Photons and Gravitons in S-Matrix Theory: Derivation of Charge Conservation and Equality of Gravitational and Inertial Mass*

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We give a purely S-matrix-theoretic proof of the conservation of charge (defined by the strength of soft photon interactions) and the equality of gravitational and inertial mass. Our only assumptions are the Lorentz invariance and pole structure of the S matrix, and the zero mass and spins 1 and 2 of the photon and graviton. We also prove that Lorentz invariance alone requires the S matrix for emission of a massless particle of arbitrary integer spin to satisfy a "mass-shell gauge invariance" condition, and we explain why there are no macroscopic fields corresponding to particles of spin 3 or higher.

I. INTRODUCTION

It is not yet clear whether field theory will continue to play a role in particle physics, or whether it will ultimately be supplanted by a pure S-matrix theory. However, most physicists would probably agree that the place of local fields is nowhere so secure as in the theory of photons and gravitons, whose properties seem indissolubly linked with the space-time concepts of gauge invariance (of the second kind) and/or Einstein's equivalence principle.

The purpose of this article is to bring into question the need for field theory in understanding electromagnetism and gravitation. We shall show that there are no general properties of photons and gravitons, which *have* been explained by field theory, which cannot also be understood as consequences of the Lorentz invariance and pole structure of the S matrix for massless particles of spin 1 or 2.[1] We will also show why there can be no macroscopic fields whose quanta carry spin 3 or higher.

What are the special properties of the photon or graviton S matrix, which might be supposed to reflect specifically field-theoretic assumptions? Of course, the usual version of gauge invariance and the equivalence principle cannot even be stated, much less proved, in terms of the S matrix alone. (We decline to turn on external fields.) But there are two striking properties of the S matrix which *seem* to require the assumption of gauge invariance and the equivalence principle:

(1) The S matrix for emission of a photon or graviton can be written as the product of a polarization "vector" ϵ^μ or "tensor" $\epsilon^\mu\epsilon^\nu$ with a covariant vector or tensor amplitude, and it vanishes if any ϵ^μ is replaced by the photon or graviton momentum q^μ.

(2) Charge, defined dynamically by the strength of soft-photon interactions, is additively conserved in all reactions. Gravitational mass, defined by the strength of soft graviton interactions, is equal to inertial mass for all nonrelativistic particles (and is twice the total energy for relativistic or massless particles).

Property (1) is actually a straightforward consequence of the well-known[2,3] Lorentz transformation properties of massless particle states, and is proven in Sec. II for massless particles of arbitrary integer spin. (It has already been proven for photons by D. Zwanziger.[4])

Property (2) does not at first sight appear to be derivable from property (1). Even in field theory (1) does not prove that the photon and graviton "currents" $J_\mu(x)$ and $\theta_{\mu\nu}(x)$ are conserved, but only that their matrix elements are conserved for light-like momentum transfer, so we cannot use the usual argument that $\int d^3x J^0(x)$ and $\int d^3x \theta^{0\mu}(x)$ are time-independent. And in pure S-matrix theory it is not even possible to define what we mean by the operators $J^\mu(x)$ and $\theta^{\mu\nu}(x)$.

We overcome these obstacles by a trick, which replaces the operator calculus of field theory with a little simple polology. After defining charge and gravitational mass as soft photon and graviton coupling constants in Sec. III, we prove in Sec. IV that if a reaction violates charge conservation, then the same process with inner bremsstrahlung of a soft extra photon would have an S matrix which does not satisfy property (1), and hence would not be Lorentz invariant; similarly, the inner bremstrahlung of a soft graviton would violate Lorentz invariance if any particle taking part in the reaction has an anomalous ratio of gravitational to inertial mass.

Appendices A, B, and C are devoted to some technical problems: (A) the transformation properties of polarization vectors, (B) the construction of tensor amplitudes for massless particles of general integer spin, and (C) the presence of kinematic singularities in the conventional $(2j+1)$-component "M functions."

A word may be needed about our use of S-matrix theory for particles of zero mass. We do not know whether it will ever be possible to formulate S-matrix

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[1] Some of the material of this article was discussed briefly in a recent letter [S. Weinberg Phys. Letters **9**, 357 (1964)]. We will repeat a few points here, in order that the present article be completely self-contained.

[2] E. P. Wigner, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 59. We have repeated Wigner's work in Ref. 3.

[3] S. Weinberg, Phys. Rev. **134**, B882 (1964).

[4] D. Zwanziger, Phys. Rev. **113**, B1036 (1964). Zwanziger omits some straightforward details, which are presented here in Appendix B.

theory as a complete dynamical theory even for strong interactions alone, and the presence of massless particles will certainly add a formidable technical difficulty, since every pole sits at the beginning of an infinite number of branch cuts. All such "infrared" problems are outside the scope of the present work. We shall simply make believe that there does exist an S-matrix theory, and that one of its consequences is that the S matrix has the same poles that it has in perturbation theory, with residues that factor in the same way as in perturbation theory. (We will lapse into the language of Feynman diagrams when we do our 2π bookkeeping in Sec. IV, but the reader will recognize in this the effects of our childhood training, rather than any essential dependence on field theory.)

When we refer to the "photon" or the "graviton" in this article, we assume no properties beyond their zero mass and spin 1 or 2. We will not attempt to explain why there should exist such massless particles, but may guess from perturbation theory that zero mass has a special kind of dynamical self-consistency for spins 1 and 2, which it would not have for spin 0.

Most of our work in the present article has a counterpart in Feynman-Dyson perturbation theory. In a future paper we will show how the Lorentz invariance of the S matrix forces the coupling of the photon and graviton "potentials" to take the same form as required by gauge invariance and the equivalence principle.

II. TENSOR AMPLITUDES FOR MASSLESS PARTICLES OF INTEGER SPIN

Let us consider a process in which a massless particle is emitted with momentum $\mathbf{q}$ and helicity $\pm j$. We shall call the S-matrix element simply $S_{\pm j}(\mathbf{q},p)$, letting p stand for the momenta and helicities of all other particles participating in the reaction. The Lorentz transformation property of S can be inferred from the well-known transformation law for one-particle states[2]; we find that

$$S_{\pm j}(\mathbf{q},p)=(|\Lambda\mathbf{q}|/|\mathbf{q}|)^{1/2}\times\exp[\pm ij\Theta(\mathbf{q},\Lambda)]S_{\pm j}(\Lambda\mathbf{q},\Lambda p). \quad (2.1)$$

The angle Θ is given in Appendix A as a function of the momentum $\mathbf{q}$ and the Lorentz transformation $\Lambda^\mu{}_\nu$.

We prove in Appendix B that, in consequence of (2.1), it is always possible for integer j to write $S_{\pm j}$ as the scalar product of a "polarization tensor" and what Stapp[5] would call an "M function":

$$S_{\pm j}(\mathbf{q},p)=(2|\mathbf{q}|)^{-1/2}\epsilon_\pm{}^{\mu_1 *}(\mathbf{q})\cdots\times\epsilon_\pm{}^{\mu_j *}(\mathbf{q})M_{\pm\mu_1\cdots\mu_j}(\mathbf{q},p) \quad (2.2)$$

with M a symmetric tensor,[6] in the sense that

$$M_\pm{}^{\mu_1\cdots\mu_j}(\mathbf{q},p)=\Lambda_{\nu_1}{}^{\mu_1}\cdots\Lambda_{\nu_j}{}^{\mu_j}M_\pm{}^{\nu_j\cdots\nu_j}(\Lambda\mathbf{q},\Lambda p). \quad (2.3)$$

The polarization $\epsilon_\pm{}^\mu(\hat{q})$ is defined by

$$\epsilon_\pm{}^\mu(\hat{q})\equiv R(\hat{q})^\mu{}_\nu\epsilon_\pm{}^\nu, \quad (2.4)$$

where $R(\hat{q})$ is a standard rotation that carries the z axis into the direction of $\mathbf{q}$, and $\epsilon_\pm{}^\mu$ is the polarization for momentum in the z direction:

$$\epsilon_\pm{}^\mu\equiv\{1,\pm i,0,0\}/\sqrt{2}. \quad (2.5)$$

Some properties of $\epsilon_\pm{}^\mu(\hat{q})$ are obvious:

$$\epsilon_{\pm\mu}{}^*(\hat{q})\epsilon_\pm{}^\mu(\hat{q})=1, \quad (2.6)$$

$$\epsilon_{\pm\mu}(\hat{q})\epsilon_\pm{}^\mu(\hat{q})=0, \quad (2.7)$$

$$\epsilon_\pm{}^{\mu *}(\hat{q})=\epsilon_\mp{}^\mu(\hat{q}), \quad (2.8)$$

$$\epsilon_\pm{}^0(\hat{q})=0, \quad (2.9)$$

$$q_\mu\epsilon_\pm{}^\mu(\hat{q})=0, \quad (2.10)$$

$$\sum\nolimits_\pm \epsilon_\pm{}^\mu(\hat{q})\epsilon_\pm{}^{\nu *}(\hat{q})=\Pi^{\mu\nu}(\hat{q})\equiv g^{\mu\nu}+(\bar{q}^\mu q^\nu+\bar{q}^\nu q^\mu)/|\mathbf{q}|^2, \quad [\bar{q}^\mu\equiv\{-\mathbf{q},|\mathbf{q}|\}], \quad (2.11)$$

$$\sum\nolimits_\pm \epsilon_\pm{}^{\mu_1}(\hat{q})\epsilon_\pm{}^{\mu_2}(\hat{q})\epsilon_\pm{}^{\nu_1 *}(\hat{q})\epsilon_\pm{}^{\nu_2 *}(\hat{q}) =\tfrac{1}{2}\{\Pi^{\mu_1\nu_1}(\hat{q})\Pi^{\mu_2\nu_2}(\hat{q})+\Pi^{\mu_1\nu_2}(\hat{q})\Pi^{\mu_2\nu_1}(\hat{q}) -\Pi^{\mu_1\mu_2}(\hat{q})\Pi^{\nu_1\nu_2}(\hat{q})\}. \quad (2.12)$$

We also note the very important transformation rule, proved in Appendix A,

$$(\Lambda_\nu{}^\mu-q^\mu\Lambda_\nu{}^0/|\mathbf{q}|)\epsilon_\pm{}^\nu(\Lambda\hat{q})=\exp\{\pm i\Theta[\mathbf{q},\Lambda]\}\epsilon_\pm{}^\mu(\hat{q}), \quad (2.13)$$

with Θ the same angle as in (2.1).

If it were not for the q^μ term in (2.13), the polarization "tensor" $\epsilon_\pm{}^{\mu_1}\cdots\epsilon_\pm{}^{\mu_j}$ would be a true tensor, and the tensor transformation law (2.3) for $M_\pm{}^{\mu_1\cdots\mu_j}$ would be sufficient to ensure the correct behavior (2.1) of the S matrix. But $\epsilon_\pm{}^\mu$ is not a vector,[7] and (2.3) and (2.13) give the S-matrix transformation rule

$$S_{\pm j}(\mathbf{q},p)=(2|\mathbf{q}|)^{-1/2}\exp\{\pm ij\Theta(\mathbf{q},\Lambda)\}\times[\epsilon_\pm{}^{\mu_1}(\Lambda\hat{q})-(\Lambda q)^{\mu_1}\Lambda_\nu{}^0\epsilon_\pm{}^\nu(\Lambda\hat{q})/|\mathbf{q}|]^*\cdots\times[\epsilon_\pm{}^{\mu_j}(\Lambda\hat{q})-(\Lambda q)^{\mu_j}\Lambda_\nu{}^0\epsilon_\pm{}^\nu(\Lambda\hat{q})/|\mathbf{q}|]^*\times M_{\pm,\mu_1\cdots\mu_j}(\Lambda\mathbf{q},\Lambda p). \quad (2.14)$$

For an infinitesimal Lorentz transformation $\Lambda^\mu{}_\nu=\delta^\mu{}_\nu+\omega^\mu{}_\nu$, we can use (2.2) and the symmetry of M to put (2.14) in the form

$$S_{\pm j}(\mathbf{q},p)=(|\Lambda\mathbf{q}|/|\mathbf{q}|)^{1/2}\exp\{\pm ij\Theta(\mathbf{q},\Lambda)\}S_{\pm j}(\Lambda\mathbf{q},\Lambda p)-j(2|q|^3)^{-1/2}(\omega_\nu{}^0\epsilon_\pm{}^{\nu *}(\hat{q}))q^{\mu_1}\epsilon_\pm{}^{\mu_2 *}(\hat{q})\cdots\times\epsilon_\pm{}^{\mu_j *}(\hat{q})M_{\pm,\mu_1\cdots\mu_j}(\mathbf{q},p). \quad (2.15)$$

Hence the necessary and sufficient condition that (2.14) agree with the correct Lorentz transformation property (2.1), is that $S_\pm$ vanish when one of the $\epsilon_\pm{}^\mu$ is replaced with q^μ:

$$q^{\mu_1}\epsilon_\pm{}^{\mu_2 *}(\hat{q})\cdots\epsilon_\pm{}^{\mu_j *}(\hat{q})M_{\pm,\mu_1\cdots\mu_j}(\mathbf{q},p)=0. \quad (2.16)$$

For $j=1$ this may be expressed as the conservation

[5] M functions for massive particles were introduced by H. Stapp, Phys. Rev. **125**, 2139 (1962). See also A. O. Barut, I. Muzinich, and D. N. Williams, Phys. Rev. **130**, 442 (1963).

[6] We use a real metric, with signature $\{+++-\}$. Indices are raised and lowered in the usual way. The inverse of the Lorentz transformation $\Lambda^\mu{}_\nu$ is $[\Lambda^{-1}]^\mu{}_\nu=\Lambda_\nu{}^\mu$.

[7] The transformation rule (2.13) shows that $\epsilon_\pm{}^\mu(\hat{q})$ transforms according to one of the infinite-dimensional representations of the Lorentz group discussed by V. Bargmann and E. P. Wigner, Proc. Natl. Acad. Sci. **34**, 211 (1948).

condition

$$q_\mu M_\pm{}^\mu(\mathbf{q},p)=0. \tag{2.17}$$

For $j=2$ we conclude that

$$q_\mu M_\pm{}^{\mu\nu}(\mathbf{q},p)\propto q^\nu. \tag{2.18}$$

However, (2.7) shows that the subtraction of a term proportional to $g^{\mu\nu}$ from $M_\pm{}^{\mu\nu}$ will not alter the S matrix (2.2), so $M_\pm{}^{\mu\nu}$ can always be defined in such a way that (2.18) becomes

$$q_\mu M_\pm{}^{\mu\nu}(\mathbf{q},p)=0. \tag{2.19}$$

The condition (2.16) may look empty, since it can always be satisfied by a suitable adjustment of $M_\pm{}^{0\mu_2\cdots\mu_j}$, which in light of (2.9) will have no effect on the S matrix. But we cannot play with the time-like components of $M_\pm{}^{\mu_1\cdots\mu_j}$ and still keep it a tensor in the sense of (2.3). Neither (2.3) nor (2.16) is alone sufficient for Lorentz invariance, and together they constitute a nontrivial condition on $M_\pm{}^{\mu_1\cdots\mu_j}$.

Condition (2.16) may, if we wish, be described as "mass-shell gauge invariance," because it implies that the S matrix is invariant under a regauging of the polarization vector

$$\epsilon_\pm{}^\mu(\hat{q})\rightarrow\epsilon_\pm{}^\mu(\hat{q})+\lambda_\pm(\mathbf{q})q^\mu, \tag{2.20}$$

with $\lambda_\pm(\mathbf{q})$ arbitrary. It was purely for convenience that we started with the "Coulomb gauge" in (2.4), (2.5). [However, the theorem in Sec. III of Ref. 3 shows that it is *not* possible to construct an $\epsilon_\pm{}^\mu(\hat{q})$ which would satisfy (2.13) without any q^μ term.]

The S matrix for emission and absorption of several massless particles can be treated in the same way, except that $\epsilon^{\mu*}$ is replaced by ϵ^μ when a massless particle is absorbed.

III. DYNAMIC DEFINITION OF CHARGE AND GRAVITATIONAL MASS

We are going to define the charge and gravitational mass of a particle as its coupling constants to very-low-energy photons and gravitons, with "coupling constant" understood in the same sense as the Watson-Lepore pion-nucleon coupling constant. In general, such definitions are based on the fact that the S matrix has poles, corresponding to Feynman diagrams in which a virtual particle is exchanged between two sets of A and B of incoming and outgoing particles, with four-momentum nearly on its mass shell. The residue at the pole factors into Γ_A and Γ_B, the two "vertex amplitudes" Γ_A and Γ_B depending respectively only upon the quantum numbers of the particles in sets A and B, and of the exchanged particle. Hence it is possible to give a purely S-matrix-theoretic definition of the vertex amplitude Γ for any set of physical particles, as a function of their momenta and helicities; the coupling constant or constants define the magnitude of Γ. (As discussed in the introduction, we will not be concerned in this article with whether the above remarks can be proven rigorously in S-matrix theories involving massless particles, or with the related question of whether $m=0$ poles can really be separated from the branch cuts on which they lie. Our purpose is to explore the implications of the generally accepted ideas about the pole structure.)

Let us first consider the vertex amplitude for a very-low-energy massless particle of integer helicity $\pm j$, emitted by a particle of spin $J=0$, mass m (perhaps zero), and momentum $p^\mu=\{\mathbf{p},E\}$, with $E=(\mathbf{p}^2+m^2)^{1/2}$. (We are restricting ourselves here to very soft photons and gravitons, because we only want to define the charge and gravitational mass, and not the other electromagnetic and gravitational multipole moments.) The only tensor which can be used to form $M_\pm{}^{\mu_1\cdots\mu_j}$ is $p^{\mu_1}\cdots p^{\mu_j}$ [note that terms involving $g^{\mu\mu'}$ do not contribute to the S matrix, because of (2.7)] so the tensor character of $M_\pm{}^{\mu_1\cdots\mu_j}$ dictates the form of the vertex amplitude as

$$p_{\mu_1}\cdots p_{\mu_j}\epsilon_\pm{}^{\mu_1*}(\hat{q})\cdots\epsilon_\pm{}^{\mu_j*}(\hat{q})/2E(\mathbf{p})(2|\mathbf{q}|)^{1/2}. \tag{3.1}$$

If the emitting particle has spin $J>0$, with initial and final helicities σ and σ' then (3.1) still gives a tensor M function if we multiply it by $\delta_{\sigma\sigma'}$; this is because the unit matrix has the Lorentz transformation property

$$\delta_{\sigma\sigma'}\rightarrow D_{\sigma\sigma''}{}^{(J)}(\mathbf{p},\Lambda)D_{\sigma'\sigma'''}{}^{(J)*}(\mathbf{p},\Lambda)\delta_{\sigma''\sigma'''}=\delta_{\sigma\sigma'}, \tag{3.2}$$

where $D^{(J)}(\mathbf{p},\Lambda)$ is the unitary spin-J representation of the Wigner rotation[8] (or its analog,[2] if $m=0$) associated with momentum $\mathbf{p}$ and Lorentz transformation Λ. However, the vertex amplitude so obtained is not unique. For instance if $J=\frac{1}{2}$ and $m>0$ then we get (3.1) times $\delta_{\sigma\sigma'}$ if we use a "current"[9]

$$\bar{\psi}\{\gamma_{\mu_1}p_{\mu_2}\cdots p_{\mu_j}+\text{permutations}\}\psi, \tag{3.3}$$

while using $\gamma_5\gamma_\mu$ in place of γ_μ would give a helicity-flip vertex amplitude.

At the end of the next section we will see that these other possibilities are prohibited by the Lorentz invariance of the total S matrix. Indeed, the only allowed vertex functions for soft massless particles of spin j are of the form (3.1) times $\delta_{\sigma\sigma'}$ for $j=1$ and $j=2$ (and none at all for $j\geq 3$). We may therefore define the soft photon coupling constant e, by the statement that the $j=1$ vertex amplitude is[10]

$$\frac{2ie(2\pi)^4\delta_{\sigma\sigma'}p_\mu\epsilon_\pm{}^{\mu*}(\hat{q})}{(2\pi)^{9/2}[2E(\mathbf{p})](2|\mathbf{q}|)^{1/2}}, \tag{3.4}$$

[8] E. P. Wigner, Ann. Math. **40**, 149 (1939). For a review, see S. Weinberg, Phys. Rev. **133**, B1318 (1964).

[9] For $j=2$, see I. Y. Kobsarev and L. B. Okun, Dubna (unpublished).

[10] Proper Lorentz invariance alone would allow different charges $e_\pm$ for photon helicities ± 1. Parity conservation would normally require that $e_+=e_-$ (with an appropriate convention for the photon parity). However if space inversion takes some particle into its antiparticle then its "right charge" e_+ will be equal to the "left charge" $\bar{e}_-$ of its antiparticle, and we will see in the next section that this gives $e_+=\bar{e}_-=-e_-$. In this case we speak of a magnetic monopole rather than a charge. The same conclusions can be drawn from CP conservation. We will not consider magnetic monopoles in this paper, though in fact none of our work in Sec. IV will depend on any relation between e_+ and e_-. Time-reversal invariance allows us to take e as real.

the factors 2, i, and π being separated from e in obedience to convention. And in the same way we may define a "gravitational charge" f, by the statement that the $j=2$ vertex amplitude is[11]

$$\frac{2if(8\pi G)^{1/2}(2\pi)^4\delta_{\sigma\sigma'}(p_\mu\epsilon_\pm{}^{\mu*}(\hat{q}))^2}{(2\pi)^{9/2}[2E(\mathbf{p})](2|\mathbf{q}|)^{1/2}}, \tag{3.5}$$

the extra factor $(8\pi G)^{1/2}$ (where G is Newton's constant) being inserted to make f dimensionless.

In order to see how e and f are related to the usual charge and gravitational mss, let us consider the near forward scattering of two particles with masses m_a and m_b, spins J_a and J_b, photon coupling constants e_a and e_b, and graviton coupling constants f_a and f_b. As the invariant momentum transfer $t=-(p_a-p_a')^2$ goes to zero, the S matrix becomes dominated by its one-photon-exchange and one-graviton-exchange poles. An elementary calculation[12] using (2.11) and (2.12) shows that for $t\to 0$, the S matrix becomes

$$\frac{\delta_{\sigma_a\sigma_a'}\delta_{\sigma_b\sigma_b'}}{4\pi^2E_aE_bt}[e_ae_b(p_a\cdot p_b)+8\pi Gf_af_b\{(p_a\cdot p_b)^2-m_a^2m_b^2/2\}]. \tag{3.6}$$

If particle b is at rest, this gives

$$\frac{\delta_{\sigma_a\sigma_a'}\delta_{\sigma_b\sigma_b'}}{\pi t}\left[-\frac{e_ae_b}{4\pi}+Gf_a\left\{2E_a-\frac{m_a^2}{E_a}\right\}f_bm_b\right]. \tag{3.7}$$

Hence we may identify e_a as the *charge* of particle a, while its effective *gravitational mass* is

$$\tilde{m}_a=f_a\{2E_a-(m_a^2/E_a)\}. \tag{3.8}$$

If particle a is nonrelativistic, then $E_a\cong m_a$, and (3.8) gives its gravitational rest mass as

$$\tilde{m}_a=f_am_a. \tag{3.9}$$

On the other hand, if a is massless or extremely relativistic, then $E_a\gg m_a$ and (3.8) gives

$$\tilde{m}_a=2f_aE_a. \tag{3.10}$$

[Formulas (3.8) or (3.10) should not of course be understood to mean anything more than already stated in (3.7). However, they serve to remind us that the response of a massless particle to a static gravitational field is finite, and proportional to f.]

The presence of massless particles in the initial or final state will also generate poles in the S matrix, which, like that in (3.7), lie on the edge of the physical region. It is therefore possible to measure the coupling constants e and f in a variety of process, such as Thomson scattering or soft bremsstrahlung, or their analogs for gravitons. All these different experiments will give the same value for any given particle's e or f, for purely S-matrix-theoretic reasons. The task before us is to show how the e's and f's are related for different particles.

IV. CONSERVATION OF e AND UNIVERSALITY OF f

Let $S_{\beta\alpha}$ be the S matrix for some reaction $\alpha\to\beta$, the states α and β consisting of various charged and uncharged particles, perhaps including gravitons and photons. The same reaction can also occur with emission of a very soft extra photon or graviton of momentum $\mathbf{q}$ and helicity ±1, or ±2, and we will denote the corresponding S-matrix element as $S_{\beta\alpha}{}^{\pm1}(\mathbf{q})$ or $S_{\beta\alpha}{}^{\pm2}(\mathbf{q})$.

These emission matrix elements will have poles at $\mathbf{q}=0$, corresponding to the Feynman diagrams in which the extra photon or graviton is emitted by one of the incoming or outgoing particles in states α or β. The poles arise because the virtual particle line connecting the photon or graviton vertex with the rest of the diagram gives a vanishing denominator

$$\begin{aligned} 1/[(p_n+q)^2+m_n^2]&=1/2p_n\cdot q \quad \text{(particle } n \text{ outgoing)},\\ 1/[(p_n-q)^2+m_n^2]&=-1/2p_n\cdot q \quad \text{(particle } n \text{ incoming)}. \end{aligned} \tag{4.1}$$

For $|\mathbf{q}|$ sufficiently small, these poles will completely dominate the emission-matrix element. The singular factor (4.1) will be multiplied by a factor $-i(2\pi)^{-4}$ associated with the extra internal line, a factor

$$\frac{2ie[p_n\cdot\epsilon_\pm{}^*(\hat{q})](2\pi)^4}{(2\pi)^{3/2}(2|\mathbf{q}|)^{1/2}} \tag{4.2}$$

or

$$\frac{2if(8\pi G)^{1/2}[p_n\cdot\epsilon_\pm{}^*(\hat{q})]^2(2\pi)^4}{(2\pi)^{3/2}(2|\mathbf{q}|)^{1/2}} \tag{4.3}$$

arising from the vertices (3.4) or (3.5), and a factor $S_{\beta\alpha}$ for the rest of the diagram. Hence the S matrix for soft photon or graviton emission is given in the limit

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[11] Proper Lorentz invariance alone would not rule out different values for the gravitational charges $f_\pm$ for gravitons of helicity ±2. Parity conservation (with an appropriate convention for the graviton parity) requires that $f_+=f_-$. This conclusion holds even for the magnetic monopole case discussed in footnote 10, since then $f_+=\bar{f}_-$, and we will see in Sec. IV that the antiparticle has "left gravitational charge" $\bar{f}_-$ equal to f_-. The same conclusions can be drawn from CP conservation. Time-reversal invariance allows us to take f as real.

[12] The residue of the pole at $t=0$ can be most easily calculated by adopting a coordinate system in which $q\equiv p_{a'}-p_a=p_b-p_{b'}$ is a finite real light-like four-vector, while p_a, p_b, $p_{a'}$, $p_{b'}$ are on their mass shells, and hence necessarily complex. Then the gradient terms in (2.11) and (2.12) do not contribute, because $q\cdot p_a=q\cdot p_b=0$, so that $\Pi_{\mu\nu}$ may be replaced by $g_{\mu\nu}$, yielding (3.6). We are justified in using (3.6) in the physical region (where p_a, p_b, $p_{a'}$, $p_{b'}$ are real and q is small, though *not* in the direction of the light cone) because Lorentz invariance tells us that the matrix element depends only upon s and t. Lorentz invariance is actually far from trivial in a perturbation theory based on physical photons and gravitons, since then the Coulomb force and Newtonian attraction must be explicitly introduced into the interaction in order to get the invariant S matrix (3.6). (Such a perturbation theory will be discussed in an article now in preparation.) The Lorentz-invariant extrapolation of (3.6) into the physical region of small t is the analog, in S-matrix theory, of the introduction of the Coulomb and Newton forces in perturbation theory.

$\mathbf{q}\to 0$ by[13–15]

$$S_{\beta\alpha}^{\pm 1}(\mathbf{q})\to(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}\times\left[\sum_n \eta_n e_n \frac{[p_n\cdot\epsilon_\pm^*(\hat{q})]}{(p_n\cdot q)}\right]S_{\beta\alpha} \quad (4.4)$$

or

$$S_{\beta\alpha}^{\pm 2}(\mathbf{q})\to(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}(8\pi G)^{1/2}\times\left[\sum_n \eta_n f_n \frac{[p_n\cdot\epsilon_\pm^*(q)]^2}{(p_n\cdot q)}\right]S_{\beta\alpha}, \quad (4.5)$$

the sign η_n being $+1$ or -1 according to whether particle n is outgoing or incoming.

These emission matrices are of the general form (2.2), i.e.,

$$S_{\beta\alpha}^{\pm 1}(\mathbf{q})\to(2|\mathbf{q}|)^{-1/2}\epsilon_\pm^{\mu *}(\hat{q})M_\mu(\mathbf{q},\alpha\to\beta), \quad (4.6)$$

$$S_{\beta\alpha}^{\pm 2}(\mathbf{q})\to(2|\mathbf{q}|)^{-1/2}\epsilon_\pm^{\mu *}(\hat{q})\epsilon_\pm^{\nu *}(\hat{q})M_{\mu\nu}(\mathbf{q},\alpha\to\beta), \quad (4.7)$$

where M_μ and $M_{\mu\nu}$ are tensor M functions

$$M^\mu(\mathbf{q},\alpha\to\beta)=(2\pi)^{-3/2}\left[\sum_n \eta_n e_n p_n^\mu/(p_n\cdot q)\right]S_{\beta\alpha}, \quad (4.8)$$

$$M^{\mu\nu}(\mathbf{q},\alpha\to\beta)=(2\pi)^{-3/2}(8\pi G)^{1/2}\times\left[\sum_n \eta_n f_n p_n^\mu p_n^\nu/(p_n\cdot q)\right]S_{\beta\alpha}. \quad (4.9)$$

However, we have learned in Sec. II that the covariance of M_μ and $M_{\mu\nu}$ is not sufficient by itself to guarantee the Lorentz invariance of the S matrix; Lorentz invariance also requires the vanishing of (2.2) when any one $\epsilon_\pm^\mu(\hat{q})$ is replaced with q^μ. For photons this implies (2.17), i.e.,

$$0=q^\mu M_\mu(\mathbf{q},\alpha\to\beta)=(2\pi)^{-3/2}\left[\sum_n \eta_n e_n\right]S_{\beta\alpha}, \quad (4.10)$$

so if $S_{\beta\alpha}$ is not to vanish, *the transition* $\alpha\to\beta$ *must conserve charge*, with

$$\sum_n \eta_n e_n=0. \quad (4.11)$$

For gravitons Lorentz invariance requires (2.18), which here takes the simpler form (2.19)

$$0=q_\mu M^{\mu\nu}(\mathbf{q},\alpha\to\beta)=(2\pi)^{-3/2}(8\pi G)^{1/2}\left[\sum_n \eta_n f_n p_n^\nu\right]S_{\beta\alpha}. \quad (4.12)$$

But the p_n^μ are arbitrary four-momenta, subject only to the condition of energy momentum conservation:

$$\sum_n \eta_n p_n^\mu=0. \quad (4.13)$$

The requirement that (4.12) vanish for all such p_n^μ, can be met if and only if *all particles have the same gravitational charge.* The conventional definition of Newton's constant G is such as to make the common value of the f_n unity, so

$$f_n=1 \quad (\text{all } n) \quad (4.14)$$

and (3.8) then tells us that any particle with inertial mass m and energy E has effective gravitational mass

$$\tilde{m}=2E-m^2/E. \quad (4.15)$$

In particular, a particle at rest has gravitational mass $\tilde{m}$ equal to its inertial mass m.

It seems worth emphasizing that our proof also applies when some particle n in the initial or final state is itself a graviton. Hence the graviton must emit and absorb single soft gravitons (and therefore respond to a uniform gravitational field) with gravitational mass $2E$. It would be conceivable to have a universe in which all f_n vanish, but since we know that soft gravitons interact with matter, they must also interact with gravitons.

Having reached our goal, we may look back, and see that no other vertex amplitudes could have been used for $\mathbf{q}\to 0$ except (3.4) and (3.5). A helicity-flip or helicity-dependent vertex amplitude could never give rise to the cancellations between different poles [as in (4.10) and (4.12)] needed to satisfy the Lorentz invariance conditions (2.17) and (2.19). It is also interesting that such cancellations cannot occur for massless particles of integer spin higher than 2. For suppose we take the vertex amplitude for emission of a soft massless particle of helicity $\pm j$ ($j=3, 4, \cdots$) as

$$\frac{2ig^{(j)}(2\pi)^4(\epsilon_\pm^*(\hat{q})\cdot p)^j\delta_{\sigma\sigma'}}{(2\pi)^{9/2}[2E(\mathbf{p})](2|\mathbf{q}|)^{1/2}} \quad (4.16)$$

in analogy with (3.4) and (3.5), the S matrix $S_{\beta\alpha}^{\pm j}(\mathbf{q})$ for emission of this particle in a reaction $\alpha\to\beta$ will be given in the limit $\mathbf{q}\to 0$ by

$$S_{\beta\alpha}^{\pm j}(\mathbf{q})\to(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}\times\left[\sum_n \eta_n g_n^{(j)}[p_n\cdot\epsilon_\pm^*(\hat{q})]^j/(p_n\cdot q)\right]S_{\beta\alpha}. \quad (4.17)$$

This is only Lorentz invariant if it vanishes when any one $\epsilon_\pm^\mu$ is replaced with q^μ, so we must have

$$\sum_n \eta_n g_n^{(j)}[p_n\cdot\epsilon_\pm^*(\hat{q})]^{j-1}=0. \quad (4.18)$$

But there is no way that this can be satisfied for all momenta p_n obeying (4.13), unless $j=1$ or $j=2$. This

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[13] Formula (4.4) is well known to hold to all orders in quantum electrodynamic perturbation theory. See, for example, J. M. Jauch and F. Rohrlich, *Theory of Photons and Electrons* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1955), p. 392, and F. E. Low, Ref. 14.

[14] It has been shown by F. E. Low, Phys. Rev. **110**, 974 (1958), that the next term in an expansion of the S matrix in powers of $|\mathbf{q}|$ is uniquely determined by the electromagnetic multipole moments of the participating particles and by $S_{\beta\alpha}$. However, this next (zeroth-order) term is Lorentz-invariant for any values of the multipole moments.

[15] Relations like (4.4) and (4.5) are also valid if $S_{\beta\alpha}^{\pm 1}(\mathbf{q})$, $S_{\beta\alpha}^{\pm 2}(\mathbf{q})$, and $S_{\beta\alpha}$ are interpreted as the effective matrix elements for the transition $\alpha\to\beta$, respectively, with or without one extra soft photon or graviton of momentum $\mathbf{q}$, *plus* any number of unobserved soft photons or gravitons with total energy less than some small resolution ΔE. [For a proof in quantum-electrodynamic perturbation theory, see, for example, D. R. Yennie and H. Suura, Phys. Rev. **105**, 1378 (1957). The same is undoubtedly true also for gravitons, and in pure S-matrix theory.]

is not to say that massless particles of spin 3 or higher cannot exist, but only that they cannot interact at zero frequency, and hence cannot generate macroscopic fields. And similarly, the uniqueness of the vertex amplitudes (3.4) and (3.5) does not show that electromagnetism and gravitation conserve parity, but only that parity must be conserved by zero-frequency photons and gravitons.

The crucial point in our proof is that the emission of soft photons or gravitons generates poles which individually make non-Lorentz-invariant contributions to the S matrix. Only the sum of the poles is Lorentz-invariant, and then only if e is conserved and f is universal. Just as the universality of f can be expressed as the equality of gravitational and inertial mass, the conservation of e can be stated as the equality of charge defined dynamically, with a quantum number defined by an additive conservation law. But, however, we state them, these two facts are the outstanding dynamical peculiarities of photons and gravitons, which until now have been proven only under the *a priori* assumption of a gauge-invariant or generally covariant Lagrangian density.

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APPENDIX A: POLARIZATION VECTORS AND THE LITTLE GROUP

In this Appendix we shall discuss the "little group"[2] for massless particles, with the aim of defining the angle $\Theta(\mathbf{q},\Lambda)$, and of determining the transformation properties of the polarization vectors $\epsilon_\pm(\hat{q})$.

The little group is defined as consisting of all Lorentz transformations $\mathcal{R}^\mu{}_\nu$, which leave invariant a standard light-like four-vector K^μ:

$$\mathcal{R}^\mu{}_\nu K^\nu = K^\mu\,, \tag{A1}$$

$$K^1=K^2=0\,,\quad K^3=K^0=\kappa>0\,. \tag{A2}$$

It is a matter of simple algebra to show that the most general such $\mathcal{R}^\mu{}_\nu$ can be written as a function of three parameters Θ, X^1, X^2:

$$\mathcal{R}^\mu{}_\nu=\begin{bmatrix} \cos\Theta & \sin\Theta & -X_1\cos\Theta-X_2\sin\Theta & X_1\cos\Theta+X_2\sin\Theta \\ -\sin\Theta & \cos\Theta & X_1\sin\Theta-X_2\cos\Theta & -X_1\sin\Theta+X_2\cos\Theta \\ X_1 & X_2 & 1-X^2/2 & X^2/2 \\ X_1 & X_2 & -X^2/2 & 1+X^2/2 \end{bmatrix}. \tag{A3}$$

$$X^2\equiv X_1^2+X_2^2.$$

(The rows and columns are in order 1, 2, 3, 0.) Wigner[2] has noted that this group is isomorphic to the group of rotations (by angle Θ) and translations (by vector $\{X_1,X_2\}$) in the Euclidean plane. In particular the "translations" form an invariant Abelian subgroup, defined by the condition $\Theta=0$, and are represented on the physical Hilbert space by unity. It is possible to factor any $\mathcal{R}^\mu{}_\nu$ into

$$\mathcal{R}^\mu{}_\nu=\begin{bmatrix} \cos\Theta & \sin\Theta & 0 & 0 \\ -\sin\Theta & \cos\Theta & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}\begin{bmatrix} 1 & 0 & -X_1 & X_1 \\ 0 & 1 & -X_2 & X_2 \\ X_1 & X_2 & 1-X^2/2 & X^2/2 \\ X_1 & X_2 & -X^2/2 & 1+X^2/2 \end{bmatrix}. \tag{A4}$$

The representation of $\mathcal{R}^\mu{}_\nu$ on physical Hilbert space is determined solely by the first factor, so

$$U[\mathcal{R}]=\exp(i\Theta[\mathcal{R}]J_3)\,. \tag{A5}$$

In discussing the transformation rules for massless particles it is necessary to consider members of the little group defined by

$$\mathcal{R}(\mathbf{q},\Lambda)=\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{q})\,. \tag{A6}$$

Here Λ is an arbitrary Lorentz transformation, and $\mathcal{L}(\mathbf{q})$ is the Lorentz transformation:

$$\mathcal{L}^\mu{}_\nu(\mathbf{q})=R^\mu{}_\rho(\hat{q})B^\rho{}_\nu(|\mathbf{q}|)\,, \tag{A7}$$

where $B(|\mathbf{q}|)$ is a "boost" along the z axis, with nonzero components

$$\begin{aligned} B^1{}_1&=B^2{}_2=1\,,\\ B^3{}_3&=B^0{}_0=\cosh\varphi\,,\\ B^3{}_0&=B^0{}_3=\sinh\varphi\,,\\ \varphi&\equiv\log(|q|/\kappa)\,, \end{aligned} \tag{A8}$$

and $R(\hat{q})$ is the rotation introduced in (2.4), which takes the z axis into the direction of $\mathbf{q}$. The transformation $\mathcal{L}(\mathbf{q})$ takes the standard four-momentum K^μ [see (A2)] into $q^\mu\equiv\{\mathbf{q},|\mathbf{q}|\}$:

$$\mathcal{L}^\mu{}_\nu(\mathbf{q})K^\nu=q^\mu \tag{A9}$$

so therefore,

$$\begin{aligned}\mathcal{R}^\mu{}_\nu(\mathbf{q},\Lambda)K^\nu&=[\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}]^\mu{}_\nu(\Lambda q)^\nu\\ &=[\mathcal{L}^{-1}(\mathbf{q})]^\mu{}_\nu q^\nu=K^\mu.\end{aligned} \tag{A10}$$

Hence $\mathcal{R}(\mathbf{q},\Lambda)$ does belong to the little group.

It was shown in Ref. 3 that, as a consequence of (A5), the S matrix obeys the transformation rule (2.1), with $\Theta(\mathbf{q},\Lambda)$ given as the Θ angle of $\mathcal{R}(\mathbf{q},\Lambda)$:

$$\Theta(\mathbf{q},\Lambda)=\Theta[\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{q})]. \quad \text{(A11)}$$

We now turn to the polarization "vectors" $\epsilon_\pm{}^\mu(\hat{q})$, defined in Sec. II by

$$\epsilon_\pm{}^\mu(\mathbf{q})=R^\mu{}_\nu(\hat{q})\epsilon_\pm{}^\nu, \quad \text{(A12)}$$

$$\epsilon_\pm{}^\mu\equiv\{1,\pm i,0,0\}/\sqrt{2}. \quad \text{(A13)}$$

Observe that we could just as well write (A12) as

$$\epsilon_\pm{}^\mu(q)=\mathcal{L}^\mu{}_\nu(\mathbf{q})\epsilon_\pm{}^\nu \quad \text{(A14)}$$

since $B(|\mathbf{q}|)$ has no effect on $\epsilon_\pm$.

An arbitrary $\mathcal{R}^\mu{}_\nu$ of the form (A3) will transform $\epsilon_\pm{}^\nu$ into

$$\mathcal{R}^\mu{}_\nu\epsilon_\pm{}^\nu=\exp(\pm i\Theta[\mathcal{R}])\epsilon_\pm{}^\mu+X_\pm[\mathcal{R}]K^\mu, \quad \text{(A15)}$$

where

$$X_\pm[\mathcal{R}]=\frac{X_1[\mathcal{R}]\pm iX_2[\mathcal{R}]}{\kappa\sqrt{2}}. \quad \text{(A16)}$$

If we let $\mathcal{R}$ be the transformation (A6), and use (A14), then (A15) gives

$$[\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}]^\mu{}_\nu\epsilon_\pm{}^\nu(\Lambda\mathbf{q})=\exp[\pm i\Theta(\mathbf{q},\Lambda)]\epsilon_\pm{}^\mu+X_\pm(\mathbf{q},\Lambda)K^\mu, \quad \text{(A17)}$$

where

$$X_\pm(\mathbf{q},\Lambda)\equiv\frac{X_1[\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{q})]\pm iX_2[\mathcal{L}^{-1}(\mathbf{q})\Lambda^{-1}\mathcal{L}(\Lambda\mathbf{q})]}{\kappa\sqrt{2}}. \quad \text{(A18)}$$

Multiplying (A17) by $\mathcal{L}(\mathbf{q})$, we have the desired result

$$\Lambda_\nu{}^\mu\epsilon_\pm{}^\nu(\Lambda\mathbf{q})=\exp[\pm i\Theta(\mathbf{q},\Lambda)]\epsilon_\pm{}^\mu(\mathbf{q})+X_\pm(\mathbf{q},\Lambda)q^\mu. \quad \text{(A19)}$$

Note that it is the "translations" which at the same time make the little group non-semi-simple, and which yield the gradient term in (A19).

The quantity $X_\pm(\mathbf{p},\Lambda)$ may be found in terms of $\epsilon_\pm(\mathbf{q})$ by setting $\mu=0$ in (A19):

$$X_\pm(\mathbf{q},\Lambda)|\mathbf{q}|=\Lambda_\nu{}^0e_\pm{}^\nu(\Lambda\mathbf{q}). \quad \text{(A20)}$$

Hence we may rewrite (A19) as a homogeneous transformation rule:

$$(\Lambda_\nu{}^\mu-\Lambda_\nu{}^0q^\mu/|\mathbf{q}|)\epsilon_\pm{}^\nu(\Lambda\mathbf{q})=\exp[\pm i\Theta(\mathbf{q},\Lambda)]\epsilon_\pm{}^\mu(\hat{q}) \quad \text{(A21)}$$

or, recalling that $\epsilon_\pm{}^0\equiv0$,

$$(\Lambda_i{}^j-\Lambda_i{}^0\hat{q}^j)e_\pm{}^i(\Lambda\mathbf{q})=\exp[\pm i\Theta(\mathbf{q},\Lambda)]\epsilon_\pm{}^j(\hat{q}).$$

This also incidentally shows that $\Theta(\mathbf{q},\Lambda)$ does not depend on $|\mathbf{q}|$.

We have not had to define the rotation $R(\hat{q})$ any further than by just specifying that it carries the z axis into the direction of q. However, the reader may wish to see explicit expressions for the polarization vectors, so we will consider one particular standardization of $R(\hat{q})$. Write $\hat{q}$ in the form

$$\hat{q}=\{-\sin\beta\cos\gamma,\sin\beta\sin\gamma,\cos\beta\} \quad \text{(A22)}$$

and let $R(\hat{q})$ be the rotation with Euler angles $0,\beta,\gamma$:

$$R^\mu{}_\nu(\hat{q})=\begin{bmatrix}\cos\beta\cos\gamma & \sin\gamma & -\sin\beta\cos\gamma & 0\\ -\cos\beta\sin\gamma & \cos\gamma & \sin\beta\sin\gamma & 0\\ \sin\beta & 0 & \cos\beta & 0\\ 0 & 0 & 0 & 1\end{bmatrix}. \quad \text{(A23)}$$

Then (2.4) and (2.5) give

$$\epsilon_\pm{}^\mu(\hat{q})=\{\cos\beta\cos\gamma\pm i\sin\gamma,\ -\cos\beta\sin\gamma\pm i\cos\gamma,\sin\beta,0\}/\sqrt{2}\quad(\mu=1,2,3,0). \quad \text{(A24)}$$

We can easily check (2.6)–(2.12) explicitly for (A24).

APPENDIX B: CONSTRUCTION OF TENSOR AMPLITUDES

We consider a reaction in which is emitted a massless particle of momentum $\mathbf{q}$ and integer helicity $\pm j$, all other particle variables being collected in the single symbol p. Let us first divide the set of all possible $\{\mathbf{q},p\}$ into disjoint equivalence classes, $\{\mathbf{q},p\}$ being equivalent to $\{\mathbf{q}',p'\}$ if one can be transformed into the other by a Lorentz transformation. (This is an equivalence relation, because the Lorentz group is a group.) The axiom of choice allows us to make an arbitrary selection of one set of standard values $\{\mathbf{q}_c,p_c\}$ from each equivalence class, so any $\{\mathbf{q},p\}$ determines a unique standard $\{\mathbf{q}_c,p_c\}$, such that for some Lorentz transformation $L^\mu{}_\nu$ we have

$$\mathbf{q}=L\mathbf{q}_c,\quad p=Lp_c. \quad \text{(B1)}$$

It will invariably be the case in physical processes that the only $\Lambda^\mu{}_\nu$ leaving both $\mathbf{q}$ and p invariant is the identity $\delta^\mu{}_\nu$, so the $L^\mu{}_\nu$ in (B1) is uniquely determined by $\mathbf{q}$ and p. (This is true, for instance, if p stands for two or more general four-momenta.) Hence the arguments $\{\mathbf{q},p\}$ stand in one-to-one relation to the variables $\{\mathbf{q}_c,p_c,L\}$.

Now let us construct an $M_\pm{}^{\mu_1\cdots\mu_j}(\mathbf{q}_c,p_c)$ satisfying (2.2) for each standard $\{\mathbf{q}_c,p_c\}$. A suitable choice is

$$M_\pm{}^{\mu_1\cdots\mu_j}(\mathbf{q}_c,p_c)\equiv(2|\mathbf{q}_c|)^{1/2}\epsilon_\pm{}^{\mu_1}(\hat{q}_c)\cdots\epsilon_\pm{}^{\mu_j}(\hat{q}_c)S_{\pm j}(\mathbf{q}_c,p_c), \quad \text{(B2)}$$

which satisfies (2.2) because of (2.6). The tensor amplitude for a general $\mathbf{q}$, p is then defined by

$$M_\pm{}^{\mu_1\cdots\mu_j}(\mathbf{q},p)\equiv L^{\mu_1}{}_{\nu_1}(\mathbf{q},p)\cdots L^{\mu_j}{}_{\nu_j}(\mathbf{q},p)M_\pm{}^{\nu_1\cdots\nu_j}(\mathbf{q}_c,p_c), \quad \text{(B3)}$$

where $\mathbf{q}_c$, p_c, and $L(\mathbf{q},p)$ are the standard variables and Lorentz transformation defined by (B1). With this definition we can easily show that $M_\pm{}^{\mu_1\cdots\mu_j}$ is a tensor, because

$$M_\pm{}^{\mu_1\cdots\mu_j}(\mathbf{q},p)=L^{\mu_1}{}_{\nu_1}(\mathbf{q},p)\cdots L^{\mu_j}{}_{\nu_j}(\mathbf{q},p)L_{\rho_1}{}^{\nu_1}(\Lambda\mathbf{q},\Lambda p)\cdots L_{\rho_j}{}^{\nu_j}(\Lambda\mathbf{q},\Lambda p)M_\pm{}^{\rho_1\cdots\rho_j}(\Lambda\mathbf{q},\Lambda p)$$
$$=\Lambda_{\rho_1}{}^{\mu_1}\cdots\Lambda_{\rho_j}{}^{\mu_j}M_\pm{}^{\rho_1\cdots\rho_j}(\Lambda\mathbf{q},\Lambda p),\quad\text{(B4)}$$

the latter equality holding because $L(\mathbf{q},p)L^{-1}(\Lambda\mathbf{q},\Lambda p)$ induces the transformation $\{\Lambda\mathbf{q},\Lambda p\}\rightarrow\{\mathbf{q}_c,p_c\}\rightarrow\{\mathbf{q},p\}$ and hence must be just Λ^{-1}.

We must now show that (B.3) satisfies (2.2) for all $\{\mathbf{q},p\}$. The Lorentz transformation property (2.13) of $\epsilon_\pm{}^\mu$ can be written as

$$\epsilon_\pm{}^\mu(\hat{q})=\exp\{\mp i\Theta(\hat{q},L^{-1}(\mathbf{q},p))\}[L^\mu{}_\nu(\mathbf{q},p)-q^\mu L^0{}_\nu(\mathbf{q},p)/|\mathbf{q}|]\epsilon_\pm{}^\nu(\hat{q}_c).$$

Hence, (B.3) gives

$$\epsilon_\pm{}^{\mu_1*}(\hat{q})\cdots\epsilon_\pm{}^{\mu_j*}(\hat{q})M_{\mu_1\cdots\mu_j}(\hat{q},p)=\exp\{\pm ij\Theta(\hat{q},L^{-1}(\hat{q},p))\}[\epsilon_\pm{}^{\mu_1}(\hat{q}_c)-q_c{}^{\mu_1}\epsilon_\pm{}^{\nu_1}(\hat{q}_c)L^0{}_{\nu_1}(\mathbf{q},p)/|\mathbf{q}|]^*\cdots$$
$$\times[\epsilon_\pm{}^{\mu_j}(\hat{q}_c)-q_c{}^{\mu_j}\epsilon_\pm{}^{\nu_j}(\hat{q}_c)L^0{}_{\nu_j}(\mathbf{q},p)/|\mathbf{q}|]^*M_{\mu_1\cdots\mu_j}(\mathbf{q}_c,p_c).\quad\text{(B5)}$$

But (B2) and (2.10) show that all $q_c{}^\mu$ terms may be dropped, because

$$q_c{}^{\mu_r}M_{\mu_1\cdots\mu_j}(\mathbf{q}_c,p_c)=0,\quad\text{(B6)}$$

so (B5) simplifies to

$$\epsilon_\pm{}^{\mu_1*}(\hat{q})\cdots\epsilon_\pm{}^{\mu_j*}(\hat{q})M_{\mu_1\cdots\mu_j}(\mathbf{q},p)=\exp\{\pm ij\Theta(\hat{q},L^{-1}(\mathbf{q},p))\}\epsilon_\pm{}^{\mu_1*}(\hat{q}_c)\cdots\epsilon_\pm{}^{\mu_j*}(\hat{q}_c)M_{\mu_1\cdots\mu_j}(\mathbf{q}_c,p_c)\quad\text{(B7)}$$

or, using (B2) and (2.6),

$$(2|\mathbf{q}|)^{-1/2}\epsilon_\pm{}^{\mu_1*}(\hat{q})\cdots\epsilon_\pm{}^{\mu_j*}(\hat{q})M_{\mu_1\cdots\mu_j}(\mathbf{q},p)=(|\mathbf{q}_c|/|\mathbf{q}|)^{1/2}\exp\{\pm ij\Theta(\hat{q},L^{-1}(\mathbf{q},p))\}S_{\pm j}(q_c,p_c).\quad\text{(B8)}$$

The right-hand side is just the formula for $S_{\pm j}(q,p)$ obtained by setting $\Lambda=L^{-1}(\mathbf{q},p)$ in (2.1), so (B8) gives finally

$$S_{\pm j}(\mathbf{q},p)=(2|\mathbf{q}|)^{-1/2}\epsilon_\pm{}^{\mu_1*}(\hat{q})\cdots\times\epsilon_\pm{}^{\mu_j*}(\hat{q})M_{\mu_1\cdots\mu_j}(\mathbf{q},p).\quad\text{(B9)}$$

It should be noted that (B2) is *not* valid for all $\mathbf{q}$, p, since then $M_\pm{}^{0\mu_2\cdots\mu_j}(\mathbf{q},p)$ would vanish in all Lorentz frames, and $M_\pm$ could hardly then be a tensor.

APPENDIX C: $(2j+1)$-COMPONENT M FUNCTIONS

It has become customary[5] to write the S matrix for massive particles of spin j in terms of $2j+1$-component M functions, which transform under the $(j,0)$ or $(0,j)$ representation of the homogeneous Lorentz group. In contrast, the symmetric-tensor M functions used here transform according to the $(j/2, j/2)$ representation. The massless-particle S matrix could also have been written in terms of a conventional $(2j+1)$-component M function, but only at the price of giving the M function a very peculiar pole structure.

To see what sort of peculiarities can occur for zero mass, let us consider the emission of a very soft photon in a reaction like Compton scattering, in which there is only one charged particle in the initial state α and in the final state β. The S-matrix element is then given by (4.4) as

$$S_{\beta\alpha}{}^{\pm1}(\mathbf{q})\rightarrow(2\pi)^{-3/2}(2|\mathbf{q}|)^{-1/2}\times e\left[\frac{p_\mu}{(p\cdot q)}-\frac{p'_\mu}{(p'\cdot q)}\right]\epsilon_\pm{}^{\mu*}(\hat{q})S_{\beta\alpha},\quad\text{(C1)}$$

where p and p' are the initial and final charged-particle momenta. This may be rewritten as

$$S_{\beta\alpha}{}^\pm(\mathbf{q})\rightarrow(2|\mathbf{q}|)^{-1/2}M_{[\mu,\nu]}(\mathbf{q},\alpha\rightarrow\beta)\times\{q^{\nu*}\epsilon_\pm{}^\mu(\hat{q})-q^\mu\epsilon_\pm{}^{\nu*}(\hat{q})\},\quad\text{(C2)}$$

where $M_{[\mu,\nu]}$ is a $(1,0)\oplus(0,1)$ M function

$$M_{[\mu,\nu]}(\mathbf{q},\alpha\rightarrow\beta)=\frac{e[p_\mu p'_\nu-p_\nu p'_\mu]S_{\beta\alpha}}{(2\pi)^{3/2}(p\cdot q)(p'\cdot q)}.\quad\text{(C3)}$$

It can be shown that $S_{\beta\alpha}{}^+$ and $S_{\beta\alpha}{}^-$ receive contributions, respectively, only from the self-dual and anti-self-dual parts of $M_{[\mu,\nu]}$, which transform according to the three-component (0,1) and (1,0) representations. But (C3) shows that *these conventional M functions have a double pole*, arising simultaneously from the incoming and outgoing charged particle propagators. This singularity is partly kinematic, since the S matrix (C1) involves a sum of single poles, but certainly no double pole. The presence of kinematic singularities in $M_{[\mu,\nu]}$ makes it an inappropriate covariant photon amplitude. Similar remarks apply to gravitons, but not to any other massless particles like the neutrino, for which there is no analog to charge.