bearing on the transfer problem are not closely concerned with the part of the wave function which has an immediate bearing on the transfer process, viz., that consisting of the residual nucleus function in a specified state and the function of relative motion of the nucleon with its separation energy rather than the energy in an auxiliary central field used in introducing a set of functions in shell-model calculations.

The antisymmetry of wave functions in protons and neutrons has been considered to some extent in BE-I in connection with the $p$-shell wave functions. The resonance effects mentioned toward the end of Sec. VI of the present paper contribute in general Heitler-London type forces to the interaction of nuclei $a+c$ with $b$, and of $a$ with $b+c$, which should be considered as part of the combined problem of nucleon transfer and of nucleus-nucleus scattering.

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[23] G. Breit, Phys. Rev. **102**, 549 (1956).

[24] D. H. Wilkinson, in *Proceedings of the Rutherford Jubliee International Conference, Manchester, 1961* (Heywood and Company, Ltd., London, 1961), p. 339.