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Octet Mass Vector

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It is shown that the concept of the octet mass vector, introduced previously by Glashow, may be used to provide an unambiguous criterion for the accuracy of the various mass sum rules that have been proposed on the basis of unitary symmetry. This criterion is applied to the experimental baryon and pseudoscalar meson masses. The electromagnetic mass splittings are discussed in detail. It is shown that the observed mass splittings correspond closely to the types that are favored in a simple bootstrap model, if the F/D interaction angle for the meson-baryon interactions is set equal to a value suggested by other considerations.

RECENTLY, arguments based on unitary symmetry have been used to derive linear sum rules for the masses (or the squares of the masses) of the particles in the baryon octet and the particles in the pseudoscalar meson octet.[1,2] The purposes of the present note are to define a quantitative criterion for the accuracy of any such linear sum rule, and to propose an extension of the Coleman-Glashow sum rule for the mass differences within isotopic multiplets.[2,3]

The group theoretical arguments used to derive the Gell-Mann–Okubo mass formula do not make it clear whether the masses or their squares are more appropriate variables.[1] However, only the squares of the meson masses occur in dispersion relations, while the baryon masses and their squares both occur. Consequently, we will apply the sum rules both to the baryon masses and their squares, and to the squares of the meson masses. The symbol β_j is used to represent either the mass or the square of the mass of the particle j.

We imagine an eight-dimensional vector space and eight mutually orthogonal unit vectors $\mathbf{e}_j$, each $\mathbf{e}_j$ corresponding to one of the members of the SU_3 octet. The physical β_j values for the baryon octet form a vector $\boldsymbol{\beta}$ in this space, i.e.,

$$\boldsymbol{\beta}=(\tfrac{1}{8})^{1/2}\sum_j \beta_j \mathbf{e}_j. \qquad (1)$$

The factor $(\frac{1}{8})^{1/2}$ is used so that $|\boldsymbol{\beta}|$ corresponds to the physical β if the octet is degenerate. The fact that it may be useful to treat the masses of the members of a multiplet as components of a vector was first pointed out by Glashow,[4] and has been discussed extensively by Cutkosky and Tarjanne.[5] This formalism may be used to give an elegant and unambiguous criterion for the accuracy of mass sum rules. Any mass formula of the type $\sum_j c_j\beta_j=0$ may be written in vector notation, $\mathbf{c}\cdot\boldsymbol{\beta}=0$; the angle between the physical $\boldsymbol{\beta}$ vector and the forbidden $\mathbf{c}$ direction measures the accuracy of the formula. By contrast, the conventional method of measuring the accuracy in terms of the difference from zero of the sum $\sum_j c_j\beta_j$ is ambiguous since there is no obviously preferred manner of normalizing the c_j factors.

The vectors $\mathbf{e}_j$ of Eq. (1) are not eigenfunctions of R conjugation, charge reflection, or other physically meaningful transformations, and thus are not a convenient basis. We introduce the following alternate, orthonormal basis:

$$\begin{aligned}
\mathbf{E}_0&=(1/8)^{1/2}\textstyle\sum_i \mathbf{e}_i\,,\\
\mathbf{E}_1&=(1/20)^{1/2}(p+n+\Xi^-+\Xi^0+2\Lambda-2\Sigma^+-2\Sigma^--2\Sigma^0)\,,\\
\mathbf{E}_2&=(1/4)^{1/2}(p+n-\Xi^--\Xi^0)\,,\\
\mathbf{E}_3&=(1/120)^{1/2}(\Sigma^++\Sigma^-+\Sigma^0+9\Lambda\\
&\qquad -3p-3n-3\Xi^--3\Xi^0)\,, \qquad (2)\\
\mathbf{E}_4&=(1/6)^{1/2}(\Sigma^++\Sigma^--2\Sigma^0)\,,\\
\mathbf{E}_5&=(1/4)^{1/2}(p-n+\Xi^--\Xi^0)\,,\\
\mathbf{E}_6&=(1/12)^{1/2}(p-n-\Xi^-+\Xi^0+2\Sigma^+-2\Sigma^-)\,,\\
\mathbf{E}_7&=(1/6)^{1/2}(p-n-\Xi^-+\Xi^0-\Sigma^++\Sigma^-)\,,
\end{aligned}$$

where the baryon symbol j is an abbreviation for $\mathbf{e}_j$. One method of obtaining the $\mathbf{E}_i$ basis is to regard the baryon octet as a compound of the baryon and meson octets, with Clebsch-Gordan coefficients corresponding to unitary symmetry. These coefficients are given in terms of the F/D interaction angle θ by Cutkosky.[6] The general matrix element Π_{ij} of the probability matrix $\mathbf{\Pi}$ is defined to be the probability of the baryon j in the wave function of the baryon i. Because of the crossing symmetry of the pseudoscalar mesons, $\mathbf{\Pi}$ is symmetric (as well as real), so that a complete set of orthogonal eigenvectors may be found. The basis vectors $\mathbf{E}_0$, $\mathbf{E}_3$, $\mathbf{E}_4$, and $\mathbf{E}_7$, are eigenvectors of $\mathbf{\Pi}$ for all values of the interaction angle θ ($\mathbf{E}_3$ and $\mathbf{E}_4$ are degenerate for all θ). The (12) and (56) planes are invariant subspaces for all θ, but the eigenvectors within these subspaces depend

[1] M. Gell-Mann, Phys. Rev. **125**, 1067 (1962), and California Institute of Technology Synchrotron Report, CTSL-20, 1961 (unpublished); S. Okubo, Progr. Theoret. Phys. (Kyoto) **27**, 949 (1962).
[2] S. Coleman and S. L. Glashow, Phys. Rev. Letters **6**, 423 (1961).
[3] It has been pointed out to the author that the extended sum rule has been proposed and discussed by S. Coleman and S. L. Glashow (to be published).
[4] S. L. Glashow, Phys. Rev. **130**, 2132 (1963).
[5] R. E. Cutkosky and Pekka Tarjanne, Phys. Rev. **132**, 1354 (1963).
[6] R. E. Cutkosky, Ann. Phys. (N. Y.) **23**, 415 (1963), Eq. (16). These equations may be separated into equations for the eight baryons if use is made of the ordinary isotopic-spin Clebsch-Gordan coefficients.

TABLE I. Pseudoscalar meson and baryon octet mass vectors.

i	$\mathbf{m}_B \cdot \mathbf{E}_i$ (MeV)	$\mathbf{m}_B{}^2 \cdot \mathbf{E}_i$ (10^4 MeV^2)	$\mathbf{m}_p{}^2 \cdot \mathbf{E}_i$ (10^4 MeV^2)	Ξ^0 Error (10^4 MeV^2)
0	1151	134.4	16.77	
1	−32.3	−6.34	11.63	
2	−134.2	−30.29		
3	2.3	−0.83	−0.602	0.05
4	0.34	0.082	0.036	
5	0.62	0.181	−0.137	0.09
6	−1.96	−0.473		0.05
7	0.07	0.008		0.07
ϕ_0	6.8°	13.0°	34.7°	
ϕ_{12}	1.3°	1.8°	3.0°	
ϕ_{56}	9.5°	9.2°	14.8°	

on θ. The basis vectors within these planes are chosen to be eigenvectors of R conjugation.

The mass vector for the P-meson octet (π,K,η) is defined in an analogous fashion. In this case the equality of the masses of particle-antiparticle pairs limits $\boldsymbol{\beta}$ to the subspace (01345).

The components of the baryon mass vector $\mathbf{m}_B$, baryon mass-squared vector $\mathbf{m}_B{}^2$ and meson mass-squared vector $\mathbf{m}_P{}^2$ along the directions $\mathbf{E}_i$ are listed in the second, third, and fourth columns of Table I. The experimental masses are taken from the compilation of Roos.[7] The numbers in the fifth column represent the changes in some of the $\mathbf{m}_B{}^2 \cdot \mathbf{E}_i$ that would result from a change of 2 MeV in the Ξ^0 mass. The symbol ϕ_0 denotes the angle between the $\boldsymbol{\beta}$ vector and the $\mathbf{E}_0$ axis; $\phi_0=0$ corresponds to a degenerate octet. Conservation of isotopic spin is equivalent to the condition that $\boldsymbol{\beta}_{4567}$ vanish ($\boldsymbol{\beta}_{i\cdots j}$ denotes the projection of $\boldsymbol{\beta}$ in the $i\cdots j$ subspace). The Gell-Mann–Okubo sum rule is the statement that $\boldsymbol{\beta}_3$ should vanish. We define ϕ_{12} to be the angle between the mass-splitting vector $\boldsymbol{\beta}_{1234567}$ and its projection $\boldsymbol{\beta}_{12}$ in the Gell-Mann–Okubo plane. It is seen from the ϕ_{12} row of Table I that the mass formula is satisfied almost as well for the squares of the baryon masses as for the baryon masses themselves. The angle ϕ_{56} is defined later. The sum rule of Coleman and Glashow, concerning the violation of isotopic spin symmetry caused by electromagnetic interactions, is the statement that $\boldsymbol{\beta}_7$ vanish.

In order to discuss a possible extension of the Coleman-Glashow rule, we use the technique introduced by Okubo, and extended by Glashow and by Cutkosky and Tarjanne, to investigate the mass splitting.[1,4,5] The transformation properties of the different terms in the octet mass operator must correspond to the hypercharge 0, $I_z=0$ states that occur in the direct product of the particle and antiparticle octets. There are ten such states, one for each particle-antiparticle pair and two of the $\Lambda\bar{\Sigma}^0$ and $\Sigma^0\bar{\Lambda}$ type. The vectors $\boldsymbol{\beta}_0$, $\boldsymbol{\beta}_1$, $\boldsymbol{\beta}_2$, and $\boldsymbol{\beta}_3$ correspond to the $I=0$ states in the representations of dimensions 1, 8, 8, and 27, respectively. There are five states of $I=1$, $I_z=0$, corresponding to the representation 8 (twice), 10, 10*, and 27. These correspond to the three vectors $\boldsymbol{\beta}_5$, $\boldsymbol{\beta}_6$, and $\boldsymbol{\beta}_7$. The two extra $I=1$ transformations can lead to physical Λ and Σ^0 (or η and π^0) states that are not pure eigenfunctions of isotopic spin. Because of the fact that the electromagnetic mass differences are small compared to the $\Lambda-\Sigma^0$ and $\eta-\pi^0$ mass differences, the effect of this type of admixing on the masses of the particles is expected to be small, even compared to the average electromagnetic mass differences. Only $\boldsymbol{\beta}_5$ and $\boldsymbol{\beta}_6$ are related to the $I=1$, octet mass transformation; $\boldsymbol{\beta}_7$ is associated with a linear combination of the representations 10 and 10*. The vector $\boldsymbol{\beta}_4$ corresponds to the $I=2, I_z=0$ state of the representation 27. The Okubo rule is that the $I=0$ part of the mass-splitting vector corresponds only to the representation 8; the Coleman-Glashow rule is that the $I=1$ part corresponds only to the 8. We propose the following generalization of these rules: All appreciable contributions to the mass-splitting vector should correspond to the representation 8.[3] This generalization implies the additional condition $\boldsymbol{\beta}_4=0$, or

$$\beta_{\pi^0}=\beta_{\pi^\pm}, \tag{3a}$$

$$\beta_{\Sigma^0}=\tfrac{1}{2}(\beta_{\Sigma^+}+\beta_{\Sigma^-}). \tag{3b}$$

The above formula for the Σ masses resembles the magnetic moment formula of Marshak, Okubo, and Sudarshan, but is of different origin.[8] Unlike magnetic moments, electromagnetic corrections to masses are of second order in e, so that Eq. (3) does not follow from the absence of an isotopic tensor part of the electromagnetic current. A simple model that does lead to Eq. (3) is that introduced originally by Gell-Mann, in which the octet is formed from the direct product of the triplet representations 3 and 3*.[1] If we make the quantum numbers of the particles in the fundamental triplets explicit by labeling the particles $(pn\Lambda)$ and $(K^-\bar{K}^0\eta)$, the wave functions for the Σ particles are,

$$\begin{gathered}\Sigma^+=(p\bar{K}^0)\,,\quad \Sigma^-=(nK^-)\,,\\ \Sigma^0=2^{-1/2}[(pK^-)-(n\bar{K}^0)].\end{gathered} \tag{4}$$

If the β of each Σ is an average of that of its constituents, Eq. (4) leads to Eq. (3b). The Coleman-Glashow rule may also be obtained in this manner.

One of the most obvious properties of Eq. (3a) is its disagreement with experiment. However, it may be seen from Table I that the components $|\boldsymbol{\beta}_4|$ are much smaller than $(\boldsymbol{\beta}_5{}^2+\boldsymbol{\beta}_6{}^2)^{1/2}$ for both the baryon and meson octets. The smallness of the angle ϕ_{56} of Table I is a measure of the validity of the generalized sum rule for the I-spin violating mass splitting; ϕ_{56} is defined as the angle between $\boldsymbol{\beta}_{4567}$ and its projection $\boldsymbol{\beta}_{56}$ on the "octet" (56) plane.

One possible mechanism that may lead to the gener-

[7] Matts Roos, Rev. Mod. Phys. **35**, 314 (1963).

[8] R. Marshak, S. Okubo, and G. Sudarshan, Phys. Rev. **106**, 599 (1957).

alized sum rule for the baryons results from a simple bootstrap model. In order to illustrate this point we consider an idealized picture in which the B octet is a compound of the B and P octets, and the β of each compound state is "almost" given by the average of the β of the constituent baryons, i.e.,

$$\beta_i=\sum_j \Pi_{ij}\beta_j+C_i, \tag{5}$$

where C_i represents symbolically all quantities influencing the mass of i other than those contained in the Π term. If an eigenvector $\boldsymbol{\beta}_k$ of the probability matrix $\mathbf{\Pi}$ is considered, Eq. (5) may be written in the form,

$$\boldsymbol{\beta}_k\cdot\mathbf{E}_k=C_k/(1-a_k), \tag{6}$$

where $\mathbf{\Pi}\boldsymbol{\beta}_k=a_k\boldsymbol{\beta}_k$. The eigenvalues a_k are easily computed; the values corresponding to the interaction angle $\theta=35°$ are given below.[9]

$$\begin{aligned} &a_0=1, \qquad a_3=a_4=0.02, \quad a_7=-0.27, \\ &a_{12}^+=0.77, \quad a_{12}^-=-0.31, \\ &a_{56}^+=0.69, \quad a_{56}^-=-0.20. \end{aligned} \tag{7}$$

For $\theta=35°$ the normalized eigenvectors corresponding to a_{12}^+ and a_{56}^+ are

$$\mathbf{E}_{12}^+=0.502\mathbf{E}_1+0.864\mathbf{E}_2,$$
$$\mathbf{E}_{56}^+=-0.461\mathbf{E}_5+0.888\mathbf{E}_6.$$

If the β_j represent the squares of the baryon masses, the I-spin conserving projection and I-spin violating projection of the mass-splitting vector are aligned fairly closely to the axes corresponding to the two large eigenvalues a_{12}^+ and a_{56}^+. The angle between $\boldsymbol{\beta}_{123}$ and $\mathbf{E}_{12}^+$ is 18.5°, while the angle between $\boldsymbol{\beta}_{4567}$ and E_{56}^+ is 11.3°. It is of interest to note that the largest component of $\boldsymbol{\beta}_{4567}$ that is normal to $\mathbf{E}_{56}^+$ is the component $\boldsymbol{\beta}_4$. The angle between $\boldsymbol{\beta}_{56}$ and $\mathbf{E}_{56}^+$ is only about $6\frac{1}{2}°$. We assume that for the I-spin violating projections, the C_k in Eq. (6) results from electromagnetic effects. It is plausible that the large experimental $(\boldsymbol{\beta}_5^2+\boldsymbol{\beta}_6^2)/\boldsymbol{\beta}_4^2$ ratio does not result from a large difference in the electromagnetic "driving forces" C_k, but rather from the difference in the $1-a_k$ factors, i.e., from the fact that the octet is relatively stable against mass-splitting perturbations that do not transform as members of the representation 8.

A similar argument has been given for a model of a vector meson octet by Cutkosky and Tarjanne.[5] They also find that octet-type deviations from degeneracy are favored, but for different reasons than those applying in the present model. In the vector meson model, only mass splitting in the (1345) subspace is allowed, and the interaction is pure F type. The large octet-type eigenvalues of Eq. (7) above correspond to an interaction that is mostly D type.

A similar argument may be given for the P octet, if it is regarded as a compound of the P- and V-meson octets, and the mass splitting is assumed to depend primarily on the mass differences of the P mesons in the wave functions.[10] In this case the octet type deviations from degeneracy are favored for essentially the same reasons as in the model of Ref. 5. The eigenvalues of the probability matrix corresponding to octet-type mass splitting are $a_1=\frac{1}{2}$, $a_5=\frac{1}{6}$, and those related to the 27-fold representation are $a_3=a_4=-\frac{1}{3}$.

The author would like to thank Professor R. G. Sachs for an interesting conversation regarding sum rules.

[9] Several theoretical and experimental arguments have been given for a value of $\theta\sim35°$. See Ref. 6; also A. W. Martin and K. C. Wali, Phys. Rev. **130**, 2455 (1963); and S. L. Glashow and A. H. Rosenfeld, Phys. Rev. Letters **10**, 192 (1963).

[10] A more detailed treatment of this model, in which mass splittings in the P and V octets are both considered, is given by R. H. Capps, Phys. Rev. **134**, B460 (1964).