by (2.44). This shows that our S matrix (3.4) is identical with Faddeev's [defined by the conjugate of the second unnumbered equation after (9.81) of Ref. 4, and Eqs. (5.3), (5.44), and remark before Eq. (9.11) of the same paper].

Finally, we consider the S matrices for bound-state scattering and rearrangement collisions. The equation we need here is

$$W_{\alpha\beta}(s)=(\delta_{\alpha\beta}-1)[T_\alpha(s)G_0(s)T_\beta(s)+T_\alpha(s)G_0(s)V_\beta G_0(s)\times T_\beta(s)]+T_\alpha(s)G_0(s)U_{\alpha\beta}{}^+(s)G_0(s)T_\beta(s)\,, \quad \text{(A11)}$$

which is obtained by iterating (A5). Applying the argument that led to (A10) we find the residue at the double pole on the energy shell

$$s=q_\alpha{}^2-E_{\alpha n}=q_\beta{}^2-E_{\alpha m} \quad \text{(A12)}$$

to be

$$(\delta_{\alpha\beta}-1)\psi_{\alpha n}(p_\alpha)\times\left[g_{\beta m}(\mathbf{p}_\beta')+\int dp_\beta''V_\beta(\mathbf{p}_\beta',\mathbf{p}_\beta'')\psi_{\beta m}(\mathbf{p}_\beta'')\right]+\int d^3p_\alpha\int d^3p_\beta'\psi_{\alpha n}(\mathbf{p}_\alpha)\times U_{\alpha\beta}{}^+(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}_\beta',\mathbf{q}_\beta';q_\alpha{}^2-E_{\alpha n})\psi_{\beta m}{}^*(\mathbf{p}_\beta')\,. \quad \text{(A13)}$$

The first term cancels by (2.44). This shows that our S matrices (3.3) are identical with Faddeev's [defined by the third unnumbered equation after (9.81), and (5.3) of Ref. 4].

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Unification of Photoproduction and Electroproduction

S. M. Berman

Stanford Linear Accelerator Center, Stanford University, Stanford, California*

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Gauge invariance and the vector nature of the photon are exploited in order to factor expressions for cross sections of photon-induced reactions into a purely kinematical part and a purely dynamical part. Detailed studies of two- and three-body final states are considered and it is shown how this separation into kinematical and dynamical aspects provides a useful and general procedure by which to compare experiment and theory.

I. INTRODUCTION

The fact that the photon is a zero-mass vector particle coupled to a conserved current allows for a separation, in expressions for the cross section for photon-induced reactions, between purely dynamical aspects and kinematical features such as gauge invariance and the vector nature of the photon. We show explicitly how this separation can be accomplished for an arbitrary photon-induced reaction where the photon can either be real as in photoproduction or virtual as in electroproduction (or mu-production). Detailed discussions of two-body and three-body final states are given. In these experimentally more accessible cases we show that a great deal can be learned about photoproduction by just analyzing the data in terms of the above-mentioned separation into kinematical and dynamical aspects. This is similar to the separation into the electric and magnetic form factors in the case of electron-nucleon scattering, but applied to cross sections rather than matrix elements.

In order to experimentally carry out the separation of photon-induced reactions into its kinematical and dynamical aspects it is necessary to perform either "coincidence" electroproduction experiments (simultaneous observation of the scattered electron and produced strong particles) or experiments using polarized photons. In the work presented here we are primarily interested in comparing photoproduction and electroproduction and thus consider principally electroproduction in the region of small photon mass.

In dealing with photoproduction and electroproduction at high energies (energies greater than approximately 1 BeV) the question arises as to what is a convenient and useful procedure for analyzing the data. For energies greater than 1 BeV, the dispersion theoretic treatment of Chew, Low, Goldberger, and Nambu[1] is not expected to hold and furthermore one would expect many multipoles to be contributing to photoproduction processes so that a multipole analysis of the data appears quite complicated and lengthy.

On the other hand, theoretical studies of photoproduction at high energies are often made in terms of simple models, e.g., one pion exchange. It therefore appears useful to have a completely general description of photoproduction which at the same time can be easily accommodated to testing ideas and models concerned with these processes. The separation into kinematical and dynamical aspects affords just such a

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[1] G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).

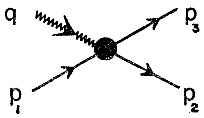

FIG. 1. Diagram showing photoproduction of a two-body final state by a photon of momentum q.

simple and direct means for comparing theory and experiment.

Because the photon is a vector particle with polarization vector e_μ, the differential cross section for photoproduction can be expressed in the form[2]

$$e_\mu e_\nu{}^* T_{\mu\nu} \tag{1a}$$

and for electroproduction, in the form

$$[2k_{1\mu}k_{1\nu}-k_{1\mu}q_\nu-q_\mu k_{1\nu}+(q^2/2)g_{\mu\nu}]T_{\mu\nu}(1/q^4) = L_{\mu\nu}T_{\mu\nu}/q^4, \tag{1b}$$

where $L_{\mu\nu}$, the tensor multiplying $T_{\mu\nu}$ in the electroproduction case, is the usual average over initial and final lepton spins of the vector interaction of photons with leptons and where k_1 and q are the four-momenta of the initial electron and photon, respectively.

The tensor $T_{\mu\nu}$ which contains all the nuclear dynamics is defined in terms of the electromagnetic current operator $J_\mu(0)$ as

$$T_{\mu\nu}=\langle f|J_\mu(0)|i\rangle\langle f|J_\nu(0)|i\rangle^*. \tag{2a}$$

Since the electromagnetic current is conserved we have that

$$q_\mu T_{\mu\nu}=q_\nu T_{\mu\nu}=0,$$

where q_μ is the four-momentum of the photon. Furthermore, it follows from its definition that

$$T_{\mu\nu}=T_{\nu\mu}{}^*. \tag{2b}$$

In the case of photoproduction with unpolarized photons or with linearly polarized photons, as well as in electroproduction, the tensor multiplying $T_{\mu\nu}$ is real and symmetric in the indices μ and ν. Thus in these cases only the real and symmetric part of $T_{\mu\nu}$ will contribute.

For the case of circularly polarized photons $T_{\mu\nu}$ need not be real but because of (2b) the imaginary parts must be antisymmetric in the indices μ and ν. It will be shown below that to have a nonzero antisymmetric part there must be at least two independent vectors in the final state so that these antisymmetric parts arise in three-body reactions when only momenta are measured or in two-body reactions when momenta as well as at least one polarization are measured.

II. TWO-BODY FINAL STATES

Photoproduction and electroproduction are shown in Figs. 1 and 2 where q and p_1 are the momenta of the photon and initial nucleon, respectively, p_2 and p_3 are the momenta of the produced particles, respectively (say nucleon and pion) and where k_1 and k_2 are the momenta of the initial and final electrons, respectively.

Consider the case when neither initial nor final nucleon spins are measured; then $T_{\mu\nu}$ depends on the various momenta. Making use of the requirement of gauge invariance, $T_{\mu\nu}$ can be cast into the form

$$\begin{aligned}T_{\mu\nu}=&A_1(s_0,t_0,q^2)\left[g_{\mu\nu}-\frac{q_\mu q_\nu}{q^2}\right]\\ &+A_2(s_0,t_0,q^2)[\epsilon_{\mu\alpha\sigma\tau}q_\alpha p_{1\sigma}p_{2\tau}\epsilon_{\nu\beta\lambda\rho}q_\beta p_{1\lambda}p_{2\rho}]\\ &+A_3(s_0,t_0,q^2)\left[p_{1\mu}-\frac{(p_1\cdot q)q_\mu}{q^2}\right]\left[p_{1\nu}-\frac{(p_1\cdot q)q_\nu}{q^2}\right]\\ &+A_4(s_0,t_0,q^2)\left[p_{2\mu}-\frac{(p_2\cdot q)q_\mu}{q^2}\right]\left[p_{2\nu}-\frac{(p_2\cdot q)q_\nu}{q^2}\right]\\ &+iA_5(s_0,t_0,q^2)\{[p_{1\mu}-(p_1\cdot q)q_\mu/q^2][p_{2\nu}-(p_2\cdot q)q_\nu/q^2]\\ &-[p_{2\mu}-(p_2\cdot q)q_\mu/q^2][p_{1\nu}-(p_1\cdot q)q_\nu/q^2]\}.\end{aligned} \tag{3}$$

The five real functions or form factors[3] $A_{1\ldots5}$ are functions of the energy $s_0=(p_1+q)^2$, the momentum transfer $t_0=(p_1-p_2)^2$ and the photon mass q^2.

In (3) we have performed the separation into the dynamical aspects of the problem which are incorporated in the form factors and the kinematical aspects which are explicitly displayed by the functions which multiply the form factors.

In order that there be no singularity[4] as $q^2\to0$ the form factors A_3, A_4, and A_5 must be proportional to q^2 in the limit as $q^2\to0$, i.e.,

$$\lim_{q^2\to0}A_{3,4,5}=q^2a_{3,4,5} \tag{4}$$

and also

$$\lim_{q^2\to0}A_1=\{(p_1\cdot q)^2\}a_3+\{(p_2\cdot q)^2\}a_4. \tag{5}$$

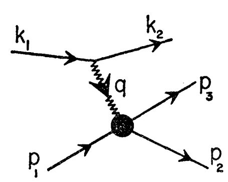

FIG. 2. Diagram showing electroproduction with two strongly interacting particles in the final state of momenta p_2 and p_3. The initial and final electron momenta are k_1 and k_2, respectively.

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[2] We use a metric such that $a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}$ so that $e^2=-1$. Cross sections are defined with $\hbar=c=1$; $\alpha\approx1/137$. $\epsilon_{\mu\nu\sigma\gamma}$ is the completely antisymmetric tensor of the fourth rank with $\epsilon_{0123}=+1$.

[3] For purposes of calculational convenience the coefficient of A_2 has been defined in (3) in terms of the four dimensional antisymmetric tensor ϵ rather than the more obvious form

$$\{[p_{1\mu}-(p_1\cdot q)q_\mu/q^2][p_{2\nu}-(p_2\cdot q)q_\nu/q^2] +[p_{2\mu}-(p_2\cdot q)q_\nu/q^2][p_{1\nu}-(p_1\cdot q)q_\nu/q^2]\}.$$

The above form is equal to a linear combination of the coefficients of A_1, A_2, A_3, and A_4.

[4] There can be no singularity in q^2 as $q^2\to0$ since from (2a) $T_{\mu\nu}$ is defined in terms of the physical matrix elements of the current. Arguments similar to this in connection with total cross sections have been made by S. D. Drell and J. D. Walecka, Ann. Phys. (N.Y.) (to be published).

A_2 cannot be singular as $q^2 \to 0$ since the coefficient of A_2 in (3) is nonzero in this limit. Thus only A_1 and A_2 contribute to photoproduction (since $q^2=0$ for real photons) and will be the leading terms in electroproduction for small q^2.

Because A_1 and A_2 depend on both s_0 and t_0, averaging over initial photon polarizations in photoproduction yields only one linear combination of these form factors. Another equation is needed to solve separately for both A_1 and A_2 and this can be provided by electroproduction. The form factors A_1 and A_2 will appear differently in photoproduction and electroproduction because of the factor $k_{1\mu}k_{1\nu}$ in the lepton term multiplying $T_{\mu\nu}$. This factor $k_{1\mu}k_{1\nu}$ provides the virtual photon with a linear polarization whose effects are measurable if the final electron is measured in "coincidence" with one of the final strongly interacting particles.

More explicitly we have for photoproduction with unpolarized photons in the s_0 center of mass

$$\left(\frac{d\sigma}{d\Omega}\right)_{\mathrm{c.m.}} = (1/16\pi)^2\left(-\frac{g_{\mu\nu}T_{\mu\nu}}{s_0}\right) \times \frac{[(s_0-M_2^2-M_3^2)^2-4M_2^2M_3^2]^{1/2}}{[s_0-M_1^2]}, \quad (6)$$

where in $T_{\mu\nu}$ only the form factors A_1 and A_2 are nonzero and are evaluated at $q^2=0$.

Similarly the differential cross section for electroproduction in the limit as $q^2 \to 0$ can be determined in terms of A_1 and A_2 and takes the form

$$\frac{d^3\sigma}{dt_0 ds_0 dq^2} = \left(\frac{1}{16\pi}\right)^2 (q^{-4}) \left(\frac{\alpha/\partial\pi}{M^2\omega_L k_L}\right) d\varphi_{p_2} \times [(s_0-M_1^2-q^2)^2-4M_1^2q^2]^{-1/2} \times [(q^2/2)g_{\mu\nu}+2k_{1\mu}k_{1\nu}]T_{\mu\nu}, \quad (7)$$

where ω_L and k_L are the initial electron energy and momentum, respectively. To first order in q^2 and neglecting the lepton mass the two terms $g_{\mu\nu}T_{\mu\nu}$ and $k_{1\mu}k_{1\nu}T_{\mu\nu}$ can be easily evaluated.

In the s_0 center-of-mass sytem we have directly from (3) that

$$g_{\mu\nu}T_{\mu\nu} = 2A_1 - A_2 s_0 \mathbf{q}^2\mathbf{p}_2^2 \sin^2\theta_{p_2q}, \quad (8)$$

$$k_{1\mu}k_{1\nu}T_{\mu\nu} = \frac{2(s-M_1^2)(s-s_0)}{(s_0-M_1^2)^2} q^2 A_1 + s_0\mathbf{q}^2\mathbf{p}_2^2 \sin^2\theta_{p_2q}\mathbf{k}_1^2 \sin^2\theta_{k_1q} \cos^2\varphi_{p_2} A_2, \quad (9)$$

where θ_{p_2q} is the angle between the photon momentum and the outgoing nucleon; θ_{k_1q} is the angle between the photon momentum and incident electron in the s_0 center-of-mass system; $\mathbf{k}_1$, $\mathbf{p}_2$, and $\mathbf{q}$ are the initial electron, final nucleon, and photon momenta in the s_0 center-of-mass respectively,[5] and where the lepton mass has been neglected. The angle φ_{p_2} is the azimuth angle of the final nucleon with respect to the plane containing $\mathbf{k}_1$ and $\mathbf{q}$ (or $\mathbf{k}_1$ and $\mathbf{k}_2$) in the s_0 center-of-mass system. The above angles are shown in Fig. 3.

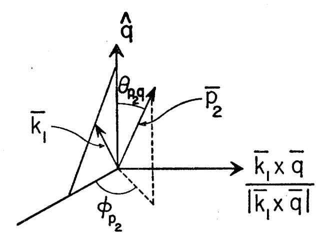

FIG. 3. Kinematics for the two-body final state in the s_0 center-of-mass system ($\mathbf{q}+\mathbf{p}_1=0$). The angle φ_{p_2} is measured with respect to the plane containing the vectors $\mathbf{q}$ and $\mathbf{k}_1$. The unit vector $\hat{q}$ is taken as the polar axis.

The fact that θ_{k_1q} vanishes in the limit $q^2 \to 0$ means that the term $k_1^2 \sin^2\theta_{k_1q}$ will be of order q^2 in this limit. In fact to order q^2 and neglecting the lepton mass we have that

$$k_1^2 \sin^2\theta_{k_1q} = q^2\frac{k_1}{q_4^2}(q_4-k_1) = \frac{q^2(s_0-s)(s-M_1^2)}{(s_0-M_1^2)^2}, \quad (10)$$

where q_4 is fourth component of q_μ in the s_0 center-of-mass system and $s=(p_1+k_1)^2$.

A simple proportionality between photoproduction with unpolarized photons and electroproduction is obtained by averaging over the azimuthal angle φ_{p_2} in (9).[6] This means no "coincidence" between scattered electron and produced strong particle. By averaging over φ_{p_2} we have from (8) and (9) that

$$[q^2/2g_{\mu\nu}T_{\mu\nu}+2k_{1\mu}k_{1\nu}]T_{\mu\nu} = q^2/2\left[1+\frac{2(s-s_0)(s-M_1^2)}{(s_0-M_1^2)^2}\right]g_{\mu\nu}T_{\mu\nu} \quad (11)$$

and that electroproduction in the limit $q^2 \to 0$ and neglecting lepton mass is related to photoproduction

[5] The quantities $|\mathbf{k}_1|$, $|\mathbf{p}_2 \cos\theta_{p_2q}|$ and $|\mathbf{q}|$ can be expressed invariantly as

$$2|\mathbf{k}_1| = (q^2+s-M_1^2)/s_0^{1/2}$$

$$-2|\mathbf{p}_2|\cos\theta_{p_2q} = (t_0/|\mathbf{q}|) + \frac{(s_0-M_1^2)(s_0-M_2^2)-q^2(s_0+M_2^2-M_3^2)-M_3^2(s_0+M_1^2)}{2s_0|\mathbf{q}|},$$

where

$$4\mathbf{q}^2 = [(s_0-M_1^2-q^2)^2-4M_1^2q^2]/s_0 \quad \text{and} \quad s=(p_1+k_1)^2.$$

[6] That no new information is obtained in comparing electroproduction at $q^2 \approx 0$ with photoproduction for two-body final states was first stated by R. H. Dalitz and D. R. Yennie, Phys. Rev. **105**, 1598 (1957). See also the subsequent work of L. N. Hand, Phys. Rev. **129**, 1834 (1963); and M. Gourdin, Nuovo Cimento **21**, 1094 (1961).

with unpolarized photons by the equation

$$\frac{d^3\sigma}{dq^2 ds_0 dt_0}=\left(\frac{\alpha}{2\pi}\right)\frac{(s_0-M_1^2)}{(s-M_1^2)^2}\left(\frac{1}{|q^2|}\right)\times\left[1+\frac{2(s-M_1^2)(s-s_0)}{(s_0-M_1^2)^2}\right]\left\{\frac{d\sigma(s_0,t_0)}{d|t_0|}\right\}_{\gamma p_1\to p_2 p_3}. \quad (12)$$

Separation between the two form factors A_1 and A_2 can be accomplished by not averaging over the azimuthal angle φ_{p_2}. This requires a "coincidence" experiment between the scattered electron and strongly produced particle. This "coincidence" defines a plane in which the virtual photon is linearly polarized. Thus a comparison of the azimuthally averaged cross section with the "coincidence" cross section allows for the separate determination of both A_1 and A_2.

Note that since q^2, t_0, and s_0 are all independent variables, one can check that q^2 was indeed small enough to be in the $q^2\to 0$ limit by keeping s_0 and t_0 fixed, then varying q^2 in the vicinity of $q^2\approx 0$ and checking that one has the same A_i. A second and perhaps simpler way for checking that one is in the $q^2\approx 0$ limit is to note that (9) has no linear term in $\cos\varphi_{p_2}$. We show below that the terms linear in $\cos\varphi_{p_2}$ are proportional to $(q^6)^{1/2}$ and thus the absence of such a term guarantees that one is indeed in the limit of small q^2.

As a simple example of a model we note that in the $q^2\to 0$ limit the one-pion-exchange approximation for the reaction $\gamma p\to\pi^+ n$ would predict $A_1=0$ and $A_2\neq 0$.[7] This same prediction also prevails if all particles but the photon are treated as spin-zero particles. For the reaction $\gamma p\to\pi^0 p$ with one vector meson exchange both A_1 and A_2 are nonzero and in general independent, while

$$A_4=A_1 q^2[(p_2\cdot q)^2-p_2^2 q^2]^{-1}$$

and $A_3=0$. However, if the vector meson is coupled to nucleons with only charge coupling nonzero, then A_1 and A_2 are related by the equation

$$A_1=(p_1-p_3)^2[q^2 p_2^2-(p_2\cdot q)^2]A_2,$$

where p_2 and p_3 are the final pion and nucleon momenta, respectively.

Another application of this method arises when the final p_2p_3 state is in resonance. In such cases the explicit t_0 dependence of A_1 and A_2 can be given since this dependence is completely determined by the spin of the resonance. This follows immediately from the fact that the most general couplings between photon, nucleon, and a spin J baryon resonance are functions only of the masses of these particles, respectively. For example, for $S_{1/2}$ and $P_{1/2}$ resonances only A_1 is nonzero (in the $q^2\to 0$ limit) while for a $P_{3/2}$ or $D_{3/2}$ resonance both A_1 and A_2 are nonzero. Thus analyzing the second pion-nucleon resonance $N^*(1512)$ in terms of the form factors A_1 and A_2 could shed light on whether this resonance is either $P_{1/2}$ or $D_{3/2}$. In particular, if the resonance is pure $P_{1/2}$ (or $S_{1/2}$) the differential cross section is given by Eqs. (6) and (7) with $A_2=0$ and with A_1 having no t_0 dependence and evaluated at $S_0=M^{*2}$, where M^* is the nucleon resonance mass.

For pure $D_{3/2}$ (or $P_{3/2}$) the form factor A_1 factors into a function of s_0 and a function of $\cos\theta_{p_2q}$.[8] Parity conservation and the fact that the resonance has spin 3/2 means that the $\cos\theta_{p_2q}$ dependence must be of the form $a+b\cos^2\theta_{p_2q}$. Thus for pure $D_{3/2}$ (or $P_{3/2}$) we have $A_1(s_0,t_0)=a(s_0)+b(s_0)\cos^2\theta_{p_2q}$ while A_2 reduces to a function of s_0 alone. Interfering $P_{1/2}$ and $D_{3/2}$ resonances give rise to both A_1 and A_2 and furthermore allow a linear term in $\cos\theta_{p_2q}$ for A_1, i.e., in this case A_1 is of the form $a(a_0)+b(s_0)\cos^2\theta_{p_2q}+c(s_0)\cos\theta_{p_2q}$ and A_2 is again a function of s_0 only. If the production, in terms of multipoles, is pure $M_1(3/2)$ then A_1 and A_2 are simply related in order that (8) take the form $[2+3\sin^2\theta_{p_2q}]$.

Also, since the t_0 dependence for the production of a baryon resonance is explicit, both form factors A_1 and A_2 can be determined from either unpolarized photoproduction or non-"coincidence" electroproduction alone by a study of the final πN angular distribution.

The next order term, of order $(q^6)^{1/2}$, can be determined in a straightforward manner similar to (8) and (9). To this order, there will be contributions from A_1, A_3, no contribution from A_2, while the contribution of A_4 can be related to A_3 and A_1 by using (5). Thus keeping terms to order $(q^6)^{1/2}$ we add to (9) the expression linear in $\cos\varphi_{p_2}$

$$(p_2\sin\theta_{p_2q})(k_1\sin\theta_{k_1q})(\cos\varphi_{p_2})\times\frac{(2s-s_0-M_1^2)(s_0-M_1^2)}{(s_0+t_0-M_1^2-M_2^2)}\left[A_3-\frac{4q^2A_1}{(s_0-M_1^2)^2}\right].$$

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[7] Since photoproduction in the one-pion-exchange (OPE) approximation is not gauge invariant, we define the gauge invariant OPE model by adding to the simple OPE term the minimum factor which makes it gauge invariant. The gauge invariant OPE matrix element in the $q^2\to 0$ limit is then of the form

$$\left[\frac{2p_{2\mu}-q_\mu}{(2p_2\cdot q-q^2)}-\frac{2p_{1\mu}+q_\mu}{(2p_1\cdot q+q^2)}\right]f(s_0,t_0,q^2),$$

where p_2 is the final pion momentum, and where $f(s_0,t_0,q^2)$ is an arbitrary function which is nonsingular as

$$[2(p_2\cdot q)-q^2]\to 0 \quad\text{and}\quad [2(p_1\cdot q)+q^2]\to 0,$$

[8] The most general gauge invariant coupling Γ_μ between a spin 1/2 particle of mass M_1 and four-momentum p_1, and a particle of spin $3/2^-$, mass M_2, and four-momentum p_2 and a photon (real or virtual) of four-momentum q is of the form

$$\begin{aligned}\Gamma_\mu=&q^2F_1(q^2)q_\alpha\bar{u}_\alpha(p_2)[\gamma_\mu-q_\mu(M_2-M_1)/q^2]u(p_1)\\&+F_2(q^2)[(M_2-M_1)\bar{u}_\mu(p_2)u(p_1)-q_\alpha\bar{u}_\alpha(p_\alpha)\gamma_\mu u(p_1)]\\&F_3(q^2)q_\alpha\bar{u}_\alpha(p_2)[(p_1+p_2)_\mu-(M_1+M_2)\gamma_\mu]u(p_1),\end{aligned}$$

where $u_\alpha(p_2)$ is the free spin 3/2 particle wave function and $u(p_1)$ the free spin 1/2 particle wave function. The form factors $F_{1,2,3}$ are arbitrary functions of q^2. The matrix element for $3/2^+$ is obtained by replacing $\bar{u}_\alpha$ by $\bar{u}_\alpha\gamma_5$ and M_2 by $-M_2$. See M. Gourdin and Ph. Salin, Nuovo Cimento **27**, 193 (1963); and **27**, 309 (1963).

This expression is of order $(q^6)^{1/2}$ since $k_1 \sin\theta_{k_1q}$ is of order $(q^2)^{1/2}$ and the term in brackets is of order q^2. Measurement of this term leads to the determination of A_3 which cannot be measured with real photons and which could be interesting in its own right.

We conclude this section by remarking that in terms of the more conventional language the form factors A_1 and A_2 can be thought of as arising from transverse photons, while A_4 may be thought of as the result of interference between transverse and longitudinal photons and A_3 pure longitudinal photons.

III. THREE-BODY FINAL STATES

In this section we consider processes where there are three strongly interacting bodies in the final state which we label according to momenta as $q+p_1 \to p_2 + p_3 + p_4$. Special cases of interest might be where there are two final pions in resonance or one final pion and nucleon in resonance accompanied by a second pion.

The most general form for $T_{\mu\nu}$ for the case of three final bodies of four-momenta p_2, p_3, p_4 has nine terms and can be cast into the form

$$\begin{aligned}T_{\mu\nu}= &+B_1\epsilon_{\mu\alpha\beta\lambda}\epsilon_{\nu\rho\sigma\tau}q_\alpha p_{3\beta}p_{4\lambda}q_\rho p_{3\sigma}p_{4\tau} \\ &+B_2\epsilon_{\mu\alpha\beta\lambda}\epsilon_{\nu\rho\sigma\tau}q_\alpha p_{1\beta}p_{2\lambda}q_\rho p_{1\sigma}p_{2\tau} \\ &+B_3\epsilon_{\mu\alpha\beta\lambda}\epsilon_{\nu\rho\sigma\tau}q_\alpha p_{2\beta}p_{3\lambda}q_\rho p_{2\sigma}p_{3\tau} \\ &+B_4[p_{1\mu}-(p_1\cdot q)q_\mu/q^2][p_{1\nu}-(p_1\cdot q)q_\nu/q^2] \\ &+B_5[p_{2\mu}-(p_2\cdot q)q_\mu/q^2][p_{2\nu}-(p_2\cdot q)q_\nu/q^2] \\ &+B_6[p_{3\mu}-(p_3\cdot q)q_\mu/q^2][p_{3\nu}-(p_3\cdot q)q_\nu/q^2] \\ &+iB_7\{[p_{1\mu}-(p_1\cdot q)q_\mu/q^2][p_{2\nu}-(p_2\cdot q)q_\nu/q^2] \\ &-[p_{2\mu}-(p_2\cdot q)q_\mu/q^2][p_{1\nu}-(p_1\cdot q)q_\nu/q^2]\} \\ &+iB_8\{[p_{1\mu}-(p_1\cdot q)q_\mu/q^2][p_{3\nu}-(p_3\cdot q)q_\nu/q^2] \\ &-[p_{3\mu}-(p_3\cdot q)q_\mu/q^2][p_{1\nu}-(p_1\cdot q)q_\nu/q^2]\} \\ &+iB_9\{[p_{2\mu}-(p_2\cdot q)q_\mu/q^2][p_{3\nu}-(p_3\cdot q)q_\nu/q^2] \\ &-[p_{3\mu}-(p_3\cdot q)q_\mu/q^2][p_{2\nu}-(p_2\cdot q)q_\nu/q^2]\}\,, \quad (13)\end{aligned}$$

where q and p_1 are the photon and initial nucleon, respectively. [*Note added in proof*. Equation (13) as given in an unpublished version of this work (see SLAC Pub. 27 Stanford Linear Accelerator Center, 1964) was given incorrectly. The author is indebted to Dr. J. K. Randolph for calling this point to his attention.] The form factors B_i will in general depend on five variables, two energies and an angle in the final three-body center of mass, the initial energy and the photon mass. It is convenient to take as these five variables the covariant quantities

$$x=(p_3+p_4-p_1)^2;\quad y=(p_4-q)^2;\quad z=(p_2+p_3)^2;\quad s_0=(p_1+q)^2;\quad q^2.$$

Just as in the two-body case the requirement that $T_{\mu\nu}$ be nonsingular as $q^2\to 0$ assures that $B_{4,5,6}$ are of order q^2 as $q^2\to 0$ and thus only $B_{1,2,3}$ contribute to real photoproduction with unpolarized or linearly polarized photons, and will be the leading terms in the limit of small q^2 for electroproduction. Again the apparent singularity in $B_{4,5,6}$ will cancel out because of the relation

$$\lim_{q^2\to 0}[(p_1\cdot q)^2/q^2]B_4+[(p_2\cdot q)^2/q^2]B_5 +[(p_3\cdot q)^2/q^2]B_6=0 \quad (14)$$

and $B_{4,5,6}\to 0$ as $q^2\to 0$ which is the analog of (5).

Because the q^{-4} terms cancel automatically in the antisymmetric terms, $B_{7,8,9}$ are not required to be proportional to q^2 as $q^2\to 0$. However, to cancel the q^{-2} singularity there are two relations among $B_{7,8,9}$ and thus there is only one independent antisymmetric form in the $q^2\to 0$ limit. Such imaginary antisymmetric terms can arise from the interference between resonance and background or between two different resonances and require circularly polarized photons in order to be detected.[9]

We next show, as in Sec. II, that if the final electron is undetected then electroproduction in the limit $q^2\to 0$ is proportional to photoproduction with unpolarized photons. This is true regardless of the number of final bodies and the proof given below is valid for arbitrary final states.

From (13) we see that the proof of this statement requires evaluating the expression $k_1^2\sin^2\theta_{k_1q}$ in an arbitrary coordinate system (not necessarily the c.m. system as in Sec. II) when the final electron is undetected. Let p_s be the four-momentum in whose rest system $k_1^2\sin^2\theta_{k_1q}$ is to be evaluated. Then using $2k_1\cdot q=q^2$, we have to first order in q^2

$$k_1^2\sin^2\theta_{k_1q}=\frac{q^2(p_s\cdot k_1)}{(p_s\cdot q)^2}[(p_s\cdot q)-(p_s\cdot k_1)]. \quad (15)$$

The quantity $(p_s\cdot k_1)$ can be expressed in the over-all center-of-mass system of the final strong particles (s_0 c.m. system) as

$$p_s\cdot k_1=k_1[E_s-|\mathbf{p}_s|(\cos\theta_{k_1q}\cos\theta_{p_2q}+\sin\theta_{k_1q}\sin\theta_{p_sq}\cos\varphi_{p_s})].$$

Since only first order in q^2 is desired, $\sin\theta_{k_1q}$ is negligible in the above expression and $\cos\theta_{k_1q}$ can be taken to be unity. Using these facts in (15) gives immediately that

$$k_1^2\sin^2\theta_{k_1q}=q^2(k_1/q_4^2)(q_4-k_1)\,, \quad (10')$$

where k_1 and q_4 are expressed in the over-all center of mass. This result is the same as (10) and hence we have (11) *independent of the number of final particles*.[10]

[9] These terms could also be detected in electroproduction if either initial or final electron polarization is measured. The contribution of these terms to the cross section is, however, proportional to the lepton mass and would be very difficult to detect when the initial energy of the leptons is high enough to have a three-body production.

[10] The result is the same even if nucleon polarizations are included as measureables. For example, in Sec. II if the initial proton is polarized one adds additional terms to (3) of the form

$$\epsilon_{\mu\alpha\beta\lambda}\epsilon_{\nu\rho\sigma\tau}q_\alpha p_{1\beta}p_{2\lambda}q_\rho p_{1\sigma}W_\tau,$$

where W_τ is the four-vector which reduces to the target polarization in its rest system. Applying the same procedure as in going from (15) to (10′) yields the stated result.

The differential cross section for photoproduction of three final bodies of four-momenta p_2, p_3, and p_4 can be expressed in terms of invariants as

$$\frac{d^3\sigma}{dxdydz}=\left(\frac{1}{16\pi}\right)^3|g_{\mu\nu}T_{\mu\nu}|4[s_0-M_1{}^2]^{-2}\times[s_0+y-M_1{}^2-M_4{}^2]^{-1}, \quad (16)$$

where $T_{\mu\nu}$ is given by (13). Similarly the differential cross section for electroproduction of the same process is given by

$$\frac{d^5\sigma}{dxdydzds_0dq^2}=\left(\frac{1}{16\pi}\right)^3\left(\frac{2\alpha}{\pi^3}\right)\frac{L_{\mu\nu}T_{\mu\nu}d\varphi_{p_4}d\varphi_{p_2}{}'}{(s-M_1{}^2)^2q^4s_0|\mathbf{p}_1||\mathbf{Q}|}, \quad (17)$$

where φ_{p_4} is the azimuthal angle of $\mathbf{p}_4$ with respect to the plane containing $\mathbf{q}$ and $\mathbf{k}_1$ in the s_0 rest system; $\varphi_{p_2}{}'$ is the azimuthal angle of $\mathbf{p}_2$ with respect to the plane containing $\mathbf{q}$ and $\mathbf{k}_1$ in the z rest system; $|\mathbf{p}_1|$ and $|\mathbf{Q}|$ are the magnitudes of the initial nucleon and initial photon three-momenta in the s_0 and z rest systems, respectively. These may be expressed covariantly as

$$4s_0\mathbf{p}_1{}^2=[(s_0-M_1{}^2-q^2)^2-4M_1{}^2q^2],$$
$$4z\mathbf{Q}^2=[(s_0+q^2+y-M_1{}^2-M_4{}^2)^2-4zq^2].$$

The differential $d\varphi_2{}'$ is readily expressed in terms of the differential $d\varphi_{p_2}$, where φ_{p_2} is the azimuthal angle of $\mathbf{p}_2$ in the s_0 system as

$$(d\varphi_{p_2}{}')=(d\varphi_{p_2})\left[\frac{\epsilon_{\mu\nu\sigma\tau}p_{1\mu}q_\nu k_{1\sigma}p_{2\tau}}{\epsilon_{\mu\nu\sigma\tau}p_{3\mu}q_\nu k_{1\sigma}p_{2\tau}}\right]\left(\frac{z}{s_0}\right)^{1/2}\frac{|\mathbf{Q}|}{|\mathbf{p}_1|}. \quad (18)$$

In the limit as $q^2\to 0$ it is possible, just as in the case of a two-body final state, to separate the three form factors $B_{1\cdots 3p}$ by the azimuthal dependences of the differential cross section. For each x, y, z, and s_0 the distribution in φ_{p_2} gives one equation of the form $a+b\cos^2\varphi_{p_2}$ and the distribution in φ_{p_4} gives another equation of the form $c+d\cos^2\varphi_{p_4}$. The coefficients a, b, c, d are then linearly related to the $B_{1\cdots 3p}$ thus overdetermining these three form factors. Again as in Sec. II lack of a linear $\cos\varphi_{p_2,p_4}$ dependence is evidence that the region is indeed in the neighborhood of $q^2\approx 0$.

As an example of a special three-body fin .l state consider the state $\pi\pi N$ where the two pions are in resonance. Similarly to the resonance case considered in Sec. II there will be a reduction in the number of variables that the various form factors B_i depend on. Instead of the general case of five variables the form factors will depend only on three variables s_0, q^2, and y, where for convenience p_4 is taken to be the recoil nucleon momentum. The dependence on x is no longer arbitrary but depends on the spin of the $\pi\pi$ resonance (x is linearly related to the cosine of the angle of the decay pion).

In general there is no reduction in the number of form factors if the spin of the $\pi\pi$ resonance is one or greater. For the special case of a spin-zero resonance only B_1 is nonvanishing in the limit as $q^2\to 0$ (in this case p_2 is taken as the recoil nucleon momentum). Also if the two-pion resonance is produced predominantly by one-pion exchange there will be a reduction in the number of form factors. For example, if the $\pi\pi$ resonance is the ρ meson then in the one-pion-exchange approximation only B_3 is nonvanishing in the $q^2\to 0$ limit.

IV. THE GENERAL CASE

The separation of the cross section into the kinematical functions and form factors for the case of an arbitrary number of final particles can be easily accomplished following the case of the three-body final state. We observe, that in order to span the Minkowski space four linearly independent four-vectors are required one of which is space like. For three or more bodies in the final state we may take as these vectors the momenta q, p_1, p_2, p_3. From these vectors we can construct the most general covariant tensor $T_{\mu\nu}$ which satisfies gauge invariance. But this tensor is precisely given already by (13). Thus because $T_{\mu\nu}$ is a second rank tensor in the four-dimensional Minkowski space the most general decomposition into the kinematical and dynamical aspects is given by (13) regardless of the number of final bodies, momenta, and polarizations included. The only difference for more than three bodies is that the form factors B_i will depend on more scalar variables, including polarizations, the exact number depending on the number of variables which are measured in the reaction. Since (13) is the most general expression for the tensor $T_{\mu\nu}$ the argument used in Sec. III relating electroproduction at small q^2 to photoproduction is valid for the general case and we see that these two processes are proportional at small q^2 when the final electron angle is not observed. The factor of proportionately is given by (11) and is independent of the final variables of the photoproduced particles and depends only on the energy loss of the scattered electron.

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