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Weak-Coupling Limit for Scattering by Strongly Singular Potentials*

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The validity of certain cutoff procedures, which have lately been employed in the treatment of termwise divergent perturbative expansions in nonrenormalizeable field theories, is investigated in the context of nonrelativistic scattering from strongly singular repulsive potentials. For the cases considered the heuristic cutoff prescription indeed yields the correct expression for the weak-coupling limit of the phase shift.

I. INTRODUCTION

HEURISTIC computational schemes for extracting meaningful results in the framework of nonrenormalizeable field-theoretic models have come under considerable discussion recently.[1,2] Although the perturbative expansions represented by sets of Feynman graphs are termwise divergent, one supposes that this is only an artifact introduced by an improper expansion in powers of a coupling constant g, and that an appropriate representation of the sum must exist in which divergences do not appear. To achieve such a representation, one introduces a cutoff parameter, Λ say, in the Feynman integrals, rendering them termwise convergent. The supposition is that the sum over graphs is itself finite as $\Lambda\to\infty$ and that in this limit it represents the physical answer. In practice, the summation cannot be fully carried out in closed form. Often, however, one is content to find the leading term in an asymptotic expansion for small values of the coupling constant g. Here the further supposition is then made that the leading term can be obtained by summing the leading contributions (as $\Lambda\to\infty$) for each order of g. Even this sum over leading terms cannot in general be carried out in closed form; hence, one cannot in general confirm that it indeed leads to a finite result as $\Lambda\to\infty$. But on the assumption that the sum *does* in fact exist, a power counting analysis then yields information about the nature of the leading term in an asymptotic expansion for small values of g.

Since, as said, the legitimacy of these procedures cannot be easily investigated in a field-theoretic context, we propose in this note to study similar improper perturbative expansions arising in nonrelativistic potential scattering theory.

II. CUTOFF PROCEDURE FOR SINGULAR POTENTIALS

In this section we describe how the cutoff procedure would be used in connection with the study of scattering by a strongly singular repulsive potential. Consider the radial Schrödinger equation for the lth partial wave

$$\frac{d^2\varphi}{dr^2}+\left[k^2-\frac{l(l+1)}{r^2}-gV(r)\right]\varphi=0. \qquad (1)$$

The integral equation for the regular solution of (1) is

$$\varphi(k,r)=s_l(kr)-\frac{1}{k}gs_l(kr)\int_r^\infty c_l(kr')V(r')\varphi(k,r')dr' -\frac{1}{k}gc_l(kr)\int_0^r s_l(kr')V(r')\varphi(k,r')dr', \qquad (2)$$

where

$$s_l(kr)=(\tfrac{1}{2}\pi kr)^{1/2}J_{l+\frac{1}{2}}(kr),$$
$$c_l(kr)=-(\tfrac{1}{2}\pi kr)^{1/2}Y_{l+\frac{1}{2}}(kr).$$

The phase shift for the lth partial wave is determined by

$$\tan\delta_l=-(g/k)\int_0^\infty s_l(kr)V(r)\varphi(k,r)dr. \qquad (3)$$

Now we shall always suppose that $rV\to 0$ as $r\to\infty$. But suppose, in addition, that $r^2V\to 0$ as $r\to 0$. Then the Born series solution of (2) will exist and will converge for small enough values of the coupling constant g. However, if $gV\to gr^{-\beta}$ as $r\to 0$, $\beta>2$, $g>0$, then the nth interation of (2) will behave near the origin like

$$\varphi^{(n)}\underset{r\to 0}{\longrightarrow} g^nk^{l+1}r^{l+1-n(\beta-2)},$$

so that the iteration integrals diverge for order n larger than $(2l+1)/(\beta-2)$. The Born series does not exist for any value of g other than zero. This naturally suggests that there is a branch point at $g=0$ and we seek a procedure for obtaining an asymptotic expansion of the solution in the limit of small g. Let us see how we can set up a heuristic method for obtaining the leading term in the asymptotic expansion by use of a cutoff. We replace the actual potential V by a cutoff potential, e.g.,

$$V_\epsilon(r)=\theta(r-\epsilon)V(r).$$

The Born series corresponding to this potential exists for small enough g and we can compute the phase shift as a power-series expansion in g. Since we are later going to take the limit $\epsilon\to 0$, we may expect that

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[1] T. D. Lee and C. N. Yang, Phys. Rev. **128**, 885 (1962); T. D. Lee, *ibid.* **128**, 899 (1962).

[2] G. Feinberg and A. Pais, Phys. Rev. **131**, 2724 (1963).

the leading term in the asymptotic expansion for small g can be obtained by retaining, for every order of g, only the leading terms for $\epsilon \to 0$.

For example, in the case of S-wave scattering by the potential

$$gV = gr^{-4} + gV', \quad r^4 V' \underset{r\to 0}{\longrightarrow} 0,$$

this procedure leads to the result

$$\tan\delta_\epsilon = -k[g + \tfrac{1}{3}g^2\epsilon^{-3} + \tfrac{1}{5}g^3\epsilon^{-5} + \cdots]; \tag{4}$$

and it is easily shown that this series converges to

$$\tan\delta_\epsilon = -k(g)^{1/2}\tanh(g^{1/2}/\epsilon). \tag{5}$$

For finite ϵ this expression is analytic in the coupling constant g. Passing now to the limit $\epsilon \to 0$, we find

$$\tan\delta = -k(g)^{1/2}, \tag{6}$$

i.e., there is now a branch point at the origin of the g plane. The cutoff procedure for obtaining the leading term in an asymptotic expansion for small g evidently works, in the sense that it gives a finite result. We shall see later on that the result is in fact the correct one.

In the above example, we could explicitly sum the power series in g for finite cutoff ϵ. Even where this cannot be done in closed form a power counting analysis permits one to infer the nature of the leading term in the asymptotic expansion for small g. Thus, if we find for finite ϵ the series of leading terms

$$\tan\delta_\epsilon = k\sum_{n=0}^{\infty} a_n g^n \epsilon^{-nb+1},$$

we can rewrite this as

$$\tan\delta_\epsilon = kg^{1/b}[(g\epsilon^{-b})^{-1/b}\sum_n a_n(g\epsilon^{-b})^n]. \tag{7}$$

If we now suppose that the bracketed expression exists in the limit $\epsilon \to 0$ we infer that $\tan\delta \to \text{const}\times kg^{1/b}$, as $g \to 0$.

In the example worked out above in connection with the result (5), all Born terms in the expansion for $\tan\delta$ were divergent in the limit $\epsilon \to 0$. In the general case, with $V \to r^{-\beta}$ as $r \to 0$, the cutoff procedure gives the following results: If $(2l+1)/(\beta-2)$ is not an integer and if n is the greatest integer less than $(2l+1)/(\beta-2)$, we have the form

$$\tan\delta = \sum_{m=0}^{n} a_m(k)g^m + ak^{2l+1}g^{(2l+1)/(\beta-2)} + \text{higher order in } g. \tag{8}$$

If $(2l+1)/(\beta-2) = n$ is an integer we have

$$\tan\delta = \sum_{m=0}^{n-1} a_m(k)g^m + ak^{2l+1}g^n \ln g + \text{higher order in } g. \tag{9}$$

In both cases the coefficient a is independent of k and g.

It should be noted that in a formulation of the scattering problem in momentum space, certain quantities—as, for example, the matrix elements $\langle k',l|V|k,l\rangle$ of the potential operator—may not exist. In particular, if $l<\frac{1}{2}(\beta-3)$ the Lippman-Schwinger integral equation for the T matrix cannot be formulated without a cutoff. A similar situation arises in nonrenormalizeable field-theoretic models. Thus, the kernel of a Bethe-Salpeter-type equation may be so singular on the light cone that its Fourier transform does not exist.[3] Nevertheless, the cutoff technique can in principle always be introduced. In the following we shall investigate the validity of this cutoff procedure for the problem of scattering by singular repulsive potentials of a certain class.

III. THE ASYMPTOTIC EXPANSION

We consider repulsive potentials of the form ($g=\alpha^2$)

$$gV(r) = \alpha^2 r^{-2-2/\nu}, \quad 0<\nu<\infty. \tag{10}$$

With $\lambda = l+\frac{1}{2}$ the radial equation is

$$\frac{d^2\varphi}{dr^2} + \left[k^2 - \frac{\lambda^2-\frac{1}{4}}{r^2} - \alpha^2 r^{-2-2/\nu}\right]\varphi = 0. \tag{11}$$

In the limit of zero energy ($k=0$) this can be reduced to the Bessel equation, whose regular solution is[4]

$$\varphi_0 = \alpha^{\lambda\nu} r^{1/2} K_{\lambda\nu}(\nu\alpha r^{-1/\nu}). \tag{12}$$

In order to obtain a convergent expression for the regular solution of (10) when $k\neq 0$, we set

$$\varphi(r) = \varphi_0(r)Z(r)$$

so that $Z(r)$ satisfies the equation

$$\frac{d}{dr}\left[\varphi_0^2\frac{dZ}{dr}\right] + k^2\varphi_0^2 Z = 0.$$

Imposing the boundary condition $Z(r)\to 1$ as $r\to 0$, we obtain the Volterra equation

$$Z(r) = 1 + k^2\int_0^r W(r,r')Z(r')dr', \tag{13}$$

where

$$W(r,r') = -\int_{r'}^{r} [\varphi_0^2(r')/\varphi_0^2(t)]dt.$$

From the asymptotic behavior of the K function we see that

$$\varphi_0(r) \underset{r\to 0}{\longrightarrow} r^{(\nu+1)/2\nu} e^{-\nu\alpha r^{-1/\nu}};$$

and it is then as easy matter to show that there exists a

[3] R. F. Sawyer (to be published).

[4] In terms of the Bessel functions $J_p(x)$ we have

$$k_p(x) = \pi(2\sin p\pi)^{-1}\{e^{ip\pi/2}J_{-p}(ix) - e^{-ip\pi/2}J_p(ix)\}.$$

The factor $\alpha^{\lambda\nu}$ has been included to ensure a finite limit for $\alpha\to 0$.

positive number B, independent of r, r', λ, and g, such that

$$|W(r,r')| < Br'$$

for

$$|\arg g| \leqslant \pi. \tag{14}$$

Thus, the iteration solution of (13) converges uniformly in λ and g in the cut plane defined by (14). From the fact that $K_p(x)$ is an even entire function of p and analytic in the x plane cut along the negative real axis, it follows that $Z(r)$, hence $\varphi(r)$, is analytic in the g plane cut along the negative real axis and that it is an entire function of λ^2 considered as a complex variable. Since the Jost solution of (11) can be shown by standard methods to be entire in g and λ^2, we conclude that the scattering amplitude is meromorphic in the cut g plane and meromorphic in λ^2 (apart from a factor $e^{i\pi\lambda}$ in the S-matrix element). This generalizes the results announced by Regge and Predazzi[5] for a special example of a singular potential.

Now $\tan\delta$ has a branch point at $g=0$ and we are interested in finding the leading term in an asymptotic expansion for small g. For this purpose, define

$$v(r)=r^{\lambda+\frac{1}{2}}Z(r),$$

so that

$$\frac{d^2v}{dr^2}+\left[k^2-\frac{\lambda^2-\frac{1}{4}}{r^2}\right]v=\left[\frac{2\lambda+1}{r}-2\frac{\varphi_0'}{\varphi_0}\right]r^{\lambda+\frac{1}{2}}\frac{d}{dr}\left(\frac{v}{r^{\lambda+\frac{1}{2}}}\right). \tag{15}$$

Since the iteration solution of (13) can be differentiated termwise to yield a uniformly convergent series for dZ/dr we may write

$$\frac{d^2v}{dr^2}+\left[k^2-\frac{\lambda^2-\frac{1}{4}}{r^2}\right]v=k^2\left[2\frac{\varphi_0'}{\varphi_0}-\frac{2\lambda+1}{r}\right]r^{\lambda+\frac{1}{2}}$$
$$\times\varphi_0^{-2}(r)\int_0^r\varphi_0^2(r')r'^{-(\lambda+\frac{1}{2})}v(r')dr',$$
$$\equiv B(\alpha,r). \tag{16}$$

The regular solution satisfies the integral equation

$$v(r)=s_{\lambda+\frac{1}{2}}(kr)-\frac{1}{k}s_{\lambda+\frac{1}{2}}(kr)\int_r^\infty c_{\lambda+\frac{1}{2}}(kr')B(\alpha,r')dr'$$
$$-\frac{1}{k}c_{\lambda+\frac{1}{2}}(kr)\int_0^r s_{\lambda+\frac{1}{2}}(kr')B(\alpha,r')dr'. \tag{17}$$

We now distinguish three cases:

(i) If $\lambda\nu>1$ we have

$$2\frac{\varphi_0'}{\varphi_0}-\frac{2\lambda+1}{r}=-\frac{\nu\alpha^2}{\lambda\nu-1}r^{-1-2/\nu}$$
$$+\text{terms of higher order in } \alpha.$$

[5] E. Predazzi and T. Regge, Nuovo Cimento **24**, 518 (1962).

It can readily be verified that the integrations in (17) converge if $B(\alpha,r')$ is replaced by $\alpha^2\lim_{\alpha\to0}\alpha^{-2}B(\alpha,r')$. Thus, the first iteration of (17) is proportional to α^2 and the remainder is of higher order in α. As expected, therefore, the leading term of φ in an asymptotic expansion for small g is just given by the first Born iteration, which in this case is covergent. For $\tan\delta$ the leading term for small g is thus proportional to g, being given by the first Born approximation.

(ii) For $\lambda\nu<1$ we have

$$2\frac{\varphi_0'}{\varphi_0}-\frac{2\lambda+1}{r}=4\lambda\left(\frac{\nu}{2}\right)^{2\lambda\nu}\frac{\Gamma(1-\lambda\nu)}{\Gamma(1+\lambda\nu)}\alpha^{2\lambda\nu}r^{-1-2\lambda}$$
$$+\text{terms of higher order in } \alpha.$$

Again the first iteration of (17) gives the leading term in the asymptotic expansion for small g. Correspondingly, the leading term for $\tan\delta_\lambda$ is given by

$$\tan\delta_\lambda \underset{g\to0}{\longrightarrow} -\frac{\pi\lambda}{2^{2\lambda}}\left(\frac{\nu}{2}\right)^{2\lambda\nu}\frac{\Gamma(1-\lambda\nu)}{\Gamma(1+\lambda\nu)}\frac{1}{\Gamma^2(1+\lambda)}\alpha^{2\lambda\nu}k^{2\lambda}. \tag{18}$$

In particular, for the case $\lambda=\frac{1}{2}$, $\nu=1$, which corresponds to the example worked out by the cutoff procedure in Sec. II, we recover the result obtained there, namely $\tan\delta=-\alpha k$.

(iii) For $\lambda\nu=1$ we have

$$2\frac{\varphi_0'}{\varphi_0}-\frac{2\lambda+1}{r}=-(\nu\alpha^2\ln\alpha)r^{-1-2\lambda}$$
$$+\text{terms of higher order in } \alpha.$$

Again the leading term is obtained by the first iteration of (17) and this leads to a result for $\tan\delta$ which is proportional to $\alpha^2\ln\alpha$:

$$\tan\delta_\lambda \underset{g\to0}{\longrightarrow} 2^{-2-2\lambda}(\pi k^{2\lambda}/\lambda\Gamma^2(1+\lambda))\alpha^2\ln\alpha. \tag{19}$$

As discussed in Sec. II, the same result is obtained by the cutoff technique, the present case corresponding to a situation in which the first Born term in the iteration of (2) diverges logarithmically.

IV. DISCUSSION

It is not difficult to understand the reasons for the success of the cutoff prescription for the determination of the leading term of $\tan\delta$ as $g\to0$. In the case of the simple power-law potentials $gV(r)=gr^{-2-2/\nu}$ which we have been considering, the cutoff procedure, insofar as it retains only the most singular terms in ϵ^{-1} for every order, in effect amounts to a replacement of the integral equation (2) by the equation[6]

$$\varphi_\epsilon(k,r)=\frac{\pi^{1/2}}{\Gamma(\lambda+1)}\left(\frac{k}{2}\right)^{\lambda+\frac{1}{2}}r^{\lambda+\frac{1}{2}}-\frac{g}{2\lambda}r^{\lambda+\frac{1}{2}}\int_r^\infty r'^{-\lambda+\frac{1}{2}-2-2/\nu}\varphi_\epsilon(k,r')\theta(r'-\epsilon)dr'-\frac{g}{2\lambda}r^{-\lambda+\frac{1}{2}}\int_\epsilon^r r'^{\lambda+\frac{1}{2}-2-2/\nu}\varphi_\epsilon(k,r')dr'. \quad (20)$$

The solution, apart from a k-dependent normalization, is just the regular solution of (1) for $k=0$ and $V=gr^{-2-2/\nu}\theta(r-\epsilon)$. For dimensional reasons, the corresponding expression for $\tan\delta_{\lambda,\epsilon}$ obtained from Eq. (3) is necessarily of the form

$$(\epsilon k)^{2\lambda}f(g\epsilon^{-2/\nu}),$$

where $f(x)$ is analytic at $x=0$. Therefore, if the limit

$$\lim_{\epsilon\to 0}(\epsilon k)^{2\lambda}f(g\epsilon^{-2/\nu}) \quad (21)$$

exists, it must coincide with

$$\lim_{\epsilon\to 0}\tan\delta_{\lambda,\epsilon}=-\frac{g}{k}\frac{\pi^{1/2}}{\Gamma(\lambda+1)}\left(\frac{k}{2}\right)^{\lambda+\frac{1}{2}}\int_0^\infty r^{\lambda+\frac{1}{2}-2-2/\nu}\phi_0(k,r)dr, \quad (22)$$

where $\varphi_0(k,r)$ is the properly normalized zero-energy solution of (1). [The normalization is determined by the first term on the right-hand side of (20).]

But the expression (22) is nothing other than the leading term (for $g\to 0$) of (3), which we rewrite here

$$\tan\delta_\lambda=-\frac{g}{k}\int_0^\infty s_{\lambda+\frac{1}{2}}(kr)r^{-2-2/\nu}\varphi(k,r)dr.$$

Indeed, in this integral the contribution from the range outside any neighborhood of the origin is proportional to $g=\alpha^2$. Being interested in the leading term, which vanishes more slowly than this as $g\to 0$, we may replace $s_{\lambda+\frac{1}{2}}(kr)$ and $\varphi(k,r)$ by their asymptotic forms for $r\to 0$. But from (13) we have seen that $\varphi(k,r)\to\varphi_0(k,r)$ as $r\to 0$, where φ_0 is the zero-energy solution. It can be directly verified that the expressions for $\tan\delta_\lambda$ which one obtains from (22) on use of the true zero-energy solution agree with the results obtained in the preceding section.

It should be remarked that this argument does not depend on the form of cutoff function which is employed, provided that for positive values of the cutoff parameter ϵ the potential is regular enough to ensure the existence of the Born series for small enough g and provided that the limit (21) exists.

In summary, we can argue apart from the considerations of Sec. III that the leading term in the asymptotic expansion of $\tan\delta_\lambda$ is given by the approximation (22) [recall that we are now discussing for simplicity the most serious case, where the first Born iteration of (2) is already divergent]. In (22), $\varphi_0(k,r)$ is the zero-energy solution of the Schrödinger equation, properly normalized. We have argued that it must be correctly given by the cutoff procedure, at least for the class of potentials under discussion. The direct results of Sec. III confirm this and also confirm that no delicacy has been overlooked in these plausibility remarks, i.e., they confirm that the limit (21) indeed exists. That the limit (21) is not analytic in g should not be surprising, since the limiting process $\epsilon\to 0$ is not uniform with respect to g. A simple illustration of this phenomenon has already been provided by the example of Sec. II, where for $V=gr^{-4}$, $l=0$, we found $\tan\delta=-\lim_{\epsilon\to 0}(g)^{1/2}\tanh(g^{1/2}/\epsilon)=-(g)^{1/2}k$.

In our discussion so far, we have considered simple potentials of the form $gV=gr^{-2-2/\nu}$. For the more general case,

$$gV=g[r^{-2-2/\nu}+V_1],$$

where $r^{2+2/\nu}V_1\to 0$ as $r\to 0$ and where V_1 is independent of g, we would expect that the weak-coupling limit for $\tan\delta$ is unaffected by the presence of the less singular addition V_1. It would certainly be ignored in the cutoff procedure. This expectation would in fact be justified if one could show that the integral in (22) in fact converges (it is only convergence at the lower limit that would be in question). In (22) φ_0 is the zero-energy solution of the full Schrödinger equation. Since the integral in fact converges when $V_1=0$ there can be little doubt that it exists when V_1 is present. That convergence is enough, we can argue dimensionally. Consider, for example, the case

$$V(r)=gr^{-2-2/\nu}(1+\gamma r^\rho),\quad \lambda\nu<1,\quad \rho>0,$$

where γ is a fixed parameter independent of g. For dimensional reasons, the existence of (22) implies that

$$\tan\delta\underset{g\to 0}{\longrightarrow}(kg^{\nu/2})^{\lambda+\frac{1}{2}}f(g\gamma^{2/\rho})\to(kg^{\nu/2})^{\lambda+\frac{1}{2}}f(0),$$

independent of γ. As to the convergence of (22), this can be inferred in every particular case from the asymptotic form of φ_0 for small r. For example, if $\nu=1$ and $\rho=\frac{1}{2}$ we find

$$\phi_0\underset{r\to 0}{\longrightarrow}e^{-g^{1/2}(r^{-1}+\alpha r^{-1/2})}r^{1+\frac{1}{8}(g)^{1/2}\gamma^2}[1+0(r^{1/2})]$$

and (22) indeed converges.

[6] For simplicity, we consider only the case $\lambda\nu<1$, where already the first Born term is divergent for $\epsilon\to 0$.