lished de Haas–van Alphen data of Joseph and Gordon, indicates that the two bands cross at a point about 0.04 $Å^{-1}$ from the ΓKM plane.

The transition region of magnetic breakdown across the small energy gap between these two bands has been studied in detail. The quantum oscillations which are observed in the galvanomagnetic properties are a transition region phenomena resulting from a perturbation of magnetic breakdown by the Landau levels of the needle. The structure of the oscillations indicates that these Landau levels are split into discrete spin levels having a very large effective g factor. The data are insufficient at present to allow us to determine if the quasiparticle states predicted by Pippard contribute significantly to σ_{11}.

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Macroscopic Theory of Helicons*

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Methods for treating boundary-value problems involving helicon waves (whistlers in solids) are developed and used for infinite plates and cylinders. The magnetoplasma inside the solid is assumed to be "driven" by means of external coils, which set up an oscillatory field with sinusoidal variation along the two coordinates tangential to the surface of the sample. The results show that in surfaces parallel to the external magnetic field an unusual surface mode is present; in this mode (for small resistivities) the power absorption due to Joule heating fails to decrease as the resistivity is decreased, until the limit of anomalous skin effect is reached, in which limit the lossy mode disappears. Several remarks are made concerning the various geometrical and physical properties of helicons.

1. INTRODUCTION

MAGNETOPLASMA oscillations obeying the same equations as atmospheric radio whistlers[1] were first reported in solids (sodium) by Bowers, Legéndy, and Rose[2]; in the context of solid-state physics they are known as *helicons*. The name is due to Aigrain,[3] who first proposed achievable experiments to detect them in solids.[4]

Sets of resonant frequencies in various materials, in addition to Na, were observed by Cotti, Wyder, and Quattropani[5,6] (In, Al, and Cu); Chambers and Jones[7] (Li, Na, K, Al, In, and InSb); Taylor, Merrill, and Bowers[8] (Cu, Ag, Au, Pb); Libchaber and Veilex[9] (InSb, at microwave frequencies); Kanai[10] (PbTe, at radio frequencies); and Khaikin, Edelman, and Mina[11] (Bi, at microwave frequencies). Detailed experimental studies of the mode structure in rectangular parallelepipeds were made by Rose, Taylor, and Bowers[12] (Na), and, with more refined detection techniques, by Merrill, Taylor, and Goodman[13] (Na). Cotti, Wyder, and Quattropani[5,6] attempted a theoretical justification for the semiempirical rule[12] obeyed by the resonant frequencies, however, the present author disagrees with their formulation of the boundary-value problem. Chambers and Jones[7] exploited the helicon resonance

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phenomenon as a means of measuring Hall coefficients with high precision. An abstract proof for the existence of helicon modes (in samples with zero resistivity) was given by Legéndy.[14]

The macroscopic treatment of helicons may be conveniently started from the equation

$$\mathbf{E}+R\mathbf{j}\times\mathbf{B}=\rho\mathbf{j}\,, \tag{1.1}$$

where $\mathbf{E}$, $\mathbf{B}$, and $\mathbf{j}$ are the electric field, magnetic field, and electric current; R and ρ are the Hall coefficient and resistivity, respectively (all in rationalized mks units).

This article is concerned with the consequences of (1.1). Equation (1.1) will be assumed to hold true inside the sample carrying helicons; the field outside the sample will be approximated by the product of a static field and the time-dependent factor $\exp(i\omega\tau)$. Equation (1.1) was derived from the Boltzmann equation by Cotti, Quattropani, and Wyder.[6]

Besides the standard assumptions ensuring that (1.1) correctly relates steady currents and fields, we underline, for emphasis, the assumptions ensuring that it correctly relates currents and fields depending on position and time. These assumptions require the wavelength and time period of helicons to be much larger than the relevant parameters of the microscopic conduction mechanism. [The article of Chambers and Jones[7] contains a thorough list of the assumptions involved in (1.1).]

When the wavelength becomes small, the helicon phenomenon becomes dependent on the microscopic properties of the medium. This case is beyond the scope of the present paper, it is treated in Refs. 15–26.

The dispersion law for short-wavelength helicons propagating along the magnetic field was derived by Sheard,[15] starting from the results of Rodriguez[16] and Kjeldaas[17] obtained for acoustic absorption. A thorough treatment of the short-wavelength and high-frequency limit was given by Kaner and Skobov.[18] Taylor, Merrill, and Bowers[19] observed an edge in the absorption of short-wavelength helicons in sodium and explained it in terms of the Doppler-shifted cyclotron resonance predicted by Stern[20] (similar to the Doppler-shifted cyclotron resonance connected with ultrasonic waves, discussed by Kjeldaas[17]). Kaner and Skobov,[18] Miller,[21] and Quinn[22] predicted giant quantum oscillations in the absorption of helicons; Stern and Callen[23] predicted interactions between helicons and magnons; Kaner and Skobov,[18] Langenberg and Bok,[24] and Quinn and Rodriguez[25] predicted interactions between helicons and phonons. The latter interaction was observed by Grimes[26] in potassium.

The present paper is organized as follows: Sec. 2 deals with helicons in an infinite medium; in Secs. 3–6 boundaries are introduced. In Sec. 5 the boundary-value problem is solved for an infinite plate perpendicular to the external magnetic field, an infinite plate, and an infinite cylinder parallel to the external magnetic field. In each case an oscillatory "driving field," sinusoidally varying along the two coordinates tangential to the boundary, is assumed, and the response field is computed as a function of the frequency and the tangential wave vector. In Sec. 6 is it shown that, ignoring anomalous skin effect, the Ohmic loss in the boundary surfaces parallel to the external field does not tend to zero in the limit of zero resistivity. Under anomalous skin-effect conditions the surface mode responsible for the loss disappears.

For the sake of symmetry and simplicity in what follows, we shall call all fields inside the sample helicon fields, instead of restricting the term to the freely propagating component.

2. HELICON WAVES IN AN INFINITE MEDIUM

Write $\mathbf{B}=\mathbf{B}_0+\mathbf{b}(\mathbf{r},t)$, where $\mathbf{B}_0$ is the (uniform and constant) external magnetic field. Then, by assuming $\mathbf{B}_0\gg\mathbf{b}$, linearize (1.1):

$$\mathbf{E}+R\mathbf{j}\times\mathbf{B}_0=\rho\mathbf{j}\,. \tag{2.1}$$

Take the curl of both sides and combine with Maxwell's equations, neglecting displacement current. Letting the z axis point along the field $\mathbf{B}_0$ the result is:

$$\frac{\mu_0}{\rho}\frac{\partial\mathbf{b}}{\partial t}+u\nabla\times\frac{\partial\mathbf{b}}{\partial z}+\nabla\times(\nabla\times\mathbf{b})=0\,, \tag{2.2}$$

$$u\equiv -B_0R/\rho\cong\omega_c\tau\,.$$

Assuming plane-wave solutions of the form

$$\begin{aligned}\mathbf{b}&=\mathbf{b}(0)\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})]\,,\\ \mathbf{k}&=(\alpha,\beta,\gamma)\,,\\ \mathbf{r}&=(x,y,z)\,,\end{aligned} \tag{2.3}$$

Eq. (2.2) becomes:

$$(\mu_0/\rho)i\omega\mathbf{b}-u\gamma\mathbf{k}\times\mathbf{b}+k^2\mathbf{b}=0\,,\qquad k^2=\alpha^2+\beta^2+\gamma^2\,. \tag{2.4}$$

Written out in detail, (2.4) is a set of three coupled homogeneous linear equations for the three components of the constant vector $\mathbf{b}(0)$. The secular equation has

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one root leading to a physically unacceptable solution; dividing out that root, the secular equation becomes:

$$[(\mu_0/\rho)\omega - ik^2]^2 - u^2k^2\gamma^2 = 0. \tag{2.5}$$

From Eq. (2.4), aside from an arbitrary constant factor,

$$\begin{aligned}(b_0)_x &= \alpha\gamma + i(\beta/u\gamma)[(\mu_0/\rho)\omega - ik^2],\\ (b_0)_y &= \beta\gamma - i(\alpha/u\gamma)[(\mu_0/\rho)\omega - ik^2],\\ (b_0)_z &= -\alpha^2 - \beta^2.\end{aligned} \tag{2.6}$$

Factorizing (2.5), the dispersion relation may be rewritten in the following simpler form[5,6]:

$$\omega = (\rho/\mu_0)(uk\gamma + ik^2), \tag{2.7}$$

and the solution[2]:

$$\mathbf{b} = (\alpha\gamma + ik\beta, \beta\gamma - ik\alpha, -\alpha^2 - \beta^2)\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})]. \tag{2.8}$$

In the limit $\rho \to 0$ the product $\rho u = -B_0R$ remains unaltered and the second term in (2.7) tends to zero, thus (2.7) becomes[2–4]

$$\omega = -\mu_0^{-1}B_0Rk\gamma. \tag{2.9}$$

If k_ξ, k_η, k_ζ are any orthogonal components of $\mathbf{k}$, the component γ can be expressed in terms of these, and so can k^2. Thus (2.5) interrelates the four (complex) quantities ω, k_ξ, k_η, k_ζ, and if any three of these are specified, it can be solved for the fourth. In the cases to be treated below, ω, k_ξ, k_η are given real numbers; Eq. (2.5) is a quartic equation in k_ζ, and therefore in general it yields four different complex roots. For each of these, k^2 is well defined, but k has two values. The one to be used in (2.8) is the one satisfying (2.7). [Because each of the four k_ζ satisfies (2.5), and (2.5) is merely the square of (2.7), one and only one square root of k^2 for each k_ζ necessarily satisfies (2.7).]

The above discussion should replace the remarks connected with a "±" alternative in Eq. (3) of Bowers, Legéndy, and Rose[2]; the discussion concerning this point in Ref. 2 is confusing.

In the remainder of this section we shall make several simple remarks pertaining to helicons.

Direct computation from (2.8) shows that for a plane wave of helicons

$$\nabla\times\mathbf{b} = k\mathbf{b},$$
$$\mathbf{j} = \mu_0^{-1}\nabla\times\mathbf{b} = \mu_0^{-1}k\mathbf{b}. \tag{2.10}$$

Thus, when k is real, the current associated with a single plane wave is everywhere parallel to the magnetic field. Let us multiply Eq. (2.2) through by ρ/μ_0, let $\rho\to 0$, replace the operator $\partial/\partial z$ by $-i\gamma$ and the operator ∇ by the vector $-i\mathbf{k}$. Then (2.2) becomes:

$$\partial\mathbf{b}/\partial t = \boldsymbol{\omega}\times\mathbf{b}, \tag{2.11}$$
$$\boldsymbol{\omega} = -\mu_0^{-1}B_0R\gamma\mathbf{k} = \text{constant vector.}$$

Equation (2.11) can be recognized as the precession equation. There are two ways in which the vector $\boldsymbol{\omega}$ can be real: If all components of $\mathbf{k}$ are purely real, or if all components of $\mathbf{k}$ are purely imaginary. A glance at the expression for $\boldsymbol{\omega}$ shows that when $R<0$, the scalar product $\boldsymbol{\omega}\cdot\mathbf{B}_0$ is positive in the former case and negative in the latter. This means that if the vectors rotate around the field lines in a sense agreeing with the cyclotron rotation of the carriers, the waves propagate freely; if they rotate oppositely, the waves are exponentially damped.

When $\mathbf{k}$ is real, the instantaneous spacial pattern of fields might form a right-handed screw, or a left-handed screw; the screw sense is determined by the sign of k. To show this, let $\mathbf{a}=(0,0,k^2)$, $\hat{k}=\mathbf{k}/k$, $\mathbf{G}_1=\hat{k}\times\mathbf{a}=(k\beta,-k\alpha,0)$, and $\mathbf{G}_2=\hat{k}\times(\hat{k}\times\mathbf{a})=(\alpha\gamma,\beta\gamma,-\alpha^2-\beta^2)$. Clearly, when k and the components of $\mathbf{k}$ are real, $\mathbf{G}_1$ and $\mathbf{G}_2$ are two real vectors of equal length, perpendicular to each other and to $\mathbf{k}$. Now, Eq. (2.8) can be written as follows:

$$\mathbf{b} = (\mathbf{G}_2 + i\mathbf{G}_1)\exp[i(\omega t - \mathbf{k}\cdot\mathbf{r})].$$

This is the standard form of a circularly polarized wave; the screw sense of the instantaneous pattern is determined by the right- or left-handedness of the Cartesian coordinate system $\mathbf{G}_1\mathbf{G}_2\mathbf{k}$. Note that changing the sign of k, changes the sign of $\mathbf{G}_1$, but leaves $\mathbf{G}_2$ unchanged. One can see at once that when k is negative, the screw sense is right-handed; when k is positive, the screw sense is left-handed. [Note that, by Eq. (2.10), $\mathbf{j}$ and $\mathbf{b}$ are antiparallel when the screw sense is right-handed and parallel when it is left-handed.]

Of course, when the wave vector is made complex, the geometrical clarity of the situation fades.

So far, no mention has been made of the electric field patterns. If the current density is specified, (2.1) gives the electric field explicitly; when $\rho\to 0$, this relation takes the form

$$\mathbf{E} = -R\mathbf{j}\times\mathbf{B}_0. \tag{2.12}$$

As seen from (2.10), the currents in any given plane-wave form a pattern identical to the magnetic field pattern, except for a multiplicative constant. Thus all remarks made for magnetic fields can be repeated for currents. However, they cannot be repeated for electric fields. Equation (2.12) shows that when $\rho\to 0$, $\mathbf{E}$ cannot have a component along $\mathbf{B}_0$. Thus, unless $\alpha=\beta=0$, the electric field has a longitudinal part, i.e., a part with $\nabla\cdot\mathbf{E}\neq 0$, as well as a transverse part. If the reader has not encountered a similar situation before, he may wonder if this is compatible with the assumption of neglecting displacement currents. The latter assumption brings one of Maxwell's equations to the form $\nabla\times\mathbf{H}=\mathbf{j}$; therefore, the electric current is represented as the curl of a vector and it can have no divergence. This means that space charges cannot periodically build up and disappear, thus the electric field cannot have a longitudinal component. The paradox disappears in the light of the actual magnitudes of the quantities involved. The current does have a longitudinal component, but it is about 10^{18} times smaller than the transverse component (the ratio between conduction current and displacement current densities at 10 cps is about 10^{18}). Thus, space charges do periodically appear and disappear, but they are very small. The reason why the longitudinal component of electric field is still of the same order of magnitude as the transverse component is that **E** itself is very small; it is about 10^9 times smaller than that which would correspond with the same magnetic field in a freely propagating vacuum wave. The number 10^9 is the ratio of the speed of light to the helicon phase velocity at 10 cps.

The dispersion relation (2.7) does not contain the assumption that the resistivity ρ is small. For one can rearrange Eq. (2.7) to read as follows:

$$k=\left[\frac{-i\omega\mu_0/\rho}{1+iu(\gamma/k)}\right]^{1/2}.$$

In the limit $u\cong\omega_c\tau\rightarrow 0$ this reduces to the standard skin-effect formula.

The following remark[27] concerns helicons whose amplitude is not small. Suppose $\mathbf{b}$ is not negligible compared to $\mathbf{B}_0$. Then a nonlinear term

$$-(\mu_0 R/\rho)\nabla\times(\mathbf{j}\times\mathbf{b}) \tag{2.13}$$

must be added to the left-hand side of (2.2). And yet, a single plane wave of form (2.8), obeying the dispersion relation (2.7), still satisfies the equation. For, by virtue of (2.10) in such a wave $\mathbf{j}=\mu_0^{-1}k\mathbf{b}$, and in the nonlinear term (2.13)

$$\mathbf{j}\times\mathbf{b}=(1/\mu_0)k\mathbf{b}\times\mathbf{b}\equiv 0.$$

Thus, (2.13) identically vanishes. The sum of two solutions is, as usual in nonlinear equations, not necessarily a solution.

3. BOUNDARIES

The problem of dealing with boundaries has been first considered by Cotti, Wyder, and Quattropani.[5] They assumed that the boundary condition to be satisfied is that all three components of **E** must be continuous at the boundary, and no electric current should cross the boundary. In their second paper[6] the authors drop the former condition and retain only the latter. (Indeed, the normal component of **E** is, in general, discontinuous.) Chambers and Jones,[7] in their treatment of driven oscillations (in an infinite slab) use the condition that the tangential components of magnetic field must be continuous across the boundary; in calculating frequencies of free oscillation they use the current condition (i.e., the requirement that currents do not cross the boundary).

We wish to make a few comments on these boundary conditions. Since all three articles deal with nonferromagnetic materials of finite conductivity, there can be no surface currents in either, and all three components of the magnetic fields must be continuous across the boundary. This boundary condition, together with the assumption that the vacuum fields are static, implies that the current condition is satisfied. (For, if a vector is continuous across a boundary, the normal component of its curl is also continuous, but inside the sample $\nabla\times\mathbf{H}$ is the current; outside the sample $\nabla\times\mathbf{H}$ is zero.) However, the assumption that a field satisfies the current condition clearly does not ensure the continuity of **H**. Furthermore, in Sec. 5, we shall be able to construct a solution satisfying the current condition in a finite cylinder for any given frequency. The latter construction dramatizes the criticism against identifying a frequency as a frequency of resonance merely on the ground that at that frequency there exists a helicon field satisfying the current condition.

In the present article we shall use the boundary condition that all components of the magnetic field are continuous. Since the problem is quasistatic, this implies that the boundary conditions on electric field are automatically satisfied. The latter statement may be verified as follows:

Assume that the field $\mathbf{b}(\mathbf{r})\exp(i\omega t)$ satisfies (2.2) inside the sample, satisfies $\nabla\times\mathbf{b}=0$ and $\nabla\cdot\mathbf{b}=0$ outside the sample, and is continuous at the boundary. Construct an electric field $\mathbf{E}_1$ defined outside the sample such that $\nabla\times\mathbf{E}_1=-i\omega\mathbf{b}(\mathbf{r})$. (This can always be done by use of the Green's function for the curl operator, i.e., in parallelism with the elementary calculation of a static magnetic field from the current distribution.) The electric field inside the sample, $\mathbf{E}_{\text{ins}}$, is uniquely determined from $\mathbf{b}$ through (2.10) and (2.1); one can easily check that automatically, $\nabla\times\mathbf{E}_{\text{ins}}=i\omega\mathbf{b}$. Write the electric field outside the sample as $\mathbf{E}_1+\mathbf{E}_2$. The boundary condition requires that the tangential components of electric field be continuous at the boundary; thus, with $\mathbf{E}_1$ and $\mathbf{E}_{\text{ins}}$ given, $(\mathbf{E}_2)_{\text{tang}}$ is specified at the boundary: $(\mathbf{E}_2)_{\text{tang}}=(\mathbf{E}_{\text{ins}}-\mathbf{E}_1)_{\text{tang}}$. From the continuity of the normal component of $\mathbf{b}$ it follows that the line integral of $\mathbf{E}_2$ over any closed curve lying on the surface vanishes. Thus a scalar potential φ can be defined on the surface in such a way that $(\mathbf{E}_2)_{\text{tang}}=(-\nabla\varphi)_{\text{tang}}$. (The definition can be made unique by taking into account the net charge on the sample.) The problem then reduces to extending φ into all space in such a way that $\nabla^2\varphi=0$ throughout and $\varphi\rightarrow 0$ at infinity. This problem is a well-known case of the Dirichlet problem,[28] and can always be solved.

The problem of driven oscillations will be formulated in parallelism with the standard problem of "reflection and refraction" in optics.

Imagine the vacuum field decomposed into an "incident" component or "driving field" defined as the field set up by the driving currents alone (i.e., as if the sample were removed), and a "reflected" component

[27] E. F. Johnson (private communication).

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due to the currents and charges in the sample. (Note that the former may be singular at infinity, the latter may be singular inside the sample but not vice versa.) The "transmitted wave" is the helicon field in the sample so chosen as to satisfy the boundary conditions at the samples surface.

As indicated in the Introduction, in Sec. 5 we shall consider three types of infinite samples: A plate perpendicular to the magnetic field, a plate parallel to the magnetic field, and a cylinder parallel to the magnetic field. For all three of these the driving field will be assumed to be a sinusoidal function of the two coordinates parallel to the surface; the corresponding components of the wave vector are the two given quantities referred to in the previous section as k_ξ and k_η. With these specified, the field equations restrict the third component to a choice of two in the vacuum and a choice of four in the conductor. Of the former two, one leads to singular behavior at infinity—that must be chosen as the incident wave; of the latter four, some may lead to singular behavior inside the sample—those must be disregarded. The complex constants multiplying the allowed fields (counting the reflected field too) are the only unknowns of the problems, and for them the boundary conditions provide the necessary and sufficient number of equations.

From the results it is then possible, if desired, to obtain "resonance curves" by fixing k_ξ and k_η and varying the frequency.

4. REFLECTION AND REFRACTION

The following two simple boundary-value problems shall serve to illustrate the method outlined in Sec. 3. The solutions in Sec. 5 consist of straightforward synthesis of the observations made in 4A and 4B below.

A. Conducting Front in the x,y Plane; α, β, ω Specified

Suppose the region $z\leqq 0$ is filled with conductor of resistivity ρ, satisfying (2.1), and the rest of space is vacuum. Assume the driving field has a frequency ω and its variation along x and y is wavelike; α, β, and ω are specified by the problem. The component α is real; without loss of generality we can set $\beta=0$.

The dependence of the fields on x, y, and t is described by the factor $\exp i(\omega t-\alpha x)$. We remark that the tangential phase velocity ω/α is not the speed of light in vacuo, but is many times smaller. (In a typical experiment in sodium[2] it is of order 10^9 times smaller.) The frequency ω is dictated solely by the oscillator connected to the driving coils, and the wave number α by the geometrical configuration of these coils. Since the problem has translational symmetry along x, y, and t, and is governed by linear equations, standard symmetry argument shows that the reflected field and the helicon field must have the same sinusoidal variation as the incident field. We shall denote the incident and reflected field by $\mathbf{b}_0$ and $\mathbf{b}_r$ respectively.

Since $\mathbf{b}_0$ cannot become infinite as $z\rightarrow -\infty$ and $\mathbf{b}_r$ cannot become infinite as $z\rightarrow\infty$, $\mathbf{b}_0$ and $\mathbf{b}_r$ must have the form:

$$\begin{Bmatrix}\mathbf{b}_0\\ \mathbf{b}_r\end{Bmatrix}=\begin{Bmatrix}b_0\\ b_r\end{Bmatrix}\times\xi(-i\alpha,0,\pm|\alpha|)e^{\pm|\alpha|z},\qquad(4.1)$$

$$\xi=e^{i(\omega t-\alpha x)},$$

where the upper line corresponds with the upper sign, and the lower line with the lower sign. The scalar quantities b_0 and b_r are constants (possibly complex); b_0 is given, b_r is unknown.

The components of wave vector called k_ξ and k_η in Secs. 2 and 3 are here α and β; and, as was said there, for the third component k_ζ (in the present case γ) the field equations allow four different values which can be found from (2.5). They are

$$\begin{Bmatrix}\gamma_1=-\gamma_3\\ \gamma_2=-\gamma_4\end{Bmatrix}=\alpha[-p\pm(p^2-q)^{1/2}]^{1/2},\qquad(4.2\text{a})$$

$$p=\frac{1+iu(\omega/\omega_1)+\frac{1}{2}u^2}{1+u^2},\quad q=\frac{[u(\omega/\omega_1)-i]^2}{1+u^2},$$

$$\omega_1=-\mu_0^{-1}B_0R\alpha^2$$

$$\operatorname{Im}\gamma_1\geqq 0,\quad \operatorname{Im}\gamma_2\geqq 0.\qquad(4.2\text{b})$$

To write down the helicon solutions (2.8) it is necessary to evaluate the quantities k corresponding to each γ. These are "defined" in (2.7), which can be rearranged as follows:

$$k_n=(1/u\gamma_n)[(\mu_0/\rho)\omega-ik_n^2],\quad n=1,2,3,4.\quad(4.3)$$

From (4.2) and (4.3) it is seen that the four values k_n are pairwise connected by the relations

$$k_3=-k_1,\quad k_4=-k_2.$$

With these and (2.8), the four helicon waves are

$$\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_3\end{Bmatrix}=\begin{Bmatrix}b_1\\ b_3\end{Bmatrix}\times\xi(\pm\gamma_1,\mp ik_1,-\alpha)e^{\mp i\gamma_1 z},$$

$$\begin{Bmatrix}\mathbf{b}_2\\ \mathbf{b}_4\end{Bmatrix}=\begin{Bmatrix}b_2\\ b_4\end{Bmatrix}\times\xi(\pm\gamma_2,\mp ik_2,-\alpha)e^{\mp i\gamma_2 z},\qquad(4.4)$$

$$\xi=e^{i(\omega t-\alpha x)}.$$

One can check, by going to the limit $\rho\rightarrow 0$, that $\mathbf{b}_1$ and $\mathbf{b}_2$ differ in their direction of circular polarization; so do $\mathbf{b}_3$ and $\mathbf{b}_4$. By definition [see (4.2)], γ_1 and γ_2 have positive imaginary parts. (For the case $\rho=0$, when γ_1 and γ_3 are purely real, γ_1 is chosen by means of a limiting procedure $\rho\rightarrow 0^+$. Below, we shall assume that ρ is never strictly zero.) Therefore $\mathbf{b}_3$ and $\mathbf{b}_4$

diverge at $z\to-\infty$ and must be excluded. (In Sec. 5A, where the sample is finite in the z direction, we shall have need for all four solutions.) The constants b_1 and b_2 together with b_r in (4.1) are the unknowns of the problem. For the three unknowns the boundary condition that $\mathbf{b}$ is continuous at $z=0$ furnishes three equations:

$$\begin{aligned}\gamma_1 b_1+\gamma_2 b_2+i\alpha b_r&=-i\alpha b_0,\\ -ik_1b_1-ik_2b_2&=0,\\ \alpha b_1+\alpha b_2+|\alpha|b_r&=|\alpha|b_0.\end{aligned}$$

The solution is:

$$b_1=\frac{-2\alpha}{|\alpha|A+A'}\frac{b_0}{k_1},\quad b_2=\frac{2\alpha}{|\alpha|A+A'}\frac{b_0}{k_2},\quad b_r=\frac{-|\alpha|A+A'}{|\alpha|A+A'},$$

$$A=\frac{1}{k_1}-\frac{1}{k_2},\quad A'=-\frac{i\gamma_1}{k_1}+\frac{i\gamma_2}{k_2}.$$

B. Conducting Front in the y, z Plane; β, γ, ω Specified

Let the region $x\leqq 0$ be filled with conductor, and the rest of space be vacuum. Assume that, similarly to the case in 4A, the driving field imposes a sinusoidal dependence of the fields on the tangential coordinates and time; the variation is characterized by the three real quantities, β, γ, and ω. Note that, because the z direction is singled out by the vector $\mathbf{B}_0$, neither β nor γ can be set equal to zero without loss of generality. By the same arguments that lead to (4.1), the incident and reflected wave are

$$\begin{Bmatrix}\mathbf{b}_0\\ \mathbf{b}_r\end{Bmatrix}=\begin{Bmatrix}b_0\\ b_1\end{Bmatrix}\times\xi(\pm\kappa,-i\beta,-i\gamma)e^{\pm\kappa x},$$

$$\kappa=(\beta^2+\gamma^2)^{1/2},$$

$$\xi=e^{i(\omega t-\beta y-\gamma z)}.$$

Equation (2.5) is most conveniently solved for k^2; the expression for k^2 only involves γ and not β.

$$k^2=-\tfrac{1}{4}u^2\gamma^2\left[1\pm\left(1+i\frac{4}{u}\frac{\omega}{\omega_2}\right)^{1/2}\right]^2,\tag{4.5}$$

$$\omega_2=-\mu_0^{-1}B_0R\gamma^2;$$

from this,

$$\begin{Bmatrix}\alpha_1=-\alpha_3\\ \alpha_2=-\alpha_4\end{Bmatrix}=\left\{-\kappa^2-\tfrac{1}{4}u^2\gamma^2\left[1\pm\left(1+i\frac{4}{u}\frac{\omega}{\omega_2}\right)^{1/2}\right]^2\right\}^{1/2}$$

$$\mathrm{Im}\alpha_1\geqq 0,\quad \mathrm{Im}\alpha_2\geqq 0.\tag{4.6}$$

The corresponding four values of k are computed by means of (4.3); they are pairwise related as follows:

$$k_1=k_3,\quad k_2=k_4.$$

For the purposes of the present arrangement it is desirable to replace (2.8) by

$$(-\kappa^2,\alpha\beta+ik\gamma,\alpha\gamma-ik\beta)\exp[i(\omega t-\mathbf{k}\cdot\mathbf{r})],\tag{4.7}$$

which differs from (2.8) only in a constant factor $-\kappa^2/(\alpha\gamma+ik\beta)$. The four helicon waves are:

$$\begin{aligned}\begin{Bmatrix}\mathbf{b}_1\\ \mathbf{b}_3\end{Bmatrix}&=\begin{Bmatrix}b_1\\ b_3\end{Bmatrix}\times\xi(-\kappa^2,\pm\alpha_1\beta+ik_1\gamma,\pm\alpha_1\gamma-ik_1\beta)e^{\mp i\alpha_1 x},\\ \begin{Bmatrix}\mathbf{b}_2\\ \mathbf{b}_4\end{Bmatrix}&=\begin{Bmatrix}b_2\\ b_4\end{Bmatrix}\times\xi(-\kappa^2,\pm\alpha_2\beta+ik_2\gamma,\pm\alpha_2\gamma-ik_2\beta)e^{\mp i\alpha_2 x}.\end{aligned}\tag{4.8}$$

Of these, $\mathbf{b}_3$ and $\mathbf{b}_4$ diverge when $x\to-\infty$ and must be dropped. (In Sec. 5B, where the sample has a finite extension in the x direction, there will be need for all four solutions.) The coefficients b_1 and b_2 together with b_r are the unknowns of the problem; for them the boundary conditions furnish the following three equations:

$$\begin{aligned}(-\kappa^2)b_1+(-\kappa^2)b_2-(-\kappa)b_r&=\kappa b_0,\\ (\alpha_1\beta+ik_1\gamma)b_1+(\alpha_2\beta+ik_2\gamma)b_2-(-i\beta)b_r&=-i\beta b_0,\\ (\alpha_1\gamma-ik_1\beta)b_1+(\alpha_2\gamma-ik_2\beta)b_2-(-i\gamma)b_r&=-i\gamma b_0.\end{aligned}$$

The solutions are

$$b_1=\frac{-2}{\kappa A+A'}\frac{1}{k_1}b_0,\quad b_2=\frac{2}{\kappa A+A'}\frac{1}{k_2}b_0,\quad b_r=\frac{-\kappa A+A'}{\kappa A+A'},$$

$$A=\frac{1}{k_1}-\frac{1}{k_2},\quad A'=\frac{-i\alpha_1}{k_1}+\frac{i\alpha_2}{k_2}.$$

Finally, consider Eq. (4.6) in the limit $u\gg 4\omega/\omega_2$. At the frequency $\omega=(\kappa/\gamma)\omega_2$ the quantities α_2 and α_4 vanish, which means that the helicon wave vector is tangential to the boundary. Above this frequency α_2 and α_4 are real, but below it they are imaginary. The phenomenon is recognized as a phenomenon familiar from geometrical optics: total reflection; below the frequency $(\kappa/\gamma)\omega_2$ the tangential phase velocity of the artificial vacuum wave becomes lower than can be matched by helicons.

The same does not occur when the conducting surface is parallel to the x,y plane. In the limit $u\to\infty$, $\rho\to 0$ the phase velocity is given by Eq. (2.9): $\omega/k=-\mu_0^{-1}B_0R\gamma$. When γ is fixed by the driving field, the phase velocity is fixed, but if only α and β are fixed by the driving field, the phase velocity can be made smaller than any arbitrary quantity by choosing γ small enough.

5. THREE SIMPLE RESONANCE PROBLEMS

Sections 5A, 5B, and 5C contain the solutions of three resonance problems that can be solved exactly. The term "resonance problem" is intended to underline the fundamental difference between these three problems and the two described in Sec. 4. It can be verified at once that, in each of the problems below, for a fixed wave vector of the driving field there exist nonzero (complex) frequencies for which the secular determinant vanishes; the same is not true in 4A and 4B.

A. Infinite Plate Perpendicular to B_0

Let the region $-a\leqq z\leqq a$ be filled with conductor and let the rest of space be vacuum. As in Sec. 4A, the two specified components of the wave vector are α and β; α is real; without loss of generality we set $\beta=0$. The allowed values of γ are those given in (4.2).

Suppose the incident field is of the form

$$\mathbf{b}_0=b_0\xi(-i\alpha\cosh\alpha z,0,\alpha\sinh\alpha z),\quad |z|\geqq a,\qquad \xi=e^{i(\omega t-\alpha x)}. \tag{5.1}$$

Arguments similar to those used in the previous section show that the reflected field in the two vacuum regions $z>a$ and $z<-a$ is, respectively:

$$\begin{aligned}\mathbf{b}_r&=b_{r1}\xi(-i\alpha,0,-|\alpha|)e^{-|\alpha|(z-a)},&&\text{in region } z>a,\\ &=b_{r2}\xi(-i\alpha,0,|\alpha|e^{|\alpha|(z+a)},&&\text{in region } z<-a,\end{aligned} \tag{5.2}$$

where b_{r1} and b_{r2} are constants, as yet undetermined. Of the four helicon fields (4.4) all four will be needed; their amplitudes b_1, b_2, b_3, b_4 are further unknowns of the problem. For the six unknown constants the boundary conditions furnish six linear equations; three express the condition that at the surface $z=a$, all three components of the magnetic field are continuous; the other three express the same for the surface $z=-a$. For the sake of illustration, the six equations are written out below:

$$\begin{aligned}&\gamma_1e^{-i\gamma_1a}b_1+\gamma_2e^{-i\gamma_2a}b_2-\gamma_1e^{i\gamma_1a}b_3-\gamma_2e^{i\gamma_2a}b_4\\&\qquad-i\alpha(-b_{r1})=-i\alpha\,(\cosh\alpha a)b_0,\\&-ik_1e^{-i\gamma_1a}b_1-ik_2e^{-i\gamma_2a}b_2+ik_1e^{i\gamma_1a}b_3\\&\qquad+ik_2e^{i\gamma_2a}b_4=0,\\&-\alpha e^{-i\gamma_1a}b_1-\alpha e^{-i\gamma_2a}b_2-\alpha e^{i\gamma_1a}b_3-\alpha e^{i\gamma_2a}b_4\\&\qquad-|\alpha|(-b_{r1})=\alpha\,(\sinh\alpha a)b_0,\\&\gamma_1e^{i\gamma_1a}b_1+\gamma_2e^{i\gamma_2a}b_2-\gamma_1e^{-i\gamma_1a}b_3-\gamma_2e^{-i\gamma_2a}b_4\\&\qquad-i\alpha(-b_{r2})=-i\alpha\,(\cosh\alpha a)b_0,\\&-ik_1e^{i\gamma_1a}b_1-ik_2e^{i\gamma_2a}b_2+ik_1e^{-i\gamma_1a}b_3\\&\qquad+ik_2e^{-i\gamma_2a}=0,\\&-\alpha e^{i\gamma_1a}b_1-\alpha e^{i\gamma_2a}b_2-\alpha e^{-i\gamma_1a}b_3-\alpha e^{-i\gamma_2a}b_4\\&\qquad+|\alpha|(-b_{r2})=-(\sinh\alpha a)b_0.\end{aligned} \tag{5.3}$$

It can be seen at once that the trial relationships $b_1=-b_3$, $b_2=-b_4$, $b_{r1}=b_{r2}$ split the set into two identical sets of three equations. The physical explanation of this is given in the following symmetry argument.

A boundary value problem involving helicons can be said to possess reflection symmetry about a plane if reflection of the sample and $\mathbf{B}_0$ about the plane, followed by reversal of the vector $\mathbf{B}_0$, turns both the sample and $\mathbf{B}_0$ into itself. The reason for including $\mathbf{B}_0$ as a part of the system rather than the field is that in the equations [namely in (2.1)] $\mathbf{B}_0$ appears merely as a geometrical property of the system, rather than a part of the magnetic field; however, the pseudovector nature of $\mathbf{B}_0$ shows up in the vectorial product—hence the reversal of sign upon reflection. When a problem has reflection symmetry about some plane, it possesses solutions symmetric and solutions antisymmetric under reflection about that plane. Because all our solutions relate to magnetic fields, we shall arbitrarily use the terms "even" and "odd" to denote solutions in which reflection leaves the magnetic fields $\mathbf{b}$ unchanged and changes the signs of the magnetic fields $\mathbf{b}$, respectively. (The symmetry of the currents and electric fields is opposite to the symmetry of the magnetic fields $\mathbf{b}$.) The problem of this section is clearly symmetric about the plane $z=0$. (Note that it is not symmetric about the plane $x=0$, because the aforementioned transformation reverses $\mathbf{B}_0$ instead of leaving it unchanged.) Because the driving field (5.1) is even, so must be all the other fields, which explains the simplification of the set (5.3).

Solving the three equations is quite straightforward; the solution is conveniently written in the following form:

$$\mathbf{b}_H=b_H\xi(A'(z),-iA_0(z),i\alpha A(z)), \tag{5.4}$$

where $\xi=e^{i(\omega t-\alpha x)}$,

$$A(z)=\frac{\sin\gamma_1z}{k_1\cos\gamma_1a}-\frac{\sin\gamma_2z}{k_2\cos\gamma_2a},$$

$$A'(z)=\frac{d}{dz}A(z),$$

$$A_0(z)=\frac{\cos\gamma_1z}{\cos\gamma_1a}-\frac{\cos\gamma_2z}{\cos\gamma_2a}.$$

The constants b_H in (5.4) and b_{r1}, b_{r2} in (5.2) are

$$b_H=-i\frac{\alpha\cosh\alpha a+|\alpha|\sinh\alpha a}{|\alpha|A(a)+A'(a)}b_0,$$

$$b_{r1}=b_{r2}=\frac{-|\alpha|A(a)\cosh\alpha a+(\alpha/|\alpha|)A'(a)\sinh\alpha a}{|\alpha|A(a)+A'(a)}b_0.$$

In the special case $\alpha=0$, (4.2) and (4.3) yield

$$\gamma_1=-\gamma_2{}^*=k_1=ik_2{}^*=-(\omega\mu_0/\rho u)^{1/2}(1+i/u)^{-1/2},$$

and if

$$\mathbf{b}_0=(1,0,0)e^{i\omega t},$$

the helicon field is

$$\mathbf{b}_H=(\tfrac{1}{2}A'(z),(i/2)A_0(z),0)e^{i\omega t}.$$

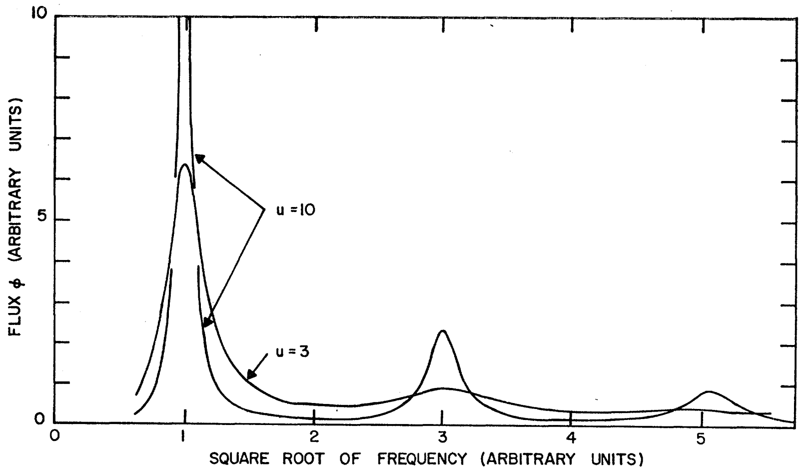

FIG. 1. Resonance curves for an infinite plate parallel to the x,y plane, with $\alpha=0$. The frequency scales have been adjusted to bring the first peaks into coincidence (courtesy of M. T. Taylor).

The average of the x component taken over the whole slab is

$$\varphi=\frac{1}{4a}\int_{-a}^{a}A'(z)dz=\frac{A(a)}{2a}=\frac{\tan\gamma_1 a}{2\gamma_1 a}+\left(\frac{\tanh\gamma_1 a}{2\gamma_1 a}\right)^*$$

in agreement with the corresponding result of Chambers and Jones.[7] (We remark that these authors ignored the existence of a "reflected field," but in the special case $\alpha=0$ this leaves the shape of the frequency response curves unaffected, and the theoretical curves of Chambers and Jones are in good agreement with experiment.) The imaginary part of φ, plotted against frequency, goes through maxima, corresponding to resonances (Fig. 1). One can see from the above expression that φ is infinite at those (complex) frequencies where $\gamma_1 a=\pm\pi/2, \pm 3\pi/2, \cdots$, or $\gamma_2 a=\pm\pi/2, \pm 3\pi/2, \cdots$; these are the roots of the determinant of (5.3). When $\alpha\neq 0$, the roots of this determinant are the roots of $|\alpha|A(a)+A'(a)$, the common denominator in the expressions for b_H, b_{r1}, and b_{r2}. We calculate the correction to the root $\gamma_1 a=\pi/2$ to first order in αa. When $\alpha a\ll 1$, $A'(a)\sim 2$, and $A(a)\sim(\tan\gamma_1 a)/\gamma_1 + a(\tanh\frac{1}{2}\pi)/(\frac{1}{2}\pi)=(\tan\gamma_1 a)/\gamma_1+0.58a$, the desired root $\gamma_1 a$ is given as the first root of the transcendental equation

$$\tan\gamma_1 a/\gamma_1 a=-2/\alpha a-0.58.$$

For a square plate with a ratio 15:1 between edge length and thickness, $\alpha a\sim\pi/15\sqrt{2}$ and $\gamma_1 a\sim 1.025(\frac{1}{2}\pi)$. This corresponds to a 5% correction in frequency which may explain the discrepancy between theory and experiment reported by Chambers and Jones.[7]

Unfortunately, because of the necessity to deal with complex numbers, plotting graphs such as Fig. 1, or computing roots to the secular determinant is usually extremely lengthy; when αa and ρ are not small, such calculations call for numerical work. When the driving field is odd in z, i.e.,

$$\mathbf{b}_0=b_0\xi(-\alpha\sinh\alpha z, 0, \alpha\cosh\alpha z), \quad |z|\geq a.$$

The equations (5.2) and (5.4) still correctly describe the fields b_r and b_H but b_H, b_{r1}, b_{r2}, $A(z)$, $A_0(z)$, must be redefined:

$$A(z)=\frac{\cos\gamma_1 z}{k_1\sin\gamma_1 a}-\frac{\cos\gamma_2 z}{k_2\sin\gamma_2 a},$$

$$A'(z)=\frac{d}{dz}A(z),$$

$$A_0(z)=-\frac{\sin\gamma_1 z}{\sin\gamma_1 a}+\frac{\sin\gamma_2 z}{\sin\gamma_2 a},$$

$$b_H=-i\frac{\alpha\sinh\alpha a+|\alpha|\cosh\alpha a}{|\alpha|A(a)+A'(a)}b_0,$$

$$b_{r1}=-b_{r2}\frac{-|\alpha|A(a)\sinh\alpha a+(\alpha/|\alpha|)A'(a)\cosh\alpha a}{|\alpha|A(a)+A'(a)}b_0.$$

B. Infinite Plate Parallel to B_0

Let the region $-a\leq x\leq a$ be filled with conductor, and let the rest of space be vacuum. As in Sec. 4B, the two specified components of the wave vector are β and γ, both real; neither of them can be set equal to zero without restriction of generality. The allowed values of α are those given in (4.6).

One can check by direct computation that to a driving field

$$\mathbf{b}_0=b_0\xi(\kappa\sinh\kappa x, -i\beta\cosh\kappa x, -i\gamma\cosh\kappa x), \quad |x|\geq a,$$
$$\kappa=(\beta^2+\gamma^2)^{1/2}, \quad \xi=e^{i(\omega t-\beta y-\gamma z)}.$$

The response is

$$\mathbf{b}_H=b_H\xi(i\kappa^2 A(x), \beta A'(x)+\gamma A_0(x), \gamma A'(x)-\beta A_0(x)), \quad \text{in} \quad -a\leq x\leq a,$$

$$\mathbf{b}_r=b_{r1}\xi(-\kappa,-i\beta,-i\gamma)e^{-\kappa(x-a)}, \quad \text{for} \quad x\geq a,$$
$$=b_{r2}\xi(\kappa,-i\beta,-i\gamma)e^{\kappa(x+a)}, \quad \text{for} \quad x\leq -a,$$

where

$$b_H=-i\frac{\sinh\kappa a+\cosh\kappa a}{\kappa A(a)+A'(a)}b_0,$$

$$b_{r1}=b_{r2}=\frac{-\kappa A(a)\cosh\kappa a+A'(a)\sinh\kappa a}{\kappa A(a)+A'(a)}b_0,$$

$$A(x)=\frac{\sin\alpha_1 x}{k_1\sin\alpha_1 a}-\frac{\sin\alpha_2 x}{k_2\sin\alpha_2 a},$$

$$A'(x)=\frac{d}{dx}A(x),$$

$$A_0(x)=\frac{\sin\alpha_1 x}{\sin\alpha_1 a}-\frac{\sin\alpha_2 x}{\sin\alpha_2 a}.$$

It can be seen at a glance that the helicon field $\mathbf{b}_H$ is a linear combination of the four helicon fields (4.8).

When the driving field is antisymmetric,

$$\mathbf{b}_0=b_0\xi(\kappa\cosh\kappa x,-\beta\sinh\kappa x,-\gamma\sinh\kappa x),\quad |x|\geqq a.$$

$A(z)$, $A'(z)$, $A_0(z)$, b_H, b_{r1}, b_{r2}, must be redefined as follows:

$$b_H=-i\frac{\sinh\kappa a+\cosh\kappa a}{\kappa A(a)+A'(a)}b_0,$$

$$b_r=\frac{-\kappa A(a)\sinh\kappa a+A'(a)\cosh\kappa a}{\kappa A(a)+A'(a)}b_0,$$

$$A(x)=\frac{\cos\alpha_1 x}{k_1\cos\alpha_1 a}-\frac{\cos\alpha_2 x}{k_2\cos\alpha_2 a},$$

$$A'(x)=\frac{d}{dx}A(x),$$

$$A_0(x)=\frac{\sin\alpha_1 x}{\sin\alpha_1 a}-\frac{\sin\alpha_2 x}{\sin\alpha_2 a}.$$

Note that the reflected field possesses a symmetry about the plane $x=0$ which corresponds with the symmetry of the incident field, but the helicon field has no such symmetry.

C. Infinite Cylinder Parallel to B_0 [Given γ, $n(=k_\varphi)$]

Adaptation of the formula (2.8) useful to problems of cylindrical symmetry, can be obtained by formally summing solutions (2.8) over all $\psi=\tan^{-1}(\beta/\alpha)$ with a weighting factor $\exp(in\psi)$. Through the formula

$$\int_0^{2\pi}e^{i(n\theta-z\sin\theta)}d\theta=2\pi(-1)^nJ_n(z),$$

this introduces Bessel functions. Expressed in terms of k_r, n, γ, and the cylindrical components, the solutions have the form

$$\xi\mathbf{R}(k_rr)=\xi(R_r,R_\varphi,R_z),$$
$$\xi=e^{i(\omega t+n\varphi-\gamma z)},$$

where

$$R_r=(k-\gamma)J_{n-1}(k_rr)+(k+\gamma)J_{n+1}(k_rr),$$
$$R_\varphi=i(k-\gamma)J_{n-1}(k_rr)-i(k+\gamma)J_{n+1}(k_rr),$$
$$R_z=-2ik_rJ_n(k_rr),$$
$$k^2=k_r^2+\gamma^2.$$

Independent solutions are obtained by replacing the Bessel functions by Neumann functions; but these are singular at $r=0$ and are of no interest to us.

Let the cylindrical region $r\leqq a$ be filled with conductor and let the rest of space be vacuum. The driving field fixes the quantities γ and n; γ is real, $n=0,\pm1,\pm2,\cdots$. The allowed values of k^2 and k_r are identical with the allowed values of k^2 and α given in (4.5) and (4.6) with the substitution $\beta=0$ used in the latter equation corresponding to the fact that $k^2=k_r^2+\gamma^2$. Because of the symmetry properties of Bessel functions, reversing the sign of k_r leaves the whole solution unchanged except at most for sign, thus, rather than four, there are only two independent acceptable helicon solutions. There is only one possible incident field and one reflected field satisfying the usual requirements:

$$\mathbf{b}_0=b_0\xi(\gamma I_n'(\gamma r),(n/r)I_n(\gamma r),-i\gamma I_n(\gamma r)),$$
$$\mathbf{b}_r=b_r\xi(\gamma K_n'(\gamma r),(n/r)K_n(\gamma r),-i\gamma K_n(\gamma r)),$$

where

$$I_n(z)=\frac{\pi}{2}i^{n+1}[J_n(iz)+iN_n(iz)],\quad I_n'(z)=\frac{d}{dz}I_n(z),$$

$$K_n(z)=i^{-n}J_n(iz),\quad K_n'(z)=\frac{d}{dz}K_n(z).$$

Thus the problem is reduced to three equations expressing that at $r=a$ the magnetic field is continuous. Letting b_1 and b_2 denote the amplitudes of the two allowed helicons we have at $r=a$:

$$R_{r1}b_1+R_{r2}b_2-\gamma K_n'b_r=\gamma I_n'b_0,$$

$$R_{\varphi1}b_1+R_{\varphi2}b_2-\frac{n}{a}K_nb_r=\frac{n}{a}I_nb_0,$$

$$R_{z1}b_1+R_{z2}b_2+i\gamma K_nb_r=-i\gamma I_nb_0.$$

Again the roots of the determinant of the set give the conditions for oscillation or free propagation.

The problem of 5C has reflection symmetry (in the sense of 5A) about all planes perpendicular to the z axis; therefore, if, under reflection about any such plane the incident magnetic field is antisymmetric, all of the magnetic field will be antisymmetric, and all the currents symmetric. Reflection symmetry of currents about a plane means that the currents never cross that plane. Choosing the incident field to be the sum of two fields of form (5.5) differing in the sign of γ, it is possible to set up a standing wave pattern in the z direction; such a pattern possesses a set of fixed planes about which the said reflection symmetry exists at all times, i.e., planes which are never crossed by currents. Thus, as promised in Sec. 3, for any given frequency we can construct a helicon field satisfying the "current condition" (i.e., the requirement that currents do not cross the boundary) for a finite cylinder.

In a similar way, the infinite plate of 5B can be driven so as to possess a similar set of planes perpendicular to z. However, because no problem involving helicons has symmetry about a plane parallel to the external magnetic field, analogously chosen driving field cannot achieve a similar set of planes parallel to z in the problem of 5A or of 5B. Thus, the current condition for a finite rectangular box cannot be satisfied by applying an appropriately chosen driving field to the plates of 5A or 5B.

Cotti, Wyder, and Quattropani[5] obtained solutions satisfying the current condition for an infinite rectangular bar, finite along y and z, by adding four freely propagating plane waves of helicons (or, more precisely, four helicons of that mode which in the limit $\rho\to 0$ is undamped). Chambers and Jones[7] obtained solutions approximately satisfying the current condition for a finite rectangular box, thin in the z direction, by adding eight such helicon waves. By using plane waves of both polarizations (sixteen plane waves in all) one can satisfy the current condition for a rectangular box exactly. As we pointed out in Sec. 3, these solutions do not, as a rule, correspond to proper modes of oscillation.

6. SURFACE LOSSES

A closer look at the fields associated with the problems of Secs. 4B, 5B, and 5C reveals some unusual results.

In the problem of 4B, make the simplifying assumption $\beta=0$ and consider the limit of $\rho\to 0$ (with $u\to\infty$, $\rho u=$const). Since in (4.6) and (4.8), $u\gg 4\omega/\omega_2$, one can write

$$\alpha_1\sim iu\gamma\,,\quad k_1\sim iu\gamma\,,$$
$$\alpha_2\sim p\gamma\,,\quad k_2\sim(\omega/\omega_2)\gamma\,,\tag{6.1a}$$

where

$$\begin{aligned}\omega_2&=-\mu_0^{-1}B_0R\gamma^2\,,\\ p&=-[(\omega/\omega_2)^2-1]^{1/2}\quad\text{for }\omega/\omega_2\geqq 1\\ &=i[1-(\omega/\omega_2)^2]^{1/2}\quad\text{for }\omega/\omega_2\leqq 1\,.\\ \mathbf{b}_1&\sim b_1(0,1,-i)e^{i(\omega t-\gamma z)}e^{u\gamma x}\,,\\ \mathbf{b}_2&\sim b_2(i,\omega/\omega_2,-ip)e^{i(\omega t-\gamma z)}e^{-ip\gamma x}\,,\end{aligned}\tag{6.1b}$$

$$\begin{Bmatrix}\mathbf{b}_0\\ \mathbf{b}_r\end{Bmatrix}=\begin{Bmatrix}b_0\\ b_r\end{Bmatrix}(\pm 1,0,-i(\gamma/|\gamma|))e^{i(\omega t-\gamma z)}e^{\pm|\gamma|x}\,,$$

and

$$\begin{aligned}b_1/b_0&=(1/D)(-2\epsilon\omega/\omega_2)\,,\\ b_2/b_0&=(1/D)(2\epsilon)\,,\\ b_r/b_0&=(1/D)[p-\omega/\omega_2-i\epsilon]\,,\\ \epsilon&=\gamma/|\gamma|\,,\quad D=p-(\omega/\omega_2)+i\epsilon\,.\end{aligned}\tag{6.2}$$

Recalling Eq. (2.10), we have for a single plane wave of helicons

$$\mathbf{j}=\mu_0^{-1}k\mathbf{b}\,.$$

The current corresponding to $\mathbf{b}_1$ is

$$\mathbf{j}_1=(1/\mu_0)(iu\gamma)b_1(0,1,i)e^{i(\omega t-\gamma z)}e^{u\gamma x}\tag{6.3}$$

and the Joule loss (per unit area) in a surface layer of unit thickness due to $\mathbf{j}_1$ is

$$\begin{aligned}W&=\int_{-1}^{0}\tfrac{1}{2}\rho(\mathbf{j}_1^*\cdot\mathbf{j}_1)dx\sim\int_{-\infty}^{0}\tfrac{1}{2}\rho(\mathbf{j}_1^*\cdot\mathbf{j}_1)dx=C\rho u\,,\\ C&=\frac{2\gamma}{\mu_0^2}|b_0|^2\frac{(\omega/\omega_2)^2}{1+[p-(\omega/\omega_2)]^2}\,,\end{aligned}\tag{6.4}$$

where star denotes complex conjugate. This loss is confined to a narrow region near the surface $x=0$ [see (6.3) and note $u\to\infty$]; it may therefore be termed "surface loss." The surprising feature of (6.4) is that if b_0, ω, and γ are fixed, so is C; and, if then, the limit $\rho\to 0$ is approached (in such a way that ρu remains constant), W does not tend to zero. (Note that surface loss does not occur if the "tangential wavelength" $2\pi/\gamma$ is infinite.) The other two terms in the expression for the Joule loss (per unit area) in a surface layer of unit thickness, $\int_{-1}^{0}\frac{1}{2}\rho(\mathbf{j}_2^*\cdot\mathbf{j}_2)$ and the "cross term" $\int_{-1}^{0}\frac{1}{2}\rho(\mathbf{j}_1^*\cdot\mathbf{j}_2+\mathbf{j}_1\cdot\mathbf{j}_2^*)$, both tend to zero. In the approximation (6.1) the integrand of the former term does not fall off toward the interior of the sample; in the same approximation the "cross term" is always negligible compared to the surface loss.

Suppose the direction of $\mathbf{B}_0$ is changed to make an angle θ with the conducting surface. It can be shown that for small enough θ the normal component of $\mathbf{k}_1$ becomes

$$\alpha_1\sim\frac{iu\gamma}{1-iu\tan\theta}\tag{6.5}$$

and in all other respects, including amplitudes, $\mathbf{b}_1$, $\mathbf{b}_2$, and $\mathbf{b}_r$ remain unchanged. (Note the sensitive dependence of α_1 on the angle between the conducting surface and the magnetic field, when u is large.) Denote the solution reducing to $-\alpha_1$ when $\theta=0$ by α_3 [see (4.6)]. One can show that for $\theta\neq 0$ the component α_3 is not $-\alpha_1$ but becomes

$$\alpha_3\sim iu\gamma/(-1-iu\tan\theta)\,,$$

thus furnishing an example in which the pairwise "degeneracy except for sign" in the allowed values of the wave vector's normal component is split up.

In (6.4) the integral from $(-\infty)$ to 0 remains unchanged, as θ is made to be different from zero, but its contributions come from a layer of thickness $(1+u^2\tan^2\theta)/2u\gamma$ rather than $1/2u\gamma$ and in the limit $u\rightarrow\infty$ this thickness tends to infinity rather than zero. The approximation equating the two integrals in (6.4) becomes invalid and in fact $W\rightarrow 0$ when $\rho\rightarrow 0$.

However, for $\theta=0$ Eq. (2.1) combined with Maxwell's equations yields a finite loss per unit volume even in the limit of perfect conduction. This unlikely result is no longer obtained when the microscopic processes of conduction are considered more carefully. When the resistivity becomes so low that the thickness of the surface layer becomes smaller than the cyclotron radius, the finite surface loss disappears. It is easy to show that the details of the surface loss mechanism are the following. Right inside the surface there is an oscillating dipole layer consisting of surface charges on the boundary and space charge of opposite sign exponentially falling off (with skin depth $1/u\gamma$) toward the interior of the conductor. Between the two "charge layers" there is a strong electric field (perpendicular to the surface), which in turn gives rise (because of $\mathbf{B}_0$) to a sheet of strong current parallel to the surface. The electric field as well as the current density are proportional to u, the thickness of the current sheet to u^{-1}, the resistivity to u^{-1}; and the current squared, times thickness, times resistivity remains constant, as $u\rightarrow\infty$. However, in the extreme anomalous limit $|\alpha_1|r_c=\gamma l\gg 0$ (where $r_c=$ cyclotron radius, $l=$ mean free path) the only electrons contributing to electrical conduction are those moving parallel to the conductor's surface.[29] The effect of the electric field on these is expressed by a term proportional to $\mathbf{E}\cdot\mathbf{v}$ in the Boltzmann equation, which is zero when $\mathbf{E}$ is perpendicular and $\mathbf{v}$ is parallel to the conductor's surface. Then the strong electric field between the two charge layers mentioned above can have no first-order effect on the conduction thus it cannot bring about the strong currents responsible for surface loss, and the surface loss (or at least surface loss through the previously described mechanism) disappears. Deciding what actually does happen is beyond the scope of this paper.

This does not mean that the above remarks on surface loss can be disregarded. In the purest sodium samples available (of residual resistance ratio 8000), at liquid-helium temperatures, and magnetic fields around 50 kG, the value of u is 100 and the cyclotron radius is around 3 microns. For a "tangential wavelength" $2\pi/\gamma=1$ cm the thickness of the surface layer is about 15 microns, thus the extreme anomalous limit is not yet reached. It is in fact quite easy to make experimental samples in which most of the loss is surface loss.[30]

One may mention the following anomaly, which directly follows from the macroscopic equations: the energy in the electric fields at the surface tends to infinity as $\rho\rightarrow 0$. For, as was said above, the currents in the surface layer increase as u; thus, from (2.1) (in which $\rho\rightarrow 0$) so does the electric field. The square of the electric field multiplied by the thickness of the surface layer increases linearly with u. However, in a typical laboratory situation[2] the magnetic field energy density in the surface layer dominates by about a factor of 10^{14} over the electric-field energy density, so that the anomaly of electric fields is merely of academic interest.

The narrowness of the surface layer confining the mode $\mathbf{b}_1$ (of order 10 μ in Ref. 30) and the fact that asymptotically $(\mathbf{b}_1)_y=i(\mathbf{b}_1)_z$, can be used to advantage in reducing the number of equations necessary for solving boundary-value problems. Instead of using the ordinary boundary condition that all three components of $\mathbf{b}$ are continuous everywhere on the boundary one can assume that this is true everywhere except at surfaces parallel to $\mathbf{B}_0$ at which only b_x and b_y-ib_z are continuous (x being the direction of the outward normal), while a third quantity b_y+ib_z may suffer discontinuity. For the wave vector inside only α_2 (and possibly α_4) is allowed; α_1 and α_3 are discarded. This viewpoint amounts to altogether disregarding the mode $\mathbf{b}_1$ and retaining it on the record only to provide explanation for the surface singularity: "There are surface currents and therefore the tangential component of $\mathbf{b}$ may be discontinuous."

For the problem of Eq. (6.1) the remaining two boundary conditions provide only two equations, but, also, there are only two unknowns: b_2 and b_r. The method is applicable to the cylinder of 5C whenever $b_{r1}\gg 1$. The fact that the surface layer may be approximated by an infinite plane manifests itself in the asymptotic form of the Bessel function $J_n(ip)$ for large, real p: $J_n(ip)\sim i^n(2\pi p)^{-1/2}e^p$ which is essentially an exponential function; $J_n(ip)$ and $J_{n+1}(ip)$ differ only in a factor i.

7. SUMMARY AND CONCLUDING REMARKS

By combining the equation $\mathbf{E}+R\mathbf{j}\times\mathbf{B}_0=\rho\mathbf{j}$ with Maxwell's equations, we arrived at the differential equation governing helicons, found its plane-wave solutions and dispersion relation. We pointed out that for plane waves the differential equation can be written in the form of a precession equation. When the medium is a perfect conductor, the mode rotating in the sense of the cyclotron rotation propagates freely (the other mode is exponentially damped), and the current and magnetic field are always either parallel or antiparallel (in the former case the spatial configuration forms a left-handed screw, in the latter case it forms a right-handed screw). Despite the incompressibility of the electron gas, the electric field has a longitudinal component of the same order of magnitude as the transverse component; the apparent contradiction was

[29] M. Ia. Azbel' and E. A. Kaner, Zh. Eksperim. i Teor. Fiz. **32**, 896 (1956) [English transl.: Soviet Phys.—JETP **5**, 730 (1957)].

[30] J. M. Goodman and C. R. Legéndy (unpublished work).

resolved by simply noting that both field components were very small. The dispersion relation for helicons reduces to the ordinary skin-effect formula, if the magnetic field is made to be small, or the resistivity large. At the beginning of the calculation the equations have to be linearized by assuming that the helicon field is small, but, it turns out, single plane waves can propagate even if their amplitudes are large (though several plane waves would usually interact).

It was shown that if the field outside the sample is quasistatic, the boundary condition that all components of the magnetic field are continuous at the boundary implies that: (i) the electric current does not cross the boundary, and (ii) for the vacuum region it is possible to construct an electric field whose curl is $-\partial \mathbf{B}/\partial t$ and whose tangential components join continuously with the field inside the sample.

We formulated the problem of driven oscillations in terms of a "driving field," a "reflected field" and a "transmitted field" (which is the helicon field); the first of these is the field that would be set up by driving coils if the sample were removed, the second and third can be determined by means of symmetry arguments, etc., except for multiplicative constants; these constants are determined from the boundary conditions. We solved the problem of driven oscillations for a few simple cases: Infinite plates, infinite cylinders, and semi-infinite regions. The response of plate samples was found to be quite sensitive to dependence of the driving field on the tangential coordinates; the approximate value of the fundamental resonance frequency for a square plate whose dimensions along x, y, z compare as 15:15:1, was found to be 5% higher than it would be if the first two dimensions were infinite. The effect is still more drastic in plates parallel to $\mathbf{B}_0$. In this case any variation along the z direction brings about a surface loss, which for small resistivities fails to decrease as the resistivity is decreased, until the limit of anomalous skin effect is reached, in which limit the loss disappears. However, for the purest of sodium samples presently available, at 4.2°K, and in a magnetic field of 50 kG, the anomalous skin effect only becomes marked for tangential wavelengths well below 1 cm. The surface mode causing the loss involves a thin oscillating dipole layer with a strong electric field and strong electric current between the charge layers; ignoring anomalous skin effect, the energy per cm^2 in the electric field tends to infinity as $\rho \to 0$. When the dipole layer is thin enough, the surface mode may be represented by an "equivalent boundary condition" on magnetic fields, and thereby the number of equations describing the boundary-value problem is reduced. The surface mode only appears in boundaries parallel to $\mathbf{B}_0$.

The present work has three obvious limitations: (i) No attempt has been made to treat resonances in finite samples exactly. (ii) We have not translated the theoretical results into immediately useable graphs. (iii) The treatment of anomalous skin effect with a nonzero wave vector along the conducting surface has been bypassed and replaced by a simple *reductio ad absurdum* argument.

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