The first factor in expression (12) comes from the interaction of the pseudoscalar charge density with the electric field. This factor rotates the spin of the incident particle about the axis $(\mathbf{j}\times\mathbf{M})$ through the angle

$$\delta\rho=-\frac{aEm}{p[m+(1-a^2)^{1/2}W]}x'\sin\rho,$$

where $(\mathbf{j}\cdot\mathbf{M})=\cos\rho$. The second factor in (12) comes from the interaction of the scalar charge density with the electric field. This factor rotates the spin of the incident beam about the z axis through angle $\delta\theta_z$. At the low-energy limit, the ratio of the two angles $\delta\rho$ and $v\theta_z$ is

$$\delta\rho/\delta\theta_z=-a(1-a^2)^{-\frac{1}{2}}(c/v)\sin\rho,$$

where c and v are the velocity of light and of the incident beam. Thus, there is a possibility at very low energy that $\delta\rho>\delta\theta_z$; that is, as far as the spin rotation is concerned, the effect of the pseudoscalar charge density is larger than the effect of the scalar charge density.

Thus, it has been shown that the pseudoscalar charge density is an observable.

Note added in proof. It was shown in I and II that $a^2\leq\frac{1}{3}$. Here we shall improve the upper limit of a^2. Among the spectral functions ρ_i, the inequality

$$(u^2+v^2+w^2-2uv-2auw+2avw)\rho_1+\frac{2u}{x}(v+aw)\rho_2-\{2vw+a(-u^2+v^2+w^2)\}\rho_3\geq 0$$

holds, where u, v, and w are any real numbers. From this inequality one obtains

$$\begin{aligned}\rho_1\pm\rho_3&\geq 0,\\(2x\rho_1-\rho_2)\rho_2-x^2\rho_3{}^2&\geq 0,\\\rho_2&\geq 0.\end{aligned}\qquad(\alpha)$$

Since $a^2<1$, $\rho_1\pm a\rho_3\geq 0$. From the first equation of (α) and the expressions

$$Z_2^{-1}=\int_0^\infty dx^2[\rho_1-a\rho_3]\quad\text{and}\quad\frac{a}{1-a^2}=-Z_2\int_0^\infty dx^2\rho_3,$$

one can prove

$$1>Z_2\geq 0\quad\text{and}\quad a^2\leq\tfrac{1}{4}.$$

ACKNOWLEDGMENTS

The author would like to thank Dr. M. Hamermesh and Dr. M. Peshkin for their warm hospitality at the Argonne National Laboratory and Dr. F. E. Throw for careful reading of the manuscript.

PHYSICAL REVIEW VOLUME 134, NUMBER 4B 25 MAY 1964

Mixing of Elementary Particles

SIDNEY COLEMAN*
Lyman Laboratory of Physics, Harvard University, Cambridge, Massachusetts
AND
HOWARD J. SCHNITZER†
Department of Physics, Brandeis University, Waltham, Massachusetts
(Received 3 October 1963; revised manuscript received 12 February 1964)

Particle mixing is studied in a field-theoretic context, as a further approximation to the pole approximation. Although particle mixing is well suited for treating spinless particles, another approximation, also a further approximation to the pole approximation, called vector mixing, is better for treating particles of spin one. Vector mixing is applied to several processes involving the mixing of the ω and the ϕ by the interaction that breaks unitary symmetry.

I. INTRODUCTION

PARTICLE mixing approximations in elementary particle physics have been used by Gell-Mann and Pais[1] (neutral K-meson mixing due to the weak interactions), Glashow[2] (ρ-ω mixing due to electromagnetism), and Okubo[3] (ω-ϕ mixing due to the unknown interaction that breaks unitary symmetry.) All these authors have discussed particle mixing within the framework of a Schrödinger equation acting on the space of one-particle states; the relation of the approximation to the usual approximations of elementary particle physics, derived from field theory or dispersion relations, is by no means clear. It is our intent here to discuss particle mixing within a field-theoretic context, as a further approximation to the pole approximation.

* Work supported in part by the U. S. Air Force Office of Scientific Research Contract No. 49 (638) 589.
† Work supported in part by the National Science Foundation.
[1] M. Gell-Mann and A. Pais, Phys. Rev. **97**, 1387 (1955).
[2] S. L. Glashow, Phys. Rev. Letters **7**, 469 (1961).
[3] S. Okubo, Phys. Letters **5**, 165 (1963). See also S. L. Glashow, Phys. Rev. Letters **11**, 48 (1963); J. J. Sakurai, Phys. Rev. **132**, 434 (1963); R. Dashen and D. Sharp, Phys. Rev. **133**, 1585 (1964).

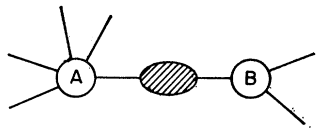

FIG. 1. The propagator approximation. The blobs labeled A and B represent vertices; the shaded blob represents a propagator. We assume that all dependence of the matrix element on the momentum transfer along the propagator comes from the momentum dependence of the propagator, and none from the vertices. Further, when we consider broken symmetry, we only consider the effect of the symmetry-breaking interaction on the propagator, and not the effect on the vertices. This corresponds to considering only those terms in the perturbation series that are enhanced by the presence of small-energy denominators.

We argue that although particle mixing is a reasonable approximation for scalar mesons, it is a most unreasonable one for vector mesons. If we naively attempt to calculate the effect of particle mixing on the electromagnetic form factors of the baryons, we will find that it can alter the charge of the proton. We introduce a new approximation, called vector mixing, which does not share this deficiency: it is designed to preserve the conservation of charge. If the force mixing the particles is truly weak, particle mixing and vector mixing are indistinguishable. (In this sense the relation between the two approximations is much like that between perturbative calculation of the S matrix and perturbative calculation of the K matrix.)

We apply vector mixing to deduce a modified form of the Gell-Mann–Okubo mass formula,[4,5] and also to calculate the effect of ω-ϕ mixing on the form factors of the strange baryons. Interestingly enough, the predictions of unitary symmetry for the strange baryon magnetic moments are not altered, even though the shapes of the form factors are changed considerably. [There is a simple physical reason for this. ω-ϕ mixing effects the electric and magnetic form factors in the same proportion. We know it cannot alter $F_1(0)$ because of the conservation of charge. Thus, it cannot alter $F_2(0)$. Of course, effects of the symmetry-breaking interaction other than particle mixing may act differently on F_1 and F_2 and alter the magnetic moments.] We also discuss the effects of ω-ϕ mixing on nucleon-nucleon scattering and on the decays of the ϕ. In an Appendix we consider massless vector meson.

II. GENERAL THEORY

We begin by considering what we will call the propagator approximation. Consider a process of the sort shown in Fig. 1. For definiteness, let us assume that the external lines attached to vertex A represent two incoming and two outgoing nucleons, the internal line represents a ρ meson, and the external lines attached to vertex B represent two pions. The diagram then describes two-pion production in nucleon-nucleon inelastic scattering. The propagator approximation consists of only considering diagrams of the type shown in Fig. 1, and, further, of assuming that the two vertices may be replaced by their value at fixed t, where t is the square of the four-momentum associated with the internal ρ-meson line. That is to say, all of the dependence of the matrix element on t comes from the propagator. A further approximation, the pole approximation, consists of replacing the propagator by its pole term and the two vertices by their values at the pole. In the particular example, this is known to be a good approximation for two-pion production near the pole.

When we have a field theory with a high degree of symmetry, we usually have the possibility, in any given process, of exchanging several kinds of mesons. In this case we must use a matrix propagator; otherwise the propagator and pole approximations are defined as above. If we introduce a symmetry-breaking perturbation, we will only consider its effect on the propagator, not on the vertices. This can be justified by looking at the nonrelativistic perturbation expansion: The terms we retain include all those terms that are enhanced by the presence of small-energy denominators.

In fact, all of the approximations we will consider are not only approximations to the propagator approximation, but also approximations to the pole approximation. Since the pole approximation may be obtained from analytic S-matrix theory, presumably all of our work could be done without recourse to entities defined off the mass shell. Nevertheless, because we find certain sum rules and symmetry properties derived from the propagator useful, we prefer to consider our approximations as special cases of the propagator approximation.

Structure of the Matrix Propagator

Let us now consider the structure of the propagator. For the moment let us assume the particles involved are scalar. Then the propagator, $\mathbf{D}(k^2)$, a matrix, is defined by

$$T\langle 0|A_i(x)A_j(y)|0\rangle = -i\int d^4k(2\pi)^{-4}e^{-ik\cdot(x-y)}[\mathbf{D}(k^2)]_{ij}, \quad (1)$$

where i runs from 1 to n, and n is the number of particles in the channel of interest. Inserting a complete set of intermediate states, we find

$$\mathbf{D}(k^2)=\int da^2\boldsymbol{\varrho}(a^2)(k^2-a^2+i\epsilon)^{-1}, \quad (2)$$

where

$$[\boldsymbol{\varrho}(a^2)]_{ij}=\sum_n \langle 0|A_i(0)|n)(n|A_j(0)|0\rangle\delta(P_n{}^2-a^2). \quad (3)$$

---

[4] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962).
[5] S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).

From its definition, $\boldsymbol{\varrho}$ is a Hermitian, positive definite matrix. Furthermore, if we assume invariance under the antiunitary *TCP* operator Ω, which has the properties

$$\Omega^2 = I\,, \tag{4}$$

$$\Omega|0\rangle = |0\rangle\,, \tag{5}$$

and

$$\Omega A_i(x)\Omega = A_i(-x)\,, \tag{6}$$

then

$$\begin{aligned}[\boldsymbol{\varrho}(a^2)]_{ij} &= \sum_n \langle 0|A_i(0)|n\rangle\langle n|A_j(0)|0\rangle\delta(P_n{}^2 - a^2) \\ &= \sum_m \langle m|A_i(0)|0\rangle\langle 0|A_j(0)|m\rangle\delta(P_m{}^2 - a^2) \\ &= [\boldsymbol{\varrho}(a^2)]_{ji}\,. \end{aligned} \tag{7}$$

That is to say, *TCP* invariance implies $\boldsymbol{\varrho}$ is symmetric. Since we already know $\boldsymbol{\varrho}$ is Hermitian, this means $\boldsymbol{\varrho}$ must be real.

The Pole Approximation

Let us suppose there are n stable particles in the same channel as the n fields we have discussed above. In that case we may separate out the contributions from the one-particle states.

$$\mathbf{D}(k^2) = \sum_{r=1}^{n} [\boldsymbol{\varrho}^{(r)}/(k^2 - m_r{}^2)] + \text{continuum terms}\,, \tag{8}$$

where

$$\boldsymbol{\varrho}^{(r)} = {}^*\mathbf{e}^{(r)}\mathbf{e}^{(r)\dagger}\,, \tag{9}$$

$$e_j{}^{(r)} = \langle 0|A_j(x)|r,p\rangle e^{ip\cdot x}\,, \tag{10}$$

and $|r,p\rangle$ indicates a state of one particle of the rth kind in an eigenstate of momentum with eigenvalue p. Since the $\boldsymbol{\varrho}^{(r)}$ must all be real symmetric matrices, it must be possible to choose the phases of the one-particle states so the $\mathbf{e}^{(r)}$ are all real. The pole approximation consists of the propagator approximation with the neglect of the continuum terms in Eq. (8). [Of course, we could also have obtained the pole approximation from the viewpoint of dispersion relations, without using the propagator approximation—in fact, without talking about fields at all. However, we have found it more convenient to prove the reality and symmetry of the $\boldsymbol{\varrho}^{(r)}$ by this method.]

We may also write $\mathbf{D}(k^2)$ in another way. We may define the matrix $\boldsymbol{\Pi}(k^2)$, which we call the self-energy matrix, by

$$(k^2 - \boldsymbol{\Pi}(k^2))^{-1} = \mathbf{D}(k^2)\,. \tag{11}$$

We prove in Appendix I that the pole approximation is equivalent to assuming that $\boldsymbol{\Pi}$ is a linear function of k^2.

$$\boldsymbol{\Pi}(k^2) = \mathbf{M} + (\mathbf{I} - \mathbf{Z})k^2\,, \tag{12}$$

where

$$\mathbf{M}^{-1} = \sum_{r=1}^{n} (\boldsymbol{\varrho}^{(r)}/m_r{}^2)\,, \tag{13}$$

and

$$\mathbf{Z}^{-1} = \sum_{r=1}^{n} \boldsymbol{\varrho}^{(r)}\,. \tag{14}$$

$\mathbf{M}$ and $\mathbf{Z}$ are, by definition, real, symmetric positive-definite matrices.

We note that the pole approximation involves n arbitrary components for each $\mathbf{e}^{(r)}$ and n arbitrary masses—a total of n^2+n arbitrary parameters. We obtain the same number of parameters by counting the components of the two unknown real symmetric matrices $\mathbf{M}$ and $\mathbf{Z}$.

If the $\mathbf{e}^{(r)}$ are all independent, we may find a matrix $\mathbf{S}$ such that

$$[\mathbf{S}\mathbf{e}^{(r)}]_i = \delta_i{}^r\,. \tag{15}$$

Then

$$\mathbf{S}\mathbf{D}(k^2)\mathbf{S}^\dagger = (k^2 - \boldsymbol{\Lambda})^{-1}\,. \tag{16}$$

where $\boldsymbol{\Lambda}$ is a diagonal matrix with $m_r{}^2$ in the rth entry.

If the particles involved are unstable, then the poles are on the second sheet and the m_r are complex numbers whose imaginary parts are connected in the usual way with the lifetime of the unstable state. The $\boldsymbol{\varrho}^{(r)}$ are complex symmetric matrices of rank one, and $\mathbf{M}$ and $\mathbf{Z}$ are arbitrary symmetric matrices. Everything is as before, except that we now have n^2+n complex parameters instead of n^2+n real ones.

If the particles are of spin one, nothing is altered except that the analysis above applies to the transverse part of the propagator only, which is the only part that contributes to processes of physical interest.

Broken Symmetry

Let us suppose that the Hamiltonian of the world is such that there is an absolute selection law forbidding transitions from one of our n-particle types to another. Then, by normalizing the fields, we may write the propagator in the pole approximation in the form

$$\mathbf{D}(k^2) = (k^2 - \mathbf{M}_0)^{-1}\,, \tag{17}$$

where M_0 is diagonal. Now let us suppose we introduce some perturbation that allows the particles to mix. The propagator then assumes the form

$$\mathbf{D}(k^2) = (k^2 - \mathbf{M}_0 - \boldsymbol{\delta}_m - \boldsymbol{\delta}_z k^2)^{-1}\,, \tag{18}$$

where $\boldsymbol{\delta}_m$ and $\boldsymbol{\delta}_z$ are unknown, real symmetric matrices. This expression involves n^2+n unknown parameters. This is more than we can reasonably determine from the crude experimental data that is usually accessible to us; therefore it is desirable to introduce further approximations to reduce the number of independent parameters. Below we shall discuss three such.

Particle-Mixing Approximation

In this approximation we assume the propagator to be of the form

$$\mathbf{D}(k^2)=(k^2-\mathbf{M})^{-1}. \tag{19}$$

If we are interested only in processes that go on in a limited range of energy—for example, in multipion production in the neighborhood of the ρ and ω masses—we may choose $\mathbf{M}$ to be the value of $\mathbf{\Pi}(k^2)$ somewhere in that range. It then does not seem an excessively drastic truncation of the pole approximation to neglect the dependence of $\mathbf{\Pi}$ on k^2.

We want to show the connection of this formulation with the usual formulation of particle mixing theories, in which we solve a Schrödinger equation for a Hamiltonian acting on a Hilbert space in which the only states are one-particle states.[1–3] It is characteristic of such a method of calculation that, as a consequence of the conservation of probability, the sum of the residues of the perturbed propagator must be the same as the sum of the residues of the unperturbed propagator. That is to say, since the only states allowed are one-particle states,

$$\sum_{r=1}^{n} \boldsymbol{\varrho}^{(r)}=\mathbf{I}. \tag{20}$$

Equation (20) is the shadow of an equation that exists in the full theory. It is easy to show,[6] that if the interaction does not effect the vacuum-expectation value of the equal-time commutators of the fields and their first time derivatives,

$$\int da^2 \boldsymbol{\varrho}(a^2)=\int da^2 \boldsymbol{\varrho}^0(a^2), \tag{21}$$

where $\boldsymbol{\varrho}^0$ is the weight function for the unperturbed propagator. However, in the full theory, residue may be transferred from the continuum to the poles.

Applying to Eq. (20) the matrix S defined by Eq. (15) we find

$$\mathbf{S}\sum_{r=1}^{n} \boldsymbol{\varrho}^{(r)}\mathbf{S}^{\dagger}=\mathbf{S}\mathbf{S}^{\dagger}=\mathbf{I}, \tag{22}$$

that is to say, $\mathbf{S}$ is unitary. Equations (22) and (16) imply that

$$\mathbf{D}(k^2)=(k^2-\mathbf{M})^{-1}, \tag{23}$$

where

$$\mathbf{M}=\mathbf{S}^{\dagger}\mathbf{\Lambda}\mathbf{S}. \tag{24}$$

It is easy to see by direct comparison that $\mathbf{M}$ is the mass matrix of Glashow[2] and of Feinberg and Bernstein.[7]

If the interaction satisfies somewhat more stringent conditions than those that are necessary for Eq. (21), we have another sum rule,[6]

$$\int da^2 a^2 \boldsymbol{\varrho}(a^2)=\int da^2 a^2 \boldsymbol{\varrho}^0(a^2). \tag{25}$$

We assume in the sequel that the symmetry-breaking interactions always preserve (21) and (25). A typical interaction that preserves these sum rules is $\delta g\bar{\Psi}\gamma_5\Psi\phi$. Some interactions that violate them are $\delta\mu^2\phi^2$ and $\delta Z\partial_\mu\phi\partial_\mu\phi$.

If we attempt to apply Eq. (25) to the pole approximation alone, we obtain, in analogy to (20),

$$\sum_{i=1}^{n} \boldsymbol{\varrho}^{(i)}m_i^2=\mathbf{M}_0. \tag{26}$$

We cannot use both (20) and (26); together they imply that the new propagator is the same as the old one. Clearly, we must choose (20); the sum rule (21) is much more dependent on the low-mass part of the weight function than the sum rule (25).

We remark that the particle-mixing approximation involve $\frac{1}{2}(n^2+n)$ parameters, half as many as the pole approximation.

Vector-Mixing Approximation

Satisfactory as it is for many purposes, the particle-mixing approximation violates an important property of vector-meson dynamics, the transversality of the vector mesons (or, equivalently, the conservation of the current to which the vector mesons are coupled). We would like any approximation we use for vector-meson theories to preserve this condition—one of the phenomena in which vector mesons play a large role is the form factors of elementary particles; if we use an approximation that violates current conservation, we are liable to find charge disappearing from the proton. The reflection of current conservation in the propagator is that the corrections to the propagator vanish at zero-momentum transfer,[8]

$$\int \frac{\boldsymbol{\varrho}(a^2)da^2}{a^2}=\int \frac{\boldsymbol{\varrho}^0(a^2)da^2}{a^2}. \tag{27}$$

Just as in the scalar case, if the interaction obeys certain additional conditions, there is another sum rule,

$$\int \boldsymbol{\varrho}(a^2)da^2=\int \boldsymbol{\varrho}^0(a^2)da^2. \tag{28}$$

We assume in the sequel that the symmetry-breaking interactions always preserve (27) and (28). A typical interaction that preserves these sum rules is $\delta e\bar{\Psi}\gamma_\mu\Psi A_\mu$. Some interactions that violate them are $\delta\mu^2A_\mu A_\mu$ and $\delta Z(\partial_\nu A_\mu-\partial_\mu A_\nu)(\partial_\nu A_\mu)$. In contrast to the scalar case, the sum rule (27) is more strongly dependent on the

[6] H. Lehmann, Nuovo Cimento **11**, 342 (1954).

[7] J. Bernstein and G. Feinberg, Nuovo Cimento **25**, 1343 (1962).

[8] K. Johnson, Nucl. Phys. **25**, 435 (1961).

low-mass part of the weight function than the sum rule (28). Therefore, the condition we must apply to the propagator is

$$\sum_{i=1}^{n} (\varrho^{(r)}/m_r^2) = \mathbf{M}_0^{-1}, \qquad (29)$$

which implies

$$\mathbf{D}(k^2) = (k^2 - \mathbf{M}_0 + \boldsymbol{\delta} k^2)^{-1}. \qquad (30)$$

We call this form for the propagator the vector mixing approximation; it involves $\frac{1}{2}(n^2+3n)$ unknown parameters.

Off-Diagonal Vector Mixing

Vector mixing contains more parameters than particle mixing. We would like to describe here an approximation which we call off-diagonal vector mixing that shares many of the desirable features of vector mixing but has fewer parameters. The approximate expression for the propagator is obtained from (30) by replacing the diagonal elements of $\boldsymbol{\delta}$ by constants. Thus,

$$\mathbf{D}(k^2) = (k^2 - \mathbf{M} + \boldsymbol{\delta} k^2)^{-1}, \qquad (31)$$

where $\mathbf{M}$ is diagonal and $\boldsymbol{\delta}$ is symmetric and off diagonal. This approximation will not violate charge conservation as long as the original symmetry group contains no transformations that exchange the vector mesons mixed by the perturbation. We shall not use off-diagonal vector mixing in the subsequent parts of this paper, but it is, for example, a suitable approximation for the ρ-ω mixing problem of Glashow.[2] Off-diagonal vector mixing involves $\frac{1}{2}(n^2+n)$ parameters.

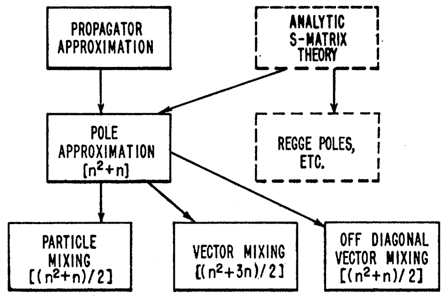

FIG. 2. The relation between some approximations which occur in elementary particle physics. The number in square brackets is the number of arbitrary parameters in the approximation, for a process that may proceed through n intermediate states, each with the same quantum numbers. The approximation at the head of an arrow is a further approximation to the approximation at the tail. We discuss in this paper only the approximations inclosed in solid lines.

Figure 2 shows the relation between the various approximations we have discussed in this section.

III. APPLICATIONS

The Strongly Interacting Vector Mesons

We wish to apply the vector mixing approximation described in the preceding section to the strongly interacting vector mesons. Under unitary symmetry, these nine particles form an octet and a singlet; the symmetry-breaking interaction causes them to decompose into two singlets (ϕ and ω), two doublets (the K^*'s) and a triplet (ρ).[9] If we assume the symmetry-breaking part of the Lagrangian transforms like part of an octet, then arguments similar to those which lead to the Gell-Mann–Okubo mass formula[4,5] tell us that the propagator is of the form

$$[\mathbf{D}(k^2)]^{-1} = \begin{pmatrix} (1+\epsilon)k^2 - M_1 & 0 & 0 & 0 \\ 0 & (1-2\epsilon)k^2 - M_1 & 0 & 0 \\ 0 & 0 & (1+2\epsilon)k^2 - M_1 & k^2\beta \\ 0 & 0 & k^2\beta & k^2 - M_2 \end{pmatrix}, \qquad (32)$$

where the rows and columns correspond to the fields K^*, ρ, ω, ϕ, in that order, and M_1, M_2, β, and ϵ are unknown constants. Using the known masses of the K^* and the ρ, it is simple to determine M_1 and ϵ from

$$m_{K^*}^2 = M_1/(1+\epsilon),$$

and

$$m_\rho^2 = M_1/(1-2\epsilon). \qquad (33)$$

These yield

$$\epsilon = -0.12,$$

and

$$M_1 = 0.69 \text{ (BeV)}^2. \qquad (34)$$

The remaining two parameters are determined by the requirement that D has poles at the observed ω and ϕ masses. This implies

$$[(1+2\epsilon)m_\omega^2 - M_1](m_\omega^2 - M_2) - \beta^2 m_\omega^4 = 0,$$

and

$$[(1+2\epsilon)m_\phi^2 - M_1](m_\phi^2 - M_2) - \beta^2 m_\phi^4 = 0, \qquad (35)$$

which yield

$$M_2 = 0.68 \text{ (BeV)}^2,$$

and

$$\beta = \pm 0.18. \qquad (36)$$

Note that M_1 and M_2 are equal to within experimental accuracy; in the absence of the symmetry-breaking interactions, the vector octet and the vector singlet have the same mass. This apparently accidental degeneracy was first observed by Okubo, using the particle-mixing approximation.

[9] We use the following masses for these particles: $m_\rho = 750$ MeV, $m_{K^*} = 888$ MeV, $m_\omega = 785$ MeV, and $m_\phi = 1020$ MeV. H. Barkas and A. H. Rosenfeld, University of California Radiation Laboratory Report UCRL-8030 (rev.) (unpublished).

The 2×2 isoscalar submatrix of **D** may be written in the form

$$\mathbf{D}=(k^2-m_\omega^2)^{-1}\varrho_\omega+(k^2-m_\phi^2)^{-1}\varrho_\phi\,, \tag{37}$$

where

$$[\varrho_\omega]=\frac{1}{(1+2\epsilon-\beta^2)(m_\omega^2-m_\phi^2)}\times\begin{pmatrix} m_\omega^2-M_2 & -m_\omega^2\beta \\ -m_\omega^2\beta & (1+2\epsilon)m_\omega^2-M_1\end{pmatrix}, \tag{38}$$

and ϱ_ϕ is of the same form with m_ω and m_ϕ interchanged.

Now that we have determined the parameters of the vector-meson propagator, we may apply our model to several processes in which mixing plays a significant role.

Nucleon Electromagnetic Form Factors

Let $F_i(k^2)$ $(i=1,2)$ denote the usual electric- and magnetic-nucleon form factors. We may write these quantities in the form

$$F_i^V(k^2)=[R_{i\rho}/(k^2-m_\rho^2)]+\text{remainder}\,,$$

and

$$F_i^S(k^2)=[R_{i\omega}/(k^2-m_\omega^2)]+[R_{i\phi}/(k^2-m_\phi^2)]+\text{remainder}\,,$$

where the remainder terms are free from singularities at the vector-meson masses.[10] The pole terms are given exclusively by diagrams of the type of Fig. 1, with the incoming particle a photon and the outgoing particles a nucleon and an antinucleon. Following the approximation procedure explained above, we use unitary-symmetric values for the vertices but the expression (32) for the propagator. In the absence of the symmetry-breaking interactions, let the coupling constants of the nucleon to the vector mesons be $G_{i\phi}$, $G_{i\rho}$, and $G_{i\omega}$: let the couplings of the ρ and ω to the photon be γ_ρ and γ_ω. (Under unitary symmetry, there is no ϕ-γ coupling.) Then the residues are given by

$$R_{i\rho}=G_{i\rho}\gamma_\rho/(1-2\epsilon)\,,$$

$$R_{i\omega}=\frac{\gamma_\omega[(m_\omega^2-M_2)G_{i\omega}-m_\omega^2\beta G_{i\phi}]}{(1+2\epsilon-\beta^2)(m_\omega^2-m_\phi^2)}\,,$$

and

$$R_{i\phi}=\frac{\gamma_\omega[(m_\phi^2-M_2)G_{i\omega}-m_\phi^2\beta G_{i\phi}]}{(1+2\epsilon-\beta^2)(m_\phi^2-m_\omega^2)}\,. \tag{39}$$

We emphasize that the quantities to be compared with the predictions of unitary symmetry are G and γ, *not* the experimental residues.

We obtain the experimental residues R by the following procedure. We assume the isoscalar form factor is the sum of an ω pole and a ϕ pole, and the isovector form factor is the sum of a ρ pole and a soft core of the type discussed by Hand, Miller, and Wilson[11] at 30 F^{-2}. We then determine the residues at these poles by fitting the values and first derivatives of our form factors to the values and first derivatives of the form factors given by Hand *et al.* at zero-momentum transfer. In this way we obtain form factors that agree with experiment for space-like momentum transfers about as well as those of Hand *et al.* We find that $R_{1\phi}=13$ F^{-2}, $R_{1\omega}=-16$ F^{-2}, $R_{2\phi}=10$ F^{-2}, $R_{2\omega}=-5$ F^{-2}, $R_{1\rho}=-11$ F^{-2} and $R_{2\rho}=-57$ F^{-2}.

These numbers should not be considered excessively reliable. To take one example of a way in which error might arise, if there is a hard core in addition to ω and ϕ poles in the isoscalar-electric form factor, then we may transfer large amounts of residue from the ϕ pole to the core without destroying the fit of the form factor to the data. Such a reduction of $R_{1\phi}$ would much diminish our estimate of $G_{1\phi}$.

We now have the products γG; to proceed further we must know γ_ρ and γ_ω. Unitary symmetry tells us that γ_ρ is $\sqrt{3}\gamma_\omega$, and if we assume that the ρ pole dominates the pion-electromagnetic form factor, the determination of γ_ρ is straightforward. It has been discussed in detail by Sakurai[12] (actually, he discusses the determination of the residue at the ρ pole in the pion form factor, but to obtain γ_ρ from this is trivial). The result is

$$\gamma_\rho\cong-m_\rho^2(1-2\epsilon)^{1/2}/(8\pi)^{1/2}\,. \tag{40}$$

(The sign is arbitrary.) This yields $\gamma_\rho=-3.2$ F^{-2} and $\gamma_\omega=-1.9$ F^{-2}. Combining this with Eq. (39) we find (with the above choice of sign),

$$G_{1\rho}=4\,,$$
$$G_{2\rho}=22.1\,,$$
$$G_{1\omega}=5\,,$$

and

$$G_{1\phi}=\mp21\,. \tag{41}$$

We do not bother to tabulate values for the isoscalar-magnetic coupling constants; the isoscalar-magnetic form factor is poorly known and any numbers we would obtain would be extremely inaccurate.

A Remark on Strange-Baryon Form Factors

Unitary symmetry tells us that there are only two independent coupling constants for the coupling of an octet of vector mesons to the baryon octet, and only one for the coupling of a vector-meson singlet. We are thus in a position to use Eqs. (40) and (41) to calculate the residues at the vector-meson poles in the strange-baryon electromagnetic form factors. Since measure-

[10] Note that it is the remainder terms that determine whether the form factors require subtractions. The number of subtractions can not be decided from our approximations, which are blatantly invalid at high-momentum transfers.

[11] L. N. Hand, D. G. Miller, and Richard Wilson, Rev. Mod. Phys. **35**, 335 (1963).

[12] J. J. Sakurai, in *Theoretical Physics* (International Atomic Energy Agency, Vienna, 1963), p. 227.

ment of these quantities seems remote, we shall resist the temptation; however, we remark, that as a consequence of the vector mixing-approximation, for any baryon

$$\frac{R_{i\phi}}{m_\phi^2}+\frac{R_{i\omega}}{m_\omega^2}=\frac{\gamma_\omega G_{i\omega}}{M_1}. \tag{42}$$

That is to say, ω-ϕ mixing does not affect the predictions of unitary symmetry at zero-momentum transfer. In particular ω-ϕ mixing has no effect on the predictions of unitary symmetry for the magnetic moment of the Λ.[13] (Of course, other effects of the symmetry-breaking interaction may change the Λ moment; we merely assert that those which are enhanced by the presence of small-energy denominators do not.)

Vector Resonances in Nucleon-Nucleon Scattering

We may also use our methods to obtain the residues at the vector-meson poles in nucleon-nucleon scattering. We shall denote these residues by $g_{i\rho}^2$, $g_{i\omega}^2$, and $g_{i\phi}^2$. These are related to the quantities we have determined by

$$g_{i\omega}^2=[(1+2\epsilon-\beta^2)(m_\omega^2-m_\phi^2)]^{-1}\times\{(m_\omega^2-M_2)G_{i\omega}^2-2m_\omega^2\beta G_{i\phi}G_{i\omega}+[(1+2\epsilon)m_\omega^2-M_1]G_{i\phi}^2\},$$

$$g_{i\phi}^2=[(1+2\epsilon-\beta^2)(m_\phi^2-m_\omega^2)]^{-1}\times\{(m_\phi^2-M_2)G_{i\omega}^2-2m_\phi^2\beta G_{i\phi}G_{i\omega}+[(1+2\epsilon)m_\phi^2-M_1]G_{i\phi}^2\},$$

and

$$g_{i\rho}^2=G_{i\rho}^2/(1-2\epsilon). \tag{43}$$

Scotti and Wong[14] have estimated that $g_{1\rho}^2/4\pi=1.3$, $g_{1\omega}^2/4\pi=2.8$, and $g_{1\phi}^2/4\pi=2.3$. These numbers are probably very unreliable; other investigators have obtained quite different results.[15]

We may solve these to obtain an estimate

$$G_{1\rho}=4.5,$$

$$G_{1\omega}=6.7,$$

and

$$G_{1\phi}=3.5. \tag{44}$$

There is an alternative solution to the quadratic equation for the latter two:

$$G_{1\omega}=-1.0,$$

and

$$G_{1\phi}=7.5. \tag{44'}$$

[13] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).

[14] A. Scotti and D. Y. Wong (to be published).

[15] In an earlier version of their work [Phys. Rev. Letters **10**, 142 (1963)], Scotti and Wong find $g_{1\rho}^2/4\pi=5.1$, $g_{2\rho}^2/4\pi=49.0$, $g_{1\omega}^2/4\pi=16.7$. R. Bryan, C. Dismukes, and W. Ramsey, Nucl. Phys. **45**, 353 (1963), using a different approach, find $g_{1\omega}^2/4\pi+g_{1\rho}^2/4\pi=34.0$.

The agreement of the first of these with the first of Eqs. (41) offers no confirmation of either unitary symmetry or vector mixing; it is merely the statement that $R_{1\rho}$ is approximately equal to $\gamma_\rho g_{1\rho}$, and is independent of all our analysis (except, of course, our calculation of γ_ρ). It is the second and third equations of (41) and (44) that are sensitive to our theories. Their failure to agree is not so discouraging as might first appear, since the "raw data"—the R's and the g's—used in calculating the G's are so very poorly known.

We emphasize that our uncertainty is only temporary. The near future is certain to bring far more reliable estimates of the residues in both nucleon-nucleon scattering and the electromagnetic form factors; our methods will then provide a good check on unitary symmetry.

K^* Interactions

The only strange-vector coupling constant about which there is some information is that of the K^* to the Λ and N. Using the values for the G's given by Eq. (44), we find $G_{1K^*\Lambda N}=-4.8$. Using the values given by Eq. (41), we find $G_{1K^*\Lambda N}=-6.1$. The experimentally measured coupling constant is

$$g_{1K^*\Lambda N}^2/4\pi=G_{1K^*\Lambda N}^2/4\pi(1+\epsilon), \tag{45}$$

and hence $g_{1K^*\Lambda N}^2/4\pi=3.0$, if we use Eq. (44), 0.7, if we use Eq. (44′), and 3.4, if we use Eq. (41). An estimate of this constant was made by Chan,[16] who assumed that the K^* pole dominated the process $\pi^-+p\rightarrow K^0+\Lambda$. Using his results and an experimental width of 45 MeV, we estimate $0.24\leq g_{1K^*\Lambda N}^2/4\pi\leq 0.35$. Once again the lack of agreement is disappointing, but Chan's model is so crude that it is difficult to assess the reliability of this result.

The Decay of the ϕ

We begin by discussing $\phi\rightarrow K+\bar{K}$. Under unitary symmetry $G_{\omega K\bar{K}}$ is arbitrary and $G_{\phi K\bar{K}}$ is zero; thus the matrix element for ϕ decay is given by

$$|M(\phi\rightarrow K\bar{K})|^2=\frac{G_{\omega K\bar{K}}^2}{(1+2\epsilon-\beta^2)}\frac{(m_\phi^2-M_2)}{(m_\phi^2-m_\omega^2)}. \tag{46}$$

Unitary symmetry also tells us that $G_{\omega K\bar{K}}^2=\frac{3}{4}G_{\rho\pi\pi}^2$; hence

$$\frac{\Gamma(\phi\rightarrow K\bar{K})}{\Gamma(\rho\rightarrow\pi\pi)}=\frac{3}{4}\frac{(1-2\epsilon)}{(1+2\epsilon-\delta^2)}\frac{(m_\phi^2-M_2)}{(m_\phi^2-m_\omega^2)}\times\left(\frac{q_K}{q_\pi}\right)^3\left(\frac{m_\rho}{m_\phi}\right)^2=0.027. \tag{47}$$

Using a ρ width of 100 MeV, we find $\Gamma(\phi\rightarrow K\bar{K})\approx 2.7$

[16] C.-A. Chan, Phys. Rev. Letters **6**, 383 (1961).

MeV. This is in agreement with experiment[17] and also on the same order as the predictions of Okubo and Sakurai,[3] who use particle mixing models.

We shall now discuss $\phi\to\rho+\pi$. It is known from experiment that the amplitude for this process is essentially zero,[17] while a good explanation[12] of ω decay is obtained by assuming it proceeds principally through $\omega\to\rho+\pi$. We shall show that it is always possible to choose the two independent coupling constants $G_{\phi\rho\pi}$ and $G_{\omega\rho\pi}$ such that the matrix element for ϕ decay vanishes while that for ω decay does not. Indeed, we shall show that this may be done not merely in the vector-mixing approximation but in the more general propagator approximation.

The matrix element for ϕ decay is the residue of the ϕ pole in forward ρ-π scattering;

$$|M(\phi\to\rho+\pi)|^2=\lim_{k^2\to m_\phi^2}(k^2-m_\phi^2)\times M(\rho+\pi\to\rho+\pi). \quad (48)$$

Let us denote the vector $(G_{\phi\rho\pi},G_{\omega\rho\pi})$ as $\mathbf{G}$. Then, calculating the right-hand side of Eq. (48) in the propagator approximation, and using Eq. (8) for the propagator, we find

$$|M(\phi\to\rho+\pi)|^2=\mathbf{G}^T\boldsymbol{\varrho}_\phi\mathbf{G}. \quad (49)$$

Since $\boldsymbol{\varrho}_\phi$ is a real symmetric 2×2 matrix of rank one, it must always possess an eigenvector with eigenvalue zero. If we choose G to be proportional to this eigenvector, the matrix element for ϕ decay must vanish.

Using the value for $\boldsymbol{\varrho}_\phi$ given by Eq. (38) we find

$$G_{\omega\rho\pi}/G_{\phi\rho\pi}=m_\phi^2\beta/(m_\phi^2-M_2)=\pm0.52. \quad (50)$$

IV. ALTERNATIVE FORMALISMS

In this section we discuss the relations between our techniques and some alternative approaches to the same problems.

Mixed States

In the calculations of Sec. III we made great use of the $\boldsymbol{\varrho}^{(r)}$, the residues at the poles of the propagator. We could also have done our calculation in terms of the $\mathbf{e}^{(r)}$, the characteristic vectors of the $\boldsymbol{\varrho}^{(r)}$. We did not use this alternative method because, for our purposes, it is computationally inexpedient; however, let us see what it looks like.

Let us suppose we have a process of the sort shown in Fig. 1. In the absence of the symmetry-breaking interactions, there are n amplitudes for the particles at the blob A to go into the n possible intermediate mesons. Let us assemble these amplitudes into a vector $\mathbf{a}$. Likewise, let us assemble the n amplitudes at the blob B into a vector $\mathbf{b}$. Then, using Eq. (8) for the propagator, and retaining only pole terms, we find that the transition amplitude is

$$\sum_r \mathbf{b}^T\boldsymbol{\varrho}^{(r)}\mathbf{a}(k^2-m_r^2)^{-1}. \quad (51)$$

Using Eq. (9), we can write this as

$$\sum_r \mathbf{b}^T\mathbf{e}^{(r)}(k^2-m_r^2)^{-1}\mathbf{e}^{(r)T}\mathbf{a}. \quad (52)$$

We may interpret this in the following way: the physical-particle state, with mass m_r, is a superposition of "bare-particle states" which couple symmetrically to the incoming and outgoing particles. The expansion coefficients are the components of the vector $\mathbf{e}^{(r)}$. As we explained in Sec. II, this is a plausible interpretation for the particle-mixing approximation, in which the vectors $\mathbf{e}^{(r)}$ are an orthonormal set, but for more general approximations, in which this is not the case, it acquires more of the aspect of a metaphor. In fact, in our formalism, it is never anything but a metaphor; it is clear from the definition of the $\mathbf{e}^{(r)}$ in Eq. (10) that they have very little connection with the coefficients in the expansion of physical-particle states in terms of eigenstates of the symmetric Hamiltonian.

Nevertheless, in order to compare our results with those of other workers, let us calculate the vectors $\mathbf{e}_\phi$ and $\mathbf{e}_\omega$. From Eq. (38), we find that

$$\mathbf{e}_\omega=(1+2\epsilon+\beta^2)^{-\frac{1}{2}}(m_\phi^2-m_\omega^2)^{-\frac{1}{2}}(m_\omega^2-M_2)^{-\frac{1}{2}}\times(m_\omega^2-M_2,\,-m_\omega^2\beta), \quad (53)$$

and $\mathbf{e}_\phi$ is of the same form with m_ϕ and m_ω interchanged. These vectors are almost orthogonal. This is a coincidence caused by the near equality of M_1 and M_2; however, it allows us to define a "mixing angle." It is approximately $\pm29°$. Dashen and Sharp,[3] using the particle mixing approximation in which there is always a well-defined mixing angle, find an angle of 38°. In both cases, the mixing angle is defined such that, for zero angle, the ϕ is pure octet.

Subtracted Particle Mixing

In Sec. II we pointed out that naive application of the particle-mixing approximation leads to violations of current conservation. Vector mixing is one way of avoiding this difficulty. Another method that has been suggested[18] is to use particle mixing to calculate only the imaginary part of (for example) electromagnetic form factors. The real part may then be calculated by dispersion relations, and the subtraction constants adjusted as to guarantee current conservation.

[17] L. Bertanza, V. Bisson, P. L. Connolly, E. L. Hart, I. S. Mittra, *et al.*, Phys. Rev. Letters **9**, 180 (1962); P. Schlein, W. E. Slater, L. T. Smith, D. H. Stork, and H. K. Ticho, Phys. Rev. Letters **10**, 368 (1963); P. Conley, E. L. Hart, K. W. Lai, A. London, G. C. Moneti, *et al.*, Phys. Rev. Letters **10**, 371 (1963). The most recent measurement gives $\Gamma(\phi\to\rho+\pi)/\Gamma(\phi\to K\bar{K})=0.1\pm0.1$, [G. London (private communication)].

[18] S. L. Glashow (private communication).

This is evidently equivalent to making the subtraction in the propagator; that is to say, to using a propagator of the form

$$\mathbf{D}=(k^2-\mathbf{M})^{-1}+\mathbf{M}^{-1}-\mathbf{M}_0^{-1}. \tag{54}$$

This vector-meson propagator satisfies all the consequences of transversality. Of course, it has a singularity at infinity, where the true propagator is singularity-free, but this need not bother us, since we are only concerned with low-energy approximations.

Although this is in many ways a reasonable alternative procedure, we prefer vector mixing. We have several reasons:

(1) It is possible to construct models (e.g., quantum electrodynamics in $2+1$ dimensions) in which there is no subtraction for the electric form factor, although current is still conserved. It is difficult to justify subtracted particle mixing in this case; vector mixing encounters no difficulties.

(2) It is plausible that when we examine the electromagnetic form factors at high-momentum transfers, we see the structure of the bare, noninteracting particles. (However, to our knowledge, there is no rigorous proof of this.) This structure should preserve the symmetry of the original theory. Therefore it is desirable to have symmetry-breaking effects do minimal damage to the high momentum-transfer behavior of the form factors. In this respect, vector mixing is superior to subtracted particle mixing.

(3) The current experimental data[11] on the electric form factors of the nucleons is fitted well by expressions that contain no "hard cores," that is to say, which have no contributions from distant singularities. If we use subtracted particle mixing, we find that distant singularities necessarily play an important role in the form factors of strange baryons, even though they play a negligible role in nucleon form factors. Vector mixing does not disturb us in this way. We consider this to be its greatest advantage.

Method of Dashen and Sharp

Dashen and Sharp[3] preserve current conservation for the form factors using a particle-mixing approximation without making mixing-dependent subtractions. They do this by using momentum-dependent form factors for the vector-meson-baryon coupling. To be precise, they assume that the coupling of a physical vector meson to baryons is proportional to the vector-meson mass. This is equivalent to using momentum-independent couplings and using a propagator of the form

$$\mathbf{D}=\mathbf{M}(k^2-\mathbf{M})^{-1}\mu^{-2}, \tag{55}$$

where μ is a constant with dimensions of a mass. (The equivalence is clearly seen if we adopt a set of basis fields such that M is diagonal.) But this may be written

$$\mathbf{D}=(\mathbf{Z}k^2-\mu^2)^{-1}, \tag{56}$$

where $\mathbf{Z}$ is $\mathbf{M}^{-1}\mu^2$. But, if the masses of the vector mesons are equal in the absence of the symmetry-breaking interaction (as is the case for ω-ϕ mixing), then this is nothing but vector mixing.

The results of Dashen and Sharp are not strictly equivalent to ours because they apply Gell-Mann–Okubo arguments to $\mathbf{M}$ rather than to $\mathbf{Z}$. However, due to the relatively small magnitude of the mass splitting, this does not have a large effect.

V. DISCUSSION

We have shown that ordinary particle mixing may be placed in a field-theoretic context, and that, within this context, for a large class of interactions, it is a suitable approximation for treating particles of spin zero. However, for particles of spin one, again for a large class of interactions, particle mixing is inferior to vector mixing.

The most striking deficiency of particle mixing for particles of spin one is that its naive application to ϕ-ω mixing leads to a violation of the conservation of electric charge. Vector mixing does not have this difficulty. There are, of course, other approximations that preserve the conservation of charge, some of which are closer in appearance to ordinary particle mixing than is vector mixing. We have discussed some of these in Sec. IV, and explained there why we believe them to be not as satisfactory as vector mixing.

We have left two closely related theoretical problems unsolved: We do not know how to refound our work on analytic S-matrix theory. (This is important if we wish to extend our results, which we have only shown to be valid for fundamental particles, to composite systems.) We do not know how to extend our results to systems of spin other than zero or one. We suspect that the place of the sum rule (27), which plays such an important part in our analysis, will be taken by the condition that the scattering amplitude for the jth partial wave must go to zero like k^{2j+1} near threshold.

Our attempts to apply vector mixing to the ω-ϕ system have not met with much success. The principal reason for this seems to be the unreliability of our input data (the residues at the vector-meson poles in nucleon-nucleon scattering, nucleon-electromagnetic form factors, and K production). These quantities have been calculated only on the basis of very crude models, and the values we possess for them are qute unreliable. The only quantity we have calculated that is independent of these residues, the $\phi\rightarrow K\bar{K}$ decay rate, is in good agreement with experiment. However, the near future should bring far better values; then our formulas should provide good checks of both unitary symmetry and vector mixing.

ACKNOWLEDGMENT

We would like to thank R. Socolow for many helpful criticisms.

APPENDIX I: PROOF OF A THEOREM

In the body of the paper we stated the theorem that the pole approximation for the propagator,

$$\mathbf{D}(k^2)=\sum_{r=1}^{n} \boldsymbol{\varrho}^{(r)}/(k^2-m_r^2)\,, \tag{A1}$$

where the $\boldsymbol{\varrho}^{(r)}$ are real symmetric matrices of rank one,

$$\boldsymbol{\varrho}^{(r)}=\mathbf{e}^{(r)}\mathbf{e}^{(r)T}\,, \tag{A2}$$

is equivalent to the inverse propagator being a linear function of k^2,

$$\mathbf{D}^{-1}(k^2)=\mathbf{Z}k^2-\mathbf{M}\,, \tag{A3}$$

where $\mathbf{M}$ and $\mathbf{Z}$ are real, symmetric, positive-definite matrices. In this Appendix we shall prove this statement.

First we shall show that (A1) implies (A3). We assume that the $\mathbf{e}^{(r)}$ form a complete set of vectors. Then we may introduce a reciprocal set of vectors $\mathbf{f}^{(r)}$ defined by

$$\mathbf{e}^{(r)T}\mathbf{f}^{(s)}=\mathbf{f}^{(s)T}\mathbf{e}^{(r)}=\delta_{rs}\,. \tag{A4}$$

As a consequence of Eq. (A4),

$$\textstyle\sum_r \mathbf{e}^{(r)}\mathbf{f}^{(r)T}=\sum_r \mathbf{f}^{(r)}\mathbf{e}^{(r)T}=\mathbf{I}\,. \tag{A5}$$

Using the $\mathbf{f}^{(r)}$, it is trivial to construct $\mathbf{D}^{-1}$,

$$\mathbf{D}^{-1}=\textstyle\sum_r \mathbf{f}^{(r)}\mathbf{f}^{(r)T}(k^2-m_r^2)\,. \tag{A6}$$

This clearly is equivalent to (A3) if

$$\mathbf{Z}=\sum_r \mathbf{f}^{(r)}\mathbf{f}^{(r)T}\,, \tag{A7}$$

and

$$\mathbf{M}=\sum_r \mathbf{f}^{(r)}\mathbf{f}^{(r)T}m_r^2\,. \tag{A8}$$

Now we will show that (A3) implies (A1). A well-known theorem in matrix theory states that given any two real symmetric matrices, one of which is positive definite, there exists a congruence transformation that reduces the positive-definite matrix to the identity and diagonalizes the other matrix. Let $\mathbf{Z}$ be the positive-definite matrix and $\mathbf{M}$ the other. Then the theorem is equivalent to saying that there exists a set of vectors $\mathbf{f}^{(r)}$ such that Eqs. (A7) and (A8) are true. Let us define a set of vectors $\mathbf{e}^{(r)}$ by Eq. (A4). Then it is trivial to find $\mathbf{D}$ from $\mathbf{D}^{-1}$ and we obtain Eqs. (A1) and (A2).

APPENDIX II: MASSLESS VECTOR MESONS

In the body of this paper we have followed the custom in strong interaction physics and have only treated electromagnetic phenomena to first order in e. Thus, there has been no need for us to consider the mixing of the photon with other vector mesons, since this is an effect of order e^2. In this Appendix we will obtain some results on vector mixing in the case where one of the vector mesons has zero mass. To treat this case properly, we would have to redo our entire analysis, for the formalism on which it is based, and in particular the sum rules (27) and (28), are valid only for massive vector mesons. It is notorious that massless vector mesons require a quite different treatment. Despite this, we shall simply apply the results of Sec. II. to this case; none of our formulas are infrared divergent, and, with luck, our results may be valid even if our methods are doubtful.

We begin with the formula for the propagator in the vector-mixing approximation

$$\mathbf{D}(k^2)=(k^2-\mathbf{M}_0+\boldsymbol{\delta}k^2)^{-1}. \tag{30}$$

We want this to have a pole at $k^2=0$; therefore,

$$\det\mathbf{D}^{-1}(0)=\det(-\mathbf{M}_0)=0\,. \tag{A9}$$

Since $\mathbf{M}_0$ is diagonal, this means one of its diagonal entries must be zero; without loss of generality, we may choose it to be the first. To obtain a zero physical mass one must begin with a zero bare mass, at least in this approximation.

We will now determine the residue of the pole at zero-momentum transfer, which we call $\boldsymbol{\varrho}_0$.

$$\boldsymbol{\varrho}_0=\lim_{k^2\to 0} k^2[k^2-\boldsymbol{\Pi}(k^2)]^{-1}. \tag{A10}$$

Now,

$$(k^2-\boldsymbol{\Pi})^{-1}=[\det(k^2-\boldsymbol{\Pi})]^{-1}\,\mathrm{adj}(k^2-\boldsymbol{\Pi})\,, \tag{A11}$$

where by adj$\mathbf{A}$ we denote the matrix constructed of the cofactors of $\mathbf{A}$. Det$(k^2-\boldsymbol{\Pi})$ is a polynomial in k^2, with a simple zero at $k^2=0$,

$$\det(k^2-\boldsymbol{\Pi})=(1+[\boldsymbol{\delta}]_{11})m_0k^2+O(k^4)\,, \tag{A12}$$

where

$$m_0=\sum_{i=2}^{n}[-\mathbf{M}_0]_{ii}\,. \tag{A13}$$

Likewise,

$$\mathrm{adj}[k^2-\boldsymbol{\Pi}]_{k^2=0}=\mathrm{adj}(-\mathbf{M}_0)\,. \tag{A14}$$

Since $\mathbf{M}_0$ is a diagonal matrix with one diagonal entry zero, adj $(\mathbf{M}_0)$ is zero except for its first diagonal entry,

$$[\mathrm{adj}(-\mathbf{M}_0)]_{ij}=m_0\delta_{i0}\delta_{j0}\,. \tag{A15}$$

Thus,

$$[\boldsymbol{\varrho}_0]_{ij}=(1+[\boldsymbol{\delta}]_{11})^{-1}\delta_{i0}\delta_{j0}\,. \tag{A16}$$

This means that the photon pole occurs only in the photon channel, and never in any other channel. (Of course, this must be the case, if the Coulomb force between particles is to depend only on their electric charge and not on their hypercharge or isospin.) On the other hand, there is no such constraint on the other residues, and thus it is possible to have a ρ-meson pole in the photon channel. (Of course, this must be the case, if our analysis of form factors, in the main body of the paper, certainly valid to first order in e, is not to be contradicted.)