Electron-Proton Coincidence Cross Section for He^3 and H^3†

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The cross section is given in the impulse approximation for inelastic e-He^3 and e-H^3 scattering assuming an ejected proton is counted in coincidence with the scattered electron. The process $e+He^3 \rightarrow d+p+e'$ is considered in detail. This cross section is evaluated for the Gaussian, Irving, and Irving-Gunn three-body wave functions, the deuteron being described by a Hulthén wave function. The best agreement with the preliminary experimental results is obtained using the Irving-Gunn wave function.

I. INTRODUCTION

THE structure of the three-nucleon systems He^3 and H^3 has been the subject of much recent experimental and theoretical investigation. Elastic scattering of high-energy electrons from these nuclei has been used to measure the charge and magnetic moment form factors of both He^3 and H^3.[1,2] Theoretical analysis of these form factors has given new insight regarding the wave function for the three-nucleon system.[3–5] In addition to elastic electron scattering, recent experiments on the photodisintegration of He^3 also give information on the structure of the three-nucleon system.[6] The purpose of the present paper is to show how the coincidence cross section for inelastic scattering of high-energy electrons from He^3 and H^3 may be used as a further sensitive test of the three-nucleon wave function.[7]

The three processes we wish to consider are

$$e+He^3 \rightarrow d+p+e'\,, \tag{1a}$$

$$e+He^3 \rightarrow (n+p)_{J=0}+p+e'\,, \tag{1b}$$

$$e+H^3 \rightarrow (n+n)_{J=0}+p+e'\,. \tag{1c}$$

The inelastically scattered electron and the ejected proton are to be measured in coincidence. We treat this process in the impulse approximation, keeping only those terms corresponding to the electron interacting with the ejected proton. The electron-proton interaction is treated in Born approximation using the effective Hamiltonian given by McVoy and Van Hove.[8] Interactions of the ejected proton with the final two-nucleon system are neglected, so that the relative motion of the proton and two-nucleon system is described by a plane wave.

In choosing an initial nuclear wave function we make use of a classification of the possible $T=\frac{1}{2}$, $J=\frac{1}{2}^+$ states of the three-nucleon system, given by Derrick and Blatt.[9] There are three possible ${}^2S_{1/2}$ states, one of which is symmetric in the interchange of the space coordinates of any pair of nucleons (this is the dominant state which we denote by S), one of which is space-antisymmetric, and one of which has mixed symmetry (denoted by S'). In addition, there are three ${}^2P_{1/2}$ states, one ${}^4P_{1/2}$ state, and three ${}^4D_{1/2}$ states. In Secs. II–IV of this paper, we shall be concerned with the contributions of the dominant S state. It is believed that the antisymmetric S state and the four P states are not present in the ground-state wave functions to any appreciable extent.[10] The effect of small admixtures of the S', P, and D states which are thought to be present will be discussed in Sec. V.

II. ANALYSIS

In this section we derive a formula for the cross section for inelastic electron scattering from He^3 and H^3 with the detection in coincidence of an ejected proton. For definiteness we consider the process $e+He^3 \rightarrow d+p+e'$ shown in Fig. 1. Since we treat this process in Born approximation, the incident and final electrons are described by plane-wave Dirac spinors. We use the impulse approximation, keeping only the terms corresponding to the electron interacting with the ejected proton. The effective Hamiltonian for the interaction between the electron and proton, to order q^2/M^2, can

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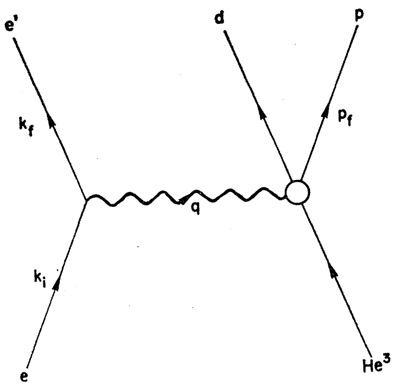

FIG. 1. Inelastic electron scattering.

be written in the form[8]

$$H'=-\frac{4\pi e^2}{q^2}\langle u_f|\sum_{j=1}^{3}\tfrac{1}{2}[1+\tau_3(j)]\Big\{F_{1p}(q^2)e^{-iq\cdot x_j}$$
$$-\frac{F_{1p}(q^2)}{2M}[(\mathbf{p}_j\cdot\boldsymbol{\alpha})e^{-iq\cdot x_j}+e^{-iq\cdot x_j}(\mathbf{p}_j\cdot\boldsymbol{\alpha})]$$
$$-i\left[\frac{F_{1p}(q^2)+\kappa_pF_{2p}(q^2)}{2M}\right]\boldsymbol{\sigma}_j\cdot(\mathbf{q}\times\boldsymbol{\alpha})e^{-iq\cdot x_j}$$
$$+\frac{q^2}{8M^2}[F_{1p}(q^2)+2\kappa_pF_{2p}(q^2)]e^{-iq\cdot x_j}\Big\}|u_i\rangle. \quad (2)$$

The notation used is as follows: k_i and k_f are the initial and final electron four momenta; $q^2=(k_i-k_f)^2$ is the four-momentum transfer to the proton; F_{1p} and F_{2p} are the Dirac and Pauli form factors of the proton; $\boldsymbol{\alpha}$ is the Dirac matrix which operates on the free-electron spinors u_i and u_f; $\mathbf{p}$ and $\boldsymbol{\sigma}$ are the momentum and spin operators for the proton, κ_p is the anomalous moment of the proton in nuclear magnetons, and M is the nucleon mass. We use units in which $\hbar=c=1$ and the metric $a\cdot b=a_0b_0-\mathbf{a}\cdot\mathbf{b}$.

It is convenient to write the matrix element for the reaction in the form

$$\mathfrak{M}=-(4\pi e^2/q^2)[\langle u_f|u_i\rangle Q-\langle u_f|\boldsymbol{\alpha}|u_i\rangle\cdot\mathbf{J}], \quad (3)$$

where

$$Q=\left[F_{1p}+\frac{q^2}{8M^2}(F_{1p}+2\kappa_pF_{2p})\right]$$
$$\times\langle\psi_f|\sum_{j=1}^{3}\tfrac{1}{2}[1+\tau_3(j)]e^{i\mathbf{q}\cdot\mathbf{x}_j}|\psi_i\rangle, \quad (4a)$$

$$\mathbf{J}=\langle\psi_f|\sum_{j=1}^{3}\tfrac{1}{2}[1+\tau_3(j)]\Big\{\frac{F_{1p}}{2M}(\mathbf{p}_je^{i\mathbf{q}\cdot\mathbf{x}_j}+e^{i\mathbf{q}\cdot\mathbf{x}_j}\mathbf{p}_j)$$
$$+i[(F_{1p}+\kappa_pF_{2p})/2M]e^{i\mathbf{q}\cdot\mathbf{x}_j}(\boldsymbol{\sigma}_j\times\mathbf{q})\Big\}|\psi_i\rangle \quad (4b)$$

with ψ_i and ψ_f the initial and final nuclear wave functions.

After squaring the matrix element and summing over electron spins we obtain

$$\frac{1}{2}\sum_{\substack{\text{electron}\\ \text{spins}}}|\mathfrak{M}|^2=\frac{(4\pi e^2)^2}{q^4}\{4E_iE_f|Q|^2-2E_fQ^*\mathbf{k}_i\cdot\mathbf{J}$$
$$-2E_iQ\mathbf{k}_f\cdot\mathbf{J}^*-2E_iQ^*\mathbf{k}_f\cdot\mathbf{J}-2E_fQ\mathbf{k}_i\cdot\mathbf{J}^*$$
$$+2\mathbf{k}_i\cdot\mathbf{J}\mathbf{k}_f\cdot\mathbf{J}^*+2\mathbf{k}_i\cdot\mathbf{J}^*\mathbf{k}_f\cdot\mathbf{J}+q^2|Q|^2-q^2\mathbf{J}\cdot\mathbf{J}^*\}. \quad (5)$$

The evaluation of the coincidence cross section is then reduced to the evaluation of the nuclear matrix elements Q and $\mathbf{J}$, which depend on the choice of initial and final nuclear wave functions. In choosing these wave functions we will be guided to some extent by the previous results on elastic electron scattering and photodisintegration of He^3.

For the initial three-body wave function ψ_i we closely follow the notation of Schiff[5] and write the dominant S-state wave function as

$$\psi_i(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=u(r_{12},r_{13},r_{23})\phi_0, \quad (6)$$

where the spatial wave function u is completely symmetric under the interchange of any pair of nucleons. The spin-isospin function ϕ_0 is defined to be

$$\phi_0=[\chi_2(1,2,3)\eta_1(1,2,3)-\chi_1(1,2,3)\eta_2(1,2,3)]/\sqrt{2}, \quad (7)$$

where the doublet spin functions are given by

$$\chi_1(1,2,3)=\sum_{m_1}(\tfrac{1}{2}1m_1m-m_1|\tfrac{1}{2}1\tfrac{1}{2}m)\chi_1^{m-m_1}(2,3)\chi_{1/2}^{m_1}(1), \quad (8a)$$

$$\chi_2(1,2,3)=\chi_0^0(2,3)\chi_{1/2}^m(1). \quad (8b)$$

The doublet isospin functions η are defined similarly.

In the final three-body wave function we describe the ejected proton by a plane wave. That is, we neglect the final-state interactions between this proton and the other nucleons. The residual two-nucleon system we assume to be left in either the ${}^3S_1$ or ${}^1S_0$ state. The final wave function is then

$$\psi_f(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\sqrt{3}\varphi_J(r_{23})\exp[-i\mathbf{p}_i\cdot(\mathbf{r}_2+\mathbf{r}_3)/2+i\mathbf{p}_f\cdot\mathbf{r}_1]$$
$$\times\chi_J(2,3)\chi_{1/2}(1)\eta_T(2,3)\eta_{1/2}(1), \quad (9)$$

where $\mathbf{p}_f$ is the momentum of the ejected proton and $\mathbf{q}=\mathbf{k}_i-\mathbf{k}_f$. The momentum of the proton in the initial state, which is the negative of the total momentum of the recoiling two-nucleon system, is denoted by $\mathbf{p}_i$. It is not necessary to explicitly antisymmetrize the final wave function, since the interaction and the initial state have the correct symmetry. The effect of antisymmetrizing the final state is to introduce the factor $\sqrt{3}$ in Eq. (9). The generalized Pauli principle requires that $J+T=1$.

Using these wave functions to evaluate the nuclear matrix elements Q and $\mathbf{J}$, the coincidence cross section for the three processes may be written, after some algebraic manipulation, in the form:

I. $e+\mathrm{He}^3 \to d+p+e'$
$$d^3\sigma/dE_f d\Omega_f d\Omega_p = \tfrac{3}{2}\sigma_0 |I_1|^2; \tag{10a}$$

II. $e+\mathrm{He}^3 \to (n+p)_{J=0}+p+e'$
$$d^3\sigma/dE_f d\Omega_f d\Omega_p = \tfrac{1}{2}\sigma_0 |I_0|^2; \tag{10b}$$

III. $e+\mathrm{H}^3 \to (n+n)_{J=0}+p+e'$
$$d^3\sigma/dE_f d\Omega_f d\Omega_p = \sigma_0 |I_0|^2; \tag{10c}$$

where

$$\sigma_0=\sigma_{\mathrm{Mott}}\frac{|\mathbf{p}_f|(\mathbf{p}_f^2+M^2)^{1/2}}{(2\pi)^3|\mathbf{k}_i/E_i-\mathbf{p}_i/M|}\Big\{F_{1p}^2-\frac{q^2}{4M^2}\kappa_p^2F_{2p}^2 +\frac{\mathbf{q}^2}{2M^2}\tan^2\frac{\theta}{2}(F_{1p}+\kappa_pF_{2p})^2+\frac{F_{1p}^2}{4M^2}\tan^2\frac{\theta}{2}(2\mathbf{p}_f-\mathbf{q})^2 +\frac{F_{1p}^2}{4M^2}\sec^2\frac{\theta}{2}[\hat{k}_i\cdot(2\mathbf{p}_f-\mathbf{q})\hat{k}_f\cdot(2\mathbf{p}_f-\mathbf{q}) -2M(\hat{k}_i+\hat{k}_f)\cdot(2\mathbf{p}_f-\mathbf{q})]\Big\}, \tag{11}$$

and

$$I_J(\mathbf{p}_i)=\int d^3\rho\int d^3r\,\varphi_J(\boldsymbol{\varrho})\exp[i(\mathbf{q}-\mathbf{p}_f)\cdot\mathbf{r}]u(\boldsymbol{\varrho},\mathbf{r}). \tag{12}$$

The vectors $\boldsymbol{\varrho}$ and $\mathbf{r}$ are related to r_{12}, r_{13}, and r_{23} through the equations $\mathbf{r}_{23}=\boldsymbol{\varrho}$, $\mathbf{r}_{12}=\mathbf{r}-\boldsymbol{\varrho}/2$, $\mathbf{r}_{13}=\mathbf{r}+\boldsymbol{\varrho}/2$, while

$$\sigma_{\mathrm{Mott}}=e^4\cos^2(\theta/2)/4E_i^2\sin^4(\theta/2). \tag{13}$$

The kinetmatical relations among the parameters appearing in Eqs. (9) to (13) are given in Appendix A.

Note that since $\mathbf{q}-\mathbf{p}_f=\mathbf{p}_i$ is the initial momentum of the ejected proton, the cross-section factors into the cross section for scattering from a proton of momentum $\mathbf{p}_i$, times the probability of finding a proton with momentum $\mathbf{p}_i$ in the initial nucleus. The angular distribution of the coincidence proton clearly provides a sensitive test of the initial three-body wave function.

III. ANALYTICAL RESULTS

In this section we give analytical results for the process $e+\mathrm{He}^3\to d+p+e'$ for some specific wave functions. We choose to discuss this process in detail since the deuteron wave function is relatively well known, allowing the three-body system to be investigated without the additional complications arising from uncertainties in one's understanding of the two-body system. We describe the deuteron by the Hulthén wave function[11]

$$\varphi_1(\boldsymbol{\varrho})=[N/(4\pi)^{1/2}](e^{-a\rho}-e^{-b\rho})/\rho, \tag{14}$$

where $a=45.8$ MeV, $b=285$ MeV, and the normalization constant is

$$N=[2ab(a+b)]^{1/2}/(b-a). \tag{15}$$

[11] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39.

The three-body wave functions we shall consider are the Gaussian and Irving[12] wave functions, which were used by Schiff[5] in his analysis of elastic $e-\mathrm{He}^3$ and $e-\mathrm{H}^3$ scattering, and the Irving-Gunn wave function,[13] which Berman, Koester, and Smith[6] used in their analysis of the photodisintegration of He³. The techniques used to perform the required integrations analytically are given in Appendix B.

A. Gaussian Wave Function

The spatially symmetric Gaussian wave function is

$$u(r_{12},r_{23},r_{13})=A\exp[-\tfrac{1}{2}\alpha^2(r_{12}^2+r_{23}^2+r_{13}^2)] =A\exp[-\alpha^2(r^2+3\rho^2/4)], \tag{16}$$

where the normalization constant A is given by

$$A=3^{3/4}\alpha^3/\pi^{3/2}. \tag{17}$$

The required integral $I_1(\mathbf{p}_i)$ is straightforward and turns out to be

$$I_1(\mathbf{p}_i)=\frac{4\pi^{5/2}AN}{3\alpha^5}\left\{\frac{a}{\sqrt{3}\alpha}e^{a^2/3\alpha^2}\left[1-\Phi\left(\frac{a}{\sqrt{3}\alpha}\right)\right] -\frac{b}{\sqrt{3}\alpha}e^{b^2/3\alpha^2}\left[1-\Phi\left(\frac{b}{\sqrt{3}\alpha}\right)\right]\right\}e^{-p_i^2/4\alpha^2}. \tag{18}$$

In Eq. (18), $\Phi(x)$ is the error function defined by

$$\Phi(x)=\frac{2}{\pi^{1/2}}\int_0^x e^{-y^2}dy. \tag{19}$$

B. Irving Wave Function

The spatially symmetric Irving wave function is defined to be

$$u(r_{12},r_{23},r_{13})=A\exp[-\tfrac{1}{2}\alpha(r_{12}^2+r_{23}^2+r_{13}^2)^{1/2}] =A\exp[-\tfrac{1}{2}\alpha(2r^2+3\rho^2/2)^{1/2}], \tag{20}$$

where the normalization constant A is given by

$$A=3^{3/4}\alpha^3/(120)^{1/2}\pi^{3/2}. \tag{21}$$

In this case, using the techniques given in Appendix B, one finds the integral $I_1(\mathbf{p}_i)$ to be

$$I_1(\mathbf{p}_i)=\frac{2560\pi^{3/2}(b^2-a^2)AN}{\sqrt{3}\alpha^6} \times\int_0^\infty\frac{k^2dk}{(a^2+k^2)(b^2+k^2)[1+(8k^2/3\alpha^2)+(2p_i^2/\alpha^2)]^{7/2}}. \tag{22}$$

While the integral in Eq. (22) can be evaluated analytically, it was found to be more convenient to compute it

[12] J. Irving, Phil. Mag. 42, 338 (1951).
[13] J. C. Gunn and J. Irving, Phil. Mag. 42, 1353 (1951).

numerically, owing to the complexity of the analytical result.

C. Irving-Gunn Wave Function

The spatially symmetric Irving-Gunn wave function is

$$u(r_{12},r_{13},r_{23}) = A \exp[-\tfrac{1}{2}\alpha(r_{12}^2+r_{23}^2+r_{13}^2)^{1/2}]/[r_{12}^2+r_{13}^2+r_{23}^2]^{1/2}$$
$$= A \exp[-\tfrac{1}{2}\alpha(2r^2+3\rho^2/2)^{1/2}]/[2r^2+3\rho^2/2]^{1/2}, \quad (23)$$

with

$$A = 3^{1/4}\alpha^2/\sqrt{2}\pi^{3/2}. \quad (24)$$

The required integral in this case is evaluated as discussed in Appendix B. The result is

$$I_1(\mathbf{p}_i) = \frac{256\pi^{3/2}(b^2-a^2)AN}{\sqrt{3}\alpha^5} \times \int_0^\infty \frac{k^2dk}{(a^2+k^2)(b^2+k^2)[1+(8k^2/3\alpha^2)+(2p_i^2/\alpha^2)]^{5/2}}. \quad (25)$$

Again the remaining integration can be performed analytically, but we leave it in this form for computational convenience.

IV. NUMERICAL RESULTS

We have evaluated the coincidence cross section numerically for the three-body wave functions described above. The experimental conditions chosen for the calculation were those at which recent data have been obtained,[14] namely $E_i=549.1$ MeV, $E_f=443.4$ MeV, and $\theta=51.68$ deg. The corresponding four-momentum transfer is $q^2=-4.75$ F^{-2}. Kinematic relations useful in performing the computations are given in Appendix A. The results are shown in Fig. 2, where the coincidence cross section is given as a function of θ_p, the angle between the ejected proton momentum and the incident electron beam. The values of the parameter α for the three cases were the following: For the Gaussian $\alpha=75.9$ MeV, for the Irving $\alpha=250$ MeV, and for the Irving-Gunn $\alpha=152$ MeV. For the Gaussian and the Irving wave function these values are those found by Schiff in his analysis of the elastic $e-He^3$ and $e-H^3$ experiments.[5] In the case of the Irving-Gunn, we use the value of α, found in the analysis of the photodisintegration of He^3, which was also found to fit the charge form factor and the Coulomb energy of He^3.[6]

Recent measurements of the $e-He^3$ coincidence cross section[14] are indicated in Fig. 2 for comparison with the cross sections calculated for the completely symmetric Gaussian, Irving, and Irving-Gunn wave functions. Comparing the calculations with the experimental

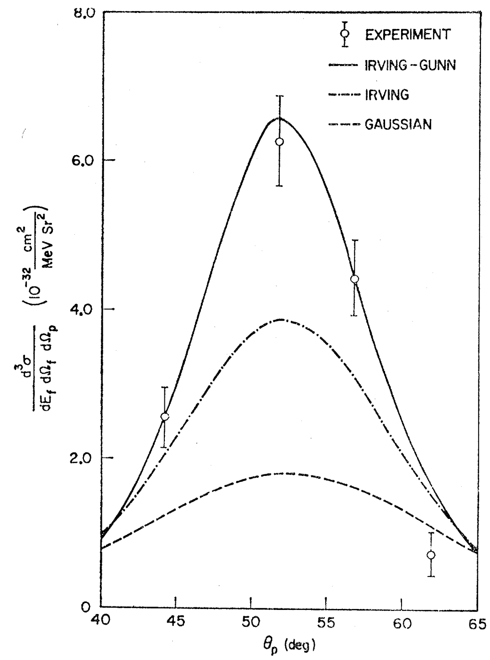

FIG. 2. The cross section $d^3\sigma/dE_fd\Omega_fd\Omega_p$ for the process $e+He^3 \to d+p+e'$ as a function of the proton scattering angle θ_p for the conditions $E_i=549.1$ MeV, $E_f=443.4$ MeV, and $\theta=51.68$ deg. The curves shown are the results obtained using Gaussian, Irving, and Irving-Gunn three-body wave functions having parameters α of 75.9, 250, and 152 MeV, respectively. The normalization is absolute.

results, one sees that the Irving-Gunn wave function gives an adequate fit to the data. The Irving and Gaussian wave functions give rather poor fits, the Gaussian being considerably worse than the Irving.

V. ADMIXTURES IN THE THREE-BODY WAVE FUNCTION

The calculations shown in Fig. 2 include only the contribution of the dominant S state of the He^3 wave function, the contribution of the other nine possible states being neglected. Before any definite conclusions may be reached regarding the coordinate dependence of the wave function, it is important to know the contribution of these other states.

Of the ten possible states which can be present, variational calculations of the binding energy of the triton indicate that the fully antisymmetric S state and the four P states are not present in the wave function to any appreciable extent.[10] These same calculations suggest that the total D-state probability may be of the order of a few percent, while the probability of the S' state was found to be of the order of or less than 1%.

In the analysis of the elastic electron scattering on He^3 and H^3, Schiff found that the difference in the charge form factors of He^3 and H^3 could be explained by an admixture of the S' state of the order of 4%.[5]

Although these additional states are present with, at most, a few percent probability, the square of the matrix

[14] A. Johansson (to be published).

element can contain an interference with the S state proportional to the amplitude, which can be important. For reasons given below only the S' state contributes an important interference term of this type.

The S state and that P state which have coordinate wave functions which are completely antisymmetric clearly cannot contribute to the matrix element since the final-state deuteron coordinate function is symmetric. Moreover, the remaining three P states cannot contribute to the matrix element. Their wave functions are proportional to $\mathbf{r}\times\boldsymbol{\rho}$, which leads to a vanishing matrix element when integrated over the azimuthal angle of either $\mathbf{r}$ or $\boldsymbol{\rho}$.

Although the contribution to the matrix element from the three D states does not vanish, its interference with the S-state matrix element vanishes when averaged over spins, assuming the target is unpolarized and neglecting the magnetic part of the interaction. Consequently, to order q^2/M^2 times the D-state amplitude, there is no contribution to the interference allowing the D state to be neglected also.

The remaining state, called S', does indeed contribute an interference term with the S state and cannot be ignored is present with a probability of a few percent. Following the notation of Schiff,[5] the three-body wave function including the S' state is

$$\psi_i(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=\cos\delta\, u(r_{12},r_{13},r_{23})\phi_0 +\sin\delta[v_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\phi_1-v_1(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)\phi_2], \quad (26)$$

where

$$\phi_1=(\chi_2\eta_2-\chi_1\eta_1)/\sqrt{2}, \quad (27a)$$

$$\phi_2=(\chi_1\eta_2+\chi_2\eta_1)/\sqrt{2}, \quad (27b)$$

ϕ_0, $\chi_{1,2}$, and $\eta_{1,2}$ being defined by Eqs. (7) and (8). The coordinate functions v_1 and v_2 can be written in terms of a single function $g(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)$ which is symmetric in its first two arguments.

$$v_1(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=[g(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)+g(\mathbf{r}_1,\mathbf{r}_3;\mathbf{r}_2) -2g(\mathbf{r}_2,\mathbf{r}_3;\mathbf{r}_1)]/\sqrt{6}, \quad (28a)$$

$$v_2(\mathbf{r}_1,\mathbf{r}_2,\mathbf{r}_3)=[g(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)-g(\mathbf{r}_1,\mathbf{r}_3;\mathbf{r}_2)]/\sqrt{2}. \quad (28b)$$

We have separately normalized the S and S' parts of ψ_i so that $\sin^2\delta$ is the S'-state probability. Using Eq. (26) for the initial wave function modifies the cross sections given in Eqs. (9) to read

I. $e+He^3\to d+p+e'$
$$d^3\sigma/dE_fd\Omega_fd\Omega_p=\tfrac{3}{2}\sigma_0|\cos\delta I_1+\sin\delta I_1'|^2; \quad (29a)$$

II. $e+He^3\to (n+p)_{J=0}+p+e'$
$$d^3\sigma/dE_fd\Omega_fd\Omega_p=\tfrac{1}{2}\sigma_0|\cos\delta I_0-\sin\delta I_0'|^2; \quad (29b)$$

III. $e+H^3\to (n+n)_{J=0}+p+e'$
$$d^3\sigma/dE_fd\Omega_fd\Omega_p=\sigma_0|\cos\delta I_0-\sin\delta I_0'|^2. \quad (29c)$$

In similar manner to Eq. (11) we define the S' integrals to be

$$I_J'(\mathbf{p}_i)=\int d^3\rho\int d^3r\,\varphi_J(\boldsymbol{\rho})\exp[i(\mathbf{q}-\mathbf{p}_f)\cdot\mathbf{r}]v_1(\boldsymbol{\rho},\mathbf{r}). \quad (30)$$

Numerical calculations for the process $e+He^3\to d+p+e'$ considered above have been carried out including the S' contribution in the case of the Irving and Irving-Gunn wave function. As can be seen from Fig. 2 there is little point in considering the Gaussian wave function further. For the Irving and Irving-Gunn wave functions the function g is given by

$$g(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)=B\exp[-\tfrac{1}{2}(\alpha^2r_{13}^2+\alpha^2r_{23}^2+\beta^2r_{12}^2)^{1/2}], \quad (31a)$$

and

$$g(\mathbf{r}_1,\mathbf{r}_2;\mathbf{r}_3)=\frac{B\exp[-\tfrac{1}{2}(\alpha^2r_{13}^2+\alpha^2r_{23}^2+\beta^2r_{12}^2)^{1/2}]}{(r_{12}^2+r_{13}^2+r_{23}^2)^{1/2}}, \quad (31b)$$

respectively. We assume β is not too different from α so that only lowest order terms in $\beta-\alpha$ need be retained. In this approximation the normalization constants B are

$$B=(\sqrt{3}/35\pi^3)^{1/2}[\alpha^4/(\alpha-\beta)], \quad (32a)$$

for the Irving wave function, and

$$B=(6\sqrt{3}/5\pi^3)^{1/2}[\alpha^3/(\alpha-\beta)], \quad (32b)$$

for the Irving-Gunn wave function.

Using techniques similar to those given in Appendix B, one finds the integrals I_1' to be

$$I_1'(\mathbf{p}_i)=\frac{71680\pi^2NB(\alpha-\beta)(b^2-a^2)}{9\sqrt{2}\alpha^9}\times\int_0^\infty\frac{(3p_i^2+4k^2)k^2dk}{(a^2+k^2)(b^2+k^2)[1+(8k^2/3\alpha^2)+(2p_i^2/\alpha^2)]^{9/2}}, \quad (33a)$$

for the Irving wave function and

$$I_1'(\mathbf{p}_i)=\frac{512\pi^2NB(\alpha-\beta)(b^2-a^2)}{3\sqrt{2}\alpha^6}\int_0^\infty\frac{k^2}{(a^2+k^2)(b^2+k^2)}\cdot\left[\frac{16}{Q^4[1+Q^2+(1+Q^2)^{1/2}]}-\frac{5}{(1+Q^2)^{7/2}}-\frac{6}{Q^2(1+Q^2)^{5/2}}-\frac{8}{Q^4(1+Q^2)^{3/2}}\right]dk, \quad (33b)$$

for the Irving-Gunn wave function where

$$Q^2=(8k^2+6p_i^2)/(3\alpha^2). \quad (34)$$

The effect on the coincidence cross section of a 4% ($\sin\delta=0.2$) S' state as indicated by the analysis of the elastic-scattering data[5] is shown in Figs. 3 and 4. The experimental conditions are the same as above, i.e., $E_i=549.1$ MeV, $E_f=443.4$ MeV, and $\theta=51.68$ deg.

The experimental data indicated in the figures are those of Johansson.[14] From Figs. 3 and 4 it is evident that such a large admixture of S' state does not improve the agreement in the case of the Irving wave function and clearly destroys the agreement in the case of the Irving-Gunn wave function.

VI. SUMMARY

In conclusion we wish to emphasize that the coincidence cross section provides a sensitive means for investigating the three-body wave function. The present calculations when compared with the experimental data indicate that the Irving-Gunn wave function is somewhat better than the Irving wave function and that the Gaussian wave function is a rather poor approximation. It is consistent with this approximate calculation to neglect the effects of admixtures of states other than the dominant S state amounting to a few percent, with the exception of the S' state. The results of including the S' state indicate that an admixture as large as 4% is inconsistent with the present data. This conclusion is corroborated by recent calculations of the slow neutron capture rate on deuterium.[15]

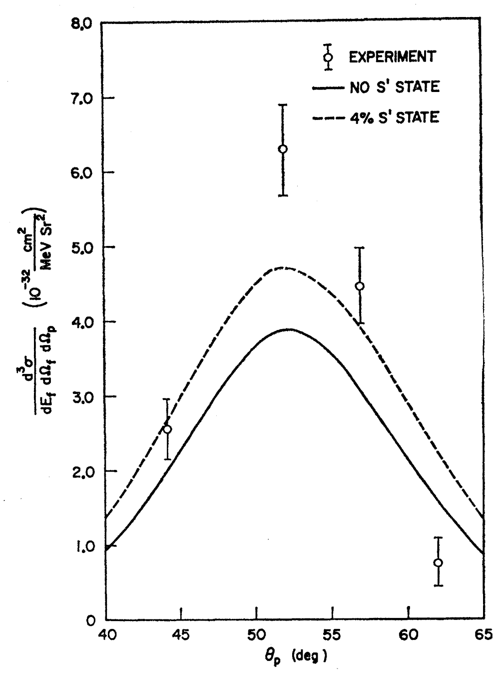

FIG. 3. The cross section $d^3\sigma/dE_f d\Omega_f d\Omega_p$ for the process $e+\mathrm{He}^3\rightarrow d+p+e'$ as a function of the proton scattering angle θ_p for the conditions $E_i=549.1$ MeV, $E_f=443.4$ MeV, and $\theta=51.68$ deg. The curves are the results obtained using the Irving wave function ($\alpha=250$ MeV) with no S' state and with a 4% admixture.

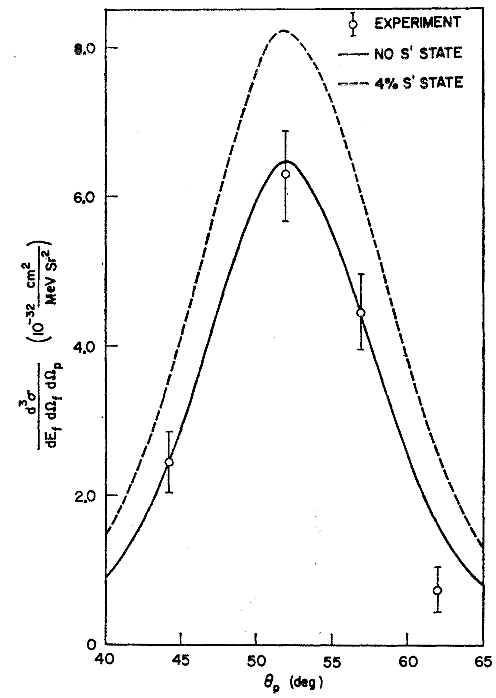

FIG. 4. The cross section $d^3\sigma/dE_f d\Omega_f d\Omega_p$ for the process $e+\mathrm{He}^3\rightarrow d+p+e'$ as a function of the proton scattering angle θ_p for the conditions $E_i=549.1$ MeV, $E_f=443.4$ MeV, and $\theta=51.68$ deg. The curves are the results obtained using the Irving-Gunn wave function ($\alpha=152$ MeV) with no S' state and with a 4% admixture.

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APPENDIX A: KINEMATICS

We denote the initial and final electron (proton) momenta in the laboratory by $\mathbf{k}_i$ and $\mathbf{k}_f$ ($\mathbf{p}_i$ and $\mathbf{p}_f$), respectively, so that momentum conservation requires

$$\mathbf{p}_i+\mathbf{k}_i=\mathbf{p}_f+\mathbf{k}_f. \tag{A1}$$

Notice that $-\mathbf{p}_i$ is also the total momentum of the recoiling two-nucleon system. Denoting the initial and final electron energies by E_i and E_f, respectively, we have

$$E_i-E_B=E_f+(\mathbf{p}_f^2/2M)+(\mathbf{p}_i^2/2M_d), \tag{A2}$$

where the binding energy $E_B=M+M_d-M_{\mathrm{He}^3}$. We define the angles θ, θ_p, and Θ such that $\cos\theta=\hat{k}_i\cdot\hat{k}_f$, $\cos\theta_p=\hat{k}_i\cdot\hat{p}_f$, and $\cos\Theta=\hat{p}_f\cdot\hat{k}_f$. The kinematic relations useful in evaluating the cross section are the following:

$$q^2=(k_i-k_f)^2=-4E_iE_f\sin^2(\theta/2), \tag{A3}$$

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[15] N. T. Meister, T. K. Radha, and L. I. Schiff, Phys. Rev. Letters **12**, 509 (1964).

$$p_f^2[1+(M_d/M)]+2p_f(k_f\cos\Theta-k_i\cos\theta_p)+\mathbf{q}^2+2M_d(E_B-E_i+E_f)=0\,, \quad (A4)$$

$$\hat{k}_i\cdot(2\mathbf{p}_f-\mathbf{q})=2p_f\cos\theta_p+k_f\cos\theta-k_i\,, \quad (A5)$$

$$\hat{k}_f\cdot(2\mathbf{p}_f-\mathbf{q})=2p_f\cos\Theta-k_i\cos\theta+k_f\,, \quad (A6)$$

$$(2\mathbf{p}_f-\mathbf{q})^2=4p_f^2+\mathbf{q}^2-4p_f(k_i\cos\theta_p-k_f\cos\Theta)\,. \quad (A7)$$

Throughout we neglect the rest mass of the electron. In the case when the incident electron, scattered electron, and ejected proton are coplanar, which corresponds to the experimental conditions considered in the text, we have simply $\Theta=\theta+\theta_p$.

APPENDIX B: EVALUATION OF INTEGRALS

The integral required in the calculation of $I_1(\mathbf{p})$ for the Gaussian wave function is

$$I_1(\mathbf{p})=AN\int d^3\rho\int d^3r\,\exp[i\mathbf{p}\cdot\mathbf{r}-\alpha^2r^2-\tfrac{3}{4}\alpha^2\rho^2]\times[(e^{-a\rho}-e^{-b\rho})/\rho]\,, \quad (B1)$$

where the normalization factors A and N are given in the text. This factors into the product of tabulated integrals giving as a result Eq. (18) of the text.

In the case of Irving and Irving-Gunn wave functions, the required integral is

$$I_1(\mathbf{p})=AN\int d^3\rho\int d^3r\frac{\exp[i\mathbf{p}\cdot\mathbf{r}-\frac{1}{2}\alpha(2r^2+\frac{3}{2}\rho^2)^{1/2}]}{(2r^2+\frac{3}{2}\rho^2)^{n/2}}\times\left(\frac{e^{-a\rho}-e^{-b\rho}}{\rho}\right), \quad (B2)$$

where n has the values 0 and 1 for the Irving and Irving-Gunn, respectively. The normalization constant A is given by either Eq. (21) or Eq. (24), whichever is appropriate. To evaluate this integral we follow the method of Schiff[5] and introduce the Fourier transform of the deuteron wave function.

$$\frac{1}{(2\pi)^3}\int d^3x e^{i\mathbf{k}\cdot\mathbf{x}}\varphi(\mathbf{x})=\frac{N(b^2-a^2)}{4\pi^{5/2}(a^2+k^2)(b^2+k^2)}\,. \quad (B3)$$

The integral then becomes

$$I_1(\mathbf{p})=\frac{AN(b^2-a^2)}{4\pi^{5/2}}\int\frac{d^3\mathbf{k}}{(a^2+k^2)(b^2+k^2)}\int d^3\rho\int d^3r\cdot\frac{\exp[i(\mathbf{p}\cdot\mathbf{r}+\mathbf{k}\cdot\boldsymbol{\rho})-\frac{1}{2}\alpha(2r^2+\frac{3}{2}\rho^2)^{1/2}]}{(2r^2+\frac{3}{2}\rho^2)^{n/2}}\,. \quad (B4)$$

Next, one transforms the two three-dimensional integrals over $\boldsymbol{\rho}$ and $\mathbf{r}$ into one six-dimensional integral with the substitutions

$$R_{1,2,3}=(\tfrac{3}{8})^{1/2}\alpha\boldsymbol{\rho}\,;\quad R_{4,5,6}=(\tfrac{1}{2})^{1/2}\alpha\mathbf{r} \quad (B5a)$$

and

$$Q_{1,2,3}=(8/3)^{1/2}(1/\alpha)\mathbf{p}\,;\quad Q_{4,5,6}=(\sqrt{2}/\alpha)\mathbf{k}\,. \quad (B5b)$$

The six-dimensional integral can then be written as

$$I_1(\mathbf{p})=\frac{AN(b^2-a^2)}{4\pi^{5/2}}\int\frac{d^3\mathbf{k}}{(a^2+k^2)(b^2+k^2)}\cdot\frac{64\alpha^n}{3^{3/2}\alpha^6 2^n}\int\frac{\exp[i\mathbf{Q}\cdot\mathbf{R}-R]}{R^n}d^6R\,. \quad (B6)$$

The angular part of the six-dimensional integral may be performed by expanding the plane wave in Gegenbauer polynomials and using their integral properties.[16] The result is

$$\int\frac{\exp[i\mathbf{Q}\cdot\mathbf{R}-R]}{R^n}d^6R=\frac{8\pi^3}{Q^2}\int_0^\infty e^{-R}J_2(QR)R^{3-n}dR\,. \quad (B7)$$

For n, either 0 or 1, the radial integral may be written in terms of elementary functions, while more generally the result may be expressed in terms of hypergeometric functions.[17] We give the result only for n equal to 0 or 1 since these are the cases of interest here.

$$\int_0^\infty e^{-R}J_2(QR)R^{3-n}dR=\frac{2^{3-n}Q^2\Gamma(\frac{7}{2}-n)}{\pi^{1/2}(1+Q^2)^{(7-2n)/2}}\,. \quad (B8)$$

Combining the results given in Eqs. (B6)–(B8) we arrive at the results given in the text for the Irving ($n=0$) and Irving-Gunn ($n=1$) wave functions.

[16] A. Sommerfeld, *Partial Differential Equations in Physics* (Academic Press Inc., New York, 1949), pp. 227–235.

[17] W. Magnus and F. Oberhettinger, *Formulas and Theorems for the Functions of Mathematical Physics* (Chelsea Publishing Company, New York, 1954), p. 131.