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μ^- Capture Rate in $(p\mu p)$*

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The probability of finding the ortho, $L=1$, $(p\mu p)$ system in an $s=\frac{1}{2}$ state at the time of μ^- capture, was calculated using the techniques developed by Weinberg. A wave function correct to first order in the muon-to-proton mass ratio was used in the calculation, giving a probability lying between 0.99933 and 0.99952. This result was due to the fact that the contact term in the spin-orbit Hamiltonian masked the effects of the other terms. The errors due to the use of the approximate wave function in this calculation were analyzed qualitatively and found to be small.

WE have studied the relative probability for μ^- capture by a proton in an $s=\frac{1}{2}$ state in the bound state $(p\mu p)^+$ system (ortho, $L=1$). Weinberg[1] has shown that the probability of the $(p\mu p)^+$ being in a spin $\frac{1}{2}$ state at the time of capture is

$$\xi=\sum_1^4 p_n\xi_n=\sum_1^4 \frac{2J_n+1}{6}\xi_n^2;\quad J_1=J_2=\tfrac{1}{2};\quad J_3=J_4=\tfrac{3}{2},$$

where

$$\xi_n=|\langle\psi_n|s=\tfrac{1}{2},\quad J_n\rangle|^2,$$

[$|\psi_n\rangle$=actual wave function of $(p\mu p)^+$ with J_n; $|s=\frac{1}{2}, J_n\rangle$=spin $\frac{1}{2}$ eigenstate of $(p\mu p)^+$ when spin interactions are neglected], and ξ_n are given in terms of the following expectation values taken over the internal coordinates:

$$E_1=e^2/m_p^2\left\{\frac{2\mu_p-1}{2}\left\langle\frac{r^2+r_{1\mu}^2-r_{2\mu}^2}{2r^2r_{1\mu}^3}-\frac{1}{r^3}\right\rangle-\mu_p\left\langle\frac{1}{r^3}\right\rangle\right\}$$

$$E_2=e^2/m_\mu m_p\left\langle\frac{r^2+r_{1\mu}^2-r_{2\mu}^2}{r^2r_{1\mu}^3}\right\rangle$$

$$E_3=\frac{8\pi}{3}\frac{e^2\mu_p}{m_\mu m_p}\langle\delta^3(r_{1\mu})\rangle$$

$$E_4=3/5\frac{e^2\mu_p}{m_\mu m_p}\left\langle\frac{1}{r_{1\mu}^3}-\frac{3(r^2+r_{1\mu}^2-r_{2\mu}^2)^2}{4r^2r_{1\mu}^5}\right\rangle'$$

$$E_5=3/5\frac{e^2\mu_p^2}{m_p^2}\left\langle\frac{1}{r^3}\right\rangle,$$

where $\mu_p=2.79$, r=interproton distance, and $r_{1\mu}$, $r_{2\mu}$ are the distances from the μ^- to protons 1 and 2, respectively.

The actual capture rate in $(p\mu p)^+$, $\omega_{p\mu p}$, is then given by

$$\omega_{p\mu p}=\xi\omega(\tfrac{1}{2})+(1-\xi)\omega(\tfrac{3}{2}),$$

where $\omega(\frac{1}{2})$ and $\omega(\frac{3}{2})$ are the capture rates in an $s=\frac{1}{2}$ and an $s=\frac{3}{2}$ state, respectively.

To get a numerical value for ξ the E_n expectation values must be obtained. The $L=1$ bound state wave function of the $(p\mu p)^+$ can be written, to first order, in the form $X_0(r)\psi_0(r,r_{1\mu},r_{2\mu})$, where ψ_0 is the normalized μ^- mesic wave function for the adiabatic case, i.e., a μ^- moving in the field of two infinitely heavy protons a distance r apart. $X_0(r)$ is the proton wave function for protons moving in the "potential well" set up by the μ^- meson. The ψ_0 used here is the exact ground state "$1s\sigma g$" mesic solution evaluated by Bates, Ledsham, and Stewart.[2] $X_0(r)$ including the self-energy correction term has been evaluated by both Gershtein and Zel'dovich,[3] and by Cohen, Judd, and Riddell.[4] The $X_0(r)$ used here is one evaluated by the author, based essentially on the Morse function method of Gershtein and Zel'dovich. The CJR wave function, done numerically, is no doubt more accurate but does not seem to be available presently. The present calculation could easily be adapted to the CJR wave function if required.

There are essentially four expectation values to be evaluated, two of which can be performed directly:

$$\left\langle\frac{1}{r^3}\right\rangle=\int_{r=0}^{\infty}\frac{X_0^2}{r^3}r^2dr\left\{\int_{\tau_\mu}\psi_0^2d\tau_\mu\right\}=\int_{r=0}^{\infty}\frac{X_0^2}{r}dr$$

and

$$\langle\delta^3(r_{1\mu})\rangle=\int_{r=0}^{\infty}X_0^2r^2\psi_0^2|_{r1\mu=0}dr.$$

The other two expectation values are

$$G\equiv\left\langle\frac{r^2+r_{1\mu}^2-r_{2\mu}^2}{r^2r_{1\mu}^3}\right\rangle;\quad F\equiv\left\langle\frac{1}{r_{1\mu}^3}-\frac{3(r^2+r_{1\mu}^2-r_{2\mu}^2)^2}{4r^2r_{1\mu}^5}\right\rangle'.$$

The prime on this last one indicates that the limiting process to be used is that of excluding a sphere about the $r_{1\mu}$ origin, and letting it shrink to zero (see, e.g., Bethe and Salpeter[5]).

These last two expectation values are evaluated in

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[1] S. Weinberg, Phys. Rev. Letters 4, 575 (1960).

[2] D. R. Bates, K. Ledsham, and A. L. Stewart, Phil. Trans. Roy. Soc. (London) **A246**, 215 (1953–1954).
[3] S. S. Gershtein and Ia. B. Zel'dovich, Zh. Eksperim. i Teor. Fiz. **35**, 649 (1958) [English transl.: Soviet Phys.—JETP **35**, 451 (1959)].
[4] S. Cohen, D. L. Judd, and R. J. Riddel, Phys. Rev. **119**, 384 (1960).
[5] H. A. Bethe and E. E. Salpeter, *Quantum Mechanics of One and Two Electron Atoms* (Academic Press Inc., New York, 1957), pp. 108, 180, etc.

spheroidal mesonic coordinates, the natural coordinates for the problem, and the ones in which ψ_0 is found. F can be broken up into three parts. One is an integral from a sphere of radius "α" about proton 1 out to a small "ϵ" shell determined by the natural spheroidal coordinates, ($\alpha \ll \epsilon$). After integrating, we let $\alpha \to 0$ and then let $\epsilon \to 0$. The second part of F can be done by partial integration in spheroidal coordinates. The last part of F, which is the nonsingular contribution, was calculated numerically on the IBM 7090 computer at Columbia. The G integral was done the same way as the last part of the F integral. In all cases the results were obtained in such a way that the final integration over "r" remained to be done by hand. Thus, any "nuclear" wave function X_0 can be readily adapted to the calculation. The numerical integrals are correct to within better than 1% of the exact integrals.

The results are tabulated below.

Letting $E_n' = (m_p/e^2)a_\mu{}^3 E_n$, where $a_\mu = \mu$ mesic Bohr radius in terms of m_μ', the reduced mesic mass relative to two protons, and everything is measured in units of $m_\mu = \hbar = c = 1$, we have

$$E_1' = -0.015 \pm 15\%$$
$$E_2' = +0.0955 \pm 2\%$$
$$E_3' = +3.76 \pm 2\%$$
$$E_4' = -0.102 \pm 6\%$$
$$E_5' = +0.025 \pm 10\%.$$

Now[1]

$$\xi_1 = 1 - \xi_2 = \tfrac{1}{2}\{1 + [1 + 2A^2]^{-1/2}\};$$
$$\xi_3 = 1 - \xi_4 = \tfrac{1}{2}\{1 + [1 + 5B^2]^{-1/2}\},$$

where

$$A = \frac{-4E_1 + 4E_2 - 5E_4 + 10E_5}{2E_1 + 7E_2 - 9E_3 - 5E_4 - 5E_5};$$

$$B = \frac{-4E_1 + 4E_2 + E_4 - 2E_5}{8E_1 + E_2 - 9E_3 + 4E_4 + 4E_5}.$$

(In Weinberg's paper the coefficient of E_5 in the denominator of A is 13.) We have

$$\xi = \tfrac{1}{3}\xi_1{}^2 + \tfrac{1}{3}(1-\xi_1)^2 + \tfrac{2}{3}\xi_3{}^2 + \tfrac{2}{3}(1-\xi_3)^2.$$

Since A and B are both $< 1/25$, we have

$$\xi_1 \sim 1 - A^2/2; \quad \xi_3 \sim 1 - 5B^2/4,$$

and

$$\xi \sim 1 - A^2/3 - 5B^2/3,$$

to 1 part in 10^5. Neglecting correlations in the errors in numerator and denominator we get an upper and lower limit for ξ:

$$\xi_{\max} = 0.99952; \quad \xi_{\min} = 0.99933.$$

The errors shown in E_2' and E_3' are due only to numerical integration. The larger error in E_4' is due to subtraction of nearly equal integrals. The errors in E_1' and E_5' take account of the fact that $\int (X^2/r)dr$ diverges in the Morse function approximation, and therefore some adjustment was made near the origin. This error could be eliminated with the CJR wave function.

An entirely different set of errors, not considered in the calculation, arise due to the limitations of the adiabatic approximation itself. The exact wave function for the $(p\mu p)^+$ can be written in the form[4]

$$\sum_0^\infty \bar{X}_i \psi_i,$$

where $\bar{X}_i$ are functions of r only and ψ_i are the complete set of adiabatic mesic orbitals of even symmetry. It can be further shown that only "σ" and "π" orbitals can contribute to the $L=1$ $(p\mu p)^+$ state. $\bar{X}_0$ differs from the X_0 used above, by order ϵ^2 (where $\epsilon = 2m_\mu/m_p$). The other $\bar{X}_i$'s are down from X_0 by at least order ϵ. For the case of the $\psi_{i\sigma}$, the effect on the E_i expectation values is essentially the same as for ψ_0. Thus the "relative" changes in the E_i's will be very small, in addition to the absolute changes being small. For the case of $\psi_{i\pi}$ we have considerably different effects. Since ψ_π states vanish along the interproton axis, they do not affect E_3, the contact term; however, they might give rise to small new $\sigma - \pi$ mixing terms from the Hamiltonian, as well as small relative shifts in E_2 and E_4. By noting the size of the contribution of E_3' to the denominators of A and B relative to the other terms, one notes that very large "relative" changes in the E_i's must occur if ξ is to be made to go below 0.9985, for example. Thus the values of ξ given above may be shifted due to the effects of higher orbitals, but this shift will be of the order of the width between the $\xi_{\max}$ and $\xi_{\min}$ already shown.

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