Practical Theory of Three-Particle States. I. Nonrelativistic*

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We use Faddeev's mathematical work on nonrelativistic three-particle systems to obtain a practical theory of three-particle states. Each of the two-particle subsystems is assumed to be dominated by a finite number of bound states and resonances. We then obtain two sets of equations. One expresses processes with three final particles in terms of unstable-particle scattering. It generalizes and justifies the isobar model. The other is a set of coupled Lippmann-Schwinger equations for the scattering of bound states and unstable particles. The potentials in these scattering equations for composite particles are nonlocal and energy-dependent, and become complex above the three-particle threshold. They are expressed in terms of the wave functions of the two-particle bound states and resonances—there are no new arbitrary constants. The solution satisfies three-particle unitarity. Unstable particles are *not* approximated by stable ones. The theory applies especially to "overlapping" final-state interactions. The equations are given in detail for the $3N$ and $N\pi\pi$ systems in the static limit. In the latter case, the nucleon is treated as an $N\pi$ bound state. The $N\pi\pi$ equations are solved approximately in closed form, giving analogs of the Chew-Low effective-range formula, for the coupled $N\pi$ and $N^*\pi$ systems. This predicts large inelasticity in the P_{11} state, in agreement with experiment. Detailed solutions will be given in subsequent papers. If all elementary particles are treated as bound states of themselves, our approach gives the basis of a complete dynamical theory of strong interactions.

1. INTRODUCTION

a.

THE saving grace of strong interactions is that they never involve zero-mass particles. This means that the forces are of finite range, and therefore only become strong in a limited number of angular momentum states at any given energy. It also means that there are energies at which only two-particle states need be considered. This low-energy region is where strong-interaction theory has had its main success so far. It is dominated by a finite number of bound states and resonances. Progress has consisted in finding rules for handling these. They are essentially quite simple—exchanged resonances are treated just like elementary particles and their contributions to the forces calculated from the Born approximation, resonances in processes that are actually being observed are treated by effective-range formulas of various types. The worst that can then happen is that a few constants have to be added to the theory, corresponding to the masses and coupling constants of these resonances, or to cutoff parameters in the effective-range formulas.

At a few hundred MeV, we can no longer ignore three-particle states. The natural approach then, is to try to extend the technique of treating resonances as particles, which worked so well at lower energies. Especially, the peripheral approximation has had some success, which led to rather extravagant claims being made for it. Again, this is essentially just a Born approximation, but with the unstable particle as one of the external lines, instead of the internal one. It is clear theoretically that it must break down when the production process becomes large, for the same reason that the ordinary Born approximation breaks down—it takes no account of unitarity. There is also plenty of recent experimental evidence that its validity is more limited than had been thought.

The next step is a big one. We must learn to use unstable particles not just in the Born approximation, but in graphs with closed loops. Efforts to do this have so far not met with much success. This is not really very surprising, since it took twenty years to learn how to use elementary particles in graphs with closed loops. Now the scattering of bound states and unstable particles is a special case of three-particle scattering. Hitherto, people have been terrified of three-particle states, and have therefore tried to study unstable particle scattering out of its natural habitat, using only the analogy with two-particle scattering. We think this is why they have had such little success.

This situation is likely to be radically changed by the brilliant work of Faddeev,[1–4] who succeeded for the first time in giving a mathematically correct theory of nonrelativistic three-particle systems. Such systems are far too treacherous to be dealt with by physical intuition. For a description of some of the traps into which a naive approach may fall, see Refs. 1, 5–7. Faddeev's work opens the way to a solution of the problem of

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[1] L. D. Faddeev, Zh. Eksperim. i Teor. Fiz. **39**, 1459 (1960) [English transl.: Soviet Phys.—JETP **12**, 1014 (1961)].
[2] L. D. Faddeev, Dokl. Akad. Nauk SSR **138**, 565 (1961) [English transl.: Soviet Phys.—Doklady **6**, 384 (1961)].
[3] L. D. Faddeev, Dokl. Akad. Nauk SSR **145**, 301 (1962) [English transl.: Soviet Phys.—Doklady **7**, 600 (1963)].
[4] L. D. Faddeev, *Mathematical Problems of the Quantum Theory of Scattering for a Three-Particle System* (Publications of the Steklov Mathematical Institute, Leningrad, 1963), No. 69. I am grateful to Dr. J. B. Sykes for translating this reference [English transl.: H. M. Stationery Office (Harwell, 1964)].
[5] C. Lovelace, in *Strong Interactions and High Energy Physics*, edited by R. G. Moorhouse (Oliver and Boyd, London, 1964). Referred to as E in the present paper.
[6] L. L. Foldy and W. Tobocman, Phys. Rev. **105**, 1099 (1957).
[7] R. Omnes, Phys. Rev. **134**, B1358 (1964); University of California, Lawrence Radiation Laboratory Reports No. UCRL 11186, and UCRL 11219, 1963–64 (unpublished).

bound-state and unstable-particle scattering in the nonrelativistic case. Having achieved this, we can then postulate a similar solution in the relativistic case, with a fair degree of confidence.

In a previous work,[5] referred to as E, we gave an account of Faddeev's theory and of some extensions to it, especially ones concerned with unstable particles. The present paper is intended to be readable independently of E, but does not supplant it. Subjects which were treated at length in E will be dealt with very briefly here. The present work is especially concerned with practical aspects of the theory. A subsequent paper[8] will be concerned with rigorous proofs, insofar as these are not contained in E, or in Faddeev's big new paper.[4]

Other approaches to three-particle states, based on somewhat similar ideas to Faddeev, have recently been suggested by S. Weinberg[9] and J. G. Taylor.[10] We have had interesting discussions with both these authors, but we think that the present approach is more practical.[11]

Section 1b explains our notations. Section 2 is concerned with preliminary results on two-particle scattering off the energy shell. In Sec. 2a, we give a compactness proof for the Lippmann-Schwinger equation. Unlike most such proofs, it is valid on the right-hand cut. The present paper is not really concerned with rigorous proofs, but we have included this one to show how easy it is. This subsection also contains a very general method for obtaining bounds on phase shifts. Nonmathematicians may be intimidated by a few sentences in Sec. 2a, in which case they are advised to skip it.

Section 2b contains an explicit solution of the off-shell unitarity equations for two-particle scattering. This is the off-shell analog of the well-known K-matrix representation, and does not appear to have been given before. In Sec. 2c, we study the structure of the off-shell scattering amplitude in a partial wave dominated by a bound state or resonance, and how it can be deduced from a knowledge of the experimental on-shell amplitude.

In Sec. 3a, we describe briefly the modified Faddeev equations obtained in E. These are mathematically quite similar to the original Faddeev equations, but have the practical advantage of being much closer to the experimentally observed quantities, especially for processes involving bound states. In Sec. 3b, we show how these equations simplify, when each of the two-particle subsystems is dominated by certain bound states and resonances. The three particle equations then reduce to equations for the scattering of these bound states and resonances by the third particle. These equations also describe bound-state rearrangement collisions, and even bound-state disintegration into three particles. Contrary to what one might suppose, they still satisfy three-particle unitarity. The kernels of the equations are very simply related to the bound-state wave functions, and there are no arbitrary constants. These equations therefore give the answer, in the nonrelativistic case, to the problem we mentioned earlier—how to use unstable particles in Feynman graphs with closed loops.

Section 3c shows how these equations simplify when all three particles are indistinguishable. In Sec. 4 we work out the equations in detail for the three-nucleon system. This gives equations not only for nd scattering, but also for $n+d\to n+n+p$, which has hitherto proved rather intractable.

Section 5 is concerned with the static $N\pi\pi$ system, with the nucleon treated as a pion-nucleon bound state. Because of the peculiar kinematics of the static limit, we can, after one slight approximation, solve the equations in closed form. Sec. 5b considers the nucleon-pion system, *treated as bound-state scattering*. This is very interesting, since it shows the relation between elementary particle scattering and bound-state scattering. What comes out is just the Chew-Low effective-range formula. The cutoff function enters in an especially natural way, as the "wave function" of the nucleon considered as a pion-nucleon bound state. In Sec. 5c, we extend this theory to the coupled $N\pi$ and $N^*\pi$ channels. Again the equations can be approximately solved in closed form, except for one integral which has to be done numerically. We then obtain a theory of the process $N+\pi\to N+\pi+\pi$ at low energies, with no parameters, apart from those already contained in the elastic πN equations. The detailed predictions and comparison with experiment will be considered in a subsequent paper.

Section 6 is a discussion. Especially, we consider the possibility that elementary particles may also be described by our bound-state scattering equations. Such a theory would fulfill part of Chew's program by treating all particles on the same footing. If our theory can be extended to states of more than three particles (this doesn't look too difficult in principle), and made relativistic, it would then provide the basis for a complete dynamical theory of strong interactions. However, it would not necessarily be a theory with no arbitrary constants, as claimed by some of von Münchhausen's[12] more extreme disciples.

In his big paper[4] Faddeev has defined S matrices for processes involving bound states, and proved that they are the S matrices of time-dependent scattering theory.

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[8] C. Lovelace (unpublished).

[9] S. Weinberg, Phys. Rev. **133**, B232 (1964).

[10] J. G. Taylor, Cambridge (unpublished); Nuovo Cimento Supplement (to be published).

[11] L. Rosenberg, Phys. Rev. **135**, B715 (1964), has suggested another similar approach.

[12] Baron von Münchhausen (1720–1797) first discovered the bootstrap mechanism. Finding himself stuck in a bog one day, and no trees within reach, he pulled himself out by his own bootstraps. His achievement was not recognized by his contemporaries. See, *The Singular Travels, Campaigns, Voyages and Sporting Adventures of Baron Munnikhousen, Commonly Pronounced Munchausen; as He Relates Them Over a Bottle When Surrounded by His Friends* (Raspe, London, 1786).

i.e., the ones which experimentalists measure. His definition is not at all convenient for calculation, and at first sight might seem quite different from ours.[5] We therefore prove in the Appendix that they are identical.

b. Notation and Kinematics

We have three particles of masses m_1, m_2, m_3 and momenta $\mathbf{k}_1$, $\mathbf{k}_2$, $\mathbf{k}_3$. We consider certain combinations of these momenta.

$$\mathbf{P}=\frac{1}{\sqrt{2}(m_1+m_2+m_3)^{1/2}}[\mathbf{k}_1+\mathbf{k}_2+\mathbf{k}_3], \quad (1.1)$$

which is the total momentum of the system,

$$\mathbf{p}_1=\frac{1}{\sqrt{2}[m_2m_3(m_2+m_3)]^{1/2}}[m_3\mathbf{k}_2-m_2\mathbf{k}_3], \quad (1.2)$$

which is the center-of-mass momentum of the (2,3) subsystem,

$$\mathbf{q}_1=\frac{1}{\sqrt{2}[m_1(m_2+m_3)(m_1+m_2+m_3)]^{1/2}} \times[m_1(\mathbf{k}_2+\mathbf{k}_3)-(m_2+m_3)\mathbf{k}_1], \quad (1.3)$$

which is the momentum of particle 1 relative to the (2,3) subsystem, all with rather peculiar normalizations. The nonrelativistic kinetic energy is given by

$$H_0=(1/2m_1)k_1^2+(1/2m_2)k_2^2+(1/2m_3)k_3^2 =P^2+p_1^2+q_1^2. \quad (1.4)$$

We shall assume that the delta function from conservation of total momentum has been factored out and that we are in the three-particle center-of-mass system. We can then forget $\mathbf{P}$.

Other sets of momenta $\mathbf{p}_2$, $\mathbf{q}_2$ and $\mathbf{p}_3$, $\mathbf{q}_3$ can be obtained by a cyclic interchange of subscripts in (1.3) and (1.4). They are appropriate to the cases when particles 3 and 1, and particles 1 and 2, respectively, are considered forming a subsystem. They are linearly dependent on $\mathbf{p}_1$, $\mathbf{q}_1$. For example,

$$\mathbf{p}_2=-\left(\frac{m_1m_2}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{p}_1 +\left(\frac{m_3(m_1+m_2+m_3)}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{q}_1, \quad (1.5)$$

$$\mathbf{q}_2=-\left(\frac{m_3(m_1+m_2+m_3)}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{p}_1 -\left(\frac{m_1m_2}{(m_1+m_3)(m_2+m_3)}\right)^{1/2}\mathbf{q}_1. \quad (1.6)$$

We will sometimes use $\mathbf{q}_\alpha$, $\mathbf{q}_\beta$, $\alpha\neq\beta$ as our independent momenta. The formulas expressing the $\mathbf{p}$'s in terms of them are, for example,

$$\mathbf{p}_1=-\left(\frac{m_1m_2}{m_3(m_1+m_2+m_3)}\right)^{1/2}\mathbf{q}_1 -\left(\frac{(m_2+m_3)(m_1+m_3)}{m_3(m_1+m_2+m_3)}\right)^{1/2}\mathbf{q}_2, \quad (1.7)$$

$$\mathbf{p}_2=\left(\frac{(m_2+m_3)(m_1+m_3)}{m_3(m\ +m_2+m_3)}\right)^{1/2}\mathbf{q}_1 +\left(\frac{m_1m_2}{m_3(m_1+m_2+m_3)}\right)^{1/2}\mathbf{q}_2. \quad (1.8)$$

In the center-of-mass frame, the kinetic energy becomes

$$H_0=p_1^2+q_1^2=p_2^2+q_2^2=p_3^2+q_3^2. \quad (1.9)$$

This can be considered as an operator—the free Hamiltonian. Our Hilbert space consists of the square-integrable functions of two momentum vectors $|\mathbf{p},\mathbf{q}\rangle$. The Green's function or resolvent of the free Hamiltonian is an operator

$$G_0(s)=[H_0-sI]^{-1}, \quad (1.10)$$

where I is the unit operator, and s a complex variable. Its momentum-space representation is, of course,

$$\langle\mathbf{p},\mathbf{q}|G_0(s)|\mathbf{p}',\mathbf{q}'\rangle=[p^2+q^2-s]^{-1} \times\delta_3(\mathbf{p}-\mathbf{p}')\delta_3(\mathbf{q}-\mathbf{q}'). \quad (1.11)$$

Besides the three-particle system, we shall also frequently consider two-particle subsystems of it. In the center-of-mass frame of a two-particle system, there will only be one independent momentum vector, so the two-particle Hilbert space will consist of square-integrable functions of *one* momentum vector $|\mathbf{p}\rangle$. We denote operators which belong to this two-particle Hilbert space by putting hats on them. Thus the free two-particle Hamiltonian is given by

$$\langle\mathbf{p}|\hat{H}_0|\mathbf{p}'\rangle=p^2\delta_3(\mathbf{p}-\mathbf{p}'). \quad (1.12)$$

We shall find it convenient sometimes to use the notation of integral kernels, instead of writing matrix elements,

$$\langle\mathbf{p}|\hat{T}(s)|\mathbf{p}'\rangle=T(\mathbf{p},\mathbf{p}';s), \quad (1.13)$$

$$\langle\mathbf{p},\mathbf{q}|U(s)|\mathbf{p}',\mathbf{q}'\rangle\equiv U(\mathbf{p},\mathbf{q},\mathbf{p}',\mathbf{q}';s) \quad (1.14)$$

(mathematically, of course, this is more correct). In the case of integral kernels, we shall not use the hat notation, since there is no ambiguity. Thus the two-particle potentials will be

$$\langle\mathbf{p}|\hat{V}|\mathbf{p}'\rangle\equiv V(\mathbf{p},\mathbf{p}'), \quad (1.15)$$

and if the potential is local

$$V(\mathbf{p},\mathbf{p}')=V(\mathbf{p}-\mathbf{p}'). \quad (1.16)$$

In Sec. 5, particles 1 and 2 will be pions, and must be treated relativistically. The kinetic energy will be then

$$(m_1^2+k_1^2)^{1/2}+(m_2^2+k_2^2)^{1/2}+(k_3^2/2m_3). \quad (1.17)$$

In the static limit, $m_1=m_2=\mu$, $m_3=M$, and $\mu/M\to 0$; (1.2) and (1.3) become

$$\mathbf{p}_1=\mathbf{k}_2/(2\mu)^{1/2}=-\mathbf{q}_2, \quad (1.18)$$

$$\mathbf{q}_1=-\mathbf{k}_1/(2\mu)^{1/2}=\mathbf{p}_2, \quad (1.19)$$

so that the kinetic energy is

$$(\mu^2+2\mu q_1^2)^{1/2}+(\mu^2+2\mu q_2^2)^{1/2}. \quad (1.20)$$

We shall frequently speak of bound state "form factors," to be defined below. Much confusion will be avoided, if it is remembered that these form factors are pionic or nucleonic, rather than electromagnetic.

2. SOME RESULTS ON TWO-PARTICLE SCATTERING

We start with the Lippmann-Schwinger equation for a single partial wave

$$\langle p|\hat{T}_l(s)|p'\rangle=\langle p|\hat{V}_l|p'\rangle - 4\pi\int_0^\infty dq\frac{\langle p|\hat{V}_l|q\rangle q^2\langle q|\hat{T}_l(s)|p'\rangle}{q^2-s-i\epsilon}. \quad (2.1)$$

It defines a function of three variables, p,p', and s. The physical scattering amplitude in this partial wave, is a function of one variable, obtained by putting

$$p^2=p'^2=s. \quad (2.2)$$

The Lippmann-Schwinger equation thus defines an extension of the T matrix off the energy shell.

a. A Compactness Proof

The Lippmann-Schwinger equation is an integral equation. To get any practical results out of it, we will have to solve it numerically. Now all numerical methods for solving integral equations depend on reducing them to matrix equations. This means that the kernel is replaced by an operator of finite rank, i.e., one which can be written as a finite sum of outer products

$$K=\sum_{n=1}^{N}\lambda_n|nA\rangle\langle nB|. \quad (2.3)$$

Operators which can be *approximated* by operators of finite rank, i.e.,

$$K=\sum_{n=1}^{\infty}\lambda_n|nA\rangle\langle nB| \quad (2.4)$$

with $|nA\rangle$, $\langle nB|$ all orthonormal, and $\lambda_n\to 0$, are called compact.[13] Thus, unless the kernel of an integral equation is a compact operator, it cannot be approximated by a matrix equation, and there is therefore little hope of solving it numerically. Another reason for preferring compact kernels is that they are the only ones for which definite statements can be made about the analytic properties of the solution.[14]

The space usually used in quantum mechanics is the Hilbert space of square-integrable functions. It is easy to prove (see, for example, E) by using the Schmidt criterion, that the kernel of the Lippmann-Schwinger equation (2.1) for generalized Yukawa potentials, is compact in this Hilbert space, *provided* s is not actually on the right-hand cut. Thus bound states can be treated in the ordinary Hilbert space, but scattering states cannot.

Noncompact integral equations are called singular. The theory of such equations[15] consists in turning them into compact ones. For this, it has just two tricks. One is to split off a bit of the kernel, which is so simple that we can solve it in terms of something we know already, and for which the remainder is compact. The other is to go into a different Banach space of functions, because operators which are not compact in one Banach space, may be compact in another. This second trick means essentially that we change the boundary conditions on the solution.

To make the kernel of the Lippmann-Schwinger equation compact on the right-hand cut, we must use a Banach space of continuous functions,[16] rather than the Hilbert space of square-integrable functions. Proofs for the integral equation in x space have been given by Hunziker[17] and Grossmann and Wu,[18] but they cannot be generalized to three-particle states. p-space proofs, which can be generalized, have been given by Faddeev[4] and Taylor,[19] but they get extremely complicated even for two particles. It is therefore worthwhile to give a compactness proof which is simple, rigorous, and valid on the right-hand cut. We first illustrate the proof in the simplest case—two-particle scattering in one-partial wave.

We consider the Banach space C_1 of bounded continuous differentiable functions, with bounded continuous derivatives, on the interval $(0,\infty)$.[16] In this space, the norm of a vector is defined as

$$|f|=\mathrm{Max}_p|f(p)|+\mathrm{Max}_p|f'(p)|. \quad (2.5)$$

[13] In general, an operator is called compact (or completely continuous) if it maps bounded sets into compact sets. Every operator which can be approximated (in the uniform topology) by an operator of finite rank is compact. In Hilbert space, the converse is also true [see R. Schatten, *Norm Ideals of Completely Continuous Operators* (Springer Verlag, Berlin, 1960), p. 18, Theorem 7]. Whether it is true in every Banach space is still an unsolved problem. However, it is certainly true for all the compact operators we shall be using since they are explicitly constructed as limits of operators of finite rank.

[14] See the theorems stated in Ref. 5 and proved in Ref. 8.

[15] N. I. Muskhelishvili, *Singular Integral Equations* (P. Noordhoff Ltd., Groningen, The Netherlands, 1953).

[16] For properties of such Banach spaces, see N. Dunford and J. Schwartz, *Linear Operators, I* (Interscience Publishers, Inc., New York, 1958), especially IV.6.2. and IV.13.36.

[17] W. Hunziker, Helv. Phys. Acta **34**, 593 (1961).

[18] A. Grossmann and T. T. Wu, J. Math. Phys. **2**, 710 (1961).

[19] J. G. Taylor, Cambridge, 1963 (unpublished).

Now everyone knows that in ordinary Hilbert space, we have two kinds of vectors, which Dirac has called bras $\langle A|$ and kets $|B\rangle$. In the Hilbert space of square-integrable functions, the bras and kets will be represented by the same objects. When we pass from Hilbert space to more general Banach spaces, one of the biggest differences is that the bras and kets belong to different Banach spaces. The space to which the bras belong is called the *adjoint space* of that to which the kets belong. It is defined as the space of continuous linear functionals on the kets. Thus, if the kets belong to the Banach space C_1 (of continuous bounded differentiable functions with continuous bounded derivatives), then the adjoint space, to which the bras belong, will contain not only ordinary functions but also some distributions, namely $\delta(p-a)$ and $\delta'(p-a)$. To see this, we need only note that

$$f'(a)=\int_0^\infty dp\delta'(p-a)f(p) \tag{2.6}$$

is a continuous linear functional of $|f\rangle$ because $|f'(a)-g'(a)|\leqslant|f-g|$, by (2.5). In fact, $\delta(p-a)$ and $\delta'(p-a)$ are easily seen to be vectors of norm 1 in the adjoint space.

We now write the kernel of the Lippmann-Schwinger equation

$$\langle p|\hat{K}_l(s)|q\rangle=4\pi V_l(p,q)q^2[q^2-s]^{-1} \tag{2.7}$$

as

$$\hat{K}_l(s)=\frac{1}{\pi}\int_0^\infty ds'\frac{\hat{\Delta}_l(s')}{s'-s}, \tag{2.8}$$

where

$$\langle p|\hat{\Delta}_l(s)|q\rangle=2\pi^2V_l(p,s^{1/2})s^{1/2}\delta(q-s^{1/2}). \tag{2.9}$$

Because the adjoint space to C_1 contains delta functions, $\hat{\Delta}_l(s)$ is a perfectly respectable linear operator in C_1 for each value of s. In fact it is a linear operator of rank 1, since $\delta(q-s^{1/2})$, with s fixed, is a vector in the adjoint space, and $2\pi V_l(p,s^{1/2})s^{1/2}$ with s fixed, is a vector in C_1, provided only that $V_l(p,q)$ and $\partial V_l(p,q)/\partial p$ are continuous and bounded in p. Therefore, $\hat{\Delta}_l(s)$ is certainly compact under these very general conditions on the potential.

It is easy to show that sums and limits (in the uniform topology) of compact operators are compact.[20] Therefore, convergent integrals over compact operators will also be compact, because integrals are defined as limits of sums.[21] To show that $\hat{K}_l(s)$ is compact in C_1, therefore, we need only show that the integral in (2.8) converges. Convergence for operator integrals is very analogous to that for ordinary ones.[21] For s not on the right-hand cut, the integral is nonsingular and will therefore converge provided

$$|\hat{\Delta}_l(s)|\leqslant\frac{C}{1+\ln^2 s}, \quad \text{uniformly in } s; \tag{2.10}$$

i.e., using the definition of the norm (2.6) and the fact that $\delta(q-s^{1/2})$ has norm 1 in the adjoint space,

$$\left.\begin{aligned}|qV_l(p,q)|&\leqslant A/(1+\ln^2 q)\\ \left|q\frac{\partial V_l(p,q)}{\partial p}\right|&\leqslant B/(1+\ln^2 q)\end{aligned}\right\} \quad \text{uniformly in } p,q. \tag{2.11}$$

To show that $\hat{K}_l(s)$ is compact for s also on the cut, we prove that the limit onto the cut exists (in the uniform-operator topology). This is done simply by extending to operators, a theorem of Muskhelishvili on singular integrals.[22]

Theorem: If $\hat{\Delta}(s)$ is an operator depending on the real variable s, and satisfying the Hölder $H(\mu)$ condition

$$|\hat{\Delta}(s)-\hat{\Delta}(s')|<C|s-s'|^\mu \tag{2.12}$$

in some neighborhood of s, then limits

$$\lim_{\epsilon\to 0}\int\frac{\hat{\Delta}(s')ds'}{s'-s\pm i\epsilon} \tag{2.13}$$

exist (in the uniform operator topology).

The proof for operators follows exactly the proof for functions given by Muskhelishvili.[22] To apply the theorem, we note that, if $\hat{\Delta}(s)$ is a differentiable[21] function of $k=s^{1/2}$ and $d\hat{\Delta}(k^2)/dk$ is uniformly bounded (in some neighborhood of each point on the right-hand cut—it need not be uniformly bounded on the entire cut), then $\hat{\Delta}(s)$ will satisfy the $H(1)$ condition, except at $s=0$ where it satisfies the $H(\frac{1}{2})$ condition.

Now $\Delta(k^2)$ is the outer product of $\delta(k^2-q^2)$, whose derivative is $(2k)^{-1}\delta'(k-q)$, which we have already seen to have norm $(2k)^{-1}$ in the adjoint space, and $4\pi^2k^2V_l(p,k)$ whose derivative is

$$8\pi^2kV_l(p,k)+4\pi^2k^2\partial V_l(p,k)/\partial k$$

provided this is in C_1. Therefore, (2.12) will be satisfied provided we add to condition (2.11)

$$\left.\begin{aligned}\left|\frac{\partial[qV_l(p,q)]}{\partial q}\right|&<C\\ \left|\frac{\partial^2[qV_l(p,q)]}{\partial p\partial q}\right|&<D\end{aligned}\right\} \quad \text{uniformly in } p,q. \tag{2.14}$$

So $\hat{\Delta}_l(s)$ satisfies the $H(\frac{1}{2})$ condition at $s=0$, and the $H(1)$ condition for $s>0$, and the kernel of the Lippmann-Schwinger equation is compact on the right-hand cut,

[20] N. Dunford and J. Schwartz, Ref. 16, VI.5.3–4.

[21] For the definition and properties of integrals and derivatives of operators, see E. Hille and R. S. Phillips, *Functional Analysis and Semigroups* (American Mathematical Society, Providence, Rhode Island, 1957), Chap. III, or Dunford and Schwartz, Ref. 16, Chap. III. The operator-valued function we are integrating in (2.8) is uniformly continuous for s not on the cut, so that the integral can then be defined in the ordinary Riemann sense (Hille-Phillips, Theorem 3.3.2).

[22] N. I. Muskhelishvili, Ref. 15, p. 38. Other theorems in this chapter can also be generalized to operators with interesting results for scattering theory.

provided the conditions (2.11) and (2.14) are satisfied. For a Yukawa potential

$$V_l(p,q)=\frac{G}{4\pi^2 pq}Q_l\left(\frac{p^2+q^2+\mu^2}{2pq}\right), \tag{2.15}$$

they are easily seen to be satisfied if $\mathrm{Re}l\geqslant 0$. [The $1/pq$ factor in (2.15) cancels with the threshold behavior of the Q function.] It appears possible to extend this proof to the region $\mathrm{Re}l\geqslant 1-n$, if we use the Banach space C_n of n-times differentiable functions, and separate out the most singular parts of the kernel before making the continuation.

In fact, the theorem quoted from Muskhelishvili can be strengthened to give actual bounds on the Lippmann–Schwinger kernel on the right-hand cut, considered as an operator in C_1, in terms of the constants A, B, C, D of Eqs. (2.11) and (2.14), or of any similar formulas. For example,

$$\begin{aligned}|\hat{K}_l(s)|&=\left|\frac{1}{\pi}P\int_0^\infty\frac{ds'\hat{\Delta}_l(s')}{s'-s}\right|\\&\leqslant 2\pi(A+B)\left[\frac{s}{\rho}-1+\frac{2}{\ln(s+\rho)}\right]\\&\quad+4\pi(A+B+C+D)\left(\frac{\rho}{s-\rho}\right)^{1/2},\end{aligned} \tag{2.16}$$

where the integral is a principal value one, and ρ is an arbitrary positive number less than s. Using the equation

$$|\tan\delta_l(k)|\leqslant\frac{2\pi^2(A+B)}{1-|\hat{K}_l(s)|},\quad\text{if } |\hat{K}_l(s)|<1, \tag{2.17}$$

which follows from (2.26) and (2.35) of the next subsection and from (2.5) above, these give rigorous bounds on the phase shift in terms of the potential in momentum space and its partial derivatives. These appear to be new. By modifying the right-hand side of (2.11) and (2.14), many similar bounds can be obtained. They are valid for nonlocal potentials, and can therefore be improved and used to obtain variational principles, by subtracting a separable potential from the given one and obtaining a bound on the remainder.

The proof can easily be generalized to the complete two-particle amplitude (without separation into partial waves). The kernel is now

$$\langle\mathbf{p}|\hat{V}|\mathbf{q}\rangle[q^2-s]^{-1}, \tag{2.18}$$

which we write

$$\begin{aligned}&\int_0^\infty ds'\int_{-1}^1 d\cos\omega\int_0^{2\pi}d\psi\langle p,\theta,\varphi|\hat{V}|s'^{1/2},\omega,\psi\rangle\\&\quad\times\delta(\cos\omega-\cos\vartheta')\delta(\psi-\varphi')[s'-s]^{-1}(s'^{1/2})\delta(q^2-s').\end{aligned}$$

Before the integration over ω, ψ, s' is performed, the kernel is of rank 1 and therefore compact. The integrations over ω and ψ, being finite, will not affect the compactness, while the integral over s' can be treated just as before.

The proof can also be generalized to the three-particle Faddeev equations (see Sec. 3 below), provided s is not on the right-hand cut. The proof is quite analogous to the Hilbert-space proof given in E. The limit onto the cut in the three-particle case, however, leads to difficulties with coincident singularities. Faddeev[4] has recently shown that by iterating the kernel a sufficient number of times, these can be smoothed sufficiently to enable the limit onto the cut to be performed.

It should be stressed that it is only in the Banach space C_1 that limits onto the cut exist in the *uniform* operator topology, so that compactness on the cut can be proved. In the ordinary Hilbert space, these limits are at best in the *strong* operator topology,[4] which is why the asymptotic condition[23] only involves strong convergence.

b. Solution of Off-Shell Unitarity

The off-shell[24] unitarity relation for the T matrix is[5,25]

$$\begin{aligned}&\mathrm{Im}\langle\mathbf{p}|\hat{T}(s+i\epsilon)|\mathbf{p}'\rangle\\&\quad=-\pi\int d^3p''\langle\mathbf{p}|\hat{T}(s+i\epsilon)|\mathbf{p}''\rangle\\&\qquad\times\delta(p''^2-s)\langle\mathbf{p}''|T(s-i\epsilon)|\mathbf{p}'\rangle.\end{aligned} \tag{2.19}$$

Defining partial waves by

$$\begin{aligned}\langle\mathbf{p}|\hat{T}(s)|\mathbf{p}'\rangle&=\sum_{l=0}^\infty(2l+1)\\&\quad\times P_l((\mathbf{p}\cdot\mathbf{p}')/pp')T_l(p,p';s),\end{aligned} \tag{2.20}$$

this gives

$$\begin{aligned}\mathrm{Im}T_l(p,p';s+i\epsilon)&=-2\pi^2(s^{1/2})T_l(p,s^{1/2};s+i\epsilon)\\&\quad\times T_l(s^{1/2},p';s-i\epsilon).\end{aligned} \tag{2.21}$$

Putting $p^2=p'^2=s$, we recover the ordinary on-shell unitarity relation. It is well known that the latter can be explicitly solved in a number of ways—by representing the T matrix in terms of the real K matrix, or the real phase shift, etc. In this section we want to obtain a similar explicit solution of off-shell unitarity (2.20). It is the off-shell analog of the K-matrix representation, and does not appear to have been given before.

[23] J. M. Jauch, Helv. Phys. Acta **31**, 127 (1958).

[24] Previous papers on off-shell scattering include I. M. Dremin, Zh. Eksperim. i Teor. Fiz. **41**, 821 (1961) [English transl.: Soviet Phys.—JETP **14**, 589 (1962)]; E. Ferrari and F. Selleri, Nuovo Cimento **21**, 1028 (1961); D. I. Fivel, *ibid.* **22**, 326 (1961); K. L. Kowalski and D. Feldman, J. Math. Phys. **2**, 499 (1961); **4**, 507 (1963). It is rather surprising that, so far as we can see, none of them seems to have hit on our Eq. (2.33). If they had done so, it would have simplified some of their arguments considerably.

[25] There is a sign wrong in some of the unitarity equations in E. We have corrected this when quoting from it.

We assume that the potential $V_l(p,q)$ is real and symmetric in p,q, as it must be, to make the Hamiltonian Hermitian. We decompose the partial-wave Lippmann-Schwinger equation (2.1) into real and imaginary parts, using the well-known relation

$$\frac{1}{s'-s-i\epsilon}=\mathrm{P}\frac{1}{s'-s}+i\pi\delta(s-s'),$$

where P means principal value. Defining

$$X_l(p,p';s)=\mathrm{Re}T_l(p,p';s+i\epsilon), \tag{2.22}$$

$$Y_l(p,p';s)=\mathrm{Im}T_l(p,p';s+i\epsilon), \tag{2.23}$$

we get

$$X_l(p,p';s)=V_l(p,p')+2\pi^2V_l(p,s^{1/2})(s^{1/2})Y_l(s^{1/2},p';s)$$
$$-2\pi\mathrm{P}\int_0^\infty dq^2\frac{V_l(p,q)qX_l(q,p';s)}{q^2-s}, \tag{2.24}$$

$$Y_l(p,p';s)=-2\pi^2V_l(p,s^{1/2})s^{1/2}X_l(s^{1/2},p';s)$$
$$-2\pi\mathrm{P}\int_0^\infty dq^2\frac{V_l(p,q)qY_l(q,p';s)}{q^2-s}. \tag{2.25}$$

Here P$\int$ means a principal-value integral. We now define $A_l(p,p';s)$ to be the solution of the Lippmann-Schwinger equation with a principal-value integral

$$A_l(p,p';s)=V_l(p,p')$$
$$-2\pi\mathrm{P}\int_0^\infty dq^2\frac{V_l(p,q)qA_l(q,p';s)}{q^2-s}. \tag{2.26}$$

$A_l(p,p';s)$ will, of course, be real. The way in which the inhomogeneous terms of (2.24) and (2.25) depend on p shows at once that

$$X_l(p,p';s)=A_l(p,p';s)+2\pi^2$$
$$\times A_l(p,s^{1/2};s)(s^{1/2})Y_l(s^{1/2},p';s), \tag{2.27}$$

$$Y_l(p,p';s)=-2\pi^2A_l(p,s^{1/2};s)(s^{1/2})X_l(s^{1/2},p';s). \tag{2.28}$$

The symmetry of the potential in p,p' can be shown to imply that $T_l(p,p';s)$ is also symmetric. Therefore we have, as well as (2.27), (2.28),

$$X_l(p,p';s)=A_l(p,p';s)+2\pi^2$$
$$\times Y_l(p,s^{1/2};s)(s^{1/2})A_l(s^{1/2},p';s), \tag{2.29}$$

$$Y_l(p,p';s)=-2\pi^2X_l(p,s^{1/2};s)$$
$$\times(s^{1/2})A_l(s^{1/2},p';s). \tag{2.30}$$

We now put $s^{1/2}=k$, and substitute (2.28) into (2.29), and (2.27) into (2.30), then using (2.22) and (2.23). This gives

$$T_l(p,q;k^2+i\epsilon)=A_l(p,q;k^2)-2\pi^2A_l(p,k;k^2)$$
$$\times[ik+2\pi^2k^2T_l(k,k;k^2+i\epsilon)]A_l(k,q;k^2). \tag{2.31}$$

Now we put $p=q=k$ and solve for $T_l(k,k;k^2+i\epsilon)$ to get

$$T_l(k,k;k^2+i\epsilon)$$
$$=A_l(k,k;k^2)/[1+2\pi^2ikA_l(k,k;k^2)]. \tag{2.32}$$

Substituting this into (2.31) gives our solution of off-shell unitarity

$$T_l(p,q;k^2+i\epsilon)=A_l(p,q;k^2)$$
$$-\frac{2\pi^2ikA_l(p,k;k^2)A_l(k,q;k^2)}{1+2\pi^2ikA_l(k,k;k^2)}, \tag{2.33}$$

where $A_l(p,q;k^2)$ is real, symmetric in p and q, and can be calculated from (2.26). It is easy to see that (2.33) satisfies the off-shell unitarity relation (2.20) identically, since A_l is real.

Comparing (2.32) with the well-known formula

$$2\pi^2T_l(k,k;k^2+i\epsilon)=1/(-k\cot\delta_l+ik), \tag{2.34}$$

we see that on the energy shell

$$A_l(k,k;k^2)=-1/2\pi^2k\cot\delta_l. \tag{2.35}$$

Thus $A_l(s)$ will have poles in s at the actual positions of resonances (not on the neighboring unphysical sheet). By the result of the previous subsection, the integral equation (2.26) for $A_l(p,p';s)$ will have a compact kernel. Therefore (see E, Theorem III), the solution will factorize in p,p' at the resonance pole. Thus in the vicinity of a resonance we have

$$A_l(p,q;k^2)\approx g(p)g(q)/(k_r^2-k^2), \tag{2.36}$$

which gives, when substituted into (2.33),

$$T_l(p,q;k^2+i\epsilon)\approx\frac{g(p)g(q)}{k_r^2-k^2+2\pi^2ik[g(k)]^2}. \tag{2.37}$$

Here k_r^2 is the resonance energy, and we shall refer to $g(p)$ as the resonance form-factor. The latter gives, among other things, the correct centrifugal barrier—it can be shown that

$$g(p)\sim p^l, \quad \text{as} \quad p\to 0. \tag{2.38}$$

c. Separable Approximations

The T matrix is related to the Green's function or resolvent of the total Hamiltonian $\hat{H}_l$,

$$\hat{G}_l(s)=[\hat{H}_l-s\hat{I}]^{-1} \tag{2.39}$$

($\hat{I}$ is the unit operator, s a complex number), by (see, for example, E)

$$\hat{T}_l(s)=\hat{V}_l-\hat{V}_l\hat{G}_l(s)\hat{V}_l. \tag{2.40}$$

By the spectral theorem for self-adjoint operators,

$$\langle p|\hat{G}_l(s)|p'\rangle=-\sum_n \frac{\psi_n(p)\psi_n(p')}{E_n+s} + \frac{1}{\pi}\int_0^\infty dqq^2\frac{\psi_l(p,q)\psi_l^*(p',q)}{q^2-s-i\epsilon}. \quad (2.41)$$

Here E_n are the binding energies, and $\psi_n(p)$ the corresponding bound-state wave functions normalized by

$$4\pi\int_0^\infty dpp^2\psi_n(p)\psi_m(p)=\delta_{nm}. \quad (2.42)$$

$\psi_l(p,q)$ are the continuum wave functions.[26] In particular, we note that the residues at the bound-state poles factorize in the off-shell variables p,p'. If a bound state is degenerate, the residue will become a sum of separable terms. The bound state contribution to the T matrix will be, by (2.40),

$$T_l(p,p';s)=\sum_n \frac{g_n(p)g_n(p')}{s+E_n}+\cdots, \quad (2.43)$$

where

$$g_n(p)=4\pi\int_0^\infty dqq^2V_l(p,q)\psi_n(q) =-(p^2+E_n)\psi_n(p), \quad (2.44)$$

since the bound-state wave function is a solution of the homogeneous Schrödinger equation. This quantity $g_n(p)$ may be called the form factor of the bound state. It can be proved[27] that the wave function has a pole at $p^2=-E_n$, which cancels out with the factor in (2.44). Thus the form factor will actually be a simpler quantity than the wave function in practice. More important, the form factor continues to exist even for resonances, whereas the wave function does not.

This can be seen from (2.37) above. Another way of treating resonances was given in E. We consider the Schrödinger equation with the complex potential $\hat{V}_l^\varphi$ defined by

$$V_l^\varphi(p,p')=e^{i\varphi}V_l(e^{i\varphi}p,e^{i\varphi}p'), \quad (2.45)$$

where $e^{i\varphi}$ is an arbitrary phase factor. For example, for a Yukawa potential we would have

$$V_l^\varphi(p,q)=(G/4\pi^2pq)e^{-i\varphi} \times Q_l[(p^2+q^2+e^{-2i\varphi}\mu^2)/2pq], \quad (2.46)$$

and we see that it will be bounded provided

$$|\varphi|<\tfrac{1}{2}\pi. \quad (2.47)$$

From this potential, we calculate a new scattering amplitude $\hat{T}_l^\varphi(s)$ by solving the Lippmann-Schwinger equation. It can be shown[8] that the matrix elements $T_l^\varphi(p,p';s)$ of this new scattering amplitude, are analytic continuations of the matrix elements of the original scattering amplitude onto the unphysical sheets in s, reached through the right-hand cut. The formula connecting them is

$$T_l(p,p',e^{2i\varphi}s)=e^{-i\varphi}T_l^\varphi(e^{-i\varphi}p,e^{-i\varphi}p';s). \quad (2.48)$$

For generalized Yukawa potentials, this is valid when $|\varphi|<\frac{1}{2}\pi$. If we take the physical sheet to be $0\leqslant\arg s\leqslant 2\pi$, then we see that (2.48) defines an extension of the scattering amplitude to the half-planes reached by continuing through the right-hand cut from either above or below: $-\pi<\arg s<0$, and $2\pi<\arg s<3\pi$. The bound states of the new potential $\hat{V}^\varphi$ will become resonances of the original potential $\hat{V}$, and their contributions to the T matrix will be of the form[5]

$$A(p)B(p')/2k_n(k_n-k), \quad (2.49)$$

where k_n is in the lower half-plane of $k=s^{1/2}$. The two "form factors" $A(p)$ and $B(p')$ are not now conjugate, because $\hat{V}^\varphi$ is non-Hermitian. However, it can easily be shown (by the Schwarz reflection principle) that there will be another conjugate pole with contribution

$$B^*(p)A^*(p')/2k_n^*(k_n^*+k). \quad (2.50)$$

(2.49) can be regarded as the contribution to the scattering amplitude from the production and decay of an unstable particle, $A(p)$ being the amplitude for its creation, $B(p')$ the amplitude for its decay, and $[2k_n(k_n-k)]^{-1}$ its propagator with complex mass k_n.

This provides a practical method of calculating the masses and decay rates of unstable particles in potential scattering. All we have to do is to find the eigenstates (in the usual sense of square integrability) of the related complex potential [defined by (2.45)]. The corresponding left and right eigenfunctions give us the form factors, by the transformation (2.48). Further details are given in Sec. 5 of E.

Thus the residues at resonance poles factorize in the off-shell variables p, p', just as do those at bound-state poles. If the scattering is dominated by bound states and resonances, then we can approximate the off-shell T matrix by a sum of such separable terms.

$$T_l(p,p';k^2)\approx\sum_n \frac{A_n(p)B_n(p')}{2k_n(k_n-k)}. \quad (2.51)$$

This approximation has the disadvantage of not being unitary. Also, applied to resonances, it requires two poles in order to give a time-reversal-invariant answer,

[26] Strictly speaking, of course, (2.41) must be rewritten as a Stieltjes integral, since the "continuum wave functions" are not in the Hilbert space.

[27] G. Wanders, Helv. Phys. Acta **30**, 417 (1957); L. Bertocchi, C. Ceolin, and M. Tonin, Nuovo Cimento **18**, 770 (1960); R. Blankenbecler and L. F. Cook, Phys. Rev. **119**, 1745 (1960); T. Sawada, Progr. Theoret. Phys. (Kyoto) **27**, 882 (1962); M. A. Braun, Zh. Eksperim. i Teor. Fiz. **42**, 1106 (1962) [English transl.: Soviet Phys.—JETP **15**, 764 (1962)].

and it does not give the centrifugal barrier correctly for higher partial waves. All these disadvantages are avoided by (2.37) of the previous subsection. This preserves the simple separable form of the off-shell T matrix, but modifies the "propagator" of the unstable particle, so that it no longer consists of a simple pole term.

Yet a third approximation can be obtained by making use of the compactness of the kernel, to approximate it by a sum of separable terms, as in (2.4). This is obviously equivalent to approximating the potential by a sum of separable potentials. The study of scattering from separable potentials has a long history,[28] though its theoretical justification has often been lost sight of. If we have just one separable potential per partial wave,

$$V_l(p,q)\approx\lambda_l g_l(p)g_l(q)\,, \tag{2.52}$$

then we can easily get a closed form for the off-shell T matrix which has been given by many authors[28]

$$T_l(p,q;s)\approx g_l(p)t_l(s)g_l(q)\,, \tag{2.53}$$

where

$$t_l(s)=\left[\frac{1}{\lambda_l}+4\pi\int_0^\infty dq\frac{q^2|g_l(q)|^2}{q^2-s-i\epsilon}\right]^{-1}. \tag{2.54}$$

The separable potential will have a bound state, if there is a point $s=-E_l$ for which

$$-\frac{1}{\lambda_l}=4\pi\int_0^\infty dq\frac{q^2|g_l(q)|^2}{q^2+E_l}\,. \tag{2.55}$$

Equation (2.54) then becomes

$$t_l(s)=\frac{1}{s+E_l}\left[4\pi\int_0^\infty dq\frac{q^2|g_l(q)|^2}{(q^2+E_l)(q^2-s-i\epsilon)}\right]^{-1}. \tag{2.56}$$

If we were to take the second factor constant, then the T matrix would reduce to the bound-state contribution (2.43), which we obtained previously. We see that the factors of the separable potential are just the bound-state form factors, provided the coupling constant λ_l of (2.52) is so defined as to give these the right normalization [(2.42) and (2.44)]

$$4\pi\int_0^\infty dp\frac{p^2|g_l(p)|^2}{(p^2+E_l)^2}=1. \tag{2.57}$$

This result also suggests a way[29] of determining the off-shell T matrix from experiment, in the case of partial waves dominated by a bound state or resonance. Experimentally, only the on-shell T matrix is given. As is well known, this can be decomposed into the form

$$T_l(s)=N_l(s)/D_l(s)\,, \tag{2.58}$$

where $D_l(s)$ has only the right-hand cut and $N_l(s)$ only the left-hand cut. This requirement determines $D_l(s)$ up to a constant factor, by the formula[30]

$$D_l(s)=C\exp\left[-\frac{s}{\pi}\int_0^\infty\frac{ds'\delta_l(s')}{s'(s'-s-i\epsilon)}\right], \tag{2.59}$$

where $\delta_l(s)$ is the experimental phase shift. From (2.54) we see that $t_l(s)$ has only the right-hand cut. Furthermore, it is known[27] that bound-state form factors $g_l(p)$ satisfy dispersion relations in p^2 with only a left-hand cut, and an analogous result can be proved for resonance form factors by the techniques of E. Therefore, we have

$$t_l(s)=1/D_l(s)\,, \tag{2.60}$$

$$g_l(p)=[N_l(p^2)]^{1/2}\,, \tag{2.61}$$

and the off-shell T matrix is given by

$$T_l(p,q;s)\approx[N_l(p^2)N_l(q^2)]^{1/2}/D_l(s)\,, \tag{2.62}$$

provided that the partial wave is dominated by a bound state or resonance. We intend to investigate elsewhere[31] how accurate this formula is for a Yukawa potential. In the case of bound states, the undetermined factor C in (2.59) must be chosen to normalize the bound-state form factor correctly, (2.57). This is so that, when we go on to three-particle systems, the bound-state scattering amplitudes of the next section will be correctly defined. In the case of resonances, we can normalize $g_l(p)$ as we wish, since resonance scattering amplitudes are not directly observable.

It should be noted that there is no simple extension of this formula to Regge poles. The Regge residue function β will of course factorize in p and p' off shell, but each

[28] Y. Yamaguchi, Phys. Rev. **95**, 1628, 1635 (1954); M. Gourdin and A. Martin, Compt. Rend. **244**, 1153, 1329, 1469 (1957); Nuovo Cimento **6**, 757 (1957); **8**, 699 (1958); A. Martin, *ibid.* **7**, 607 (1958); K. Chadan, Compt. Rend. **245**, 1597 (1957); **246**, 1513 (1958); Nuovo Cimento **10**, 892 (1958); G. Targonski, *ibid.* **14**, 1093 (1959); D. B. Fairlie, Proc. Cambridge Phil. Soc. **56**, 182 (1960); D. B. Fairlie and J. C. Polkinghorne, Nuovo Cimento **8**, 345, 555 (1958); A. N. Mitra, Phys. Rev. **123**, 1892 (1961); **127**, 1342 (1962); Nucl. Phys. **32**, 529 (1962); Phys. Rev. **131**, 832 (1963); A. N. Mitra and V. L. Narasimham, Nucl. Phys. **14**, 407 (1960); A. N. Mitra and S. P. Pandya, *ibid.* **20**, 455 (1958); **29**, 352 (1962); A. N. Mitra and J. H. Naqvi, *ibid.* **25**, 307 (1961); A. N. Mitra, V. S. Bhasin, and B. S. Bhakar, *ibid.* **32**, 316 (1962); A. N. Mitra and S. Ray, Ann. Phys. (N.Y.) **21**, 439 (1963); A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1265 (1963); V. L. Narasimham, S. K. Shah, and S. P. Pandya, Nucl. Phys. **33**, 529 (1962); J. H. Naqvi, *ibid.* **36**, 578 (1962); B. B. Dotsenko and V. M. Salasyuk, Ukr. Fiz. Zh. **7**, 563 (1962); Izv. Akad. Nauk SSSR **26**, 1097 (1962); A. H. Sytenko and O. V. Drobachenko, Ukr. Fiz. Zh. **8**, 5 (1963); S. Weinberg, Phys. Rev. **130**, 776; **131**, 440 (1963); J. T. Cushing, Nuovo Cimento **28**, 818 (1963); A. N. Mitra and J. D. Anand, Phys. Rev. **130**, 2117 (1963); M. McMillan, Nuovo Cimento **29**, 1043 (1963).

[29] This was essentially given by M. Gourdin and A. Martin, Nuovo Cimento **8**, 699 (1958), though they described it as a method of determining a separable potential from the phase shift. As we have pointed out, it is the off-shell scattering amplitude in the neighborhood of a resonance that is determined—it need not have any resemblance to the actual potential.

[30] R. Jost and W. Kohn, Phys. Rev. **87**, 977 (1952); R. G. Newton, J. Math. Phys. **1**, 319 (1960), Eq. (5.21′); G. F. Chew, in *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, London, 1960), p. 190.

[31] C. Lovelace (unpublished).

of the factors will depend on s as well, so that the Regge wave functions will be functions of two variables. This is proved by the fact that the centrifugal barrier must vary along the trajectory. For this reason, we are doubtful whether scattering by Regge poles will prove a very useful concept.

Summing up: If a partial wave is dominated by a bound state or resonance, then the off-shell T matrix can be approximated by the separable form (2.53). Here $g_l(p)$ is the "form factor" of the bound state or resonance. (This does not necessarily have any relation to the electromagnetic form factor, though there is some analogy, especially in its analytic properties.[27] The bound-state wave function is given by $-g_l(p)/(p^2+E_l)$. In (2.53) $t_l(s)$ is the "propagator" of the bound state or unstable particle. We have obtained three separate formulas for it, of varying degrees of accuracy. The simplest is just a pole term

$$t_l(s)\approx 1/(s+E_l). \tag{2.63}$$

This should be adequate for a bound state, in cases when the rescattering is known experimentally to be small. However, for resonances, or for bound states with large rescattering, we need more sophisticated formulas which satisfy unitarity. We obtained two of these: (2.54) and

$$t_l(k^2)\approx\{k_r^2-k^2+ik2\pi^2[g_l(k)]^2\}^{-1}. \tag{2.64}$$

They differ only in shape-dependent terms. For a separable potential (2.54) would be exact, and (2.64) approximate. However, it remains to be investigated whether (2.54) is actually preferable for say a Yukawa potential. In these approximations, the off-shell T matrix can be obtained from the experimental on-shell T matrix by the N/D decomposition (2.62).

3. THREE-PARTICLE STATES

a. The Faddeev Equations

In the last section we obtained some properties of the two-particle scattering amplitude off the energy shell. Of course, none of this said why we should be interested in the off-shell T matrix in the first place. To see this we must consider three-particle states. Now in three-particle systems, many different processes can go on. Bound states can be formed and they can scatter or disintegrate, or they can rearrange themselves into bound states between other pairs. Because of this, we require a large number of transition amplitudes and S matrices to represent the system correctly.[5,32]

Suppose the particles interact through pair potentials V_α, where α goes from 0 to 3, V_1 being the potential between particles 2 and 3, and $V_0=0$. Three-body forces could be included, but we shall leave them out for simplicity. Let

$$G(s)=[H-sI]^{-1}, \tag{3.1}$$

where H is the total Hamiltonian of the three-particle system. Then we can define transition operators[5]

$$U_{\alpha\beta}^{+}(s)=\sum_{\gamma\neq\alpha}V_\gamma-\sum_{\gamma\neq\alpha}\sum_{\delta\neq\beta}V_\gamma G(s)V_\delta,$$

$$U_{\alpha\beta}^{-}(s)=\sum_{\delta\neq\beta}V_\delta-\sum_{\gamma\neq\alpha}\sum_{\delta\neq\beta}V_\gamma G(s)V_\delta. \tag{3.2}$$

These describe all possible transitions in the three-particle system. For example, the S matrix for this process in which particle α is scattered from a bound state of the other two particles, and ends up with particle β free, and the remaining two particles bound, is

$$S_{\alpha n,\beta m}(\mathbf{q}_\alpha,\mathbf{q}_\beta')=\delta_{\alpha\beta}\delta_{nm}\delta_3(\mathbf{q}_\alpha-\mathbf{q}_\alpha')$$

$$-2\pi i\delta(q_\alpha^2-E_{\alpha n}-q_\beta'^2+E_{\beta m})\int d^3p_\alpha\int d^3p_\beta'$$

$$\times\psi_{\alpha n}^*(\mathbf{p}_\alpha)U_{\alpha\beta}^{\pm}(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}_\beta',\mathbf{q}_\beta';$$

$$q_\alpha^2-E_{\alpha n}+i\epsilon)\psi_{\beta m}(\mathbf{p}_\beta'). \tag{3.3}$$

Here $\psi_{\alpha m}(\mathbf{p}_\alpha)$ and $\psi_{\beta n}(\mathbf{p}_\beta)$ are the bound-state wave functions. $\mathbf{p}_\alpha$ and $\mathbf{q}_\alpha$ are defined in (1.2)–(1.3). The + and − forms of (3.3) can be proved identical, using the fact that the bound-state wave functions are solutions to the homogeneous Schrödinger equations. $U_{\alpha\beta}^{+}(s)$ and $U_{\alpha\beta}^{-}(s)$ thus represent two different continuations of the physical scattering amplitudes off the energy-shell. Both are needed in order to formulate the unitarity relations.[5]

The S matrix for the process in which particle α disintegrates a bound state of the other two particles, giving three particles in the final state, is

$$S_{\alpha n,0}(\mathbf{q}_\alpha,\mathbf{p}',\mathbf{q}')=-2\pi i\delta(q_\alpha^2-E_{\alpha n}-p'^2-q'^2)$$

$$\times\int d^3p_\alpha\psi_{\alpha n}^*(\mathbf{p}_\alpha)$$

$$\times U_{\alpha 0}^{\pm}(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}',\mathbf{q}';q_\alpha^2-E_{\alpha n}+i\epsilon). \tag{3.4}$$

These processes in which bound states are disintegrated, give us a model for production processes. For example, if we consider a system of one nucleon and two pions, then we can describe the nucleon as a pion-nucleon bound state, and $N+\pi\rightarrow N+\pi+\pi$ then corresponds to a bound-state disintegration process. We will pursue this analogy further in Sec. 5. However, we want to make the point now, because people often ask what is the relevance of three-particle states to elementary particle physics.

We prove in the Appendix that our S matrices are identical with those defined by Faddeev in his big paper,[4] and that they are therefore the ones that experimentalists would measure. (Those who have worked on bound-state scattering may appreciate just how nontrivial this statement is.) S matrices for processes with three particles in the initial state can also be defined us-

[32] H. Ekstein, Phys. Rev. **101**, 880 (1956).

ing the $U_{0\beta}{}^{\pm}(s)$, and are given in E. However, they are obviously not of much experimental interest. In E we also discuss the off-shell unitarity relations for the amplitudes $U_{\alpha\beta}{}^{\pm}(s)$.

The difficulty of three-particle theory is that the kernel of the Lippmann-Schwinger equation is not compact.[1,5–7,9] This is because the interaction contains delta functions in which one of the particles goes straight through. It is closely related to the fact that the system can have additional asymptotic states and additional right-hand cuts (bound-state scattering), which are not present in perturbation theory. Moreover, it cannot be made compact by going into a different Banach space. Therefore, we have to make it compact by our other trick—solving part of the kernel in closed form. This is the basis of the Faddeev theory.[1] We do it, by summing all the disconnected graphs in which only two of the particles interact. The sum can be expressed in terms of the exact scattering amplitudes for the two-particle subsystems (1,2), (2,3) and (3,1), which we suppose we have solved already.

In the present paper we shall use the Faddeev equations in a somewhat modified form, first given in E. The kernel is the same as in Faddeev's original equations, but we write them for a different set of amplitudes, and with a different inhomogeneous term. The advantage is that our modified equations give directly the physically observed quantities, not only for three-particle scattering, which is not experimentally interesting, but also for bound-state scattering and disintegration, which are. This form of the equations is therefore much more practical. At first sight, it might seem that there are an unnecessarily large number of amplitudes. However, this is not so—the equations boil down as we insert symmetries and other details of the specific problem. We will see this in Secs. 3c, 4, and 5. The final equations have just as many amplitudes as there are physically different transitions taking place.

The equations can easily be derived by means of Green's functions. Besides the total Hamiltonian

$$H=H_0+V_1+V_2+V_3 \tag{3.5}$$

with Green's function or resolvent (1), we have the Hamiltonians of the various subsystems

$$H_\alpha=H_0+V_\alpha, \quad \alpha=0,1,2,3. \tag{3.6}$$

and their Green's functions

$$G_\alpha(s)=[H_\alpha-sI]^{-1} \tag{3.7}$$

(remember that $V_0\equiv 0$). These satisfy the second resolvent identity

$$\begin{aligned} G(s)&=G_\beta(s)-\sum_{\delta\neq\beta} G(s)V_\delta G_\beta(s) \\ &=G_\alpha(s)-\sum_{\gamma\neq\alpha} G_\alpha(s)V_\alpha G(s) \end{aligned} \tag{3.8}$$

which can easily be checked by substituting (3.1) and (3.7) and multiplying out. Substituting (3.8) into (3.2) gives us the modified Faddeev equations

$$\begin{aligned} U_{\alpha\beta}{}^{+}(s)&=\sum_{\gamma\neq\alpha} V_\gamma-\sum_{\delta\neq\beta} U_{\alpha\delta}{}^{+}(s)G_0(s)T_\delta(s), \\ U_{\alpha\beta}{}^{-}(s)&=\sum_{\delta\neq\beta} V_\delta-\sum_{\gamma\neq\alpha} T_\gamma(s)G_0(s)U_{\gamma\beta}{}^{-}(s), \end{aligned} \tag{3.9}$$

where we have used

$$G_\alpha(s)V_\alpha=G_0(s)T_\alpha(s) \tag{3.10}$$

and $T_0(s)\equiv 0$. Here $T_3(s)$ is the amplitude for the process in which particles 1 and 2 scatter, while particle 3 goes straight through. It is not quite the same as the two-particle amplitude for particles 1 and 2, $\hat{T}_3(s)$. The relation between them is given by

$$\begin{aligned} &\langle \mathbf{p}_3,\mathbf{q}_3|\, T_3(s)\,|\mathbf{p}_3',\mathbf{q}_3'\rangle \\ &\qquad=\delta_3(\mathbf{q}_3-\mathbf{q}_3')\langle \mathbf{p}_3|\,\hat{T}_3(s-q_3{}^2)\,|\mathbf{p}_3'\rangle. \end{aligned} \tag{3.11}$$

An important consequence of (3.11) is that, if $\hat{T}_\alpha(s)$ has a bound-state pole at $s=-E_{\alpha n}$, then $T_\alpha(s)$ will have a branch point there, with a cut going from $-E_{\alpha n}$ to $+\infty$. This is the right-hand cut for the scattering of the bound state. (Remember that our energy scale is defined so that $s=0$ corresponds to the three-particle threshold for a three-particle system, and to the two-particle threshold for a two-particle system.) These bound-state scattering cuts are discussed at length in E.

The importance of the Faddeev equations is that the square of the kernel is compact. In E, this is proved in the ordinary Hilbert space for s not on the various right-hand cuts. This result enables us to prove analytic properties of the amplitudes $U_{\alpha\beta}{}^{\pm}(s)$ as in E. Faddeev[4] has recently shown that for s on the right-hand cuts, the fifth power of the kernel is compact in a certain Banach space of continuous functions.

b. The Separable Approximation

We see that the kernel of the Faddeev equations (3.9) doesn't involve the original potentials at all, but only the *solutions* of the two-particle subsystems. However, these solutions have to be known off the energy shell. This is why we were interested in off-shell potential scattering.

Now suppose that the two-particle scattering is dominated by certain bound states and resonances. Then we know, from Sec. 2c, that the off-shell T matrix will factorize in the initial and final momenta in the neighborhood of the bound state or resonance pole. Furthermore, this off-shell T matrix, in the vicinity of the pole, will be just the same as that calculated from a separable potential, chosen to give the right wave function. Since the kernel of the three-particle equations only depends on this off-shell T matrix, and not on the original potentials, this means that many of the properties of the three-particle system can be obtained correctly from separable potentials, provided we choose

these to give the two-particle bound-state wave functions correctly. Moreover, the compactness properties of the Faddeev kernel guarantee that the result will converge to the correct three-particle amplitudes as the number of separable terms is increased.[33]

Separable potentials have a long history. Especially, Mitra[34,35] has applied them to three-particle systems. At first sight, this might seem quite unjustified, since we all know that field-theoretical potentials are local and not separable. One of the most important results of the Faddeev theory is to justify this separable potential approach. The condition for it to be valid is that each of the two-particle subsystems shall be dominated by a limited number of bound states and resonances.

Let us see what the Faddeev equations reduce to, when each of the two-particle subsystems is dominated by a finite number of bound states and resonances. We then know from Sec. 2c that the two-particle amplitudes have the form

$$T_\alpha(\mathbf{p}_\alpha,\mathbf{p}_\alpha';s)\approx\sum_n g_{\alpha n}(\mathbf{p}_\alpha)t_{\alpha n}(s)g_{\alpha n}(\mathbf{p}_\alpha'), \quad (3.12)$$

which we write as

$$\hat{T}_\alpha(s)=\sum_n|\alpha n\rangle t_{\alpha n}(s)\langle\alpha n|, \quad (3.13)$$

where n denotes the different bound states and resonances within one of the two-particle subsystems. We recall three different expressions for $t_{\alpha n}(s)$ of various accuracies, (2.54), (2.63), (2.64). We assume that the bound states and resonances in any one subsystem all belong to different partial waves. Thus

$$\langle\alpha n|\alpha m\rangle=0, \quad \text{if} \quad n\neq m. \quad (3.14)$$

The two-particle amplitudes in the three-particle Hilbert space are now, by (3.11),

$$T_\alpha(s)=\sum_n|\alpha n\rangle\hat{\tau}_{\alpha n}(s)\langle\alpha n|, \quad (3.15)$$

where $\hat{\tau}_{\alpha n}(s)$ is now an operator given by

$$\langle\mathbf{q}_\alpha|\hat{\tau}_{\alpha n}(s)|\mathbf{q}_\alpha'\rangle=\delta_3(\mathbf{q}_\alpha-\mathbf{q}_\alpha')t_{\alpha n}(s-q_\alpha^2). \quad (3.16)$$

We see that $\hat{\tau}_{\alpha n}(s)$ is an operator in the Hilbert space corresponding to bound-state (or resonance) scattering by the third particle. Unlike the three-particle Hilbert space, wave functions in this bound-state-scattering Hilbert space depend only on one momentum vector. This is why we have written $\hat{\tau}_{\alpha n}(s)$ with a hat. In fact $-\hat{\tau}_{\alpha n}(s)$ is the *propagator* for bound-state (or resonance) scattering. This can be most clearly seen if we use the approximation (2.63), for then

$$-t_{\alpha n}(s-q_\alpha^2)\approx[q_\alpha^2-s-E_{\alpha n}]^{-1}, \quad (3.17)$$

where $E_{\alpha n}$ is the binding energy.

We now define *potentials* for bound-state or resonance scattering

$$\hat{Z}_{\alpha n,\beta m}(s)=(1-\delta_{\alpha\beta})\langle\alpha n|G_0(s)|\beta m\rangle, \quad (3.18)$$

using the notation of (3.13). These are also operators in the bound-state-scattering Hilbert space, their momentum representations being

$$\langle\mathbf{q}_\alpha|\hat{Z}_{\alpha n,\beta m}(s)|\mathbf{q}_\beta\rangle=(1-\delta_{\alpha\beta})\times\frac{g_{\alpha n}(\mathbf{p}_\alpha)g_{\beta m}(\mathbf{p}_\beta)}{p_\beta^2+q_\beta^2-s-i\epsilon}\left[\frac{m_\gamma(m_\alpha+m_\beta+m_\gamma)}{(m_\alpha+m_\gamma)(m_\beta+m_\gamma)}\right]^{-3/2}. \quad (3.19)$$

Here $\mathbf{p}_\alpha$ and $\mathbf{p}_\beta$ are linear combinations of $\mathbf{q}_\alpha$ and $\mathbf{q}_\beta$, given by formulas such as (1.7)–(1.8). γ is the third subscript (neither α nor β). Thus the potentials between the bound states are nonlocal, as we might expect for the interactions of composite particles. They are separable in certain combinations of the initial and final momenta, but not in general in the initial and final momenta themselves (see, however, Sec. 5). They are energy-dependent, because of the s in the denominator of (3.19), and they become complex above the three-particle threshold ($s>0$), which is again what we should expect since the bound states can then disintegrate into three particles.

We now define *scattering amplitudes* for bound states and resonances

$$\hat{X}_{\alpha n,\beta m}(s)=\langle\alpha n|G_0(s)U_{\alpha\beta}{}^+(s)G_0(s)|\beta m\rangle-\hat{Z}_{\alpha n,\beta m}(s)[\hat{I}+\lambda_{\beta m}\hat{r}_{\beta m}(s)], \quad (3.20)$$

where

$$\hat{r}_{\beta m}(s)=\langle\beta m|G_0(s)|\beta m\rangle, \quad (3.21)$$

and

$$V_\beta\approx\sum_m|\beta m\rangle\lambda_{\beta m}\langle\beta m|, \quad (3.22)$$

so that, by (2.54),

$$\hat{\tau}_{\beta m}(s)=[\lambda_{\beta m}{}^{-1}\hat{I}+\hat{r}_{\beta m}(s)]^{-1}. \quad (3.23)$$

We then find from (3.2) that

$$\hat{X}_{\alpha n,\beta m}(s)=\langle\alpha n|G_0(s)U_{\alpha\beta}{}^-(s)G_0(s)|\beta m\rangle-[\hat{I}+\lambda_{\alpha n}\hat{r}_{\alpha n}(s)]\hat{Z}_{\alpha n,\beta m}(s). \quad (3.24)$$

Now on the energy shell for bound-state scattering, we have

$$s=q_\alpha{}^2-E_{\alpha n}=q_\beta{}'^2-E_{\beta m}, \quad (3.25)$$

from which it follows that on the energy shell $-\langle\alpha n|G_0(s)$ and $-G_0(s)|\beta m\rangle$ are the bound-state wave functions [remembering (2.44), (3.12)–(3.13) and (1.11)], and

$$\lambda_{\beta m}\hat{r}_{\beta m}(s)=-1 \quad \text{on shell}, \quad (3.26)$$

[33] This is not quite immediate—besides the compactness of $G_0(s)\hat{T}_\alpha(s)$, which follows from the compactness of the two-particle Lippmann-Schwinger kernels, we also need a "stability theorem" saying that the solution of the Faddeev equations is a continuous function of the kernels, when both are considered as operators. This point will be considered subsequently.

[34] A. N. Mitra, Phys. Rev. **127**, 1342 (1962); **131**, 832 (1963); A. N. Mitra and S. Ray, Ann. Phys. (N.Y.) **21**, 439 (1963).

[35] A. N. Mitra, Nucl. Phys. **32**, 529 (1962); A. N. Mitra and V. S. Bhasin, Phys. Rev. **131**, 1264 (1963).

[in order to produce a bound-state pole in $t_{\beta m}(s)$]. Thus, on the energy shell, the second term in (3.20) or (3.24) vanishes, and the $\hat{X}_{\alpha n,\beta m}(s)$ are the correct amplitudes for bound-state scattering, as defined in the last subsection. The reason why we have introduced these extra terms is to get symmetrical equations, in which the potentials in the inhomogeneous terms are the same as the potentials in the kernels, as we shall shortly see.

By (3.18) with $\alpha=\beta$,

$$\hat{Z}_{\alpha n,\alpha m}(s)=0, \tag{3.27}$$

by definition. If we now substitute (3.15) into the Faddeev equations (3.9), we find using (3.20), (3.23), and (3.27),

$$-\langle\alpha n|G_0(s)U_{\alpha\beta}{}^+(s)=-\sum_m \hat{Z}_{\alpha n,\beta m}(s) \times\lambda_{\beta m}\langle\beta m|+\sum_{\gamma,m}\hat{X}_{\alpha n,\gamma m}(s)\hat{\tau}_{\gamma m}(s)\langle\gamma m|. \tag{3.28}$$

We recall that $-\langle\alpha n|G_0(s)$ becomes the bound-state wave function on the energy shell [see remarks after (3.25)]. Thus (3.28) expresses the amplitudes for all scattering processes with an initial bound state (i.e., all observable three-particle processes) in terms of the $\hat{X}_{\alpha n,\beta m}(s)$. In particular, for bound-state disintegration with a three-particle final state, $\beta=0$, so that the first term vanishes (since $V_0\equiv 0$). This means that, in this approximation, disintegration only occurs through scattering of the bound state into another bound state or resonance, which is off the energy shell and then disintegrates according to its form factor $\langle\gamma m|\equiv g_{\gamma m}(\mathbf{p}_\gamma)$. This provides a theoretical justification of the isobar model[36] as we shall discuss in a subsequent paper.

Substituting (3.28) into (3.20) gives us Lippmann-Schwinger equations for bound-state and resonance scattering

$$\hat{X}_{\alpha n,\beta m}(s)=-\hat{Z}_{\alpha n,\beta m}(s) -\sum_{\gamma r}\hat{X}_{\alpha n,\gamma r}(s)\hat{\tau}_{\gamma r}(s)\hat{Z}_{\gamma r,\beta m}(s). \tag{3.29}$$

These have exactly the form of many-channel two-particle Lippmann-Schwinger equations, if we identify $-\hat{Z}_{\alpha n,\beta m}(s)$ with the potentials and $-\hat{\tau}_{\gamma r}(s)$ with the propagators. As in two-particle systems, everything depends on only one initial and final momentum vector.

Thus, in the approximation in which the two-particle subsystems are dominated by bound states and resonances, the three-particle equations reduce to many-channel two-particle equations, for the scattering of these bound states and resonances. The potentials in these equations contain no arbitrary constants, and can be expressed in terms of the wave functions of the bound states (3.19). Using the fact that the wave-functions are square integrable, it can easily be proved (see E) that the equations are of the ordinary Fredholm type. Three points should be especially noted. Firstly, these equations enable us to study "overlapping" final-state interactions, i.e., the situation when particles 1 and 2 resonate, and particles 2 and 3 resonate at the same time, or even when all three pairs resonate. Such situations occur very frequently in practice and very little is known about them. It has, for example, been suggested that some bumps seen by experimentalists could be due to interference between overlapping resonances. In our equations it is the overlapping of resonances that provides the forces between them. Secondly, despite the approximations, our equations satisfy three-particle unitarity exactly. This is because they would be exact equations for separable potentials. Both these points are big improvements over models of the peripheral type, which have been the main resource of elementary particle physics so far, in dealing with three-particle states. Thirdly, the amplitudes with three final particles can be expressed in terms of these amplitudes for unstable particle scattering, leading to expressions quite similar to those *assumed* in certain phenomenological isobar models.[36]

An earlier version of these equations was given in Sec. 6 of E. Some remarks on the difference may be helpful. In E, we only used the crudest (2.63) of the three possible approximations for the bound-state propagators. Also we made a further approximation for the potentials (3.18), which we shall reintroduce in Sec. 5. This is based on noting that $\hat{Z}_{\alpha n,\beta m}(s)$ almost always (except in the inhomogeneous term) occurs in the equations multiplied to the right by $\hat{\tau}_{\beta m}(s)$. The latter has a pole at $s=q_\beta{}'^2-E_{\beta m}$. Therefore, it should be a good approximation to replace $\hat{Z}_{\alpha n,\beta m}(s)$ by its value at this pole, giving

$$\langle\mathbf{q}_\alpha|\hat{Z}_{\alpha n,\beta m}(s)|\mathbf{q}_\beta\rangle\approx(1-\delta_{\alpha\beta})g_{\alpha n}(\mathbf{p}_\alpha) \times g_{\beta m}(\mathbf{p}_\beta)[p_\beta{}^2+E_{\beta m}]^{-1}. \tag{3.30}$$

In this further approximation, the potential ceases to be energy-dependent, but exact three-particle unitarity is lost. The inhomogeneous term of (3.28), which is not multiplied to the right by $\hat{\tau}_{\beta m}(s)$, is always put back onto the energy-shell eventually, which corresponds to (3.30) anyway.

A third difference from our previous equations is in the definition of the bound-state and resonance scattering amplitudes (3.20). Our previous definition did not have the last term. Since this last term vanishes on the energy shell, the final results would have been the same. Nevertheless, the improvement is a very real one. Not only are the equations for the new amplitudes much simpler and more symmetric, but the physical interpretation of the various terms is much clearer [especially in the relation of (3.28) to the isobar model]. Anyone trying to use the equations in practice will soon notice the difference.

[36] M. Olsson and G. B. Yodh, Phys. Rev. Letters **10**, 353 (1963). References to earlier isobar models are given there.

c. Identical Particles

In this subsection we want to see what further simplifications occur when all three particles are identical. We assume that the particles are always ordered cyclically, i.e.,

$$\begin{aligned}&\alpha=1 \text{ corresponds to } (2,3)+1,\\ &\alpha=2 \text{ corresponds to } (3,1)+2,\\ &\alpha=3 \text{ corresponds to } (1,2)+3.\end{aligned} \tag{3.31}$$

We need not then distinguish between Bose and Fermi statistics.

Because of the identity of the particles, the bound-state form factors $g_{\alpha n}(\mathbf{p}_\alpha)$ will be independent of α (but not of course of n which denotes the different bound states and resonances within one subsystem). Therefore, by (3.19), the nondiagonal elements $\alpha\neq\beta$ of $\hat{Z}_{\alpha n,\beta m}(s)$ are independent of α and β. We write them as $\hat{Z}_{n,m}(s)$. The elements with $\alpha=\beta$ vanish, by (3.27). Similarly, the propagators $\hat{\tau}_{\beta m}(s)$ will be independent of β and can be written as $\hat{\tau}_m(s)$. Looking at the equations for $\hat{X}_{\alpha n,\beta m}(s)$, (3.29), we then see that all the nondiagonal elements ($\alpha\neq\beta$) will be independent of α, β, and all the diagonal elements with $\alpha=\beta$ will likewise be independent of α. However, the diagonal elements will not necessarily be equal to the nondiagonal ones. We therefore put

$$\hat{X}_{\alpha n,\alpha m}(s)=\hat{X}_{nm}{}^{D}(s), \tag{3.32}$$

$$\hat{X}_{\alpha n,\beta m}(s)=\hat{X}_{nm}{}^{N}(s), \quad \text{for} \quad \alpha\neq\beta, \tag{3.33}$$

and the equations (3.29) reduce to [remembering (3.27)]

$$\hat{X}_{nm}{}^{D}(s)=-2\sum_r \hat{X}_{nr}{}^{N}(s)\hat{\tau}_r(s)\hat{Z}_{rm}(s), \tag{3.34}$$

$$\hat{X}_{nm}{}^{N}(s)=-\hat{Z}_{nm}(s)-\sum_r \hat{X}_{nr}{}^{N}(s)\hat{\tau}_r(s)\hat{Z}_{rm}(s) -\sum_r \hat{X}_{nr}{}^{D}(s)\hat{\tau}_r(s)\hat{Z}_{rm}(s). \tag{3.35}$$

In order to decouple the equations, we form combinations of $\hat{X}_{nm}{}^{D}(s)$ and $\hat{X}_{nm}{}^{N}(s)$:

$$\hat{X}_{nm}(s)=\hat{X}_{nm}{}^{D}(s)+2\hat{X}_{nm}{}^{N}(s), \tag{3.36}$$

$$\hat{Y}_{nm}(s)=\hat{X}_{nm}{}^{D}(s)-\hat{X}_{nm}{}^{N}(s). \tag{3.37}$$

Now $\hat{X}_{nm}(s)$ correspond to the observed bound-state or resonance scattering amplitudes, since we cannot distinguish between the three final pairings of the particles ($\alpha=1, 2, 3$) but must sum over them. The other amplitudes $\hat{Y}_{nm}(s)$ do not correspond to anything observable, and, as we shall shortly see, need never be calculated.

Inserting (3.34) and (3.35) into (3.36) and (3.37), we find

$$\hat{X}_{nm}(s)=-2\hat{Z}_{nm}(s)-\sum_r \hat{X}_{nr}(s)\hat{\tau}_r(s)2\hat{Z}_{rm}(s), \tag{3.38}$$

$$\hat{Y}_{nm}(s)=\hat{Z}_{nm}(s)+\sum_r \hat{Y}_{nr}(s)\hat{\tau}_r(s)\hat{Z}_{rm}(s). \tag{3.39}$$

We thus see that the equations for $\hat{X}_{nm}(s)$ are completely independent of those for $\hat{Y}_{nm}(s)$. Equations (3.38) are our Lippmann-Schwinger equations for scattering of bound states of identical particles, n and m denoting the different bound states or resonances. The potentials have now become $-2\hat{Z}_{nm}(s)$, the propagators remain $-\hat{\tau}_m(s)$.

Now we consider the amplitude for a bound state to disintegrate in the scattering process, giving three final particles. By Eq. (3.28) and the remark after it, this is

$$\begin{aligned}&\langle\mathbf{q}_1|\langle 1n|G_0(s)U_{10}{}^{+}(s)|\mathbf{p}_1',\mathbf{q}_1'\rangle\\ &\quad=\sum_m\langle\mathbf{q}_1|\hat{X}_{1n,1m}(s)|\mathbf{q}_1'\rangle t_m(s-q_1'^2)g_m(\mathbf{p}_1')\\ &\quad+\sum_m\langle\mathbf{q}_1|\hat{X}_{1n,2m}(s)|\mathbf{q}_2'\rangle t_m(s-q_2'^2)g_m(\mathbf{p}_2')\\ &\quad+\sum_m\langle\mathbf{q}_1|\hat{X}_{1n,3m}(s)|\mathbf{q}_3'\rangle t_m(s-q_3'^2)g_m(\mathbf{p}_3').\end{aligned} \tag{3.40}$$

Using (3.36), (3.37) this becomes

$$\begin{aligned}&\tfrac{1}{3}\sum_m\{\langle\mathbf{q}_1|\hat{X}_{nm}(s)|\mathbf{q}_1'\rangle t_m(s-q_1'^2)g_m(\mathbf{p}_1')\\ &\quad+\langle\mathbf{q}_1|\hat{X}_{nm}(s)|\mathbf{q}_2'\rangle t_m(s-q_2'^2)g_m(\mathbf{p}_2')\\ &\quad+\langle\mathbf{q}_1|\hat{X}_{nm}(s)|\mathbf{q}_3'\rangle t_m(s-q_3'^2)g_m(\mathbf{p}_3')\}\\ &\quad+\tfrac{1}{3}\sum_m\{2\langle\mathbf{q}_1|\hat{Y}_{nm}(s)|\mathbf{q}_1'\rangle t_m(s-q_1'^2)g_m(\mathbf{p}_1')\\ &\quad-\langle\mathbf{q}_1|\hat{Y}_{nm}(s)|\mathbf{q}_2'\rangle t_m(s-q_2'^2)g_m(\mathbf{p}_2')\\ &\quad-\langle\mathbf{q}_1|\hat{Y}_{nm}(s)|\mathbf{q}_3'\rangle t_m(s-q_3'^2)g_m(\mathbf{p}_3')\}.\end{aligned} \tag{3.41}$$

Now experimentally we can never distinguish between the final particles. This means that the terms in the second curly bracket in (3.41) [those involving $\hat{Y}_{nm}(s)$] will cancel in any experimental quantity, and can therefore be dropped. The expression for the disintegration amplitude is therefore given by the first curly bracket in (3.41). Thus the $\hat{Y}_{nm}(s)$ never occur in any observable quantity and need never be calculated. The only equations we need are (3.38) and the first curly bracket of (3.41).

4. THE THREE-NUCLEON SYSTEM

In this section we derive the equations for the three-nucleon system in our approximation. Essentially the same equations have previously been obtained by Mitra and Bhasin[35] from separable potentials, and also by

Sitenko and Kharchenko.[37] However, our derivation is much simpler, and is also instructive as a model for other cases.

The two-nucleon system contains a bound state (the deuteron, denoted by d) with isospin 0 and, if we neglect its small D-wave component, spin 1 and orbital angular momentum 0. There is also the singlet virtual bound state, which we will denote by s, with isospin 1, spin 0, and orbital angular momentum 0. These two states dominate the low-energy nucleon-nucleon scattering, which we therefore approximate by

$$\hat{T}(s)=|d\rangle t_d(s)\langle d|+|s\rangle t_s(s)\langle s|\,. \tag{4.1}$$

Mitra and Bhasin[35] also included another separable term for the ${}^1P_1$ state. We have omitted this, because this state is far from resonant, and there is therefore no justification for approximating it by a separable potential. Rather, it should be included by the sort of modified perturbation theory described by Weinberg,[38] using field-theoretic nuclear potentials. The tensor forces should be included in the same way. The justification for omitting the tensor forces as a first approximation is that the D-wave component of the deuteron is known experimentally to be small, and that the Faddeev equations only depend on the *solution* of the two-nucleon system, not on the original forces.

The three-nucleon system will now be described by the equations for identical particles, developed in the last subsection, with two "channels," dN scattering, and sN scattering, s being the singlet virtual bound state. The equations for it are (3.38) with n,m taking on these two values. The potentials are defined by (3.19) in terms of the d and s form factors $g_d(\mathbf{p})$, $g_s(\mathbf{p})$.

Since both d and s have orbital angular momentum 0, the spin and orbital angular momentum in the $3N$ system will decouple in this approximation. We can then treat the total spin as an extra conserved quantum number, completely analogous to the isospin. Let $I=$total isospin, $S=$total spin, $l=$orbital angular momentum between the bound state and the third particle. The potentials in a particular partial wave of the coupled dN, sN systems are then given by, from (3.19),

$$-2\langle q_1|Z_{nm}{}^{ISl}(s)|q_2\rangle=\Lambda_{nm}{}^{IS}\,\tfrac{1}{2}\int_{-1}^{1}d\cos\theta \times P_l(\cos\theta)\frac{g_n(p_1)g_m(p_2)}{p_2{}^2+q_2{}^2-s-i\epsilon}\times\frac{8}{3\sqrt{3}}\,. \tag{4.2}$$

Here p_1 and p_2 are to be expressed in terms of q_1, q_2 and $\cos\theta$ (the angle between $\mathbf{q}_1$ and $\mathbf{q}_2$) by (1.7)–(1.8) with $m_1=m_2=m_3=M$, which gives

$$\begin{aligned}p_1{}^2&=\tfrac{1}{3}(q_1{}^2+4q_2{}^2+4q_1q_2\cos\theta)\,,\\ p_2{}^2&=\tfrac{1}{3}(4q_1{}^2+q_2{}^2+4q_1q_2\cos\theta)\,.\end{aligned} \tag{4.3}$$

The subscripts n and m in (4.2) each take on the two values s and d. $\Lambda_{nm}{}^{IS}$ expresses the dependence on spin and isospin, and can be calculated as follows: The nucleons each have isospin $\frac{1}{2}$. In passing from the final state to the initial state of (4.2), we therefore have to transform a (2,3) bound state with isospin i_{23} plus particle 1 with isospin $i_1=\frac{1}{2}$, into a (1,2) bound state with isospin i_{12} plus particle 3 with isospin i_3. This is a standard recoupling problem. The coefficients for it are, in the notation of Edmonds,[39] Eq. (6.1.4),

$$\langle(i_1i_2)i_{12},i_3,I|(i_2i_3)i_{23},i_1,I\rangle=(-1)^{i_2+i_3-i_{23}} \times[(2i_{12}+1)(2i_{23}+1)]^{1/2}\begin{Bmatrix}i_1 & i_2 & i_{12}\\ i_3 & I & i_{23}\end{Bmatrix}, \tag{4.4}$$

by Edmonds' Eqs. (6.1.5.) and (3.5.14), and our Eqs. (3.31) above. In our problem, we have $i_1=i_2=i_3=\frac{1}{2}$, while i_{12} and i_{23} are 1 for the d particle, or 0 for the s particle. The total isospin I can be $\frac{1}{2}$ or $\frac{3}{2}$. The spin recoupling coefficients are exactly similar, except that the roles of the s and d particles are now interchanged. The two contributions (spin and isospin) must then be multiplied together to form the $\Lambda_{nm}{}^{IS}$. There is an additional factor -2 from the left-hand side of (4.2). We then find that the nonvanishing coefficients are as follows:

$$\Lambda_{ss}{}^{\frac{3}{2}\frac{1}{2}}=1\,, \tag{4.5}$$

$$\Lambda_{dd}{}^{\frac{1}{2}\frac{3}{2}}=1\,, \tag{4.6}$$

$$\Lambda_{nm}{}^{\frac{1}{2}\frac{1}{2}}=\begin{array}{c}\\ s\\ d\end{array}\begin{array}{c}\begin{array}{cc}s & d\end{array}\\ \begin{pmatrix}-\frac{1}{2} & \frac{3}{2}\\ \frac{3}{2} & -\frac{1}{2}\end{pmatrix}\end{array}. \tag{4.7}$$

Thus, in this approximation, there is no interaction in the state with $I=S=\frac{3}{2}$. The states with $I=\frac{3}{2}$, $S=\frac{1}{2}$ or with $I=\frac{1}{2}$, $S=\frac{3}{2}$, each consist of a single channel, and will be described by a single Lippmann-Schwinger equation. The state with $I=S=\frac{1}{2}$ consists of two coupled channels, related by (4.7).

To complete the equations, we need now only decide on the radial parts of the d and s form factors. The Hulthén wave function is known to give a good description of the deuteron. (It also describes the ground state of a Yukawa potential quite well.[31]) This corresponds to taking for the form factor [related to the wave function by (2.44)]

$$g_d(p)=N_d/(p^2+\mu_d{}^2)\,. \tag{4.8}$$

Here μ_d is the average mass of the particles whose exchange produces the potential, and N_d is to be determined by the normalization of the deuteron wave function as in (2.57), which gives

$$N_d=[\mu_d\kappa_d(\mu_d+\kappa_d)^3/\pi]^{1/2}\,, \tag{4.9}$$

[37] V. F. Kharchenko, Ukr. Fiz. Zh. **7**, 573 (1962); A. G. Sitenko and V. F. Kharchenko, Nucl. Phys. **49**, 15 (1963).
[38] S. Weinberg, Phys. Rev. **131**, 440 (1963).
[39] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, New Jersey, 1957).

where $(\kappa_d)^2$ is the deuteron binding energy. The deuteron propagator then becomes

$$\langle \mathbf{q}|\hat{\tau}_d(s)|\mathbf{q}'\rangle=\delta_3(\mathbf{q}-\mathbf{q}')\times\left[\frac{1}{\lambda_d}+\frac{\pi^2 N_d{}^2}{\mu_d[\mu_d+(q^2-s-i\epsilon)^{1/2}]^2}\right]^{-1}. \quad (4.10)$$

The coupling constant λ_d and the mass parameter μ_d can be determined by fitting the experimental deuteron binding energy [using Eq. (2.55)] and the np triplet scattering length.[40]

We assume that the form factor and propagator of the singlet virtual bound state have the same Hulthén form (4.8), (4.10), but with different parameters λ_s and μ_s. These can be determined from the experimental singlet scattering length and effective range of the neutron-proton system.[40] The reason why we had to choose N_d to normalize the deuteron wave function was so that the scattering amplitudes $\hat{X}_{dd}(s)$ would be the correct ones for bound-state scattering. However, s is an unstable particle so that sN scattering is not directly observable. We can therefore normalize $g_s(p)$ as we wish, and we choose to put $N_s=N_d$ [given by (4.9)]. The complete potentials for the coupled dN, sN systems are then given by, using (4.3),

$$\begin{aligned}-2\langle q_1|\hat{Z}_{nm}{}^{ISl}(s)|q_2\rangle&=\Lambda_{nm}{}^{IS}N_d{}^2\\&\times\tfrac{1}{2}\int_{-1}^{1}d\cos\theta(24\sqrt{3})P_l(\cos\theta)\\&\times[q_1{}^2+4q_2{}^2+4q_1q_2\cos\theta+3\mu_n{}^2]^{-1}\\&\times[4q_1{}^2+q_2{}^2+4q_1q_2\cos\theta+3\mu_m{}^2]^{-1}\\&\times[4q_1{}^2+4q_2{}^2+4q_1q_2\cos\theta-3s-i\epsilon]^{-1}. \quad (4.11)\end{aligned}$$

The d propagator is (4.10), the s propagator is similar with λ_d and μ_d replaced by λ_s and μ_s. Inserting these into (3.38) gives us the final equations. There is one integral equation each for the states with $I=\frac{3}{2}$, $S=\frac{1}{2}$, and $I=\frac{1}{2}$, $S=\frac{3}{2}$, and two coupled integral equations for the state with $I=S=\frac{1}{2}$. These equations contain no arbitrary constants—λ_d, μ_d, λ_s, μ_s are all determined by the low-energy behavior of the two-nucleon system and approximate values have been given by several previous authors.[40] The amplitude for the process $d+N\to N+N+N$ is also determined by these equations, by means of the first curly bracket of (3.41). This process has hitherto proved rather intractable theoretically.

The integral in (4.11) can be done in closed form if required. We then see that it has logarithmic singularities at the three-particle threshold $s=0$. This gives the logarithmic factor in the known three-particle threshold behavior,[41] $\sim s^2\ln s$. The s^2 factor comes from the threshold behavior in the off-shell variables p, q, p', q'. These logarithmic singularities combine with the singularities of the propagators to give coincident singularities when the equations are iterated. The latter are probably related to the "triangle singularities" of production processes in perturbation theory.[42] They give rise to difficulties in performing the limit onto the right-hand cut, and could also make the numerical solution of the equations awkward above the three-particle threshold. However, Faddeev[4] has shown that, if the kernel is iterated a sufficient number of times, these singularities become sufficiently smooth to be innocuous.

Mitra and Bhasin[35] calculated the nd scattering lengths from equations essentially the same as these: they found the quartet scattering length ($I=\frac{1}{2}$, $S=\frac{3}{2}$) in good agreement with the more plausible of the two experimental sets. A simplified calculation also gave reasonable agreement for the triton binding energy. However, the doublet nd scattering length ($I=\frac{1}{2}$, $S=\frac{1}{2}$) was not in agreement. Similar conclusions were reached by Sitenko and Kharchenko.[37] It should be noted, however, that the equation for the quartet state, by (4.6), depends only on the deuteron parameters which are well determined experimentally, while the equations for the doublet state, by (4.7), depend also on the s parameters which are not so well determined by two-nucleon experiments, since the np singlet effective range is not accurately known. Being just above a bound state (the triton), the doublet scattering length is likely to be rather sensitive to these input parameters. We therefore hope that further calculations will improve the agreement with experiment.

5. THE $N\pi\pi$ SYSTEM

a. Equations and Kinematics

In the present section we will consider the nucleon-pion-pion system in the static limit. Each of the nucleon-pion subsystems will then contain a bound state—the nucleon itself—and a resonance—the (3,3). The fact that the nucleon occurs both as a bound state and as one of the elementary particles makes this system especially interesting. We will neglect the interaction between the two pions at present, in order to get simple equations whose significance can be easily seen. Experimentally, ρ production does not become important in pion-

[40] The latest are A. N. Mitra and V. S. Bhasin (Ref. 35), and A. G. Sitenko and V. F. Kharchenko (Ref. 37); others may be found among Ref. 28.

[41] W. H. Guier and R. W. Hart, Phys. Rev. **106**, 296 (1957); A. F. Grashin, Zh. Eksperim. i Teor. Fiz. **35**, 719 (1958) [English transl.: Soviet Phys.—JETP **8**, 499 (1959)]; L. M. Delves, Nucl. Phys. **9**, 391 (1959); **20**, 275 (1960); L. Fonda and R. G. Newton, Phys. Rev. **119**, 1394 (1960); V. N. Gribov, Zh. Eksperim. i Teor. Fiz. **41**, 1221 (1961) [English transl.: Soviet Phys.—JETP **14**, 871 (1962)]; A. J. Dragt and R. Karplus, Nuovo Cimento **26**, 168 (1962); N. S. Kronfli, Nuovo Cimento **30**, 1465 (1963).

[42] L. F. Cook and J. Tarski, Phys. Rev. Letters **5**, 585 (1960); J. Math. Phys. **3**, 1 (1962); P. V. Landshoff and S. B. Treiman, Nuovo Cimento **19**, 1249 (1961); Phys. Rev. **127**, 649 (1962); P. V. Landshoff, Phys. Rev. Letters **3**, 116 (1962); F. R. Halpern, Phys. Rev. **127**, 1819 (1962).

nucleon interactions till above 1 BeV,[36] where the static approximation certainly cannot be used. Thus the pion-pion interaction we are neglecting is really only that in the $I=0$ state. This will at most affect the isospin-$\frac{1}{2}$ state of the nucleon-pion-pion system since, in the other N$\pi\pi$ isostates, the $\pi\pi$ subsystem cannot be in an $I=0$ state.

We take particles 1 and 2 as being pions, and particle 3 as the nucleon. We then have a three-particle system in which two of the particles are identical and noninteracting. The discussion of Sec. 3c for three identical particles can easily be modified for this new case, and leads to the equations

$$\hat{X}_{nm}(\omega)=-\hat{Z}_{nm}(\omega)-\sum_r \hat{X}_{nr}(\omega)\hat{\tau}_r(\omega)\hat{Z}_{rm}(\omega)\,, \quad (5.1)$$

where ω has been used instead of s. The amplitudes here are those for the scattering of an $N\pi$ bound state or resonance by the third pion. n and m in (5.1) will have two values, corresponding to the two "bound-state scattering" channels—$N\pi$ and $N^*\pi$. Thus, in the present scheme, the $N\pi$ scattering amplitude goes *into* the equations as one of the assumed two-particle subsystems, and it also comes *out of* the equations as bound-state scattering. So our theory of the $N\pi\pi$ system implies a theory of the $N\pi$ system. We shall see in the next subsection just what this implied theory is.

The kinematics of Sec. 1b give in the static limit ($\mu/M\rightarrow 0$), as in (1.18)–(1.19),

$$\mathbf{p}_1=-\mathbf{q}_2\,, \quad \mathbf{p}_2=\mathbf{q}_1\,. \quad (5.2)$$

Instead of s we now use ω—the total center-of-mass energy of the two pions. The kinetic energy of the pions must be treated relativistically, as in (1.20), giving

$$H_0=(\mu^2+2\mu p_1^2)^{1/2}+(\mu^2+2\mu q_1^2)^{1/2}. \quad (5.3)$$

To conform with the notation of other authors, we redefine $\mathbf{p}_1$, $\mathbf{p}_2$, $\mathbf{q}_1$, $\mathbf{q}_2$ as $(2\mu)^{1/2}$ times the ones we have been using. Equation (5.2) is then unchanged, but the total kinetic energy becomes

$$H_0=(\mu^2+p_1^2)^{1/2}+(\mu^2+q_1^2)^{1/2}. \quad (5.4)$$

The integrals $\int d^3p$, $\int d^3q$ will acquire factors of $(2\mu)^{-3/2}$ but we redefine the scattering amplitudes $\hat{X}_{n,m}(\omega)$ and potentials $\hat{Z}_{n,m}(\omega)$ to eliminate these. It is quite safe to drop constant factors in this way, since they can be easily checked in the final formula by examining the unitarity relations. The Green's function of the free Hamiltonian will then be

$$\langle \mathbf{p},\mathbf{q}|G_0(\omega)|\mathbf{p}',\mathbf{q}'\rangle=\delta_3(\mathbf{p}-\mathbf{p}')\delta_3(\mathbf{q}-\mathbf{q}') \times[(\mu^2+p^2)^{1/2}+(\mu^2+q^2)^{1/2}-\omega]^{-1}. \quad (5.5)$$

Inserting (5.2) and (5.5) into the equations (3.18) for the potentials, we obtain

$$\langle \mathbf{q}_1|\hat{Z}_{nm}(\omega)|\mathbf{q}_2\rangle=\frac{g_n(-\mathbf{q}_2)g_m(\mathbf{q}_1)}{(\mu^2+q_1^2)^{1/2}+(\mu^2+q_2^2)^{1/2}-\omega-i\epsilon}\,. \quad (5.6)$$

Here $g_N(p)$ and $g_{N^*}(p)$ will be the pionic form factors of the nucleon and of the (3,3) resonance. We see that the potentials (5.6) are separable, except for the denominator. This is because of the peculiar kinematics (5.2) of the static limit. It makes the equations much easier to solve than the $3N$ equations of the previous section. The propagators become, instead of (3.16),

$$\langle \mathbf{q}|\hat{\tau}_n(\omega)|\mathbf{q}'\rangle=\delta_3(\mathbf{q}-\mathbf{q}')t_\alpha(\omega-(\mu^2+q^2)^{1/2}), \quad (5.7)$$

where the change is due to that in the kinetic energy (5.4).

The amplitude for the "bound-state disintegration" process

$$N+\pi\rightarrow N+\pi+\pi$$

can be calculated in terms of the $\hat{X}_{n,m}(\omega)$ by similar arguments to (3.41). It is

$$\tfrac{1}{2}\{\langle \mathbf{q}|\hat{X}_{N,N}(\omega)|\mathbf{q}_1'\rangle t_N(\omega-(\mu^2+q_1'^2))^{1/2}g_N(\mathbf{p}_1') +\langle \mathbf{q}|\hat{X}_{N,N}(\omega)|\mathbf{q}_2'\rangle t_N(\omega-(\mu^2+q_2'^2)^{1/2})g_N(\mathbf{p}_2') +\langle \mathbf{q}|\hat{X}_{N,N^*}(\omega)|\mathbf{q}_1'\rangle t_N(\omega-(\mu^2+q_1'^2)^{1/2})g_{N^*}(\mathbf{p}_1') +\langle \mathbf{q}|\hat{X}_{N,N^*}(\omega)|\mathbf{q}_2'\rangle t_N(\omega-(\mu^2+q_2'^2)^{1/2})g_{N^*}(\mathbf{p}_2')\}\,. \quad (5.8)$$

We see that it has a form similar to that assumed in the isobar model,[36] with N (off-shell) and N^* as the two isobars produced. As before, (5.8) is exact in the separable approximation (neglecting the $\pi\pi$ interaction), and satisfies three-particle unitarity if no further approximations are made.

b. $N\pi$ Amplitudes

As mentioned in the previous subsection, our theory of the $N\pi\pi$ system *implies* a theory of the $N\pi$ system. This is because the nucleon can be considered as a pion-nucleon bound state, and therefore the $N\pi$ system is a special case, corresponding to bound-state scattering, of the $N\pi\pi$ system. In the present subsection, we shall study this implied theory of $N\pi$ scattering, neglecting the $N^*\pi$ channel. The latter will be included in the next subsection.

We recall that we obtained three possible formulas for propagators in Sec. 2c—the simplest one (2.63) which just has the bound-state pole, and more sophisticated ones (2.54) and (2.64) which include the effect of the continuum states. Now the nucleon is a P_{11} bound state of the pion-nucleon system. Experimentally, the P_{11} phase shift is small at low energies, though it becomes large above 300 MeV. Therefore, we shall neglect the continuum contribution and use the simplest form for the nucleon propagator

$$t_N(\omega)=\omega^{-1}, \quad (5.9)$$

which gives, by (5.7),

$$\langle \mathbf{q}|\hat{\tau}_N(\omega)|\mathbf{q}'\rangle=\delta_3(\mathbf{q}-\mathbf{q}')/[\omega+i\epsilon-(\mu^2+q^2)^{1/2}]. \quad (5.10)$$

Now we notice that, in the scattering equations (5.1), $\hat{Z}_{NN}(\omega)$ always occurs multiplied to the right by $\hat{\tau}_N(\omega)$, except in the inhomogeneous term. As discussed in con-

nection with Eq. (3.30), this should justify us in replacing $\hat{Z}_{NN}(\omega)$ by its value at the pole of $\hat{\tau}_N(\omega)$. Equation (5.6) then becomes

$$\langle \mathbf{q}_1|\hat{Z}_{NN}(\omega)|\mathbf{q}_2\rangle = g_N(-\mathbf{q}_2)g_N(\mathbf{q}_1)/(\mu^2+q_1^2)^{1/2}. \quad (5.11)$$

This was one of the further approximations made in E. It can be derived in another way, given in E, by using (3.10) to write the Faddeev equations in terms of $G_\alpha(s)$ instead of $T_\alpha(s)$, and then replacing $G_\alpha(s)$ by the bound-state pole contribution alone [first term of (2.41)].

When we make this further approximation, the potential (5.11) becomes separable, and we can solve the scattering equations (5.1) exactly. (We are neglecting the coupled $N^*\pi$ channel in the present subsection.) First, however, we must separate out the isospin and angular momentum states, as we did for the $3N$ system in Sec. 4. Since the nucleon is a P_{11} bound state, its pionic form factor $g_N(\mathbf{p})$ will have a P-wave dependence on the angular variables, i.e., will be linear in them. Therefore, the potential (5.11) will vanish unless the initial and final $N\pi$ systems are in orbital P waves. This again is due to the peculiar kinematics of the static limit, which doesn't recouple the orbital angular variables. We therefore only have to work out the recoupling for the isospins, and for the manner in which the nucleon spin is coupled to the orbital angular momentum. These two factors will be completely identical: a "bound state" with isospin (spin) $\frac{1}{2}$ is added to a pion with isospin (orbital angular momentum) 1. We must remember one difference from the $3N$ case (Sec. 4)—the nucleon can be distinguished from the pions, and should therefore occupy the same position in the initial and final states. These are then given by, instead of (3.31),

$$\begin{aligned} &\alpha=1 \text{ corresponds to } (2,3)+1 \\ &\alpha=2 \text{ corresponds to } (1,3)+2. \end{aligned} \quad (5.12)$$

The effect of this change is to suppress the $(-1)^{i_2+i_3+i_{23}}$ factor in Eq. (4.4). The recoupling coefficient for the $N\pi$ state with isospin $\frac{1}{2}I$ and total angular momentum $\frac{1}{2}J$ can then be calculated analogously to (4.4). Including a minus sign from the left-hand side of (5.14), it is

$$\Lambda_{NN}{}^{IJ} = -4\begin{Bmatrix} 1 & \frac{1}{2}I & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{Bmatrix}\begin{Bmatrix} 1 & \frac{1}{2}J & \frac{1}{2} \\ 1 & \frac{1}{2} & \frac{1}{2} \end{Bmatrix}, \quad (5.13)$$

and the potential in the $\frac{1}{2}I$, $\frac{1}{2}J$ $N\pi$ state is

$$-\langle q_1|\hat{Z}_{NN}{}^{IJ}(\omega)|q_2\rangle = \Lambda_{NN}{}^{IJ}g_N(q_2)g_N(q_1) \times[(\mu^2+q_1^2)(\mu^2+q_2^2)]^{-1/4}. \quad (5.14)$$

Here we have symmetrized in the initial and final momenta, which had been made asymmetric by the approximation (5.11). This symmetrization corresponds to redefining $\langle \mathbf{q}_1|\hat{X}_{NN}(\omega)/\mathbf{q}_2\rangle$ by a factor

$$[(\mu^2+q_2^2)/(\mu^2+q_1^2)]^{1/4}$$

which reduces to 1 on the energy shell. $g_N(q)$ is now the radial part of $g_N(\mathbf{q})$. Equation (5.13) gives

$$\Lambda_{NN}{}^{33} = -4/9, \quad \Lambda_{NN}{}^{31} = \Lambda_{NN}{}^{13} = 2/9. \quad (5.15)$$

For the P_{11} state we must make a slight modification. This is because the nucleon can also be considered as a three-particle $N\pi\pi$ bound state. This would correspond to a pole of $\hat{X}_{NN}(\omega)$ at $\omega=0$, and if it does not come out of the equations dynamically, then it must be put in. It is more convenient to put it in as a pole in the off-shell variables $(\mu^2+q_1^2)^{1/2}$, $(\mu^2+q_2'^2)^{1/2}$ instead of the on-shell variable ω. The final result will be completely equivalent, since everything goes back on-shell in the end. The extra pole term which has to be added to (5.1) in the P_{11} state will then have the form, by (2)

$$g_N(q_1)g_N(q_2)/[(\mu^2+q_1^2)(\mu^2+q_2^2)]^{1/4}, \quad (5.16)$$

instead of

$$g_N(q_1)g_N(q_2)/\omega.$$

It must be added to the kernel, as well as to the inhomogeneous term, as can easily be seen graphically. Comparing with (5.14), we see that it has the same form as the potential (this is why we transferred it to the off-shell variables), and can be absorbed into it by adding 1 to $\Lambda_{NN}{}^{11}$. The latter then becomes

$$\Lambda_{NN}{}^{11} = 8/9. \quad (5.17)$$

This corresponds to including the *direct* nucleon pole in P_{11}, the crossed nucleon pole being already given by (5.14), as we shall discuss below.

We can now solve the scattering equations (5.1) exactly, using the potential (5.14) and the propagator (5.10), to obtain

$$\langle q_1|\hat{X}_{NN}{}^{IJ}(\omega)|q_2\rangle = g_N(q_1)x_{NN}{}^{IJ}(\omega)g_N(q_2)/[(\mu^2+q_1^2)(\mu^2+q_2^2)]^{1/4}, \quad (5.18)$$

where

$$x_{NN}{}^{IJ}(\omega) = \left\{\frac{1}{\Lambda_{NN}{}^{IJ}} + 2\pi\int_0^\infty \frac{dq'^2q'[g_N(q')]^2}{(\mu^2+q'^2)^{1/2}[(\mu^2+q'^2)^{1/2}-\omega-i\epsilon]}\right\}^{-1}. \quad (5.19)$$

This is identical with the Chew-Low effective-range formula.[43] To make the identification in detail, we must first renormalize the pion-nucleon coupling constant. This corresponds to making a subtraction in the integral in (5.19), the subtraction point being the nucleon pole $\omega=0$, and then putting the subtracted term equal to zero. Equation (5.19) then becomes, using $\omega'=(\mu^2+q'^2)^{1/2}$ as the integration variable,

$$x_{NN}{}^{IJ}(\omega) = \left\{\frac{1}{\Lambda_{NN}{}^{IJ}} + \pi\omega\int_\mu^\infty \frac{d\omega'q'[g_N(q')]^2}{\omega'(\omega'-\omega-i\epsilon)}\right\}^{-1}. \quad (5.20)$$

[43] G. F. Chew and F. E. Low, Phys. Rev. **101**, 1570 (1956).

Comparing (5.18) and (5.20) with the Chew-Low effective-range formula,[43] which is

$$[\hat{X}_{NN}{}^{IJ}(q,q;\omega)]^{-1}=\frac{\omega}{3f^2q^2v^2(q)}\times\left[\frac{1}{\Lambda_{NN}{}^{IJ}}+\frac{3\omega f^2}{\pi}\int_\mu^\infty\frac{d\omega'(q')^3v^2(q')}{(\omega)^2(\omega'-\omega-i\epsilon)}\right], \quad (5.21)$$

we see the pionic form factor of the nucleon is given by

$$g_N(q)=\sqrt{3}fqv(q)/\pi(\mu^2+q^2)^{1/2}, \quad (5.22)$$

where $v^2(q)$ is the Chew-Low cutoff function, and f^2 the renormalized pion-nucleon coupling constant.

As everyone knows, (5.21) predicts the (3,3) resonance if the cutoff function is suitable chosen. Equation (5.18) then shows that the off-shell $N\pi$ scattering amplitude in the (3,3) state factorizes in the variables q_1, q_2, and ω. Comparing with (2.53), we see that the radial part of the N^* form factor must be identical with the radial part of the N form factor, up to a constant factor. [The denominator in (5.18) is not present in (2.53). It is due to the relativistic kinematics of the *pions*. It cancels with a factor which arises in transforming $\int dq'^2$ into $\int d\omega'=\int d(\mu^2+q'^2)^{1/2}$.] Since $N^*\pi$ scattering is not directly observable, we can normalize the N^* wave function as we like, and we choose it so that the radial parts of the two form factors are identical. Thus

$$g_N{}^*(q)=\sqrt{3}fqv(q)/\pi(\mu^2+q^2)^{1/2}. \quad (5.23)$$

Comparing (5.18), (5.15), and (5.20) with (2.53) also shows that the N^* propagator is given by

$$t_{N^*}(\omega)=\left[-\frac{9}{4}+\frac{3\omega f^2}{\pi}\times\int_\mu^\infty\frac{d\omega'(q')^3v^2(q')}{(\omega')^2(\omega'-\omega-i\epsilon)}\right]^{-1}\approx\left[-\frac{9}{4}+\frac{9\omega}{4\omega_r}+3f^2i(\omega^2-\mu^2)^{3/2}v^2((\omega^2-\mu^2)^{1/2})/\omega\right]^{-1} \quad (5.24)$$

in the Chew-Low effective-range approximation, with ω_r the resonant energy.

Thus, we have shown that if the nucleon is considered as a pion-nucleon bound state, then our bound-state scattering equations lead to the Chew-Low effective-range formulas for $N\pi$ scattering, when the coupled $N^*\pi$ channel is neglected, and the approximation (5.11) is made. The significance of this result will be considered in Sec. 6.

c. $N\pi^*$ Amplitudes

In the previous subsection, we considered the $N\pi\pi$ system, assuming that the $N\pi$ subsystems were dominated by the N bound state. We now include the N^* resonance, as well as the N bound state, in the $N\pi$ subsystems. Our three-particle equations (5.1) will now contain two channels—$N\pi$ and $N^*\pi$.

From the last subsection, we already know the N^* propagator and form factor, given by (5.24) and (5.23). To derive the scattering equations, we need therefore only make the angular momentum and isospin decomposition. This is easily done by extending the argument leading to (5.13). We find

$$\Lambda_{MM'}{}^{IJ}=-[(M+1)(M'+1)]\times\begin{Bmatrix}1 & \frac{1}{2} & \frac{1}{2}M\\ 1 & \frac{1}{2}I & \frac{1}{2}M'\end{Bmatrix}\begin{Bmatrix}1 & \frac{1}{2} & \frac{1}{2}M\\ 1 & \frac{1}{2}J & \frac{1}{2}M'\end{Bmatrix}, \quad (5.25)$$

where $M=1$ for the $N\pi$ channel, $M=3$ for the $N^*\pi$ channel, and M' similarly. Again we must add 1 to $\Lambda_{NN}{}^{11}$ to allow for the direct nucleon pole ($N\pi\pi$ bound state). This gives for the coupling matrices in the different isospin and angular momentum states

$$\Lambda_{N^*,N^*}{}^{55}=-1 \quad \text{(others vanish)}, \quad (5.26)$$

$$\Lambda_{N^*,N^*}{}^{53}=\Lambda_{N^*,N^*}{}^{35}=\tfrac{2}{3} \quad \text{(others vanish)}, \quad (5.27)$$

$$\Lambda_{N^*,N^*}{}^{51}=\Lambda_{N^*,N^*}{}^{15}=-\tfrac{1}{3} \quad \text{(others vanish)}, \quad (5.28)$$

$$\Lambda_{n,m}{}^{33}=\begin{array}{c}\\ N\\ N^*\end{array}\begin{array}{c}\begin{array}{cc}N & N^*\end{array}\\ \begin{pmatrix}-4/9 & -5/9\\ -5/9 & -4/9\end{pmatrix}\end{array}, \quad (5.29)$$

$$\Lambda_{n,m}{}^{31}=\Lambda_{n,m}{}^{13}=\begin{array}{c}\\ N\\ N^*\end{array}\begin{array}{c}\begin{array}{cc}N & N^*\end{array}\\ \begin{pmatrix}2/9 & (2\sqrt{10})/9\\ (2\sqrt{10})/9 & 2/9\end{pmatrix}\end{array}, \quad (5.30)$$

$$\Lambda_{n,m}{}^{11}=\begin{array}{c}\\ N\\ N^*\end{array}\begin{array}{c}\begin{array}{cc}N & N^*\end{array}\\ \begin{pmatrix}8/9 & -8/9\\ -8/9 & -1/9\end{pmatrix}\end{array}. \quad (5.31)$$

The potentials are now

$$-\langle q_1|\hat{Z}_{n,m}{}^{IJ}(\omega)|q_2\rangle=\frac{\Lambda_{n,m}{}^{IJ}g(q_2)g(q_1)}{(q_1{}^2+\mu^2)^{1/2}+(q_2{}^2+\mu^2)^{1/2}-\omega-i\epsilon}, \quad (5.32)$$

where the subscripts n,m each have the two values N and N^*, and we have omitted the subscript from g, in view of (5.22)–(5.23).

We then obtain two coupled linear integral equations for each of the partial waves. Again, it is only the denominator of (5.32) which stops the equations being soluble in closed form. Unfortunately, we cannot just replace the denominator by its value at the N^* pole, since this is in the complex plane and will make the potentials become complex below the three-particle threshold. We therefore replace the denominator in all terms

of the potential by its value at the nucleon pole, giving after symmetrization

$$-\langle q_1|\hat{Z}_{n,m}{}^{IJ}(\omega)|q_2\rangle=\Lambda_{n,m}{}^{IJ}g(q_1)g(q_2)/[(q_1{}^2+\mu^2)(q_2{}^2+\mu^2)]^{1/4}. \quad (5.33)$$

For the $N\pi\to N\pi$ potentials this replacement is justified by the fact that they are multiplied by $\hat{\tau}_N(\omega)$ with its singularity at the nucleon pole, as in the previous subsection. The amplitudes $N\pi\rightleftharpoons N^*\pi$ are also at least multiplied on one side by this function. The approximation is much less justifiable for the $N^*\pi\to N^*\pi$ amplitudes. However, it can always be got rid of, if we are willing to solve the integral equations. When we make this extra approximation, we lose the exact three-particle unitarity of the theory. Therefore we have an *a posteriori* check on its accuracy—whether we calculate the total inelastic cross section by integrating Eq. (5.8) for the production amplitude, or by taking the imaginary part of the phase shift from Eqs. (5.39)–(5.41) below, we should get the same result. If we do not, then the extra approximation (5.33) is not justified, and we must solve the integral equations.

Once we have made this approximation, the potentials are again completely separable and we can solve in closed form. To do this, we define "effective-range functions" for $N\pi$ and $N^*\pi$ scattering

$$r_N(\omega)=\frac{3\omega f^2}{\pi}\int_\mu^\infty\frac{d\omega'(q')^3v^2(q')}{(\omega')^2(\omega'-\omega-i\epsilon)}, \quad (5.34)$$

$$r_{N^*}(\omega)=(-3f^2/\pi)\int_\mu^\infty d\omega'(q')^3v^2(q')\times\{t_{N^*}(\omega-\omega')-t_{N^*}(-\omega')\}/\omega', \quad (5.35)$$

where $t_{N^*}(\omega)$ is given by (5.24), and the subtractions at the nucleon pole ($\omega=0$) have been introduced to keep the coupling constant renormalized. Using the potentials (5.33) and the propagators (5.10) and (5.24), the scattering equations (5.1) then show that the amplitudes $\langle q|\hat{X}_{m,n}{}^{IJ}(\omega)|q'\rangle$ factorize in their three variables

$$\langle q|X_{m,n}{}^{IJ}(\omega)|q'\rangle=3f^2qv(q)q'v(q')x_{mn}{}^{IJ}(\omega)/[(\mu^2+q^2)(\mu^2+q'^2)]^{1/2}, \quad (5.36)$$

and that the $x_{mn}{}^{IJ}(\omega)$ satisfy *algebraic* equations

$$x_{mn}{}^{IJ}(\omega)=\Lambda_{mn}{}^{IJ}-\sum_t x_{mt}{}^{IJ}r_t(\omega)\Lambda_{tn}{}^{IJ}. \quad (5.37)$$

In the Chew-Low effective-range approximation, (5.34) becomes [as in (5.24)]

$$r_N(\omega)\approx(9\omega/4\omega_r)+3f^2iq^3v^2(q)/\omega. \quad (5.38)$$

Equations (5.36) and (5.37) then show that the effect of the $N^*\pi$ coupled channel on the pion-nucleon effective-range formulas is as follows

$$3f^2q^3v^2(q)\cot\delta_{33}/\omega=-\frac{9\omega}{4\omega_r}+\frac{9-4r_{N^*}(\omega)}{4+r_{N^*}(\omega)}, \quad (5.39)$$

$$3f^2q^3v^2(q)\cot\delta_{13}/\omega=-\frac{9\omega}{4\omega_r}-\frac{9+2r_{N^*}(\omega)}{2-4r_{N^*}(\omega)}, \quad (5.40)$$

$$3f^2q^3v^2(q)\cot\delta_{11}/\omega=-\frac{9\omega}{4\omega_r}-\frac{9-r_{N^*}(\omega)}{8[1-r_{N^*}(\omega)]}, \quad (5.41)$$

where $r_{N^*}(\omega)$ is given by (5.35) and (5.24), and becomes complex above the three-particle threshold. Equations (5.39)–(5.41) then enable us to calculate the imaginary part of the πp phase shifts, and thereby the inelasticities in the different partial waves. The detailed momentum and mass spectra of the production process $N+\pi\to N+\pi+\pi$ can be calculated by inserting (5.36) into (5.8). To do all this, we need only evaluate the integral in (5.39) numerically.

This will be done, and a detailed comparison with experiment will be made in a subsequent paper. For the present, we merely draw some qualitative conclusions. We see from (5.31) that the off-diagonal elements ($N\pi\to N^*\pi$) of the potential are especially large and attractive (negative) in the P_{11} state. This should therefore get most of the inelastic scattering at low energies, and become large and attractive as a result. This is in agreement with the experimental evidence.[44,45] From the analogy with (5.42), we might expect $r_{N^*}(\omega)$ to be positive for $\omega>0$. If it becomes larger than 1, (5.41) shows that there will be a pole in the P_{11} effective-range formula, which will cause the phase shift to change sign. Experimentally, this actually happens at 200 MeV.[46] It is amusing that if we suppose $r_{N^*}(\omega)\approx A\omega$, as in the Chew-Low effective-range approximation for $r_N(\omega)$, and choose the constant A to give the change of sign at 200 MeV, then (5.41) predicts a P_{11} resonance at 510 MeV (as compared with the P_{11} resonance at 556 MeV recently suggested by Roper[45] from phase-shift analysis).[46a] Another prediction is that there should be a low-energy resonance in the $I=J=\frac{5}{2}$ state of the $N\pi\pi$ system (as in the old strong-coupling theory), since (5.26) shows that the force in this state is attractive and more than twice as strong as the force producing the (3,3) resonance. At present there is, in our opinion, no convincing experimental evidence either for or against this.

Of the innumerable previous papers on inelastic πN scattering, the one closest to the spirit of the present

[44] P. Auvil and C. Lovelace, Nuovo Cimento **33**, 473 (1964).
[45] L. D. Roper, Phys. Rev. Letters **12**, 340 (1964).
[46] J. Hamilton and W. S. Woolcock, Rev. Mod. Phys. **35**, 737 (1963).
[46a] *Note added in proof.* However the width comes out too small. Also a calculation by Razmi and Roberts indicates that the decay mode of the P_{11} resonance cannot be $N^*+\pi$. Therefore the $\pi\pi$ interaction must be included to get a realistic theory of $\pi N\to\pi\pi N$. We are now doing this.

theory seems to be that of Wong and Ross.[47] However, these authors neglected the coupling between the $N\pi$ and $N^*\pi$ channels, which we have seen to be large in the P_{11} and P_{33} states.

It is not too difficult to extend the equations to include the $N+ABC$ channel. The integral equations will then be definitely nonseparable and will involve the ABC form factor (though arguing from the Okubo chain approximation,[48] we might expect the latter to be approximately constant). However, it is easy to obtain the signs of the forces. We find that $N+\pi \rightleftharpoons N+ABC$ is attractive in the $N\pi$ P_{11} state and repulsive in the $N\pi$ S_{11} state, while $N^*+\pi \rightleftharpoons N+ABC$ is attractive in what corresponds to the $N\pi$ P_{11} state, and very strongly attractive in what corresponds to the $N\pi$ D_{13} state. This last force is probably the mechanism of the second resonance.

6. DISCUSSION

We have developed a practical method for calculating all observable quantities in nonrelativistic three-particle systems. It was made possible by the rigorous theory of three-particle systems, initiated by Faddeev.[1-4] If we approximate each of the two-particle subsystems by a finite number of bound states and resonances, then the equations become sufficiently simple to solve. General theorems guarantee the convergence of this approximation scheme as the number of bound states and resonances is increased.[33] Our scheme is thus a sort of isobar model. However, it is free from the usual drawbacks of isobar models—there are no arbitrary constants, and resonances are not approximated by stable particles. The theory is applicable to "overlapping" resonances and, if no further approximations are made, the solution will satisfy three-particle unitarity exactly (see Sec. 3b).

Many people have tried to improve field theory by drawing Feynman graphs in which some of the lines were composite or unstable particles, and taking their masses and coupling constants as extra parameters. A well-known drawback of this approach is the existence of composite particles with spin 1 and greater, for which all but the simplest Feynman graphs are badly divergent. Our results tell us how to avoid this. Composite particles must be put into Feynman graphs with (nonlocal) blobs at their vertices, instead of just points. These blobs represent off-shell effects.[49] In the case of stable bound states, they are closely related to their wave functions. The fact that the wave functions are square integrable ensures that the bound-state scattering equations will be of Fredholm type[5] and therefore that the Feynman graphs containing these composite particles will converge. (Presumably this is related to the vanishing of the wave-function renormalization constant for composite particles,[50] which ensures that the bound-state form factors have nice asymptotic behavior.) The propagators of the composite particles also require some modification. We gave, in Sec. 2c, various formulas for these. We could say that the main achievement of our theory is that it tells us exactly what the propagators and vertex functions of composite and unstable particles should be, if we want to draw nontrivial Feynman graphs with them.

First we need the form factors (not electromagnetic) of the bound states and resonances. In the case of bound states, these are closely related to the wave functions by (2.44). They are not arbitrary, but can be deduced from the two-particle phase shifts, binding energies, etc., by various methods given in Sec. 2c. For example, we can use the shape dependence of a resonance to obtain, by Eqs. (2.64) and (2.53),

$$(1/2\pi^2)k[g(k)]^2 \cot\delta = k^2 - k_r^2. \tag{6.1}$$

Comparing with a phenomenological formula of Layson[51] for the (3,3) phase shift, this gives the N^* form factor (units $\mu=1$)

$$g(p) = \left(\frac{3\times 0.08}{1+q^2}\right)^{1/2} q\left[\frac{1}{1+0.073q^2}\right]. \tag{6.2}$$

In the static limit, the nucleon (pionic) form factor will be the same as that of the N^* (Sec. 5b). The first factor includes coupling constant and the effect of the P-wave centrifugal barrier, the factor in square brackets is the significant one. We note that it has only a mild dependence on the off-shell variable, quite different from what would be required to repair the peripheral model.[52]

Secondly, besides the form factors, we need the bound-state and resonance propagators. In the simplest case, these can be approximated by poles. However, when unitarity is important, more elaborate formulas should be used [(2.54) or (2.64)]. These include corrections from the continuum states, which can be expressed in terms of the form factors.

Thirdly, the method for using these form factors and propagators in Feynman graphs for bound-state scattering or for three-particle processes, is given in Sec. 3b. The modifications for identical particles are in Sec. 3c, while Secs. 4 and 5 illustrate particular cases. There are two sets of equations. One [(3.28) and following remarks] gives the amplitudes for processes with three final particles, in terms of the amplitudes for the scattering of bound states and unstable particles.

This equation justifies and generalizes the isobar model.[36] The amplitudes for the production of an unstable particle correspond to the arbitrary constants of

[47] W. N. Wong and M. Ross, Phys. Rev. Letters **3**, 398 (1959).
[48] S. Okubo, Phys. Rev. **118**, 357 (1960).
[49] It is interesting that these off-shell factors are the mechanism which cancels the Amati cut. See S. Mandelstam, Nuovo Cimento **30**, 1127 (1963).
[50] A. Salam, Nuovo Cimento **25**, 224 (1962); R. Acharya, *ibid.* **24**, 870 (1962); S. Weinberg, Phys. Rev. **130**, 772 (1963).
[51] W. M. Layson, Nuovo Cimento **20**, 1207 (1961).
[52] U. Amaldi and F. Selleri, Nuovo Cimento **31**, 360 (1964). J. D. Jackson and H. Pilkuhn, CERN, 1964 (to be published).

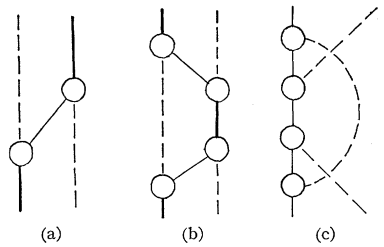

FIG. 1. Graphs illustrating the bound-state scattering equations.

the usual isobar model. They occur multiplied by the isobar propagator and form factor. The latter describes its decay. This equation should, even by itself, be valuable in developing new and better phenomenological isobar models, as we intend to discuss in a subsequent paper.

Lastly, Eq. (3.29) eliminates these arbitrary constants. It tells us that scattering processes involving bound states or unstable particles are described by a set of coupled Lippmann-Schwinger equations. The propagators can be simply expressed in terms of the bound-state and unstable-particle propagators by (3.16). The potentials are given in terms of the form factors by (3.19). They are nonlocal, energy-dependent, and become complex above the three-particle threshold. They contain no arbitrary constants. Figure 1a describes these potentials graphically. These equations justify, under certain conditions, the treatment of three-body problems by separable potentials, such as has been proposed by Mitra.[34,35] The reason why this is so, is that the Faddeev equations only depend on the *solutions* (off shell) of the two-particle subsystems, and these factorize in the off-shell variables in the neighborhood of a bound state or resonance, just as they would if the original potential had been separable.

These equations therefore solve the problem, in the nonrelativistic case, of using composite particles in Feynman graphs. Of course, in the more conventional on-shell theories, such "compositeness" effects are *in principle* taken into account by anomalous thresholds. However, the off-shell formalism presented here is very much more simple and direct. Nobody has yet been able to write down a theory of three-particle states based on anomalous thresholds which even looks as if it might be soluble.

What about making the theory relativistic? We hope to discuss this in a subsequent paper. The Lippmann-Schwinger equation for the two-particle systems would then be replaced by a Bethe-Salpeter equation. There are two difficulties. One is that no fully rigorous theory of even two-particle scattering from the Bethe-Salpeter equation has yet been given, due to difficulties connected with the Wick transformation. Thus we would probably have to forego rigorous proofs, and rely on the analogy with potential scattering. The second difficulty is that Bethe-Salpeter wave functions depend on two variables, even after the angular momentum decomposition. This means that the bound-state scattering equations would be integral equations in two variables instead of one. This sets a very much nastier numerical problem, and some form of one-time approximation would probably have to be used.

We now want to discuss deeper questions. From a theoretical point of view, one of our most remarkable results was in Sec. 5b, when we considered the nucleon as a bound state, treated pion-nucleon scattering as if it were bound-state scattering, and after some approximations came out with the Chew-Low effective-range formula. Of course, there is nothing new about the result—the interesting thing is that we got it by treating the nucleon as a bound state. This gives considerable insight into our bound-state scattering formalism, and it also suggests some new avenues for strong-interactions theory. We can understand this result if we look at it in terms of graphs.[53] Our bound-state potential (3.19) corresponds to Fig. 1a, in which a bound state between one pair of particles rearranges itself into a bound state between another pair. According to our equations, this is the only way in which bound states interact. When we iterate this potential, we get graphs like Fig. 1b, with intermediate states of a nucleon and two pions. These are topologically equivalent to Fig. 1c, which are the one-nucleon exchange graphs included in the Chew-Low effective-range formula. This means that, when we include three-particle intermediate states according to our formalism, we find that we have *automatically* included one-particle exchange at the same time for bound-state scattering.[54] Similarly, if we had four-particle intermediate states, and bound states of three particles, the graphs show that our bound-state scattering equations would automatically include the effects of two-particle exchange.

We can thus formulate an approximation scheme for strong interactions, in which we include $(n+2)$-particle intermediate states, at the same time as we include n-particle exchange. We start with various particles and resonances suggested by experiment. These are treated as bound states of themselves, like the nucleon in Sec. 5. By making an N/D decomposition of the observed two-particle scattering amplitudes, we can obtain their bound-state wave functions by (2.62). At the start, these are therefore phenomenological. However, they are not assumed for all time, we will be able to calculate some of them—the nonelementary ones—later. Next we formulate the three-particle scattering problem, taking our various particles and resonances to be two-particle bound states, as in the present paper. We then get equations for the three-particle system, from which we can predict any nonelementary three-

[53] Equations for *nd* scattering based on this set of graphs were suggested by R. D. Amado, Phys. Rev. **132**, 485 (1963), but without proof. He also discussed some of their wider implications.

[54] The Faddeev equations are written in terms of three "channels" $\alpha=1, 2, 3$, each with its own kinematics (Sec. 1b). This result shows that the transformation from one channel to another is closely related to crossing. This analogy is especially helpful for the angular momentum and isospin decomposition.

particle bound states and their wave functions. We also find that we have equations for the two-particle systems, considered as bound-state scattering. These will include the effects of one-particle exchange, i.e., the most peripheral part of the forces. This enables us to calculate the wave functions of the nonelementary particles, or at least their outermost parts, as we did for the N^* in Sec. 5b. It also enables us to calculate the two-particle interaction in the higher partial waves, not dominated by resonances. This corresponds to the residual, nonseparable part of the two-particle interactions. This is why we have not included this residual interaction as a perturbation, in the manner suggested by Weinberg.[38] If all particles are treated as bound states, this is unnecessary, since the residual interactions will all get included sooner or later anyway. The next stage would be to consider the four-particle scattering equations in the separable approximation.[54a] Since the elementary particles are being treated as bound states of themselves, they will also be "bound states of bound states" of themselves, i.e., three-particle bound states. The scattering equations for these should then include the effect of two-particle exchange, so that the next most peripheral part of the interaction can be calculated. We hope to consider this in a subsequent paper. It is plain that we have here, at least in principle, the possibility of a complete dynamical scheme for strong interactions.

The only previous dynamical scheme for strong interactions with any degree of success has been that of Mandelstam.[55] Its great drawback was that three-particle unitarity was never satisfied in any finite order. This caused the failure[56] of its most ambitious form, the strip approximation. Another difficulty was the exchange of vector particles which caused divergences and inconsistencies. The scheme we have just suggested, if it can be fully worked out, should avoid both these troubles. Three-particle unitarity is already satisfied in the first stage, as we have seen, and four-particle unitarity would be satisfied at the second stage. The off-shell form factors should cancel the divergences due to the exchange of vector mesons.[57] Compared with the Mandelstam scheme, it would include less crossing symmetry at any given stage, and more many-particle unitarity. This is probably the right order of things, since the singularities controlled by unitarity are closer to the physical region than those controlled by crossing. Also the exact crossing symmetry of the Mandelstam scheme is somewhat illusory, since almost all calculations have been forced to use the pole approximation.

Great interest has been expressed recently in the possibility that all particles might be explained as dynamic bound states of each other, with no elementary particles at all. Obviously, our results have a bearing on this. In Sec. 5, we inserted the direct nucleon pole forcibly into the equations [see argument leading to (5.17)]. If we had not done this, $\Lambda_{NN}{}^{11}$ would have been $-1/9$ in Eqs. (5.17) and (5.31), instead of 8/9. This would give an attractive force in the P_{11} state, but not strong enough to bind the nucleon, since it is weaker than the force producing the N^*. However, when we consider the coupled $N\pi$ and $N^*\pi$ channels, (5.31) shows that the off-diagonal elements of the forces will be very large and attractive. It is possible that they might be sufficient to bind the nucleon. The nucleon would then be essentially an $N^*\pi$ bound state, produced by N exchange forces. This is different from the Chew "bootstrap"[58] in which it is an $N\pi$ bound state supported by N^* exchange. The two effects would probably reinforce each other. Putting the $N+(ABC)$ channel into the equations appears to give still further attraction in this state. However, even if we could bind the nucleon this way, we are still not justified in claiming that there are no elementary particles in the theory, because we have no way of calculating the cutoff. In conventional mass-shell theories, the cutoff function appears as a *deus ex machina*. However, in our theory it has a perfectly comprehensible role. Apart from some kinematic factors, it is just the wave function of the nucleon, considered as a pion-nucleon bound state. A particle whose wave function we are unable to predict, even in principle, can hardly be said to be composite. This should be contrasted with the situation for the N^*, whose wave function we were able to calculate in Sec. 5b, in terms of that of the nucleon. Thus the situation should rather be described as in Low's model[59]—either the N or the N^* must be elementary, but we are free to decide which.

Our theory involves the extension of the S matrix off the energy shell. In relativistic theories this would correspond to an extension off the mass shell. The main point in making such an extension is that we end up with a set of linear integral equations with compact kernels, to which rigorous mathematical analysis can be applied. There is obviously a certain ambiguity in the definition of the off-shell S matrix, especially in the inhomogeneous term, since the compactness restriction makes no demands on this. We exploited this in Sec. 3, making slight changes in the off-shell extension in order to get the equations into a neater form. The essential characteristic of all our extensions is that all the right-hand singularities are in the variable s, while all the left-hand singularities and anomalous thresholds are in the other variables. To solve the equations, we need only know the analytic properties in s. The most that is required in the other variables is certain smoothness and

[54a] *Note added in proof*. There is some difficulty in the extension to four particles, since direct analogs of the Faddeev equations do not exist. Weinberg (Ref. 9) has found an equation with a compact kernel. It is not very convenient for bound-state scattering, but shows the problem is soluble in principle.

[55] S. Mandelstam, Phys. Rev. **112**, 1344 (1958).

[56] B. H. Bransden, P. G. Burke, J. W. Moffat, R. G. Moorhouse, and D. Morgan, Nuovo Cimento **30**, 207 (1963).

[57] This was suggested by J. G. Taylor (Ref. 10 and private communication).

[58] G. F. Chew, Phys. Rev. Letters **9**, 233 (1962).

[59] F. E. Low, Nuovo Cimento **25**, 678 (1962).

boundedness conditions. The fact that s has only right-hand singularities, then makes our task very easy. This extension off-shell is obviously different from that usually used in field theory, and also from that of the original Chew-Low static model.[43] It has the theoretical disadvantage of not being explicitly crossing-symmetric. Against that, there is the big practical advantage of the separation of the singularities. Also we can approximate crossing symmetry without fear of ghost states. Finally, it tells us where the cutoff function comes from, which mass-shell theories almost always have to introduce on the sly.

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APPENDIX: S MATRICES FOR COMPOSITE PARTICLES

Ekstein[32] first pointed out that the usual formulation of scattering theory, with a single S operator whose matrix elements give the S matrix, is incorrect for many channel scattering. It is necessary to define several S operators, in order to prove the asymptotic condition. The situation is worse for three-particle systems, since many plausible S operators can then be written down, and it is by no means obvious which of them satisfy the asymptotic condition. If they do not, then it means that their matrix elements do not give the cross sections that experimentalists observe. This has caused considerable difficulty in the past, in formulating an *exact* set of equations for bound-state scattering, because it is difficult to derive equations for quantities which are not precisely defined. (Of course, as long as we are content with the Born or impulse approximations, there is no trouble.) In order to show that we have really solved this problem, we must prove that our operators $U_{\alpha\beta}{}^{\pm}(s)$ satisfy the asymptotic condition. Fortunately, most of the work has already been done by Faddeev.[4] He has defined a set of S matrices for bound-state and three-particle processes, and proved that they do indeed satisfy the asymptotic condition. The reason for this Appendix is that our S matrices do not appear, at first sight, to be the same as Faddeev's. We want to show that they are identical.

Instead of our operators $U_{\alpha\beta}{}^{\pm}(s)$, defined by (3.2) in terms of the potentials and exact Green's function, Faddeev uses operators defined by

$$M_{\alpha\beta}(s)=\delta_{\alpha\beta}V_\alpha-V_\alpha G(s)V_\beta. \tag{A1}$$

$$W_{\alpha\beta}(s)=M_{\alpha\beta}(s)-\delta_{\alpha\beta}T_\alpha(s). \tag{A2}$$

By summing (A1) over all its indices, we find

$$U_{00}{}^{\pm}(s)=\sum_{\alpha,\beta}M_{\alpha\beta}(s). \tag{A3}$$

This shows immediately that Faddeev's S matrix for three initial and three final particles [Ref. 4, the unnumbered equation after (9.71)] is identical with ours [E, Eq. (136)[25]]. The difficulty comes from processes involving bound states, since Faddeev doesn't define these S matrices directly as matrix elements of any operator.

First we consider an initial bound state plus third particle, and a final state with three free particles. Comparing (3.2) and (A1) gives

$$U_{\alpha 0}{}^{+}(s)=\sum_{\gamma\neq\alpha}\sum_{\beta}M_{\gamma\beta}(s). \tag{A4}$$

The $M_{\alpha\beta}(s)$ satisfy the equations [Ref. 4, Eq. (3.13)]

$$M_{\alpha\beta}(s)=\delta_{\alpha\beta}T_\alpha(s)-T_\alpha(s)G_0(s)\sum_{\gamma\neq\alpha}M_{\gamma\beta}(s). \tag{A5}$$

Comparing with (A2) and (A4), we have

$$\sum_{\beta}W_{\alpha\beta}(s)=-T_\alpha(s)G_0(s)U_{\alpha 0}{}^{+}(s). \tag{A6}$$

The quantity

$$\sum_{\beta}W_{\alpha\beta}(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}_\beta',\mathbf{q}_\beta';s) \tag{A7}$$

will have a pole on the energy shell for bound-state scattering,

$$s=q_\alpha{}^2-E_{\alpha n}. \tag{A8}$$

Faddeev uses the residue at this pole to define the S matrix. By (2.43), (3.11), and (A6), we have

$$\sum_{\beta}W_{\alpha\beta}(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}_\beta',\mathbf{q}_\beta';s)$$
$$=\sum_{n}\int d^3p_\alpha{}''\frac{g_{\alpha n}(\mathbf{p}_\alpha)g_{\alpha n}(\mathbf{p}_\alpha'')U_{\alpha 0}{}^{+}(\mathbf{p}_\alpha'',\mathbf{q}_\alpha,\mathbf{p}',\mathbf{q}';s)}{(q_\alpha{}^2-s-E_{\alpha n})(p_\alpha{}''^2+q_\alpha{}^2-s)}$$
$$+\text{terms without such poles}. \tag{A9}$$

This can be made formally rigorous by arguments analogous to Lemma 5.1 of Ref. 4. The residue at the pole (A8) is therefore

$$g_{\alpha n}(\mathbf{p}_\alpha)\int d^3p_\alpha{}''\frac{g_{\alpha n}(\mathbf{p}_\alpha'')}{p_\alpha{}''^2+E_{\alpha n}}$$
$$\times U_{\alpha 0}{}^{+}(\mathbf{p}_\alpha'',\mathbf{q}_\alpha,\mathbf{p}',\mathbf{q}';q_\alpha{}^2-E_{\alpha n})$$
$$=-g_{\alpha n}(\mathbf{p}_\alpha)\int d^3p_\alpha{}''\psi_{\alpha n}(\mathbf{p}_\alpha'')$$
$$\times U_{\alpha 0}{}^{+}(\mathbf{p}_\alpha'',\mathbf{q}_\alpha,\mathbf{p}',\mathbf{q}';q_\alpha{}^2-E_{\alpha n}), \tag{A10}$$

by (2.44). This shows that our S matrix (3.4) is identical with Faddeev's [defined by the conjugate of the second unnumbered equation after (9.81) of Ref. 4, and Eqs. (5.3), (5.44), and remark before Eq. (9.11) of the same paper].

Finally, we consider the S matrices for bound-state scattering and rearrangement collisions. The equation we need here is

$$W_{\alpha\beta}(s)=(\delta_{\alpha\beta}-1)[T_\alpha(s)G_0(s)T_\beta(s)+T_\alpha(s)G_0(s)V_\beta G_0(s)\times T_\beta(s)]+T_\alpha(s)G_0(s)U_{\alpha\beta}{}^+(s)G_0(s)T_\beta(s), \quad \text{(A11)}$$

which is obtained by iterating (A5). Applying the argument that led to (A10) we find the residue at the double pole on the energy shell

$$s=q_\alpha{}^2-E_{\alpha n}=q_\beta{}^2-E_{\alpha m} \quad \text{(A12)}$$

to be

$$(\delta_{\alpha\beta}-1)\psi_{\alpha n}(p_\alpha)\times\left[g_{\beta m}(\mathbf{p}_\beta')+\int dp_\beta''V_\beta(\mathbf{p}_\beta',\mathbf{p}_\beta'')\psi_{\beta m}(\mathbf{p}_\beta'')\right]+\int d^3p_\alpha\int d^3p_\beta'\psi_{\alpha n}(\mathbf{p}_\alpha)\times U_{\alpha\beta}{}^+(\mathbf{p}_\alpha,\mathbf{q}_\alpha,\mathbf{p}_\beta',\mathbf{q}_\beta';q_\alpha{}^2-E_{\alpha n})\psi_{\beta m}{}^*(\mathbf{p}_\beta'). \quad \text{(A13)}$$

The first term cancels by (2.44). This shows that our S matrices (3.3) are identical with Faddeev's [defined by the third unnumbered equation after (9.81), and (5.3) of Ref. 4].

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Unification of Photoproduction and Electroproduction

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Gauge invariance and the vector nature of the photon are exploited in order to factor expressions for cross sections of photon-induced reactions into a purely kinematical part and a purely dynamical part. Detailed studies of two- and three-body final states are considered and it is shown how this separation into kinematical and dynamical aspects provides a useful and general procedure by which to compare experiment and theory.

I. INTRODUCTION

THE fact that the photon is a zero-mass vector particle coupled to a conserved current allows for a separation, in expressions for the cross section for photon-induced reactions, between purely dynamical aspects and kinematical features such as gauge invariance and the vector nature of the photon. We show explicitly how this separation can be accomplished for an arbitrary photon-induced reaction where the photon can either be real as in photoproduction or virtual as in electroproduction (or mu-production). Detailed discussions of two-body and three-body final states are given. In these experimentally more accessible cases we show that a great deal can be learned about photoproduction by just analyzing the data in terms of the above-mentioned separation into kinematical and dynamical aspects. This is similar to the separation into the electric and magnetic form factors in the case of electron-nucleon scattering, but applied to cross sections rather than matrix elements.

In order to experimentally carry out the separation of photon-induced reactions into its kinematical and dynamical aspects it is necessary to perform either "coincidence" electroproduction experiments (simultaneous observation of the scattered electron and produced strong particles) or experiments using polarized photons. In the work presented here we are primarily interested in comparing photoproduction and electroproduction and thus consider principally electroproduction in the region of small photon mass.

In dealing with photoproduction and electroproduction at high energies (energies greater than approximately 1 BeV) the question arises as to what is a convenient and useful procedure for analyzing the data. For energies greater than 1 BeV, the dispersion theoretic treatment of Chew, Low, Goldberger, and Nambu[1] is not expected to hold and furthermore one would expect many multipoles to be contributing to photoproduction processes so that a multipole analysis of the data appears quite complicated and lengthy.

On the other hand, theoretical studies of photoproduction at high energies are often made in terms of simple models, e.g., one pion exchange. It therefore appears useful to have a completely general description of photoproduction which at the same time can be easily accommodated to testing ideas and models concerned with these processes. The separation into kinematical and dynamical aspects affords just such a

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[1] G. Chew, M. Goldberger, F. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).