was 20 dB smaller than predicted by the theory. This is probably attributable to the nonuniformities in the electron density and dc magnetic-field strength in the plasma which reduce the resonance amplitude. The effects of these nonuniformities on the narrow plasma resonance peaks are naturally more pronounced than on the broad geometrical resonances. Second, an additional resonance, not predicted by the theory, was discovered in the combination frequency power as a function of the dc magnetic-field strentgh. This resonance occurs near values of the dc magnetic field for which the electron cyclotron resonance frequency is equal to the arithmetic mean of the frequencies of the incident waves. Unlike the other resonance peaks, the magnetic-field value for which this resonance occurs is independent of the electron density. The origin of this resonance is not understood at this time.

ACKNOWLEDGMENT

The authors wish to thank the members of the Plasma Physics Department of the former Palo Alto Laboratories, General Telephone and Electronics Laboratories, for many stimulating discussions during the course of this investigation.

PHYSICAL REVIEW VOLUME 135, NUMBER 2A 20 JULY 1964

Motion of a Charged Particle in a Constant Magnetic Field and a Transverse Electromagnetic Wave Propagating along the Field

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(Received 24 February 1964)

The relativistic equation of motion is examined for a charged particle in a constant magnetic field and a transverse electromagnetic wave propagating along the field. A general dicussion is given of the effects at cyclotron resonance of the magnetic field of the wave and the relativistic mass increase with energy. An exact solution to the equation of motion is found for the case of a circularly polarized wave. The solution shows that when the index of refraction of the medium in which the wave propagates is not unity, the energy of the particle is a periodic function of time, the exact relationship being expressible in terms of elliptic integrals. When the index of refraction is unity, the effect of the magnetic field of the wave just compensates for the change in mass with energy, and the energy of the particle increases indefinitely at resonance. Several possible applications of this solution to classical cyclotron resonance phenomena are pointed out. As a numerical example, the case of an electron in a constant magnetic field of 1000 G initially at resonance with microwaves whose E field is 0.1 esu is considered.

I. INTRODUCTION

THE interaction between a charged particle and an electromagnetic wave in the presence of a constant magnetic field underlies several phenomena currently under investigation concerning the Van Allen particles,[1] plasma in the earth's magnetosphere,[2] and the diagnostics, heating, and confinement of plasma in the laboratory.[3] This interaction exhibits resonance effects when the wave frequency is at or near the particle's cyclotron frequency. In this paper we study the nature of the interaction when neither the magnetic field of the wave nor the relativistic mass change of the particle with energy are neglected, and we place special emphasis on the resonance effects.

The equation of motion of a particle of rest mass m_r and charge e in a constant magnetic field $\mathbf{B}_0$ is

$$\dot{\mathbf{p}}=e[\mathbf{E}+(\mathbf{v}/c)\times\mathbf{B}+(\mathbf{v}/c)\times\mathbf{B}_0]. \tag{1.1}$$

Here $\mathbf{E}$ and $\mathbf{B}$ are the electric and magnetic fields of the electromagnetic wave, $\mathbf{p}$ the particle's momentum, $\mathbf{v}$ its velocity, and c the speed of light in vacuo. Gaussian units are used, and the dot signifies differentiation with respect to time. The electromagnetic wave is characterized by an angular frequency ω and a propagation vector $\mathbf{k}$, and in this paper we consider only the case where $\mathbf{B}_0$ and $\mathbf{k}$ are parallel and $\mathbf{k}$ and $\mathbf{E}$ are perpendicular, i.e., a purely transverse wave which propagates parallel to the constant field $\mathbf{B}_0$. For convenience, we take the direction of $\mathbf{B}_0$ and $\mathbf{k}$ to be the z direction. If the medium through which the wave propagates has an index of refraction n, then

$$n=kc/\omega=B/E. \tag{1.2}$$

[1] E. N. Parker, J. Geophys. Res. **66**, 2673 (1961); A. J. Dragt, J. Geophys. Res. **66**, 1641 (1961); D. G. Wentzel, J. Geophys. Res. **66**, 359 and 363 (1961).
[2] R. A. Helliwell, J. Geophys. Res. **68**, 5387 (1963).
[3] S. J. Buchsbaum, E. I. Gordon, and S. C. Brown, J. Nucl. Energy **C2**, 164 (1961); M. C. Baker, *et al.*, Nucl. Fusion, 1962 Suppl., Part I, 345 (1962); H. A. H. Boot and R. B. R-Sherby-Harvie, Nature **180**, 1187 (1957); R. Z. Sagdeev, *Plasma Physics and Controlled Thermonuclear Reactions*, edited by M. A. Leontovitch (Pergamon Press, Inc., New York, 1957), Vol. 3, pp. 406–422; M. Ericson, C. S. Ward, S. C. Brown, and S. J. Buchsbaum, J. Appl. Phys. **33**, 2429 (1962).

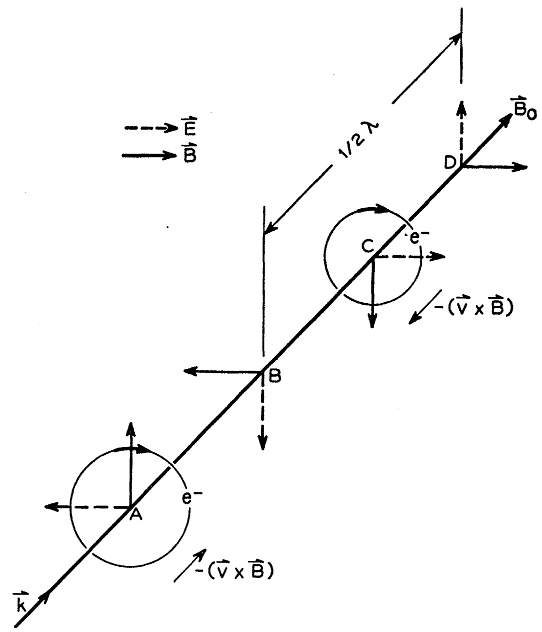

FIG. 1. An electron in a circularly polarized electromagnetic wave and a constant magnetic field.

We shall denote the total energy of the particle, rest plus kinetic, by $\mathcal{H}$ and the particle cyclotron frequency by

$$\Omega=-eB_0/mc=-eB_0(1-v^2/c^2)^{1/2}/m_rc \qquad (1.3a)$$

$$=-eB_0c/\mathcal{H}. \qquad (1.3b)$$

In dealing with Eq. (1.1) it is common practice to neglect the term involving the magnetic field of the wave and to treat the particle's mass as a constant so that $\mathbf{p}=m_r\mathbf{v}$.[4] Thus "linearized," Eq. (1.1) becomes mathematically trivial, and its solution has the following two properties: (1) The particle's velocity in the z direction is a constant; (2) at cyclotron resonance, i.e., when $\omega-k\dot{z}-\Omega=0$, the energy of the particle increases indefinitely according to the formula

$$\mathcal{H}=\mathcal{H}_0+ev_{\perp 0}Et\cos\theta_0+e^2E^2t^2/2m_r. \qquad (1.4)$$

We have defined the component of the particle's velocity perpendicular to the z direction to be $\mathbf{v}_\perp$, and θ denotes the angle between the $\mathbf{E}$ of the wave and $\mathbf{v}_\perp$. The subscript 0 appended to any variable refers to the initial value of that variable at $t=0$.

This solution to the linearized version of Eq. (1.1) fails to illuminate several important features of the true interaction in the neighborhood of cyclotron resonance. This fact has been realized by several investigators, and some approximate treatments of Eq. (1.1) including nonlinear effects have been given.[5] In this paper we present an exact solution to Eq. (1.1) for the case of a circularly polarized wave. This solution reveals the following properties of the true interaction: (1) The energy of the particle obeys a differential equation of the form $(d\mathcal{H}/dt)^2+V(\mathcal{H})=0$, where $V(\mathcal{H})$ is some function of $\mathcal{H}$. Since this is the same differential equation as that describing one-dimensional motion of a particle in the potential field $V(\mathcal{H})$, a qualitative picture of the dependence of energy upon time can be obtained by plotting the function $V(\mathcal{H})$ and imagining a particle moving on the resulting contour. (2) If the particle is initially not at resonance, i.e., $(\omega-k\dot{z}_0-\Omega_0)\neq 0$, or if the particle is initially at resonance with the index of refraction $n\neq 1$, then $V(\mathcal{H})$ has a shape similar to those shown in Figs. 3 to 6, and the energy and the particle's momentum in the z direction are periodic functions of time. (3) If the particle is initially at resonance and if $n=1$, the particle's energy and momentum in the z direction both increase indefinitely. In this case $V(\mathcal{H})$ has a shape similar to that shown in Fig. 2, there being only one finite zero. Of course, Eq. (1.1) does not include the effect of radiation damping and we would expect this effect, if included, to finally limit the energy of the particle.

Before going on to the mathematical derivation of the above-stated results, we will present a physical picture of the effects causing them. As an example we shall consider the case of the electron depicted in Fig. 1. At time $t=0$, the fields are as shown and the electron is directly above point A with its velocity antiparallel to the $\mathbf{E}$ of the wave so that initially it is gaining energy. If at this instant $\omega=\Omega_0$ so that we start from exact resonance, subsequent motion of the particle may destroy this resonance condition in two ways. First, as the electron gains energy, it becomes more massive, and, consequently, its cyclotron frequency decreases. Second, the magnetic field of the wave accelerates the particle in the direction of $\mathbf{B}_0$ and $\mathbf{k}$, and as the electron acquires some velocity in this direction it will see the wave at a Doppler-shifted frequency which is lower than ω. The relative importance of these two effects depends on the ratio $B/E=n$, the index of refraction characterizing the propagation. If $n>1$, the wave is more B than E, and the magnetically produced Doppler shift is the prime resonance destroyer. If $n<1$, the wave is more E than B, and the gain in mass is predominant. In either case the angle θ between $\mathbf{E}$ and $\mathbf{v}$, which initially was π, changes with time until it finally becomes acute. When this happens, both effects reverse; the electron now loses energy and the magnetic force has a component antiparallel to $\mathbf{B}_0$ and $\mathbf{k}$. This situation is maintained until θ once again becomes

[4] See, for example, T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), p. 9; or, W. P. Allis, S. J. Buchsbaum, and A. Bers, *Waves in Anisotropic Plasmas* (MIT Press, Cambridge, Massachusetts, 1963), p. 19.

[5] See, for example, T. H. Stix, *The Theory of Plasma Waves* (McGraw-Hill Book Company, Inc., New York, 1962), p. 162; J. W. Dungey, J. Fluid Mech. **15**, 74 (1963); S. Rand, Phys. Fluids **5**, 1237 (1962); T. Consoli and G. Mourier, Phys. Letters **7**, 247 (1963).

obtuse, and the electron reverts to gaining energy. This alternate acceleration and deceleration of the electron by the $\mathbf{E}$ of the wave accounts for the periodicity of the dependence of energy on time described in the preceding paragraph.

When $n=1$, however, so that $B=E$, a most interesting phenomenon occurs. In this case, the magnetic and mass effects just cancel one another, and $\omega-k\dot{z}-\Omega=0$ throughout the electron's motion. What happens is that as the electron gains energy and the cyclotron frequency consequently decreases, the magnetic field of the wave produces just the right velocity along $\mathbf{B}_0$ and $\mathbf{k}$ to Doppler-shift the wave frequency to the value necessary to maintain resonance. The effect is equivalent to a synchrotron which maintains its synchronism automatically. For this reason, we shall refer to the case where $n=1$ and the particle is initially at resonance as the synchronous case. That such an effect can exist was first realized by Davydovskii.[6]

In the succeeding sections the preceding highly descriptive discussion will be formalized. In Sec. II we reduce Eq. (1.1) to an ordinary differential equation for the energy of the particle as a function of time. In Sec. III, several solutions of this equation at resonance are presented for various conditions. In Sec. IV, a numerical example is presented.

II. SOLUTION OF THE EQUATION OF MOTION

We start with the expression for the rate of change of energy of a charged particle in an electromagnetic field,

$$d\mathcal{H}/dt=e(\mathbf{v}\cdot\mathbf{E}). \tag{2.1}$$

For a plane electromagnetic wave, Maxwell's equations require that

$$\mathbf{k}\times\mathbf{E}=(\omega/c)\mathbf{B}. \tag{2.2}$$

If we use Eq. (2.2) to eliminate $\mathbf{B}$ from Eq. (1.1) and then compare the z component of the resulting equation with Eq. (2.1), we arrive at the relationship

$$d\mathcal{H}/dt=(\omega/k)\dot{p}_z. \tag{2.3}$$

Equation (2.3) expresses the fact that an electromagnetic wave cannot change the energy of a particle without also changing its momentum, a relationship easily understood if one adopts a photon picture of the interaction. Equation (2.3) may be immediately integrated to give

$$\mathcal{H}(t)=\mathcal{H}_0+(\omega/k)[p_z(t)-p_{z0}]. \tag{2.4}$$

We now note that the correct condition for resonance when the particle has a component of its velocity in the z direction is

$$\omega-k\dot{z}-\Omega=0.$$

If this condition is satisfied, one finds by making an appropriate Lorentz transformation that an observer in the frame in which the particle has zero velocity in the z direction will observe the wave frequency equal to the particle cyclotron frequency. To compute the effect of the wave magnetic field on resonance we compute the quantity

$$d(\omega-k\dot{z})/dt=(kc/eB_0)d(\Omega p_z)/dt. \tag{2.5}$$

Using Eq. (2.3) and the fact that $\Omega(d\mathcal{H}/dt)=-\mathcal{H}\dot{\Omega}$, which follows directly from Eq. (1.3b), Eq. (2.5) may be expressed

$$d(\omega-k\dot{z})/dt=(kc/eB_0)(p_z-k\mathcal{H}/\omega)\dot{\Omega}. \tag{2.6}$$

Equations (1.2) and (2.4) may now be used to obtain the desired equation

$$d(\omega-k\dot{z})/dt=(d_1+1)\dot{\Omega}, \tag{2.7}$$

where d_1 is a constant determined by the initial conditions

$$d_1=(n^2\omega-k\dot{z}_0-\Omega_0)/\Omega_0. \tag{2.8}$$

Equation (2.7) may be integrated to yield

$$(\omega-k\dot{z}-\Omega)=d_1\Omega+d_2\omega, \tag{2.9}$$

where

$$d_2=1-n^2. \tag{2.10}$$

Equation (2.7) relates the effect of the magnetic field of the wave to the change in cyclotron frequency produced by the change in mass with energy. For initial cyclotron resonance, i.e., when $\omega-k\dot{z}_0-\Omega_0=0$,

$$d_1=(n^2-1)\omega/\Omega_0=-d_2\omega/\Omega_0. \tag{2.11}$$

We may now verify the statements made in Sec. I concerning the case when $n=1$, the so-called synchronous case. From Eqs. (2.10) and (2.11), $d_1=d_2=0$ when $n=1$, and therefore, by Eq. (2.9), $\omega-k\dot{z}-\Omega=0$ for all time. Thus, if a particle is initially started at resonance, it will remain at resonance indefinitely when $n=1$. In Sec. III we derive the rate of energy increase for this case.

The results derived so far are valid for any electromagnetic wave satisfying Eq. (2.2). We now confine our attention to the case of a circularly polarized wave. In this case the fields are given by

$$\begin{aligned} B_x&=B\cos(\omega t-kz), & E_x&=E\sin(\omega t-kz),\\ B_y&=B\sin(\omega t-kz), & E_y&=-E\cos(\omega t-kz),\\ B_z&=B_0, & E_z&=0, \end{aligned} \tag{2.12}$$

and the three components of Eq. (1.1) may be written out as

$$\dot{p}_x+\Omega p_y=(eE/\omega)(\omega-k\dot{z})\sin(\omega t-kz), \tag{2.13a}$$

$$\dot{p}_y-\Omega p_x=-(eE/\omega)(\omega-k\dot{z})\cos(\omega t-kz), \tag{2.13b}$$

$$\dot{p}_z=-\Omega(B/B_0)[p_x\sin(\omega t-kz)-p_y\cos(\omega t-kz)]. \tag{2.13c}$$

We show in the Appendix that Eqs. (2.13) are equiv-

[6] V. Ya. Davydovskii, Zh. Eksperim i Teor. Fiz. **43**, 886 (1962) [English transl. Soviet Phys.—JETP **16**, 629 (1963)].

alent to an ordinary differential equation for the cyclotron frequency $\Omega(t)$ or for the energy $\mathcal{H}(t)$. The equation for $\mathcal{H}(t)$ is

$$(d\mathcal{H}/dt)^2+V(\mathcal{H})=0\,, \qquad (2.14)$$

where

$$V(\mathcal{H})=\tfrac{1}{4}(\omega^2/\mathcal{H}^2)\{d_2{}^2(\mathcal{H}-\mathcal{H}_0)^4+4r_1d_2\mathcal{H}_0(\mathcal{H}-\mathcal{H}_0)^3 +4(r_1{}^2+d_2r_2)\mathcal{H}_0{}^2(\mathcal{H}-\mathcal{H}_0)^2 +8(r_1r_2-r_3)\mathcal{H}_0{}^3(\mathcal{H}-\mathcal{H}_0)-4r_4{}^2\mathcal{H}_0{}^4\}\,; \qquad (2.15)$$

$$r_1=(d_1\Omega_0+d_2\omega)/\omega\,, \qquad (2.16a)$$

$$r_2=-(\Omega_0/\omega)[Ep_{10}\Omega_0\sin\theta_0/eB_0{}^2+E^2\Omega_0/B_0{}^2\omega]\,, =(\Omega_0/\omega)[(v_{10}/c)(E/B_0)\sin\theta_0 -E^2\Omega_0/B_0{}^2\omega]\,, \qquad (2.16b)$$

$$r_3=\Omega_0{}^3E^2/\omega^3B_0{}^2\,, \qquad (2.16c)$$

$$r_4=ev_{10}E\cos\theta_0/\mathcal{H}_0\omega =-(\Omega_0/\omega)(v_{10}/c)(E/B_0)\cos\theta_0\,. \qquad (2.16d)$$

As described in Sec. I, Eq. (2.14) is just the differential equation which describes motion in the one-dimensional potential well given by $V(x)$. Since $V(\mathcal{H})$ is a sum of powers of $\mathcal{H}$, Eq. (2.14) admits a general solution in terms of elliptic integrals.[7] Several features of the motion, however, can be deduced without recourse to this rather formidable solution. We first note that $V(\mathcal{H})<0$ when $\mathcal{H}=\mathcal{H}_0$ and that $V(\mathcal{H})>0$ as $\mathcal{H}\to\pm\infty$, except when $r_1=d_2=0$ which corresponds to the special synchronous case. Thus, except in the synchronous case, $V(\mathcal{H})$ must have at least two real zeros, and since $(d\mathcal{H}/dt)^2$ in Eq. (2.14) must be positive, $\mathcal{H}$ must oscillate in the "potential well" between two of the zeros of $V(\mathcal{H})$. The maximum and minimum value acquired by $\mathcal{H}$ can therefore be found simply by finding the roots of a fourth-degree polynomial. Finally, we note that since the coefficients in this polynomial depend on θ_0 only through $\sin\theta_0$ and $\cos^2\theta_0$, the limits of the energy oscillation for initial angles θ_0 and $\pi-\theta_0$ are identical.

III. BEHAVIOR FOR INITIAL CYCLOTRON RESONANCE

In the remainder of the paper we consider only the solution of Eq. (2.14) when the particle is at resonance at $t=0$, i.e., $\omega-k\dot{z}_0-\Omega_0=0$. In this case Eqs. (2.11) and (2.16a) show that $r_1=0$. It is convenient to treat the synchronous ($n=1$) and the oscillatory cases ($n\neq1$) separately.

A. Synchronous Case

For $n=1$, Eqs. (2.10) and (2.11) show that $d_1=d_2=0$. In this case Eq. (2.15) becomes

$$V_{\rm sync}(\mathcal{H})=-\omega^2\mathcal{H}_0{}^2\{2r_3(\mathcal{H}_0/\mathcal{H}) +(r_4{}^2-2r_3)(\mathcal{H}_0/\mathcal{H})^2\}\,, \qquad (3.1)$$

[7] P. F. Byrd and M. D. Friedman, *Handbook of Elliptic Integrals for Engineers and Physicists* (Springer-Verlag, Berlin, 1954).

and the differential Eq. (2.14), subject to the boundary condition $\mathcal{H}=\mathcal{H}_0$ at $t=0$, is soluble with elementary integrals to give

$$6r_3{}^2\omega t=w^3-3(r_4{}^2-2r_3)w+2r_4(r_4{}^2-3r_3)\,, \qquad (3.2a)$$

where

$$w=[2r_3(\mathcal{H}/\mathcal{H}_0)+r_4{}^2-2r_3]^{1/2}. \qquad (3.2b)$$

Equation (3.2) expresses time as a function of energy; in order to invert it and express energy as a function of time, a cubic equation must be solved. While it is possible to do this in closed form, the resulting expression is quite involved and will not be given here. The asymptotic form of the solution as $t\to\infty$ is quite simple, however, and can be obtained directly from Eq. (3.1).

$$\mathcal{H}(t)\underset{t\to\infty}{\sim}\mathcal{H}_0(\tfrac{9}{2}r_3)^{1/3}(\omega t)^{2/3} =\mathcal{H}_0(9/2)^{1/3}(\Omega_0/\omega)[(E/B_0)\omega t]^{2/3}. \qquad (3.3a)$$

The validity of Eq. (3.3a) is limited to large times such that $(\omega t)>(2/9r_3)^{1/2}[(r_4{}^2/2r_3)-1]^{3/2}$. For special initial conditions, such that $r_4{}^2-2r_3=0$, i.e., $(v_{10}/c)^2\cos^2\theta_0=2\Omega/\omega$, the last term in Eq. (3.1) is zero and the relation

$$\mathcal{H}=\mathcal{H}_0[1+(9r_3/2)^{1/2}\omega t]^{2/3} \qquad (3.3b)$$

is valid for all time.

B. Oscillatory Case

When $n\neq1$, $d_2\neq0$, it is more convenient to rewrite Eq. (2.14) in terms of a dimensionless variable

$$u=(\mathcal{H}-\mathcal{H}_0)/\mathcal{H}_0. \qquad (3.4)$$

Equation (2.14) now becomes

$$(du/dt)^2+V_{\rm res}(u)=0\,, \qquad (3.5)$$

where

$$V_{\rm res}(u)=\tfrac{1}{4}[\omega^2/(1+u)^2] \times\{d_2{}^2u^4+4d_2r_2u^2-8r_3u-4r_4{}^2\}\,. \qquad (3.6)$$

Equation (3.5) must be solved subject to the boundary condition $u=0$ at $t=0$. It is possible to write down a general solution to Eqs. (3.5) and (3.6) which expresses t as a function of u in terms of elliptic integrals of the first and third kind.[7] This general solution is quite involved and, more importantly, it is impossible to invert the expression and obtain u as a function of t in terms of known functions. From a computational point of view, it is generally simpler to forget about elliptic integrals and use a standard numerical integration procedure to solve the differential Eq. (3.5).

There is an important set of conditions, however, when solution in terms of elliptic functions is quite profitable. First consider the limits between which u oscillates; these can be found by finding the zeros of the function $V(u)$ given in Eq. (3.6), that is, finding the roots of the fourth-degree polynomial

$$P(u)=\tfrac{1}{4}u^4+(r_2/d_2)u^2-2(r_3/d_2{}^2)u-(r_4{}^2/d_2{}^2). \qquad (3.7)$$

Since $u=0$ at $t=0$, u must oscillate between the two roots of Eq. (3.7) which flank the point $u=0$; we call the larger of these roots α, the smaller β, so that $\beta \leq u \leq \alpha$. If now $|\alpha| \ll 1$ and $|\beta| \ll 1$, as will be the case if the index of refraction n is not in the neighborhood of unity, and if the electromagnetic wave is not too strong, then to good approximation Eq. (3.6) may be replaced by

$$V_{\text{res}}(u)=d_2{}^2\omega^2 P(u). \quad (3.8)$$

Equation (3.5) may then be written

$$(d_2\omega)dt=du/[-P(u)]^{1/2} \quad (3.9a)$$

$$=2du/[(\alpha-u)(u-\beta)(u-\gamma)(u-\delta)]^{1/2}, \quad (3.9b)$$

where γ and δ are the other two roots of $P(u)=0$. The solution to Eq. (3.9) depends upon whether γ and δ are both real or are complex conjugates.

Case of two real roots. If γ and δ are complex, then[8]

$$\gamma=m+in, \quad \delta=m-in \quad (3.10)$$

and

$$u(t)=\frac{(\beta R+\alpha S)+(\beta R-\alpha S)\,\mathrm{cn}\eta(t-t_0)}{(R+S)+(R-S)\,\mathrm{cn}\eta(t-t_0)}, \quad (3.11)$$

where

$$R^2=(\alpha-m)^2+n^2, \quad S^2=(\beta-m)^2+n^2, \quad (3.12a)$$

$$\eta=\tfrac{1}{2}d_2\omega(RS)^{1/2}. \quad (3.12b)$$

The function $\mathrm{cn}(x)$ is a Jacobi elliptic function[9]; the modulus of the elliptic function is given by

$$\kappa^2=[(\alpha-\beta)^2-(R-S)^2]/4RS. \quad (3.13)$$

The constant t_0 is chosen so that at $t=0$, $u=0$, i.e.,

$$\mathrm{cn}\eta t_0=(\alpha S+\beta R)/(\alpha S-\beta R). \quad (3.14)$$

The period T for the energy oscillation may now be found from the knowledge that the period of $\mathrm{cn}x$ is $4K$, K being the complete elliptic integral of the first kind whose modulus is given by Eq. (3.13).

$$T=4K/\eta=8K[|d_2\omega|(RS)^{1/2}]^{-1}. \quad (3.15)$$

Case of four real roots. If γ and δ are real, the solution has a different form depending upon whether α and β are larger or smaller than γ and δ. When the roots are ordered so that $\alpha \geq 0 \geq \beta \geq \gamma \geq \delta$ the solution is given by[10]

$$u(t)=\frac{\beta(\alpha-\gamma)-\gamma(\alpha-\beta)\,\mathrm{sn}^2M(t-t_0)}{(\alpha-\gamma)-(\alpha-\beta)\,\mathrm{sn}^2M(t-t_0)}, \quad (3.16)$$

where

$$M=\tfrac{1}{4}d_2\omega(\alpha-\gamma)^{1/2}(\beta-\delta)^{1/2}, \quad (3.17)$$

and the modulus of the Jacobi elliptic sine function is

$$\kappa^2=(\alpha-\beta)(\gamma-\delta)/(\alpha-\gamma)(\beta-\delta). \quad (3.18)$$

[8] Reference 7, Eq. (259.00), p. 133.
[9] Reference 7, p. 18.
[10] Reference 7, Eq. (256.00), p. 120 and Eq. (252.00), p. 103.

The constant t_0 is again chosen so that $u=0$ at $t=0$, and

$$\mathrm{sn}Mt_0=[\beta(\alpha-\gamma)/\gamma(\alpha-\beta)]^{1/2}. \quad (3.19)$$

Since the period of sn^2x is $2K$, K being the complete elliptic integral of the first kind whose modulus is given by Eq. (3.18), the period of the energy oscillation in this case is

$$T=2K/M=8K[|d_2\omega|(\alpha-\gamma)^{1/2}(\beta-\delta)^{1/2}]^{-1}. \quad (3.20)$$

For the other possible ordering of the roots, $\delta \geq \gamma \geq \alpha \geq 0 \geq \beta$, equations analogous to (3.16)–(3.20) can be written.[10]

Low Initial-Energy Approximation

When the wave is sufficiently weak and the initial kinetic energy of the particle is low enough to make $(v_{\perp 0}/c)$ small, there is a useful approximation to the exact solution of Eqs. (3.5) and (3.6).

More precisely, under the conditions

$$|E/B_0| \ll 1, \quad (3.21a)$$

$$|(1-n^2)E^2/B_0{}^2|^{1/3} \ll 1, \quad (3.21b)$$

$$(v_{\perp 0}/c) \ll 2|\Omega_0/\omega|\,|E/(1-n^2)B_0|^{1/3}, \quad (3.21c)$$

the zeros of Eq. (3.6) are given to good approximation by

$$\alpha=2(\Omega_0/\omega)[E/(1-n^2)B_0]^{2/3}, \quad (3.22a)$$

$$\beta=-\tfrac{1}{2}(\omega/\Omega_0)(v_{\perp 0}/c)^2, \quad (3.22b)$$

$$\gamma=\tfrac{1}{2}\alpha(-1+i\sqrt{3}), \quad \delta=\tfrac{1}{2}\alpha(-1-i\sqrt{3}). \quad (3.22c)$$

Now if n is not very close to unity and if (Ω_0/ω) is not excessively large, then $|\beta| \ll |\alpha| \ll 1$, and we can therefore use the approximate solution (3.11). While it may seem that there are a prohibitively large number of conditions stated above, they can all be satisfied with quite reasonable values for the parameters. With the roots of Eq. (3.22), the modulus for the elliptic functions as given by Eq. (3.13) is $\kappa^2=(2-\sqrt{3})/4=0.0670$. The corresponding complete elliptic integral of the first kind is $K=1.598$ so that the period as given by Eq. (3.15) is simply

$$T=8K/|(3)^{1/4}\alpha d_2\omega|$$
$$=4.86[|\Omega_0|\,|1-n^2|^{1/3}(E/B_0)^{2/3}]^{-1}. \quad (3.23)$$

High Initial-Energy Approximation

Under the conditions

$$|E/B_0| \ll 1, \quad (3.24a)$$

$$\theta_0 \text{ not near } \pm\pi/2, \quad (3.24b)$$

$$(v_{\perp 0}/c) \gg 2|\Omega_0/\omega|\,|E/(1-n^2)B_0|^{1/3}, \quad (3.24c)$$

the four zeros of Eq. (3.6) are given to good approximation by

$$\pm[2(\Omega_0/\omega)(v_{\perp 0}/c)(E/B_0)(\sin\theta_0 \pm 1)/(n^2-1)]^{1/2}. \quad (3.25)$$

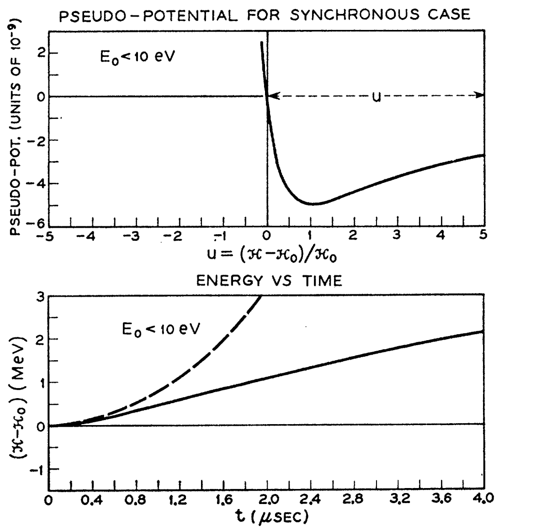

FIG. 2. Synchronous case for electron. $\omega=\Omega_0=2\pi\times2.80$ Gc/sec, $n=1.0$, $B_0=1000$ G, $E=0.1$ esu. The solid curve is the exact solution given by Eq. (3.2); the dashed curve represents Eq. (1.4), the solution in the linear approximation.

Two of the above roots are real and two are pure imaginary, and since by conditions (3.24) the magnitude of the real roots is much less than unity, we can use the approximate solution of Eq. (3.11). From Eq. (3.12a) we find

$$R=S=2[(\Omega_0/\omega)(v_{\perp0}/c)(E/B_0)/|n^2-1|]^{1/2}, \quad (3.26)$$

and the modulus for the elliptic functions is given by

$$\kappa^2=\tfrac{1}{2}(1\pm\sin\theta_0), \quad (3.27)$$

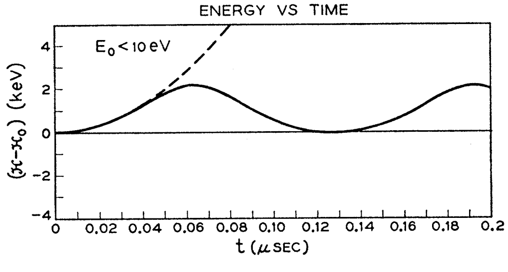

FIG. 3. Oscillatory case for electron. Low-energy approximation. $\omega=\Omega_0=2\pi\times2.80$ Gc/sec, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu. The solid curve is the exact solution given by Eq. (3.5); the dashed curve represents Eq. (1.4), the solution in the linear approximation.

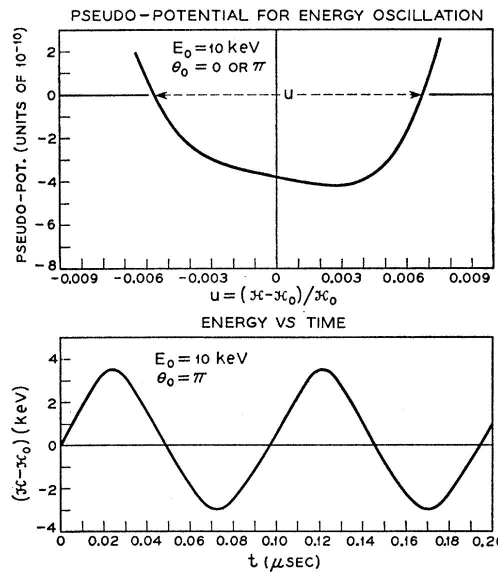

FIG. 4. Oscillatory case for electron. $\omega=\Omega_0=2\pi\times2.74$ Gc/sec, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

where the choice of sign in Eq. (3.27) corresponds to that choice in Eq. (3.25) which gives the pair of real roots. In this case, the period for the energy oscillation is given by

$$T=8K[|n^2-1||\omega|R]^{-1} \quad (3.28a)$$

$$=4K[\omega\Omega_0(v_{\perp0}/c)(E/B_0)|n^2-1|]^{-1/2}. \quad (3.28b)$$

IV. EXAMPLE

In order to illustrate more concretely the motion described by the preceding equations we have computed the results for an electron moving in a constant magnetic field of 1000 G illuminated with microwaves. We take the E field of the microwaves to be 0.1 esu. We take the velocity of the electron in the z direction to be zero at $t=0$, and the microwave frequency to equal the initial electron cyclotron frequency so that the electron starts from exact resonance. We have computed the energy as a function of time by numerically integrating the differential Eq. (3.5) for several initial kinetic energies and angles θ_0. With each graph of energy versus time we show the corresponding pseudopotential energy function $V_{\mathrm{res}}(u)/\omega^2d_2^2$ defined in Eq. (3.6).

When $n=1$, we have the synchronous case and the results are shown in Fig. 2 for an initial electron energy less than 10 eV. For this low an initial energy, the results are nearly independent of the initial angle θ_0. In the synchronous case the shape of the pseudopotential is always qualitatively similar to that shown in Fig. 2; the curve always passes through the point (0,0), goes through a minimum, and then approaches zero asymptotically as $u\rightarrow\infty$. Also shown in Fig. 2 as a dashed curve is Eq. (1.4), the energy as a function of time as obtained from the linearized equation of motion. It can be seen that the linearized equation ceases to be adequate for times greater than about 0.5 μsec, in the example chosen.

To study an oscillatory case we take $n=\sqrt{2}$. The time variation of the energy for an initial kinetic energy less than about 10 eV is shown in Fig. 3. In this case the electron's energy periodically increases to 2.2 keV every 0.13 μsec. This oscillatory behavior should be contrasted with the dashed curve, which again gives the results of the linear theory, Eq. (1.4). For such a low initial energy the approximate formulas, Eqs. (3.22) and (3.13) apply and yield results nearly identical with those shown in Fig. 3.

When the initial kinetic energy is 10 keV, the initial angle θ_0 between $\mathbf{E}$ and $\mathbf{v}_{\perp 0}$ is important, and the results are shown for $\theta_0=0(\pi)$, $\frac{1}{2}\pi$, and $-\frac{1}{2}\pi$ in Figs. 4, 5, and 6. Note that when $\theta_0=\pm\frac{1}{2}\pi$ the electron only gains energy from the wave since u never goes negative. The conditions of Figs. 4–6 are such that neither the low nor the high initial-energy approximation holds. In Fig. 7 we show the range of the energy oscillations versus initial kinetic energy for the case where $\mathbf{v}_{\perp 0}$ is either parallel or antiparallel to $\mathbf{E}$. As can be seen, as the initial kinetic energy increases, the amplitude of the energy oscillation becomes a decreasing fraction of the total energy of the particle. In Figs. 8 and 9 we show the maximum increase in kinetic energy during one period of oscillation and the period of oscillation as a function of initial kinetic energy. The large increase in period for $E_0>0.5$ MeV is due to the fact that the particle is becoming quite a bit heavier in this region due to the relativistic mass increase.

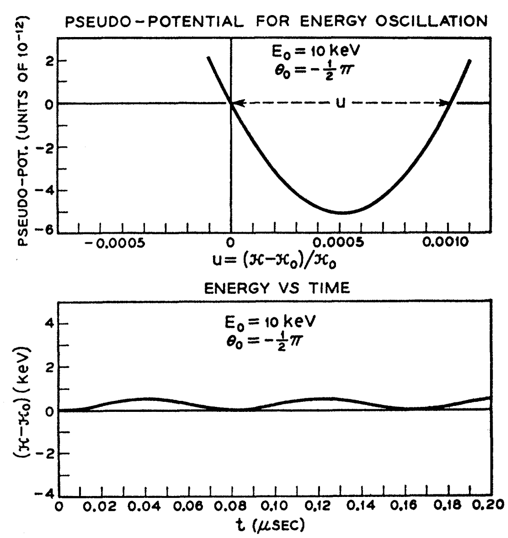

FIG. 6. Oscillatory case for electron. $\omega=\Omega_0=2\pi\times 2.74$ Gc/sec, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

V. CONCLUSION

We have shown that the solution to Eq. (1.1) can be reduced to finding the solution of a simple ordinary differential Eq. (2.23). The solution of this differential equation directly gives particle energy as a function of time; after this is known, any other parameter of the motion which may be of interest is reduced to quadratures. Using this formalism it should be possible to obtain a qualitatively and quantitatively better understanding of processes which depend upon the interaction between an electromagnetic wave and a free

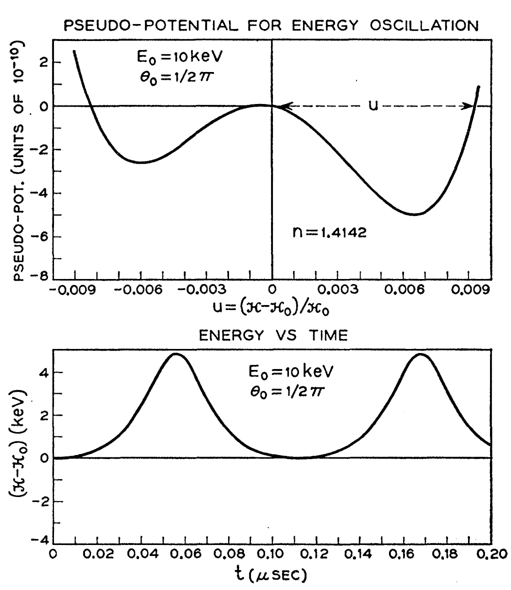

FIG. 5. Oscillatory case for electron. $\omega=\Omega_0=2\pi\times 2.74$ Gc/sec, $b=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

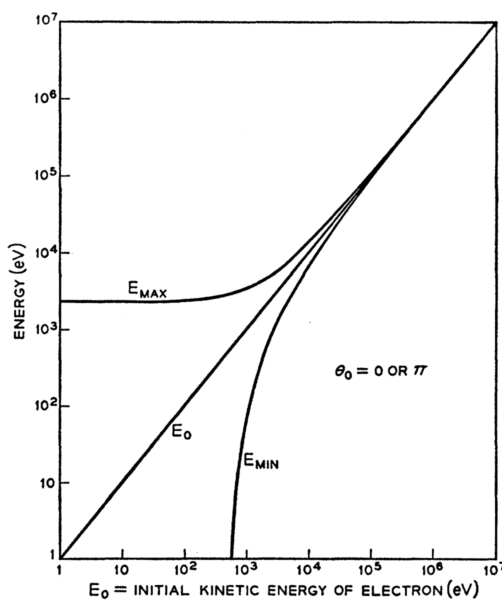

FIG. 7. Range of energy oscillation versus initial kinetic energy $\omega=\Omega_0$, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

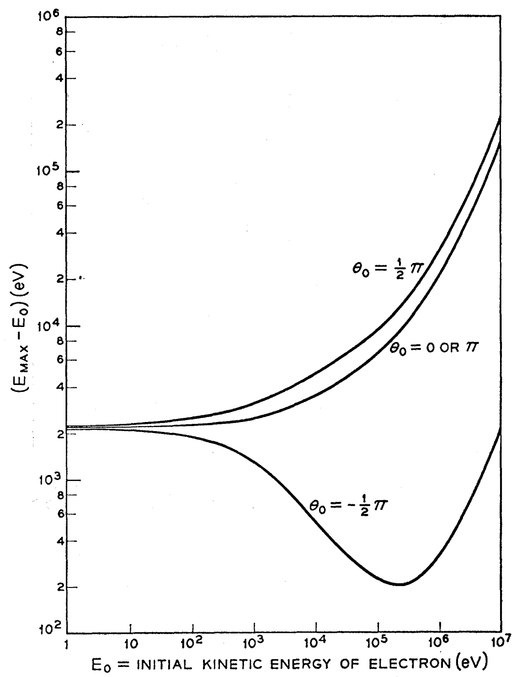

FIG. 8. Maximum increase in energy during one period of oscillation versus initial kinetic energy. $\omega=\Omega_0$, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

charged particle. A microwave oscillator, the cyclotron resonance backward-wave oscillator,[11] which has as its basis the phenomena of "magnetic bunching" produced by the B of the wave, has already been constructed. Similarly, "magnetic-bunching" has recently been advanced[12] as an explanation for whistler-triggered VLF radio emissions arising in the tenous plasma

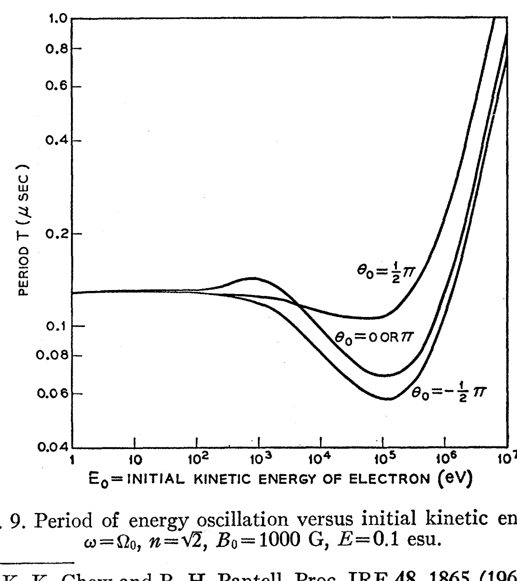

FIG. 9. Period of energy oscillation versus initial kinetic energy. $\omega=\Omega_0$, $n=\sqrt{2}$, $B_0=1000$ G, $E=0.1$ esu.

[11] K. K. Chow and R. H. Pantell, Proc. IRE **48**, 1865 (1960).

[12] N. Brice, J. Geophys. Res. **68**, 4626 (1963); S. F. Hansen, *ibid.* **68**, 5925 (1963).

surrounding the earth. Our theory relates to both these phenomena. It should also be possible to obtain an idea of how close to cyclotron resonance the linear theory of waves in a plasma is valid, since our solution enables one to get a handle on the nonlinear effects involved in the particle's motion. Work is currently underway on the use of the preceding solution in the investigation of the effects on the Van Allen particles of cyclotron-resonant VLF radio waves. Naturally occurring VLF energy in the form of whistlers may be an important loss mechanism in certain regions of the Van Allen belts, as already indicated by Dungey.[13] Finally, work is under way to determine if the synchronous acceleration of a particle, as described by Eq. (3.5), might be a possible mechanism for the acceleration of cosmic rays to their extremely high energies.

ACKNOWLEDGMENTS

The authors would like to thank Dr. W. L. Brown, Dr. J. R. Klauder, Dr. R. C. Miller, and Dr. P. M. Platzman for reading this manuscript and offering their valuable criticism. Dr. J. A. Morrison kindly pointed out to us an alternate method for deriving Eq. (A9) which does not involve the use of integral equations.

APPENDIX

In order to solve the coupled set of differential Eqs. (2.13) we define

$$\sigma(t)=\int_0^t d\tau\Omega(\tau), \tag{A1}$$

and note that Eqs. (2.13a) and (2.13b) are equivalent to the integral equations

$$p_x=p_{\perp 0}\cos[\sigma(t)+\alpha]+(eE/\omega)\int_0^t d\tau[\omega-k\dot{z}(\tau)] \times\sin[\sigma(t)-\sigma(\tau)+\omega\tau-kz(\tau)], \tag{A2a}$$

$$p_y=p_{\perp 0}\sin[\sigma(t)+\alpha]-(eE/\omega)\int_0^t d\tau[\omega-k\dot{z}(\tau)] \times\cos[\sigma(t)-\sigma(\tau)+\omega\tau-kz(\tau)]. \tag{A2b}$$

This equivalence may easily be demonstrated by differentiating Eqs. (A2a) and (A2b). The quantities $p_{\perp 0}$ and α are constants determined so that at $t=0$, $p_{x0}^2+p_{y0}^2=p_{\perp 0}^2$ and $p_{y0}/p_{x0}=\tan\alpha$. We may now substitute Eqs. (A2a) and (A2b) for p_x and p_y in Eq.

[13] J. W. Dungey, Planet. Space Sci. **11**, 591 (1963).

(2.13c) to obtain a nonlinear integrodifferential equation in z and p_z.

$$\dot{p}_z=-\Omega(B/B_0)\Big\{p_{\perp 0}\sin[\omega t-kz(t)-\sigma(t)-\alpha] + (eE/\omega)\int_0^t d\tau[\omega-k\dot{z}(\tau)]\cos[\omega t-kz(t)-\sigma(t) -\omega\tau+kz(\tau)+\sigma(\tau)]\Big\}. \quad \text{(A3)}$$

We now use Eq. (2.9) to rewrite Eq. (A3) as an equation in Ω. Integrating Eq. (2.9) from 0 to t gives

$$\omega t-kz(t)+kz_0=(d_1+1)\sigma(t)+d_2\omega t. \quad \text{(A4)}$$

From Eqs. (1.2), (1.3b), and (2.3), we have

$$\dot{p}_z=neB_0(\dot{\Omega}/\Omega^2). \quad \text{(A5)}$$

After appropriate substitution in Eq. (A3) we obtain

$$(\dot{\Omega}/\Omega^3)=-(E/eB_0^2)\Big\{p_{\perp 0}\cos[d_1\sigma(t)+d_2\omega t-\theta_0] + (eE/\omega)\int_0^t d\tau[(d_1+1)\Omega(\tau)+d_2\omega] \times\cos[d_1\sigma(t)+d_2\omega t-d_1\sigma(\tau)-d_2\omega\tau]\Big\}, \quad \text{(A6)}$$

where θ_0 is the initial angle between $\mathbf{v}_{\perp 0}$ and $\mathbf{E}$ of the wave

$$\theta_0=kz_0+\alpha+(\pi/2). \quad \text{(A7)}$$

The integrodifferential Eq. (A6) may be converted to an ordinary differential equation by noting that multiplication by $d_1\Omega(t)+d_2\omega$ makes Eq. (A6) an exact differential. Integrating this differential from 0 to t gives the equation

$$d_1/\Omega+d_2\omega/2\Omega^2 =(Ep_{\perp 0}/eB_0^2)\sin[d_1\sigma(t)+d_2\omega t-\theta_0] + (E^2/\omega B_0^2)\int_0^t d\tau[(d_1+1)\Omega(\tau)+d_2\omega] \times\sin[d_1\sigma(t)+d_2\omega t-d_1\sigma(\tau)-d_2\omega\tau] +d_1/\Omega_0+d_2\omega/2\Omega_0^2+Ep_{\perp 0}\sin\theta_0/eB_0^2. \quad \text{(A8)}$$

If we now differentiate Eq. (A6) with respect to time and subtract the resulting equation from $(d_1\Omega+d_2\omega)$ times Eq. (A8), the terms involving trigonometric functions vanish leaving the ordinary differential equation

$$\frac{d^2}{dt^2}\left(\frac{1}{2\Omega^2}\right)+\frac{d_2^2\omega^2}{2\Omega^2}+\frac{3d_1d_2\omega}{2\Omega} +\left(d_1^2-\frac{d_1d_2\omega}{\Omega_0}-\frac{d_2^2\omega^2}{2\Omega_0^2}-\frac{d_2\omega Ep_{\perp 0}\sin\theta_0}{eB_0^2}-\frac{E^2d_2}{B_0^2}\right) -\left(\frac{d_1^2}{\Omega_0}+\frac{d_1d_2\omega}{2\Omega_0^2}+\frac{d_1Ep_{\perp 0}\sin\theta_0}{eB_0^2}+(d_1+1)\frac{E^2}{\omega B_0^2}\right)\Omega=0. \quad \text{(A9)}$$

Using Eq. (1.3b), Eq. (A9) may be converted to an equivalent equation for $\mathcal{H}$. The resulting equation becomes an exact differential when multiplied by $\mathcal{H}(d\mathcal{H}/dt)$, and if we integrate this differential between 0 and t we arrive at Eq. (2.14) of Sec. II.