The quadrupole moment obtained is larger than the single particle estimate $|Q|\sim 0.05$ b. The $E2$ transition probability for the second excited state ($\frac{5}{2}^+$) to the ground state ($\frac{1}{2}^+$) transition is known to be strongly enhanced compared with the single particle estimate. The nucleus F^{19} has been treated on the rotational model by Paul[13] and Rakavy.[14] On the strong coupling theory of the rotational model, the $E2$ lifetime and the quadrupole moment are interrelated by an intrinsic quadrupole moment Q_0, and the quadrupole moment was calculated to be $|Q|\sim 0.091$ b from the measured lifetime under an assumption that the levels belong to a $K=\frac{1}{2}$ band and the particle part of the transition probability can be neglected. Paul has shown, however, that a mixture of two rotational bands is necessary to interpret the level structure of F^{19} and also that the collective part and the particle part of the $E2$ transition probability are of comparable importance.

On the other hand, Inoue, Sebe, Hagiwara, and Arima[15] have treated, recently, the s-d shell nuclei on the intermediate coupling shell model, and predicted the quadrupole moment of the state to be $Q\sim -0.097$ b by taking the effective charge $\beta\sim 0.5$.

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[13] E. B. Paul, Phil. Mag. **2**, 311 (1957).
[14] G. Rakavy, Nucl. Phys. **4**, 375 (1957).
[15] T. Inoue, T. Sebe, H. Hagiwara, and A. Arima, Nucl. Phys. (to be published).

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Nilsson Model Calculations of Mu Capture Rates in $2s-1d$ Nuclei*

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Theoretical calculations of mu capture rates in $2s$-$1d$ nuclei are compared with experiment in the hope of elucidating the coupling constants of the interaction. Working from Primakoff's closure-approximation expression for the total average capture rate, the nuclear matrix element is treated in the context of the Nilsson unified model. A Hill-Wheeler integration must be performed to avoid extraneous coordinates in the A-particle wave function. The one- and two-particle parts of the matrix element are broken up into the various shell contributions, since all of the angular momentum properties reside in the shell wave function for the nucleons outside the O^{16} core. The closed-shell matrix elements are easily treated with standard angular-momentum techniques. The method for reducing the outer-shell matrix elements to a form amenable to evaluation by a computer is given in an appendix. Radial integrals are obtained from the Ford-Wills muon wave functions. The average neutrino momentum $\bar{\nu}$ is chosen on the basis of Kaplan's Fermi-gas model for the capture process and the subsequent comparison with neutron evaporation rates. The choice of nuclear parameters for F^{19}, Ne^{20}, Si^{28}, Cl^{35}, and Cl^{37} is discussed and numerical results are given. Comparing with experimental rates, one cannot exclude the possibility that the Fermi part of the interaction is absent. If a $V-A$ theory is assumed, however, we conclude the induced pseudoscalar coupling is probably present. The induced pseudoscalar with the "wrong" sign, $g_P=-8g_A$, is definitely excluded, and the "large" pseudoscalar, $g_P=16g_A$, seems to fit the data better at $\bar{\nu}=0.75$.

I. INTRODUCTION

THE mu-capture interaction, $\mu^-+p\rightarrow n+\nu$, like beta decay and mu decay, is presumably described by the universal Fermi interaction (UFI)[1,2] given in the refined $V-A$ form of Feynman and Gell-Mann.[3] It is not very well understood experimentally, however, and we can only say at this time that it fits the UFI hypothesis to within 20% or so.[4] In this paper we present a theoretical study of this capture process in certain of the light $2s-1d$ nuclei to see whether existing

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[1] E. Fermi, Z. Physik **88**, 161 (1934).
[2] J. Tiomno and J. A. Wheeler, Rev. Mod. Phys. **21**, 153 (1949).
[3] R. P. Feynman and M. Gell-Mann, Phys. Rev. **109**, 193 (1958).
[4] R. Klein and L. Wolfenstein, Phys. Rev. Letters **9**, 408 (1962); see footnote 12.

experimental information on total average rates can elucidate the nature of the coupling constants of the interaction.

Ideally, one would like to study mu capture on free protons, e.g., in liquid hydrogen. There are a number of difficulties in such experiments, and it has only been recently that the capture has even been observed.[5] The results can only be said to be in agreement with the UFI predictions and may, perhaps, be somewhat low.[6]

For intensity reasons, most experiments on mu capture have been concerned with capture by protons within a nucleus, since the rate roughly goes as Z^4. Nuclear capture unfortunately introduces all the uncertainties of nuclear physics into the theoretical interpretation of the experiments.

There are four coupling constants to be determined, g_V, g_A, g_P, and g_{WM}, as can be seen from the most general Lorentz invariant matrix element for the interaction that can be written.[7] g_V and g_A are the usual V and A coupling constants (form factors) and UFI assumes them to be the same as those that occur in beta decay and mu decay (but evaluated at the appropriate momentum transfer). g_P is the induced pseudoscalar coupling constant[7,8] and is estimated to be $\simeq 8g_A$ from a dispersion theoretic argument. g_{WM} is the "weak magnetism" coupling constant predicted by the conserved vector current theory (CVC),[3] arising from a pion-lepton weak vertex. In principle, these four form factors could be determined by four different capture experiments.

There are, in fact, four kinds of experiments which are available: capture rates to particular final states, total capture rates, hyperfine difference effects, and neutron asymmetry. The hf experiments in F^{19} by Winston and Telegdi[9] establish that the interaction is of the $V-A$ type rather than $V+A$.[10] The neutron asymmetry experiments indicate the presence of the induced pseudoscalar term which might be somewhat larger than the theoretical estimate.[11] The ground-state to ground-state partial transition $C^{12} \rightarrow B^{12}$, a $0^+ \rightarrow 1^+$ transition, is well studied[12] and shows that the Gamow-Teller part of the interaction is as expected from UFI, as well as showing that if the induced pseudoscalar term is present, it has the expected sign. The g.s. to g.s. transition $He^3 \rightarrow H^3$, a $\frac{1}{2}^+ \rightarrow \frac{1}{2}^+$ transition which can give information regarding the Fermi part of the interaction, is in agreement with UFI.[13] A recent measurement of a $0^+ \rightarrow 0^-$ transition in capture on O^{16} indicates the induced pseudoscalar term is larger than expected,[14] $g_P \simeq 15g_A$, something which is consistent with the results on free protons[5,6] and on neutron asymmetry.[11] The results of experiments measuring total capture rates[15] will be discussed here at some length.

Since the initial ayalysis of Wheeler and Tiomno, using a Fermi-gas model of the nucleus,[2] theoretical interpretation has proceeded along two somewhat different paths. One school of thought feels that total rates should be calculated by summing up the squared matrix elements for the partial transitions to final states.[16] Most of the effort here has been in using shell-model wave functions to describe the initial and final nuclear states, some of the wave functions being quite simple-minded, some quite complex. The results leave something to be desired. If we were to accept the values of the matrix elements for Ca^{40} given by Luyten, Rood, and Tolhoek as good, then the squared coupling constant needed to fit the experimental rate would be about 50% less than the UFI value.[16] These authors indicate that this discrepancy is probably due more to a lack of detail in their nuclear wave functions than to a failure of UFI.

The other theoretical procedure, and the one to be used here, has been to avoid use of final-state wave functions by means of the closure approximation. Because of the large amount of energy available (muon mass $\cong 105$ MeV), very many of the possible final states contribute to the rate. By extending the sum over energetically allowed final states to a sum over all final states, the completeness relation reduces the expression for the rate to one which involves only the ground-state wave function of the initial nucleus. This last is presumably much better known than the wave functions for the excited final states, and we only have introduced error from the extension of the sum. However, this method requires a knowledge of the average phase-space factor which was taken out of the sum over final states.

The closure approximation was first applied by

[5] R. H. Hildebrand, Phys. Rev. Letters **8**, 34 (1962); E. Bleser, L. Lederman, J. Rosen, J. Rothberg, and E. Zavattini, *ibid.* **8**, 288 (1962); E. Bertolini, A. Citron, G. Gialanelli, S. Focardi, A. Mukhin, C. Rubbia, and S. Saporetti, in *Proceedings of the International Conference on High Energy Nuclear Physics, Geneva, 1962* (CERN Scientific Information Service, Geneva, Switzerland, 1962), p. 421.

[6] A. Fujii, Nuovo Cimento **27**, 1025 (1963).

[7] M. L. Goldberger and S. B. Treiman, Nuovo Cimento **12**, 327 (1959).

[8] L. Wolfenstein, Nuovo Cimento **8**, 882 (1958).

[9] G. Culligan, J. F. Lathrop, V. L. Telegdi, R. Winston, and R. A. Lundy, Phys. Rev. Letters **7**, 458 (1961); R. Winston, Phys. Rev. **129**, 2766 (1963).

[10] R. Silbar and H. Überall, Nuovo Cimento **22**, 864 (1961).

[11] V. Eseev, V. I. Komarov, V. Z. Kush, V. Roganov, V. Chernogorova, and M. Szymczak, Zh. Eksperim. i Teor. Fiz. **41**, 306 (1961) [English transl.: Soviet Phys.—JETP **14**, 217 (1962)].

[12] See, e.g., G. T. Reynolds, D. B. Scarl, R. A. Swanson, J. R. Waters, and R. A. Zdanis, Phys. Rev. **129**, 1790 (1963), and references cited therein.

[13] See, e.g., L. B. Auerbach, R. J. Esterling, R. E. Hill, D. A. Jenkins, J. T. Lach, and N. H. Lipman, Phys. Rev. Letters **11**, 23 (1963).

[14] R. C. Cohen, S. Devons, and A. D. Kanaris, Phys. Rev. Letters **11**, 134 (1963).

[15] See, for example, J. C. Sens, Phys. Rev. **113**, 679 (1959).

[16] J. M. Kennedy, Phys. Rev. **87**, 953 (1951), Ca^{40}; H. A. Tolhoek and J. R. Luyton, Nucl. Phys. **3**, 679 (1957), $2s$-$1d$ nuclei; J. R. Luyten, H. P. C. Rood, and H. A. Tolhoek, *ibid.* **41**, 236 (1963), O^{16}, Ca^{40}; I. Duck, *ibid.* **35**, 27 (1962), He^3, O^{16}, F^{19}; H. G. Wahsweiler, Z. Physik **170**, 574 (1962), Cl^{35-37}.

Primakoff in his model-independent calculation of the total capture rate for an "average" nucleus, specified only by Z and A.[17] He obtains a formula for the rate

$$\bar{\lambda}(Z,A)=\text{const}\times Z_{\text{eff}}{}^4\left[1-\text{const}\times\frac{A-Z}{ZA}\right], \quad (1)$$

which gives the well-known "Primakoff plot." As given, the above formula fits the $V-A$ theory quite well, expecially when the fast Auger conversion to the lower hf state is taken into account.[15,18]

Unfortunately, Klein and Wolfenstein have pointed out that one of Primakoff's approximations is unjustified and that, when this is taken into account, his formula is changed to[19]

$$\bar{\lambda}(Z,A)=\text{const}\times Z_{\text{eff}}{}^4(1-\Delta_a)\left[1-\text{const}\times\frac{A-Z}{ZA}\right]. \quad (2)$$

On fitting the experimental rates to this, one finds rather poor agreement with the UFI predictions.[19] In fact, the fitted-squared coupling constant must be twice that given by UFI. It seems that a structureless nucleus, such as Primakoff considered, does not contain sufficient detail to account for the observed rates.

The shell model calculations of Luyten, Rood, and Tolhoek give a G^2 which is too small while the structureless nuclear model of Primakoff, as modified by Klein and Wolfenstein, gives a G^2 which is too large. This might be an indication of a difference between the two methods used, summation of partial transitions and closure approximation, but the hope is that the use of a more detailed nuclear wave function will bring them both in line with UFI, and in doing so give detailed information as to its finer details.

In this paper we evaluate the nuclear matrix elements in the context of the unified Nilsson nuclear model and find a G^2 consistent with UFI. Moreover, if a $V-A$ theory is accepted, the comparison of these results with experiment indicates the presence of the induced pseudoscalar term of the expected sign.

II. EXPRESSION FOR TOTAL AVERAGE CAPTURE RATE

Let us begin with the closure approximation expression for the total average capture rate as given by Primakoff[17,20]

$$\bar{\lambda}=\frac{I+1}{2I+1}\lambda_+ +\frac{I}{2I+1}\lambda_- =\frac{Z^3}{2\pi^2 a_\mu{}^3}\frac{\bar{\nu}^2}{1+\bar{\nu}/M}\langle\Psi_{MK}{}^I|A+A'|\Psi_{MK}{}^I\rangle \quad (3)$$

as found from an effective nonrelativistic Hamiltonian for the process derived from the general Lorentz invariant matrix element.[21] Here, I is the nuclear spin, $\lambda_\pm$ being the capture rates from the upper and lower hf states respectively. M is the mass of the nucleus and the muon Bohr radius

$$a_\mu=\frac{\hbar^2}{e^2 m_\mu'}=\frac{137}{m_\mu}\left(1+\frac{m_\mu}{M}\right) \quad (4)$$

in units with $\hbar=c=m_e=1$. $\bar{\nu}$ is the average momentum of the emitted neutrino.

The nuclear matrix element in (3) is taken with the ground-state wave function of the initial nucleus. We have split the operator $\mathcal{H}^\dagger\mathcal{H}$ into one- and two-particle parts

$$A=a\sum_{i=1}^{A}\left(\frac{1+\tau_3{}^{(i)}}{2}\right)\varphi^2(r_i), \quad (5a)$$

$$A'=\sum_{i\neq j=1}^{A}\tau_+{}^{(i)}\tau_-{}^{(j)}\omega(i,j),$$
$$\omega(i,j)=(a'+a''\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j)\varphi(r_i)\varphi(r_j)j_0(\bar{\nu}r_{ij}), \quad (5b)$$

where the sums on i and j run over all the nucleons of the nucleus. The muon space wave function $\varphi(r)$ is normalized so that it goes to 1 as $Z\to 0$. j_0 is the zero-order spherical Bessel function. All dependence on the coupling constants resides in the a's.

$$a'=G_V{}^2,$$
$$a''=G_A{}^2-\tfrac{2}{3}G_AG_P+\tfrac{1}{3}G_P{}^2, \quad (6)$$
$$a=a'+3a''=G_V{}^2+3G_A{}^2-2G_AG_P+G_P{}^2,$$

where the effective G's are related to the g's under the usual UFI–CVC assumptions by[21]

$$G_V=g_V(1+\nu/2M_P),$$
$$G_A=g_A-g_V(1+\mu_p-\mu_n)\nu/2M_p, \quad (7)$$
$$G_P=[g_P-g_A-g_V(1+\mu_p-\mu_n)]\nu/2M_P.$$

Here the g's are the form factors appearing in the relativistic matrix element. μ_p and μ_n are the anomalous magnetic moments of the proton and neutron, respectively, and give, in the combination $\mu_p-\mu_n$, the effect of the "weak magnetism" term.

The two-particle operator A' gives the effect of the Pauli exclusion principle as it acts upon the produced neutron and is therefore responsible for the isotope effect, an inhibition of the capture rate due to the reduction in the number of final states available to this neutron.

III. NUCLEAR MODEL

The success of the jj-coupling shell model for a description of nuclear properties is by now well estab-

[17] H. Primakoff, Rev. Mod. Phys. **31**, 802 (1959).
[18] V. L. Telegdi, Phys. Rev. Letters **8**, 327 (1962).
[19] Reference cited in footnote 4. See, however, footnote 70a of R. Winston, Ref. 9.
[20] H. Überall, Phys. Rev. **121**, 1219 (1961).
[21] A. Fujii and H. Primakoff, Nuovo Cimento **12**, 327 (1959).

lished.[22] Nevertheless, the description is not a complete one for the $2s-1d$ nuclei, as evidenced by the poor predictions of quadrupole moments, and collective aspects must be incorporated into the nuclear wave functions. The unified approach of Bohr and Mottleson[23] together with the Nilsson model[24] has had considerable success for the $2s-1d$ nuclei.[25]

The chief collective feature of nuclei is their equilibrium deformation, which, following Nilsson, we assume to be axially symmetric. Corresponding to a splitting of the nuclear Hamiltonian into an intrinsic and a rotational part, we write the nuclear wave function

$$\Psi=\mathfrak{X}_\Omega(x')D_{MK}{}^I(\alpha,\beta,\gamma). \tag{8}$$

$\mathfrak{X}_\Omega$ is the eigenfunction of the intrinsic part of the Hamiltonian, expressed in intrinsic nuclear coordinates, and $D_{MK}{}^I$ is the eigenfunction of the rotational part, expressed in terms of Euler angles.[26] The quantum numbers I, M, and K refer to the nuclear spin and its projections along some space-fixed axis and along the nuclear body-fixed axis. Ω is a quantum number referring to the projection of the intrinsic angular momentum along the body axis.

The intrinsic wave function has as its arguments the coordinates of the A nucleons with respect to the slowly rotating nuclear axes. There are $3A$ such space coordinates. The three Euler angles, the arguments of the rotational part of the wave function, are also space coordinates. This means we have, in all, $3A+3$ space coordinates for the A nucleons. There is a problem of three extraneous coordinates in our wave function as written in (3).

The method for dealing with these superfluous coordinates is to perform a Hill-Wheeler integration over the three Euler angles.[27]

$$\Psi_{MK}{}^I=\int d\theta\,\mathfrak{X}_\Omega(x')D_{MK}{}^I(\theta), \tag{9}$$

where we use θ to abbreviate α, β, γ. Here we take $\mathfrak{X}_\Omega$, which does not have good angular momentum properties because of the lack of spherical symmetry, and project from it a good angular momentum eigenfunction $\Psi_{MK}{}^I$ by means of the weighting function $D_{MK}{}^I$. Kurath and Picman have shown that, in the $1p$ shell at least, such a generating procedure gives wave functions with very good overlap with intermediate coupling wave functions.[28]

It should be noted in (9) that the arguments of $\mathfrak{X}_\Omega$ are in body-fixed coordinates, i.e., they depend on the Euler angles θ. We can re-express this in terms of space-fixed coordinates by means of a rotation operator,

$$\mathfrak{X}_\Omega(x')=\mathbf{D}(\theta)\mathfrak{X}_\Omega(x), \tag{10}$$

where[26]

$$\begin{aligned}\mathbf{D}(\theta)&=e^{-i\alpha J_z}e^{-i\beta J_y}e^{-i\gamma J_z}\\&=e^{-i\alpha j_z(1)}e^{-i\beta j_y(1)}e^{-i\gamma j_z(1)}\cdots e^{-i\gamma j_z(A)}\\&=\partial_1(\theta)\partial_2(\theta)\cdots\partial_A(\theta),\end{aligned} \tag{11}$$

which we have factored into single-particle rotation operators $\partial(\theta)$. The rotation operators, when acting on a good angular-momentum eigenfunction, give a linear combination of angular-momentum eigenfunctions with the same j,

$$\partial(\theta)\psi_m{}^j=\sum_{m'}D_{m'm}{}^{j*}(\theta)\psi_{m'}{}^j. \tag{12}$$

The $D_{m'm}{}^j$ here are exactly the same as in (8), occurring now as the matrix elements of the rotation operator.

Our interest lies not so much in nuclear wave functions as in nuclear matrix elements. Using (9) and (10), we can express these in terms of doubly integrated theta-dependent matrix elements,

$$\begin{aligned}ME&=\langle\Psi_{MK}{}^I|\,\mathcal{O}\,|\Psi_{MK}{}^I\rangle\\&=\int d\theta'\int d\theta''D_{MK}{}^{I*}(\theta')D_{MK}{}^I(\theta'')ME(\theta',\theta''),\end{aligned} \tag{13}$$

where

$$\begin{aligned}ME(\theta',\theta'')&=\langle\mathfrak{X}_\Omega(x')|\,\mathcal{O}\,|\mathfrak{X}_\Omega(x'')\rangle\\&=\langle\mathfrak{X}_\Omega(x)|\,\mathbf{D}^\dagger(\theta')\,\mathcal{O}\,\mathbf{D}(\theta'')\,|\mathfrak{X}_\Omega(x)\rangle.\end{aligned} \tag{14}$$

The intrinsic wave function in this final form is expressed in terms of space-fixed coordinates.

The intrinsic wave function $\mathfrak{X}_\Omega$ can be written as a Slater determinant of single-particle Nilsson eigenfunctions.[24] These can be written in the form

$$\chi_\omega{}^{(r)}=\sum_j c_{j\omega}{}^{(r)}\chi_\omega{}^j. \tag{15}$$

The eigenfunctions, in addition to the component of angular momentum along the body axis ω and to a label r specifying which Nilsson level of that ω is meant, have N, the number of harmonic-oscillator quanta, as a good quantum number. For the $2s-1d$ shell, $N=2$ and the sum in (15) runs over $j=\frac{1}{2},\frac{3}{2},\frac{5}{2}$. The $c_{j\omega}{}^{(r)}$ are tabulated in Nilsson's paper for various choices of the

[22] See, e.g., M G. Mayer and J. D. H. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955).

[23] A. Bohr and B. Mottelson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **27**, No. 16 (1953); J. P. Elliott, University of Rochester Report NYO-2271, 1958 (unpublished).

[24] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, 16 (1955).

[25] G. Rakavy, Nucl. Phys. **4**, 375 (1957); H. E. Gove in *Proceedings of the International Conference on Nuclear Structure* (The University of Toronto Press, Toronto, 1960); many references are given in R. K. Sheline and R. A. Harlan, Nucl. Phys. **29**, 177 (1962), footnote 5.

[26] We use throughout this paper the conventions of A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1960), 2nd ed.

[27] J. J. Griffen and J. A. Wheeler, Phys. Rev. **108**, 311 (1957); R. E. Peierls and J. Yoccoz, Proc. Phys. Soc. (London) **70**, 381 (1957).

[28] D. Kurath and L. Picman, Nucl. Phys. **10**, 313 (1959).

nuclear deformation η. The energy eigenvalues associated with the eigenfunctions (15) are twofold degenerate in $\pm\omega$ and depend on η also. (See the energy-level diagrams given in Sec. VIII.) Knowing the deformation from other evidence, the Nilsson configuration is chosen so as to minimize the total energy.

IV. REDUCTION TO SHELL MATRIX ELEMENTS

Since N is a good quantum number of the Nilsson functions, we can take advantage of their shell-like nature to reduce the nuclear matrix elements to the contributions from the various N-shells. Following a similar reduction of Überall,[20] we write the totally antisymmetric intrinsic wave function

$$\begin{aligned}\mathfrak{X}_\Omega(1\cdots A)=&\textstyle\sum_P \epsilon_P \Phi_{N=0}{}^{p}(P1,P2)\\ &\times\Phi_{N=0}{}^{\pi}(P3,P4)\Phi_{N=1}{}^{p}(P5\cdots P10)\\ &\times\Phi_{N=1}{}^{\pi}(P11\cdots P16)\Phi_\Omega(P17\cdots PA).\end{aligned}\quad(16)$$

Here $\Phi_{N=0,1}{}^{p,\pi}$ are the *closed* neutron and proton shell functions which together make up the O^{16} core, and Φ_Ω is the function for the unfilled $2s-1d$ shell. Each of these shell functions is assumed normalized and antisymmetric in its own arguments.

The reason for doing this is that the antisymmetric closed-shell functions, which at first would be written as a Slater determinant of Nilsson single-particle eigenfunctions, can just as well be written as a Slater determinant using any other convenient set of basis functions. This is because the unitary transformation between the two sets of basis functions becomes, in the Slater determinant, the determinant of the transformation matrix, which is just a constant phase. Thus we will only have to use the Nilsson eigenfunctions in the outer shell function, Φ_Ω. The appropriate basis for the inner shell functions will become clear when we have to evaluate the contributions to the matrix element from these shells.

Another big advantage of (16) is the fact that the antisymmetry of the inner shell functions requires that each of these have a total angular momentum $J=0$. All of the angular momentum of the intrinsic wave function resides in the outer shell function Φ_Ω. The rotation operator $\mathbf{D}(\theta)$ does not act on the spherically symmetric closed shells.

Substituting (16) into the nuclear matrix element for capture occurring in (3) and using the orthogonality between the various shell functions, we can eventually recast the one-particle part in the form

$$\langle A\rangle=M_{N=0}{}^{\pi}+M_{N=1}{}^{\pi}+L^{(1)},\quad(17a)$$

$$M_{N}{}^{\pi}=a\langle\Phi_N{}^{\pi}|\sum_{i=1}^{m}\varphi^2(r_i)|\Phi_N{}^{\pi}\rangle,\quad(17b)$$

$$L^{(1)}=a\int d\theta'\int d\theta'' D_{MK}{}^{I*}(\theta')D_{MK}{}^{I}(\theta'')\times\mathfrak{L}_1{}^{(1)}(\theta',\theta'')/K,\quad(17c)$$

$$\mathfrak{L}_1{}^{(1)}=\left\langle\Phi_\Omega\left|\mathbf{D}^\dagger(\theta')\sum_{i=1}^{n}\left(\frac{1+\tau_3{}^{(i)}}{2}\right)\varphi^2(r_i)\mathbf{D}(\theta'')\right|\Phi_\Omega\right\rangle,\quad(17d)$$

where K is the normalization constant for the unfilled shell.

$$K=\int d\theta'\int d\theta'' D_{MK}{}^{I*}(\theta')D_{MK}{}^{I}(\theta'')\mathfrak{K}(\theta',\theta'').\quad(18a)$$

$$\mathfrak{K}=\langle\Phi_\Omega|\mathbf{D}^\dagger(\theta')\mathbf{D}(\theta'')|\Phi_\Omega\rangle.\quad(18b)$$

The reduction for the two-particle part is somewhat more complicated.

$$\langle A'\rangle=-\sum_{N,N',\tau,\tau'}M'_{N\tau,N'\tau'}-\sum_{N\tau}L_{N\tau}{}^{(2)}+L^{(3)},\quad(19a)$$

$$\begin{aligned}M_{N\tau,N'\tau'}{}'=&pq\langle\Phi_N{}^{\tau}(12\cdots p)\Phi_{N'}{}^{\tau'}(1'2'\cdots q')|\\ &\times\omega(1,1')|\Phi_N{}^{\tau}(1'2\cdots p)\Phi_{N'}{}^{\tau'}(12'\cdots q')\rangle,\end{aligned}\quad(19b)$$

$$L^{(2,3)}=\int d\theta'\int d\theta'' D_{MK}{}^{I*}(\theta')D_{MK}{}^{I}(\theta'')\times\mathfrak{L}^{(2,3)}(\theta',\theta'')/K,\quad(19c)$$

$$\begin{aligned}\mathfrak{L}_{N\tau}{}^{(2)}=&mn\langle\Phi_N{}^{\tau}(1\cdots)\Phi_\Omega(1'\cdots)|\mathbf{D}^\dagger(\theta')\\ &\times[\tau_+{}^{(1)}\tau_-{}^{(1')}+\tau_-{}^{(1)}\tau_+{}^{(1')}]\omega(1,1')\\ &\times\mathbf{D}(\theta'')|\Phi_N{}^{\tau}(1'\cdots)\Phi_\Omega(1\cdots)\rangle,\end{aligned}\quad(19d)$$

$$\begin{aligned}\mathfrak{L}^{(3)}=&n(n-1)\langle\Phi_\Omega|\mathbf{D}^\dagger(\theta')\tau_+{}^{(1)}\tau_-{}^{(2)}\omega(1,2)\\ &\times\mathbf{D}(\theta'')|\Phi_\Omega\rangle.\end{aligned}\quad(19e)$$

All these terms involve the exchange of a neutron and a proton and the antisymmetry of the over-all wave function leads us to expect each term to give a negative contribution. The negative sign is explicit in the inner-inner and inner-outer exchange terms and is to be expected from the antisymmetry of the shell function for the outer-outer term as well. These terms give the reduction in the rate due to the Pauli exclusion effect.

In (17) and (19) only the $N=0$ and $N=1$ inner shells are included in the closed shell contributions, i.e., only the O^{16} core. For the treatment of Cl^{35} and Cl^{37}, however, we can better work with holes in a closed Ca^{40} core. The one-particle part, in such a picture, takes the form

$$\langle A\rangle=\sum_{N=0}^{2}M_{N}{}^{\pi}-L^{(1)}\quad(20a)$$

while the two-particle part becomes

$$\langle A'\rangle=-\sum_{N,N'=0,\tau,\tau'}^{2}M_{N\tau,N'\tau'}{}'+\sum_{N=0,\tau}^{2}L_{N\tau}{}^{(2)}+L^{(3)}.\quad(20b)$$

Here the shell function Φ_Ω is composed of the Nilsson eigenfunctions for the appropriate hole configuration.

V. CLOSED-SHELL MATRIX ELEMENTS: Si^{28}

The closed-shell matrix elements in the above expressions can, as said before, be evaluated with any appropriate set of basis functions. The natural choice is the nlm_lm_s basis. The one-particle matrix elements are easily found to be

$$M_{N^\pi}=a\sum_l\sum_{m_l}\sum_{m_s}\langle nlm_lm_s|\,\varphi^2(r)\,|nlm_lm_s\rangle$$
$$=a\sum_l 2(2l+1)\langle nl|\,\varphi^2(r)\,|nl\rangle, \tag{21}$$

a simple expression in terms of the radial integrals of the squared muon wave function. The two-particle part is, by the standard techniques of angular momentum,[26]

$$M'_{N\pi,N'\nu}$$
$$=\sum_{lm_lm_s}\sum_{l'm_l'm_s'}\langle m_s,m_s'|\,a'+a''\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\,|m_s',m_s\rangle$$
$$\times\langle nlm_l,n'l'm_l'|\,\varphi(r_1)\varphi(r_2)j_0(\bar\nu r_{12})\,|n'l'm_l',nlm_l\rangle$$
$$=a\sum_{l,l'}\sum_{m_l,m_l'}2\langle nlm_l,n'l'm_l'|\,\varphi(r_1)\varphi(r_2)j_0(\bar\nu r_{12})\,|$$
$$\times n'l'm_l',nlm_l\rangle. \tag{22}$$

The collapse of the spin sums into the combination $a=a'+3a''$ for closed shells has been noted previously by Tolhoek and his collaborators.[16] To proceed further we use the identity[29]

$$j_0(\bar\nu r_{12})=\sum_{\lambda=0}^{\infty}j_\lambda(\bar\nu r_1)j_\lambda(\bar\nu r_2)$$
$$\times\sum_{\mu=-\lambda}^{\lambda}(-1)^\lambda 4\pi Y_\mu{}^\lambda(1)Y_{-\mu}{}^\lambda(2), \tag{23}$$

and find

$$M'_{N\pi,N'\nu}$$
$$=a\sum_{ll'\lambda}2\times4\pi\langle l\|Y^\lambda\|l'\rangle\langle l'\|Y^\lambda\|l\rangle$$
$$\times\langle nl|\,j_\lambda(\bar\nu r)\varphi(r)\,|n'l'\rangle\langle n'l'|\,j_\lambda(\bar\nu r)\varphi(r)\,|nl\rangle$$
$$=a\sum_{ll'\lambda}2(2l+1)(2l'+1)\mathfrak{R}_{ll'\lambda}{}^2(ll'00|\lambda0)^2, \tag{24}$$

where

$$\mathfrak{R}_{ll'\lambda}=\langle nl|\,j_\lambda(\bar\nu r)\varphi(r)\,|n'l'\rangle=\mathfrak{R}_{l'l\lambda}. \tag{25}$$

Here we have used the Wigner-Eckart theorem, the symmetries and orthogonality of Clebsch-Gordan coefficients, and the well-known result[30]

$$\langle l\|Y^\lambda\|l'\rangle=(-1)^{l'}\left(\frac{(2l+1)(2l'+1)}{4\pi}\right)^{1/2}(ll'00|\lambda0). \tag{26}$$

Note that the parity selection rule that $l+l'+\lambda$ must be even and the triangle condition $\Delta(ll'\lambda)$ severely restrict the number of λ's that contribute from the infinite sum in (23).

Within the context of the Nilsson model, we can treat the Si^{28} nucleus at this time. Here the $N=2$ shell is exactly half-full, and as such, is expected to have its quadrupole moment vanish.[31] This means the nucleus has no deformation, i.e., is spherically symmetric. That the deformation "crosses over" at this nucleus is confirmed experimentally; Al^{27} must have a positive deformation and Si^{29} a negative deformation in order to predict the correct spins from the Nilsson configurations. At zero deformation the Nilsson eigenfunctions reduce to ordinary shell-model eigenfunctions and the calculation of $\langle A+A'\rangle$ goes fairly simply.

The effect of the O^{16} core is given as before. The outer nucleons, those outside this core, are treated as constituting two filled $1d_{5/2}$ shells, for the protons and neutrons separately. The antisymmetric shell functions $\mathfrak{A}_{1d_{5/2}{}^{\nu,\pi}}$ are, as usual, orthogonal to each other and to the shell functions $\Phi_{N^{\nu,\pi}}$ of the core.

For the one-particle operator

$$\langle A\rangle_{\rm out}=a\langle\mathfrak{A}_{1d_{5/2}{}^\pi}|\sum_{i=1}^{n}\varphi^2(r_i)\,|\,\mathfrak{A}_{1d_{5/2}{}^\pi}\rangle=a\times6\mathfrak{R}_{1d}, \tag{27}$$

where the factor of 6 corresponds to the six protons in the shell.

For the two-particle operator

$$\langle A'\rangle_{\rm out}=-M'_{1d_{5/2}1d_{5/2}}-2\sum_{N=0,1}M'_{N,1d_{5/2}}, \tag{28a}$$

$$M'_{1d_{5/2},1d_{5/2}}=n^2\langle\mathfrak{A}_{1d_{5/2}{}^\pi}(1\cdots n)\mathfrak{A}_{1d_{5/2}{}^\nu}(1'\cdots n')\,|\,\omega(1,1')$$
$$\times|\,\mathfrak{A}_{1d_{5/2}{}^\pi}(1'2\cdots n)\mathfrak{A}_{1d_{5/2}{}^\nu}(12'\cdots n')\rangle, \tag{28b}$$

$$M'_{N,1d_{5/2}}=mn\langle\Phi_{N^\pi}(1\cdots m)\mathfrak{A}_{1d_{5/2}{}^\nu}(1\cdots n')\,|\,\omega(1,1')$$
$$\times|\,\Phi_{N^\pi}(1'\cdots m)\mathfrak{A}_{1d_{5/2}{}^\nu}(1\cdots n')\rangle. \tag{28c}$$

To evaluate these we must use the $nljm$ basis for the outer shell. In the inner-outer exchange term we can still use the nlm_lm_s basis for the inner shell and we find, using (23) again,

$$M'_{N,1d_{5/2}}=a\sum_\lambda 6(1l+1)\mathfrak{R}_{nl,1d,\lambda}{}^2(l200|\lambda0)^2 \tag{29}$$

[29] G. N. Watson, *Theory of Bessel Functions* (Cambridge University Press, London, 1958), 2nd ed., p. 363.

[30] A. R. Edmonds, *Angular Momentum in Quantum Mechanics* (Princeton University Press, Princeton, 1960), 2nd ed., p. 76.

[31] M. G. Mayer and J. D. H. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955), p. 106.

TABLE I. M_λ' and M_λ'' for closed $1d_{5/2}$ shell, (Si^{28}).

Inner shell	λ	M_λ'	M_λ''
$1s_{1/2}$	2	6	18
$1p_{1/2}$	1	0	48/5
$1p_{1/2}$	3	6	42/5
$1p_{3/2}$	1	36/5	12
$1p_{3/2}$	3	24/5	24
$1d_{5/2}$	0	6	42/5
$1d_{5/2}$	2	48/7	46/7
$1d_{5/2}$	4	36/5	1116/35

in much the same way as before. To collapse the spin sums and obtain the combination a, we have gone from the $nljm$ basis to the nlm_lm_s basis by means of Clebsch-Gordan coefficients.

The outer-outer exchange term $M'_{1d_{5/2},1d_{5/2}}$ not involving a closed shell of both j's for a given l, will not give the same relationship between the Fermi and the Gamow-Teller terms, viz., the a' and a'' terms. Here we must use the $nljm$ basis and write

$$M'_{nlj,n'l'j'}=\sum_\lambda (a'M_\lambda'+a''M_\lambda'')\mathcal{R}_{ll'\lambda}^2, \tag{30}$$

where[26]

$$M_\lambda'=(-1)^{j'-j}4\pi\langle j\|Y^\lambda\|j'\rangle\langle j'\|Y^\lambda\|j\rangle, \tag{31a}$$

$$M_\lambda''=(-1)^{j'-j}\sum_k(-1)^{\lambda+1-k}\times 4\pi\langle j\|T^k\|j'\rangle\langle j'\|T^k|j\rangle. \tag{31b}$$

Here T_q^k is a spherical tensor defined by

$$T_q^k=\sum_{\mu\nu}(\lambda 1\mu\nu|kq)Y_\mu^\lambda\sigma_\nu. \tag{31c}$$

The reduced matrix elements are given by

$$\langle j\|Y^\lambda\|j'\rangle=(-1)^{j+\lambda-1/2}[(2j+1)(2j'+1)/4\pi]^{1/2}\times(jj'\tfrac{1}{2}-\tfrac{1}{2}|\lambda 0), \tag{32a}$$

$$\langle j\|T^k\|j'\rangle=(-1)^{l'}[6(2j+1)(2j'+1)(2l+1)\times(2l'+1)(2k+1)/4\pi]^{1/2}\times(ll'00|\lambda 0)\begin{Bmatrix} l & l' & \lambda \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j & j' & k\end{Bmatrix}, \tag{32b}$$

the last symbol being a $9j$ symbol. Here we always couple in the order $\mathbf{l}+\frac{1}{2}=\mathbf{j}$, which accounts for the difference in phase in (33a) from the expression given by Elliott.[32] With these,

$$M_\lambda'=(2j+1)(2j'+1)(jj'\tfrac{1}{2}-\tfrac{1}{2}|\lambda 0)^2, \tag{33a}$$

$$M_\lambda''=6(2j+1)(2j'+1)(2l+1)(2l'+1)\times(ll'00|\lambda 0)^2\times\sum_k(2k+1)\begin{Bmatrix} l & l' & \lambda \\ \frac{1}{2} & \frac{1}{2} & 1 \\ j & j' & k\end{Bmatrix}^2. \tag{33b}$$

[32] J. P. Elliott, University of Rochester Report NYO-2271, 1958, p. 51 (unpublished).

The values of these quantities for the cases we need are given in Table I. We have included the inner shells as well and we see that, on summing over the inner j, we recover the result indicated in (37), viz.,

$$M_\lambda''=3M_\lambda'.[33]$$

VI. OUTER SHELL MATRIX ELEMENTS

The outer shell matrix elements, i.e., those involving the Nilsson eigenfunctions and the subsequent Hill-Wheeler integration, are of course much more difficult to evaluate. The method for doing this is given in an appendix and we only present the final results here.

For the normalization constant, we have

$$K=\sum_P \epsilon_P\Delta(P)\times j\text{ sums}, \tag{34a}$$

$$j\text{ sums}=\sum_{j1}\cdots\sum_{jn}\mathfrak{C}(j\omega rP)\times\mathcal{J}\text{ sums}, \tag{34b}$$

$$\mathcal{J}\text{ sums}=\sum_{\mathcal{J}1}\cdots\sum_{\mathcal{J}n-2}\mathcal{K}(j\mathcal{J}\omega P)\mathcal{K}(j\mathcal{J}\omega E), \tag{34c}$$

where

$$\Delta(P)=\delta_{\tau P1\tau 1}\delta_{\tau P2\tau 2}\cdots\delta_{\tau Pn\tau n}, \tag{35a}$$

$$\mathfrak{C}=c_{j_1\omega_{P1}}^{(rP1)}c_{j_1\omega_1}^{(r1)}c_{j_2\omega_{P2}}^{(rP2)}c_{j_2\omega_2}^{(r2)}\cdots c_{j_n\omega_{Pn}}^{(rPn)}c_{j_n\omega_n}^{(rn)}, \tag{35b}$$

$$\mathcal{K}(j\mathcal{J}\omega Q)=(j_1j_2\omega_{Q1}\omega_{Q2}|\mathcal{J}_1\Omega_1^{Q})(\mathcal{J}_1j_3\Omega_1^{Q}\omega_{Q3}|\mathcal{J}_2\Omega_2^{Q})\cdots\times(\mathcal{J}_{n-2}j_n\Omega_{n-2}^{Q}\omega_{Qn}|IK). \tag{35c}$$

In (34a) P is a permutation of the labels $1\cdots n$, arising from the Slater determinant for the n—particle outer shell function, where it acts on the n sets of quantum numbers of the Nilsson configuration. These quantum numbers are, as we saw, ω, r, and τ, the isospin component ($\frac{1}{2}$ for a proton, $-\frac{1}{2}$ for a neutron). $\epsilon_P=\pm 1$ depending on whether P is even or odd. $\Delta(P)$ severely restricts the nature of P so that the ith and Pith particles are either both protons or both neutrons. In (35b) we have used the fact that the Nilsson coefficients are real, and in (35c) Q is taken as P or E, the identity permutation, as required.

The expression for the one-particle matrix element is very similar, involving only additional factors of radial integrals.

$$L^{(1)}=a\sum_{i=1}^{n}\sum_P\epsilon_P\Delta_i(P)\times j\text{ sums}/K, \tag{36a}$$

$$j\text{ sums}=\sum_{j1}\cdots\sum_{jn}\mathfrak{C}(j\omega rP)\times\mathcal{R}_{l_i}\times\mathcal{J}\text{ sums}, \tag{36b}$$

$$\mathcal{J}\text{ sums}=\sum_{\mathcal{J}1}\cdots\sum_{\mathcal{J}n-2}\mathcal{K}(j\mathcal{J}\omega P)\mathcal{K}(j\mathcal{J}\omega E) \tag{36c}$$

[33] It is also interesting to note that

$$\sum_\lambda M_\lambda''=3\sum_\lambda M_\lambda',\quad j\text{ fixed}.$$

where

$$\Delta_i(P)=\delta_{\tau P1,+1/2}\delta_{\tau P1\tau_1}\cdots\delta_{\tau Pn\tau_n}, \tag{37a}$$

$$\mathcal{R}_{l_i}=\langle n_i l_i|\,\varphi^2(r_i)\,|n_i l_i\rangle, \tag{37b}$$

and the other quantities are as in (35).

The two-particle matrix elements are, as usual, more complicated. For the inner-outer exchange terms we again have a collapse of the spin summations to give the coupling constants in the combination $a=a'+3a''$.

$$L_{N\tau}{}^{(2)}=a\sum_{i=1}^{n}\sum_{P}\epsilon_P\Delta_i'(P)\times j\text{ sums}/K, \tag{38a}$$

$$j\text{ sums}=\sum_{j_1}\cdots\sum_{j_n}\Big[\sum_{\lambda}(2l+1)(ll_i00|\lambda0)^2\mathcal{R}_{ll_i\lambda}{}^2\Big]\times\mathfrak{C}(j\omega rP)\times\mathcal{J}\text{ sums}, \tag{38b}$$

$$\mathcal{J}\text{ sums}=\sum_{\mathcal{J}_1}\cdots\sum_{\mathcal{J}_{n-2}}\mathcal{K}(j\mathcal{J}\omega P)\mathcal{K}(j\mathcal{J}\omega E), \tag{38c}$$

where

$$\Delta_i'(P)=[\delta_{\tau_i,\tau_{in}-1}+\delta_{\tau_i,\tau_{in}+1}]\delta_{\tau P1\tau_1}\cdots\delta_{\tau Pn\tau_n}, \tag{38d}$$

with all other symbols as previously defined. The outer-outer exchange term does not give such a collapse, and, as in the case of Si^{28}, allows us to differentiate between a' and a''.

$$L^{(3)}=\sum_{i\neq k}\sum_{P}\epsilon_P\Delta_{ik}(P)\times ik\text{ sums}/K, \tag{39a}$$

$$ik\text{ sums}=\sum_{j_{i'}}\sum_{j_{i''}}\sum_{j_{k'}}\sum_{j_{k''}}\sum_{\mathcal{J}_1}(a'M'+a''M'')\times j\text{ sums}, \tag{39b}$$

$$j\text{ sums}=\sum_{j_1}\cdots\sum_{j_n}\mathfrak{C}_{ik}(j'j''\omega rP)\times\mathcal{J}\text{ sums}, \tag{39c}$$

$$\mathcal{J}\text{ sums}=\sum_{\mathcal{J}_2}\cdots\sum_{\mathcal{J}_{n-2}}\mathcal{K}_{ik}(j'\mathcal{J}\omega P)\mathcal{K}_{ik}(j''\mathcal{J}\omega E), \tag{39d}$$

where

$$\Delta_{ik}(P)=\delta_{\tau Pi,\tau i+1}\delta_{\tau Pk,\tau k-1}\times\delta_{\tau Pj\tau_j}\cdots\delta_{\tau Pn\tau_n} \tag{40a}$$

$$\mathfrak{C}_{ik}=c_{j_{i'}\omega_{Pi}}{}^{(\tau Pi)}c_{j_{i''}\omega_i}{}^{(\tau i)}c_{j_{k'}\omega_{Pk}}{}^{(\tau Pk)}c_{j_{k''}\omega_k}{}^{(\tau k)}\times c_{j_1\omega_{P1}}{}^{(\tau P1)}c_{j_1\omega_1}{}^{(\tau_1)}\cdots c_{j_n\omega_{Pn}}{}^{(\tau Pn)}c_{j_n\omega_n}{}^{(\tau_n)}, \tag{40b}$$

$$\mathcal{K}_{ik}(\vec{j}\mathcal{J}\omega Q)=(\vec{j}_i\vec{j}_k\omega_{Qi}\omega_{Qk}|\,\mathcal{J}_1\Omega_1{}^Q)(\mathcal{J}_1j_1\Omega_1{}^Q\omega_{Q1}|\,\mathcal{J}_2\Omega_2{}^Q)\times\cdots\times(\mathcal{J}_{n-2}j_n\Omega_{n-2}{}^Q\omega_{Qn}|\,IK), \tag{40c}$$

$$a'M'+a''M''=\sum_\lambda(a'M_\lambda'+a''M_\lambda'')\times\mathcal{R}_{l_i'l_k'\lambda}\mathcal{R}_{l_i''l_k''\lambda}, \tag{40d}$$

$$M_\lambda'=(-1)^{i_i''+i_k'+\mathcal{J}_1}\begin{Bmatrix} j_i' & \lambda & j_i'' \\ j_k'' & \mathcal{J}_1 & j_k'\end{Bmatrix}\times4\pi\langle j_i'\|Y^\lambda\|j_i''\rangle\langle j_k'\|Y^\lambda\|j_k''\rangle, \tag{40e}$$

$$M_\lambda''=\sum_\lambda(-1)^{\lambda+1-k+i_i''+i_k'+\mathcal{J}_1}\begin{Bmatrix} j_i' & k & j_i'' \\ j_k'' & \mathcal{J}_1 & j_k'\end{Bmatrix}\times4\pi\langle j_i'\|T^k\|j_i''\rangle\langle j_k'\|T^k\|j_k''\rangle. \tag{40f}$$

Here the single-particle quantum numbers are ordered $i, k, 1, \cdots, i-1, i+1, \cdots, k-1, k+1, \cdots, n$ rather than $1, 2, \cdots, n$, and the ith and kth terms are to be skipped in the "$\cdots$" of (39c), (40b), and (40c). The new symbols in (40e) and (40f) are 6-j symbols; the reduced matrix elements here are given by (32).

The formulas in this section for K, $L^{(1)}$, $L^{(2)}$, and $L^{(3)}$ are in a form amenable to computation by an electronic computer, consisting of sums over products of Nilsson coefficients, Clebsch-Gordan coefficients, radial integrals, and recoupling coefficients.

VII. RADIAL INTEGRALS

The above formulas are complete except for a knowledge of the radial integrals, $\mathcal{R}_l$ and $\mathcal{R}_{ll'\lambda}$. For these we need to know both the radial eigenfunction and the muon wave function. The former are just the eigenfunctions for the spherical harmonic oscillator[34]

$$R_{1l}(r)=(\alpha^{3/2}/\pi^{1/4})[2^{2+l}/(2l+1)!!]^{1/2}(\alpha r)^l e^{-1/2(\alpha r)^2},$$
$$R_{2s}(r)=(\alpha^{3/2}/\pi^{1/4})(2^3/3)^{1/2}[3/2-(\alpha r)^2]e^{-1/2(\alpha r)^2}. \tag{41}$$

The radial parameter α may be found by using the radial functions to calculate the expectation of r^2 and comparing this with the experimental nuclear radius.[34]

$$\langle r^2\rangle=(1/A)\textstyle\sum_i\langle r^2\rangle_{n_il_i}=(1/A\alpha^2)\sum_i[2(n-1)+l+\frac{3}{2}]_i \equiv R_{\text{expt}}{}^2=r_0{}^2A^{2/3}. \tag{42}$$

Strictly, this formula should apply only within the context of the ordinary shell model, but we see that for the $N=2$ shell, the bracket is the same for $nl=2s$, $nl=1d$. Thus

$$\alpha=\{\textstyle\sum_i[2(n-1)+l+\frac{3}{2}]_i\}^{1/2}/(r_0A^{5/6}) =[36+7(A-16)/2]^{1/2}/(r_0A^{5/6}). \tag{43}$$

The nuclear radii for a number of the $2s-1d$ nuclei consistent with muonic x-ray and electron-scattering data are given by Sens in his Table II[15] and we use these values (and interpolations between them), to find α.[35]

Ford and Wills have calculated muon wave functions for many nuclei by numerically integrating the Dirac equation, assuming a nuclear charge distribution consistent with electron scattering data.[36] We found that we could obtain a good fit to the Ford-Wills numerical values, for the nuclei of interest, by assuming cubic forms for φ and φ^2, following Überall.[20]

The one-particle radial integrals $\mathcal{R}_l$ can now be evaluated by simple integrations over r. For the two-particle radial integrals we need to deal with the spherical Bessel function $j_\lambda(\bar{\nu}r)$. The argument of these functions is $\bar{\nu}r\lesssim\bar{\nu}R\cong1.5$, which is small enough to allow us to approximate the j_λ by their expansions to order $(\bar{\nu}r)^4$.[37] With these, the integrations can be carried out in the same way as for $\mathcal{R}_l$.

[34] M. G. Mayer and J. D. H. Jensen, *Elementary Theory of Nuclear Shell Structure* (John Wiley & Sons, Inc., New York, 1955), p. 236.

[35] The α's given by Überall in Ref. 20 actually use $r_0=1.2$ instead of Sens's values, despite the statement to the contrary.

[36] K. W. Ford and J. G. Wills, Los Alamos Report LAMS-2387, 1960 (unpublished), and private communication.

[37] L. I. Schiff, *Quantum Mechanics* (McGraw-Hill Book Company, Inc., New York, 1955), 2nd ed., p. 77.

Note that these two-particle radial integrals depend on the average neutrino momentum $\bar{\nu}$. This will be of some consequence in the role the closed shells play in determining the average capture rate.

VIII. NEUTRINO MOMENTUM, NUCLEI, AND THEIR DEFORMATIONS

The average neutrino momentum $\bar{\nu}$ occurs in a very important way in the phase space factor of (3) and the correct choice of this quantity's value is critical. From the conservation of energy, the neutrino momentum for a transition from nuclear state A to state B is given by[17]

$$\nu_{AB}\cong m_\mu[1-\mathrm{BE}_\mu/m_\mu-(E_B-E_A)/m_\mu], \quad (44)$$

where we have dropped terms of order m_μ/M. The binding energy BE_μ for the low Z we are considering gives less than a 2% correction to ν_{AB} and we henceforth ignore it. Thus, we see that we know $\bar{\nu}$ if we know the average excitation of the final nucleus, $\langle E_B-E_A\rangle$.

This quantity can be calculated on the basis of a Fermi-gas model of the nucleus. Such an analysis has been done by Kaplan[38] to interpret data he had collected on neutron evaporation following mu capture. His formulas for the nondegenerate Fermi gas involve two parameters, the nuclear "temperature" and the effective mass of the nucleon within the nucleus. Results for $\bar{\nu}$ from these formulas for various choices of the parameters are presented in Table II. We see that, for given parameters, $\bar{\nu}$ does not vary much over the range of nuclei we are considering. Kaplan's data[38,39] fit his model's prediction for the neutron multiplicity for an effective mass $M^*=\frac{1}{2}M_p$ but not for $M^*=M_p$. (There is some theoretical justification for such an effective mass.) We therefore take as our best choice of average neutrino momentum $\bar{\nu}=0.75$.

Numerical calculations are presented in this paper for F^{19}, Ne^{20}, Si^{28}, Cl^{35}, and Cl^{37}. To proceed we need to know the nuclear deformation and the appropriate Nilsson configuration for each. We discuss these nuclei in turn, considering at the same time their relevance to mu capture.

F^{19}, the only stable fluorine isotope, consists of a proton and two neutrons outside the O^{16} core. It is a "meeting ground" where the three-particle shell model with configuration mixing can be compared with the Nilsson unified model. This has been done and it is found that the two different kinds of wave functions have a very good overlap.[40] Moreover, Burkhardt and Caine[41] have calculated the average capture rate in F^{19} with the Elliott-Flowers wave function, using the closure approximation in the same way as we have. A comparison of our result for the nuclear matrix element with theirs gives an indication of just how good the nuclear matrix element is.

TABLE II. Fermi-gas model results for $\bar{\nu}$.[a]

Nucleus	r_0	M^*	θ_F	$\bar{Q}$	$\bar{\nu}$
F^{19}	1.03	M_N	10	14.5	0.828
F^{19}	1.03	$\frac{1}{2}M_N$	20	22.9	0.750
Cl^{37}	1.05	$\frac{1}{2}M_N$	20	23.1	0.766

[a] M^* is the effective nucleon mass, θ_F the Fermi nuclear temperature (measured in electron masses), $\bar{Q}$ the average nuclear excitation (in MeV), and $\bar{\nu}$ the average neutrino momentum (in muon masses).

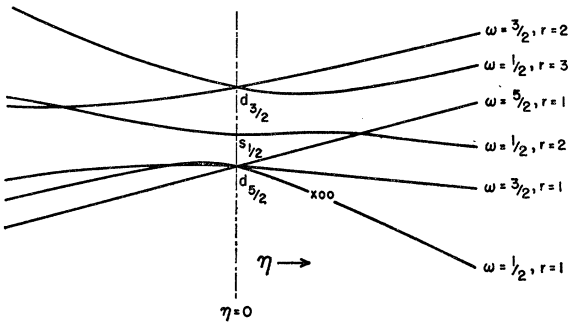

FIG. 1. Configuration for F^{19}. ×—proton, ○—neutron.

There are five recent measurements of the capture rate in F^{19}.[42] Since F^{19} has a spin of $\frac{1}{2}$ and because of the high conversion rate from the upper to the lower hf level, what is actually measured is λ_- rather than $\bar{\lambda}$. We can compare our results for $\bar{\lambda}$ with experiment, however, since the hf difference $\Delta\lambda/\bar{\lambda}$ has also been measured by Winston,[16] and, using this experimental result, we can find the experimental $\bar{\lambda}$.

Astbury (1958): $\lambda_-=2.72\pm0.20$, $\bar{\lambda}=1.73\pm0.20$;
Sens (1959): $\lambda_-=2.54\pm0.22$, $\bar{\lambda}=1.60\pm0.22$;
Eckhause (1962): $\lambda_-=2.41\pm0.18$, $\bar{\lambda}=1.53\pm0.18$;
Astbury (1962): $\lambda_-=2.40\pm0.40$, $\bar{\lambda}=1.52\pm0.40$;
Winston (1963): $\lambda_-=2.40\pm0.10$, $\bar{\lambda}=1.52\pm0.10$;

all rates given in 10^5 sec^{-1}. Here we have taken $\Delta\lambda/\bar{\lambda}=-0.78$.

The nuclear spin of $\frac{1}{2}$ indicates that F^{19} has a positive deformation and Nilsson configuration as shown in Fig. 1, the three nucleons residing in the first $\omega=\frac{1}{2}$ level. The choice of η can be made in several ways. Nilsson suggests, for one thing, that the total energy of the A-particle configuration be minimized with respect to

[38] S. N. Kaplan, University of California Lawrence Radiation Laboratory Report No. UCRL-3749, 1958 (unpublished).

[39] See also G. Groetzinger, Martin J. Berger, and Gordon W. McClure, Phys. Rev. **81**, 969 (1951); M. Widgoff, *ibid.* **90**, 892 (1953); R. D. Crouch and M. F. Sard in *Progress in Cosmic Ray Physics* (Interscience Publishers, Inc., New York, 1952), Vol. II.

[40] M. G. Redlich, Phys. Rev. **110**, 468 (1958).

[41] G. H. Burkhardt and C. A. Caine, Phys. Rev. **117**, 1375 (1960).

[42] A. Astbury, M. A. R. Kemp, N. H. Lipman, H. Muirhead, R. G. P. Voss, C. Zanger, and A. Kirk, Proc. Phys. Soc. (London) **72**, 494 (1958); J. C. Sens, reference cited in footnote 15; M. Eckhause, T. A. Filipas, R. B. Sutton, R. F. Welsh, and T. A. Romanowski, Nuovo Cimento **24**, 666 (1962); A. Astbury, I. M. Blair, M. Hussain, M. A. R. Kemp, and H. Muirhead, Proc. Phys. Soc. (London) **78**, 1149 (1962); R. Winston, reference cited in footnote 9.

the deformation parameter.[43] Another way to find η is through the quadrupole moment. A nucleus with spin $\frac{1}{2}$, as is the case here, does not have a spectroscopic quadrupole moment, but the so-called "intrinsic" quadrupole moment can be measured from the reduced transition probability for an electric quadrupole transition. Finally, the energy-level spectrum as a number of rotational bands based upon single-particle levels can be fitted to experiment. (In addition to the single-particle level spacing, the decoupling parameter depends strongly on the deformation.) All these methods for choosing η are only in rough agreement with one another, so the deformation is only approximately known. Paul finds, for F^{19}, a best value of $\eta=3.0-3.5$.[44]

Neon occurs predominately in the isotope Ne^{20} (abundance 90%), although the isotopes Ne^{21} and Ne^{22} are stable. The deformation and Nilsson configuration are much like that of F^{19}, the lowest level now filled by the additional proton.

The capture rate for naturally occurring neon has just recently been measured by a method due to Shiff,[45] in which the neon is dissolved in liquid hydrogen. Since Ne^{20} is an even-even nucleus, the experiments measure $\bar{\lambda}$ directly. The two measurements are[46]

$$\text{CERN:}\quad \bar{\lambda}=1.69\pm0.3;$$
$$\text{Columbia:}\quad \bar{\lambda}=2.01\pm0.1;$$

again in 10^5 sec^{-1}. These experimental results are reduced somewhat from the rate for Ne^{20} alone because of the presence of the heavier isotopes.

Si^{28} is a somewhat special case discussed in Section V in terms of simple shell model wave functions. An even-even nucleus, the experimental rate is[15]

$$\text{Sens:}\quad \bar{\lambda}=7.7\pm0.25,$$

again in 10^5 sec^{-1}.

Chlorine occurs in two stable isotopic forms, Cl^{35} and Cl^{37}, with an abundance of about 75 and 25%, respectively. One therefore expects an isotope effect. This has been seen experimentally.[47] Since each of the chlorine nuclei has a spin of $\frac{3}{2}$, we again have that the measured rates are the rates from the lower hf state. The correction to find $\bar{\lambda}$ in this case is in the opposite direction from that for F^{19}, because here the spin comes from a $1d_{3/2}$ single-particle state, (with $I=l-\frac{1}{2}$), while in F^{19} it comes from a $2s_{1/2}$ state (with $I=l+\frac{1}{2}$). We can estimate the amount of the correction by means of the BLYP formula[48]

$$\frac{\Delta\lambda}{\bar{\lambda}}=\frac{b}{a}\frac{1}{Z'}\frac{2I+1}{I}, \qquad I=l+\tfrac{1}{2}$$
$$=\frac{b}{a}\frac{1}{Z'}\frac{2I+1}{I+1}, \qquad I=l-\tfrac{1}{2}, \tag{45a}$$
$$Z'=(Z-1)\xi+1. \tag{45b}$$

The parameter ξ represents the effect of the Pauli exclusion principle in reducing the number of final states and can be estimated by comparing this formula with the shell-model results of Überall for F^{19}, Al^{27}, and P^{31}.[20] This comparison gives $\xi=0.50$, 0.30, 0.54, respectively, averaging to $\bar{\xi}=0.45$. For Cl^{35-37}, the above gives, using UFI values for the coupling constants,

$$\Delta\lambda/\bar{\lambda}=0.137,\quad \xi=0.30,$$
$$=0.177,\quad \xi=0.45,$$
$$=0.251,\quad \xi=0.60, \tag{46}$$

which in turn gives

$$\bar{\lambda}=1.092\,\lambda_-,\quad \xi=0.30,$$
$$=1.126\,\lambda_-,\quad \xi=0.45,$$
$$=1.187\,\lambda_-,\quad \xi=0.60. \tag{47}$$

The difference between these choices, centered about the average $\bar{\xi}$ of 0.45, is a matter of 5%, which is comparable to the experimental error of 3–4% quoted by Bertram *et al.* We combine the errors to get the experimental values for $\bar{\lambda}$ from λ_-, which are $\lambda_-^{35}=18.02\pm0.49$, $\bar{\lambda}^{35}=20.3\pm0.6$, and $\lambda_-^{37}=12.51\pm0.52$, $\bar{\lambda}^{37}=14.1\pm0.65$, all in 10^5 sec^{-1}.

The quadrupole moment and the closing of the $N=2$ shell at Ca^{40} indicate that the chlorine nuclei have negative deformations. In terms of holes, the configuration for Cl^{35} is pictured in Fig. 2. The deformation parameter η for these nuclei can be found in much the same way as for F^{19}. In this case, however, there is no fitting to the experimental spectra to help us. The

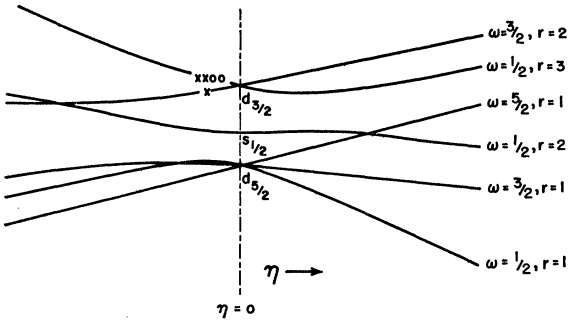

FIG. 2. Hole configuration for Cl^{35}. ×—proton hole, O—neutron hole.

[43] Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, No. 16 (1955), Appendix C; T. D. Newton, Can. J. Phys. **38**, 700 (1960).
[44] E. B. Paul, Phil. Mag. **2**, 311 (1957).
[45] M. Shiff, Nuovo Cimento **22**, 66 (1961).
[46] G. Conforto, C. Rubbia, and E. Zavattini, Phys. Letters **4**, 239 (1963); Columbia results quoted by L. Wolfenstein (private communication).
[47] W. J. Bertram, Jr., R. A. Reiter, T. A. Romanowski, and R. B. Sutton, Phys. Rev. Letters **5**, 61 (1960).
[48] J. Bernstein, T. D. Lee, C. N. Yang, and H. Primakoff, Phys. Rev. **111**, 313 (1958).

magnetic moments can give some information, as well as the spectroscopic quadrupole moment. A best value for the deformation, which is small, is $\eta=-2$.

IX. RESULTS AND CONCLUSIONS

We collect all the formulas to express the rate in the form

$$\bar{\lambda}=K(\bar{\nu},Z)\langle A+A'\rangle, \tag{48}$$

where

$$K(\bar{\nu},Z)=\frac{Z^3}{2\pi^2 a_\mu{}^3}\frac{\bar{\nu}^2}{1+\bar{\nu}/M}(g_V{}^{(\beta)})^2$$

$$\cong 50.6\bar{\nu}^2 Z^3\ \text{sec}^{-1}, \tag{49a}$$

and

$$\langle A+A'\rangle=aX+a'X'+a''X''. \tag{49b}$$

Here the X's are the nuclear matrix elements apart from the coupling constants which occur in a, a', a'', given by (6). We have taken a factor of $(g_V{}^{(\beta)})^2$ into $K(\bar{\nu},Z)$. X' and X'' arise entirely from the outer-outer exchange term $L^{(3)}$.

The values of X, X', X'' and the matrix element $\langle A+A'\rangle$ under the usual UFI assumptions are collected in Table III for various choices of $\bar{\nu}$ and η. For F^{19} we have also considered the effect of changing the nuclear radius, although we expect the Sens' values used everywhere else to be better radii.

We note the strong dependence of the matrix element on the neutrino momentum $\bar{\nu}$, a dependence which is more and more strong as we go through the shell from F^{19} to Cl^{37}. This comes about roughly because we are evaluating the quantity

$$1-j_0(\bar{\nu}r_{12})\cong(\bar{\nu}r_{12})^2\lesssim(\bar{\nu}R)^2.$$

We do not get the quadratic dependence indicated here because of the integration over $\mathbf{r}_1$, $\mathbf{r}_2$ separately and the inclusion of $(\bar{\nu}r)^4$ terms. The increase in the dependence on $\bar{\nu}$ as we approach the closing of the shell at Ca^{40} arises from the fact that the two-particle terms in the matrix element cancel more and more of the one-particle contributions, simply reflecting the Pauli exclusion-principle effect which reduces the number of neutron states available within the $N=2$ shell. The dependence of $\langle A+A'\rangle$ on $\bar{\nu}$, in addition to the $\bar{\nu}^2$ dependence of the phase-space factor, emphasizes the importance of choosing the correct average neutrino momentum.

The effect of changing the nuclear deformation is, on the whole, surprisingly small. As we can see, only the Gamow-Teller term expressing the exchange of two outer nucleons is sensitive to η. This could be expected from the fact that the one-particle term and the Fermi two-particle term do not involve, in any essential way, an angular momentum operator, as does the Gamow-

TABLE III. Nuclear matrix elements.

(a) F^{19}	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$r_0=1.03$ $\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$	$r_0=1.20$ $\bar{\nu}=0.75$
			$\eta=+2$			
X	1.5970	1.7124	1.8329	1.9579	2.0878	1.9722
X'	−0.4327	−0.3998	−0.3672	−0.3353	−0.3044	−0.3129
X''	−0.5846	−0.5413	−0.4985	−0.4566	−0.4163	−0.4270
$\langle A+A'\rangle_{\text{UFI}}$	7.845	8.614	9.410	10.230	11.075	10.383
			$\eta=+4$			
X	1.5998	1.7161	1.8371	1.9633	2.0941	1.9779
X'	−0.4368	−0.4049	−0.3733	−0.3427	−0.3130	−0.3207
X''	−0.9031	−0.8388	−0.7755	−0.7138	−0.6545	−0.6692
$\langle A+A'\rangle_{\text{UFI}}$	7.354	8.160	8.990	9.847	10.726	10.025

(b) Ne^{20}	$\eta=+2$ $\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\eta=+4$ $\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$
X	2.1627	2.2675	2.3777	2.1683	2.2743	2.3858
X'	−0.8262	−0.7637	−0.7018	−0.8304	−0.7687	−0.7078
X''	−1.2955	−1.2020	−1.1049	−1.7856	−1.6572	−1.5310
$\langle A+A'\rangle_{\text{UFI}}$	9.581	10.398	11.243	8.835	9.713	10.618

(c) Si^{28}	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$
X	3.6510	3.6999	3.7583	3.8274	3.9080
X'	−1.9757	−1.8224	−1.6736	−1.5311	−1.3962
X''	−2.7879	−2.5786	−2.3765	−2.1836	−2.0020
$\langle A+A'\rangle_{\text{UFI}}$	14.626	15.397	16.207	17.057	17.948

(d) Cl^{35-37}	Cl^{37}, $\eta=-2$ $\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$	Cl^{35}, $\eta=-2$ $\bar{\nu}=0.75$
X	1.8007	2.0264	2.2609	2.5038	2.7546	2.6132
X'	...	...	...	...	...	−0.3896
X''	...	...	...	...	...	−0.3637
$\langle A+A'\rangle_{\text{UFI}}$	10.390	11.692	13.045	14.447	15.894	14.102

TABLE IV. Rates without Fermi coupling.[a]

	F^{19}	Ne^{20}	Si^{28}	Cl^{35}
r_0	1.03	1.03	1.03	1.05
$\bar{\nu}$	0.75	0.75	0.75	0.75
η	+4	+4	0	−2
$\langle A+A'\rangle$	7.48	8.97	14.79	12.98
	6.83	8.18	13.50	11.84
$K(\bar{\nu},Z)$[b]	2.075	2.846	7.810	13.948
$\bar{\lambda}$[c]	1.55	2.55	11.55	18.15
	1.42	2.33	10.54	16.56
$\bar{\lambda}_{\text{expt}}$	1.52±0.10	1.69±0.30 2.01±0.10	7.77±0.25	20.3±0.6

[a] From Cl^{37}'s $\bar{\lambda}_{\text{expt}}=14.1\pm0.65$, w/ME at $\eta=-2$, $\bar{\nu}=0.75$; we find, if $a'=0$, $a=3a''=4.98\pm0.23=5.21$ or 4.75.
[b] In 10^4 sec^{-1}.
[c] In 10^6 sec^{-1}.

Teller term with $\boldsymbol{\sigma}_i\cdot\boldsymbol{\sigma}_j$. The breaking of the spherical symmetry will more strongly affect the matrix elements of the latter.

The isotope effect can be seen in the results for Cl^{35} and Cl^{37}.

The matrix element for F^{19} obtained by Burkhardt and Caine using the Elliott-Flowers wave function was $\langle A+A'\rangle=9.720$ at $\bar{\nu}=0.82$. We see that this compares favorably with our results at the best value of $\eta=4$ and at the same $\bar{\nu}$, and on the basis of this alone we would estimate the matrix elements calculated here are about 5% accurate.

Using these matrix elements, we now consider whether or not the interaction contains a Fermi term in addition to the Gamow-Teller term.[4] Suppose we set the Fermi coupling $a'=0$ and choose the Gamow-Teller a'' so that the theoretical and experimental rates for Cl^{37} agree. (Cl^{37} has no outer-outer exchange terms, so the only dependence on the coupling constants is through a, which in this case is $3a''$.) We can now use this value of a'' to predict the corresponding rates for the remaining nuclei. These results are compared with the experimental values in Table IV. As we can see, it is not clear that the case $a'=0$ is contradicted by the experimental evidence.

Let us assume, however, that the interaction is of the $V-A$ type. We consider and compare the results for five different possible variations:

I. Usual UFI, as above. Weak magnetism present, $g_P=8g_A$.

II. UFI, but with weak magnetism absent, $g_P=8g_A$.

III. UFI, with weak magnetism present, but $g_P=0$.

IV. UFI, with weak magnetism present, but $g_P=16g_A$.

V. UFI, with weak magnetism present, but $g_P=-8g_A$.

On the basis of the $A=12$ beta-decay experiments,[49] we do not expect that case II will reflect reality. Nevertheless, it is worthwhile to see what mu capture can say about CVC predictions. Case IV is suggested by the recent $0^+\rightarrow 0^-$ measurement in O^{16}.[14] Case V is considered because of the uncertainty in the dispersion theoretic argument[7] which obtained $g_P=8g_A$, an argument which extracts information from rates (squares of coupling constants) rather than from the coupling constants themselves. The values of a, a', and a'', in units of $(g_V^{(\beta)})^2$, are given in Table V for each of these cases.

In Table VI we present the matrix elements and rates under the five different cases for the nuclei being considered. The best values of the parameters $\bar{\nu}$ and η are indicated and the UFI values for the rates corresponding to these are given in *italics*.

The case of Si^{28}, as we have said, is somewhat special. The predicted rates are quite high compared to the experimental rates. This may result from the simplicity of the nuclear wave function in this case. While we are nominally within the context of the Nilsson model, only $1d_{5/2}$ single-particle eigenfunctions are used in making up the Slater determinant. This completely neglects the mixing of these states with the close-lying $2s_{1/2}$ states, which almost certainly takes place, and which is in fact present in the Nilsson model at a nonzero deformation. We are therefore led to discount these calculations when we come to discuss the coupling constants and henceforth concentrate our attention on the remaining nuclei, for which the outer shell contributions were calculated by electronic computer.

The UFI values given in Table VI for the preferred values of the parameters are in good agreement with the experimental rates. They are somewhat high, though, and the fit to experiment is better if we take $\bar{\nu}=0.72$, extrapolating linearly to find the matrix element at this momentum.

Are any of the cases, I through V, excluded by the comparison to experiment? It seems fairly clear that case V, for which the pseudoscalar term has the "wrong" sign, is such a case. This is the same conclusion reached by Reynolds *et al.* in their recent remeasurement of the C^{12} partial rate.[12] We might be able to go somewhat further than this and argue that case III, for which $g_P=0$, also gives results which are too large. In other words, the pseudoscalar term is probably present. The case which fits the experimental rates best at $\bar{\nu}=0.75$ appears to be case IV, for which the pseudoscalar term is large, $g_P=16g_A$, although the differences between this, the usual UFI case I, and the UFI hypothesis without the weak magnetism term II are probably not significant.

TABLE V. a, a', a'' for five choices of couplings.

	I	II	III	IV	V
a	5.77	4.84	6.73	5.26	8.17
a'	1.03	1.03	1.03	1.03	1.03
a''	1.58	1.27	1.90	1.41	2.38

[49] Y. K. Lee, L. W. Mo, and C. S. Wu, Phys. Rev. Letters **10**, 253 (1963).

TABLE VI. Matrix elements and rates.

	(a) Si^{28}	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$
	I	14.626	15.397	16.207	17.057	17.948
	II	12.095	12.756	13.449	14.174	14.934
$\langle A+A'\rangle$	III	17.239	18.124	19.055	20.033	21.059
	IV	13.238	13.949	14.695	15.476	16.295
	V	21.159	22.214	23.327	24.496	25.726
$K(\bar{\nu},Z)$[a]		6.803	7.810	8.886	10.032	11.247
	I	9.950	*12.025*	14.402	17.112	20.186
	II	8.228	9.962	11.951	14.219	16.796
$\bar{\lambda}$[b]	III	11.728	14.155	16.932	20.097	23.685
	IV	9.006	10.894	13.058	15.526	18.237
	V	14.394	17.349	20.728	24.574	28.934

		$r_0=1.03$, $\eta=+2$					$r_0=1.03$, $\eta=+4$					$r_0=1.20$, $\bar{\nu}=0.75$	
	(b) F^{19}	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$	$\eta=+2$	$\eta=+4$
	I	7.845	8.613	9.410	10.230	11.076	7.354	8.160	8.990	9.848	10.727	10.383	10.025
	II	6.541	7.189	7.860	8.551	9.263	6.146	6.824	7.522	8.243	8.982	8.681	8.393
$\langle A+A'\rangle$	III	9.191	10.084	11.010	11.964	12.946	8.601	9.539	10.506	11.504	12.528	12.139	11.710
	IV	7.130	7.832	8.560	9.309	10.081	6.692	7.427	8.185	8.968	9.770	9.449	9.130
	V	11.211	12.290	13.410	14.564	15.753	10.471	11.607	12.779	13.988	15.229	14.774	14.236
$K\bar{\nu}(,Z)$		1.807	2.075	2.361	2.665	2.989	1.807	2.075	2.361	2.665	2.989	2.705	2.705
	I	1.418	1.787	2.222	2.726	3.311	1.329	*1.693*	2.123	2.624	3.206	2.154	2.080
	II	1.182	1.492	1.856	2.279	2.769	1.111	1.416	1.776	2.197	2.685	1.801	1.742
$\bar{\lambda}$[c]	III	1.661	2.092	2.599	3.188	3.870	1.554	1.979	2.480	3.066	3.746	2.519	2.430
	IV	1.288	1.625	2.021	2.481	3.013	1.209	1.541	1.932	2.390	2.920	1.961	1.894
	V	2.026	2.550	3.166	3.881	4.709	1.892	2.408	3.017	3.728	4.552	3.066	2.954

		$\eta=+2$			$\eta=+4$		
	(c) Ne^{20}[d]	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$
	I	9.581	10.398	11.243	8.835	9.713	10.618
	II	7.971	8.662	9.376	7.372	8.111	8.874
$\langle A+A'\rangle$	III	11.288	12.190	13.170	10.345	11.366	12.419
	IV	8.698	9.446	10.219	8.033	8.834	9.662
	V	13.735	14.878	16.061	12.610	13.845	15.119
$K(\bar{\nu},Z)$		2.479	2.846	3.238	2.479	2.846	3.238
	I	2.375	2.959	3.640	2.190	*2.764*	3.438
	II	1.976	2.465	3.036	1.828	2.308	2.873
$\bar{\lambda}$[e]	III	2.798	3.469	4.264	2.565	3.235	4.021
	IV	2.156	2.688	3.309	1.991	2.514	3.129
	V	3.405	4.234	5.201	3.126	3.940	4.896

		Cl^{37}, $\eta=-2$					Cl^{35}, $\eta=-2$
	(d) Cl^{35-37}	$\bar{\nu}=0.70$	$\bar{\nu}=0.75$	$\bar{\nu}=0.80$	$\bar{\nu}=0.85$	$\bar{\nu}=0.90$	$\bar{\nu}=0.75$
	I	10.390	11.692	13.045	14.447	15.894	14.102
	II	8.715	9.808	10.943	12.118	13.332	11.785
$\langle A+A'\rangle$	III	12.119	13.638	15.216	16.851	18.539	16.495
	IV	9.472	10.659	11.892	13.170	14.489	12.831
	V	14.712	16.556	18.472	20.456	22.505	20.083
$K(\bar{\nu},Z)$		12.181	13.984	15.910	17.961	20.136	13.964
	I	12.656	*16.350*	20.755	25.948	32.004	*19.720*
	II	10.616	13.716	17.410	21.765	26.845	16.480
$\bar{\lambda}$[f]	III	14.762	19.071	24.209	30.266	37.330	23.067
	IV	11.538	14.906	18.920	23.655	29.175	17.943
	V	17.921	23.152	29.389	36.741	45.316	28.084

[a] In 10^4 sec^{-1}, holding throughout this table.
[b] In 10^6 sec^{-1}, holding throughout this table. $\bar{\lambda}_{expt}=7.77\pm0.25$ (Sens).
[c] $\bar{\lambda}_{expt}=1.52\pm0.10$ (Winston).
[d] Uncorrected for 10% Ne^{22} isotopic abundance.
[e] $\bar{\lambda}_{CERN}=1.69\pm0.3$, $\bar{\lambda}_{Columbia}=2.01\pm0.1$.
[f] Cl^{37}, $\bar{\lambda}_{expt}=14.1\pm0.65$ (Bertram); Cl^{35}, $\bar{\lambda}_{expt}=20.3\pm0.60$ (Bertram).

Apart from what we can conclude regarding coupling constants, the results above show why Tolhoek and his collaborators[16] might have obtained a rate so large for Ca^{40} that G^2 would have to be twice too small to fit experiment. First, these authors used shell-model wave functions similar to those we used for Si^{28}. We found the Si^{28} rates to be high, and this might be attributed to the simplicity of the nuclear description.

Perhaps more significant, Tolhoek's choice of $\bar{\nu}$ was 0.825, which is quite large compared with our 0.75. They arrived at this choice as follows. The nuclear matrix element was calculated in two ways—by summing over final states and by the closure approximation. The average neutrino momentum was then chosen so that the two methods gave the same result. If we look at our Cl^{37} results at this momentum, we see that the rate *is* nearly twice the experimental rate. Moreover, this should be enhanced somewhat when we go to the closed shell case of Ca^{40}. The problem here seems to be the average neutrino momentum, and Tolhoek's large $\bar{\nu}$ could easily result from not including enough final states in his summation.

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APPENDIX: OUTER SHELL MATRIX ELEMENTS, REDUCTION

As an example of the method for obtaining the computerable expressions for the outer shell matrix elements given in the text, let us consider the reduction of $L^{(1)}$.

$$L^{(1)}=a\int d\theta'\int d\theta''\,\mathfrak{L}^{(1)}(\theta',\theta'')\times D_{MK}{}^{I*}(\theta')D_{MK}{}^{I}(\theta'')/K, \quad \text{(A1)}$$

where

$$\mathfrak{L}^{(1)}=\langle\mathfrak{X}_\Omega(1\cdots n)|\,\mathbf{D}^\dagger(\theta')\sum_{i=1}^{n}\left(\frac{1+\tau_3{}^{(i)}}{2}\right)\times\varphi^2(r_i)\,\mathbf{D}(\theta'')\,|\,\mathfrak{X}_\Omega(1\cdots n)\rangle. \quad \text{(A2)}$$

Here $\mathfrak{X}_\Omega(1\cdots n)$ is the intrinsic outer shell function formed from Nilsson single-particle eigenfunctions, with its arguments given in terms of space-fixed coordinates. We have supposed it to be antisymmetric and this can be assured by means of an antisymmetrizer.

$$\mathfrak{X}_\Omega(1\cdots n)=\sum_P\epsilon_P\chi_{P1}(1)\chi_{P2}(2)\cdots\chi_{Pn}(n). \quad \text{(A3)}$$

The permutations (of n objects) act on the sets of quantum numbers of the Nilsson configuration for the nucleus being considered.

The antisymmetrizer $\mathbf{P}=\sum_P\epsilon_P P$ nominally occurs in both the bra and the ket of the matrix element given in (2). $\mathbf{P}$ is a Hermitian operator with the property that $\mathbf{P}^2=\mathbf{P}$. The operator $\mathbf{D}^\dagger(\theta')\mathbf{D}(\theta'')$ written out as a function of $\mathbf{J}=\sum_i\mathbf{j}_i$ is symmetric with respect to interchange, i.e., commutes with $\mathbf{P}$. Thus we can rewrite (A2) in the simpler form

$$\mathfrak{L}^{(1)}=\sum_{i=1}^{n}\sum_P\epsilon_P\langle\chi_{P1}(1)\chi_{P2}(2)\cdots\chi_{Pn}(n)|\times\mathbf{D}^\dagger(\theta')\left(\frac{1+\tau_3{}^{(i)}}{2}\right)\varphi^2(r_i)\,\mathbf{D}(\theta'')\times|\,\chi_1(1)\chi_2(2)\cdots\chi_n(n)\rangle. \quad \text{(A4)}$$

The Nilsson eigenfunctions, we saw, are specified by the quantum numbers ω, r, τ and are given by

$$\chi_{\omega r\tau}=Z_\tau(t)\sum_{j=1/2}^{5/2}c_{j\omega}{}^{(r)}R_{nl}{}^{(r)}\mathcal{Y}_\omega{}^{j}(\theta\varphi,s), \quad \text{(A5)}$$

where Z_τ is the isospinor and $\mathcal{Y}_\omega{}^j$ the usual spin-angle eigenfunction. Making this substitution in (A4), we get

$$L^{(1)}=\sum_{i=1}^{n}\sum_P\epsilon_P\times\delta_{\tau Pi,+1/2}\times\delta_{\tau P1\tau_1}\delta_{\tau P2\tau_2}\cdots\delta_{\tau Pn\tau_n}\times j\text{ sums}, \quad \text{(A6a)}$$

$$j\text{ sums}=\sum_{j1'}\cdots\sum_{jn'}\sum_{j1''}\cdots\sum_{jn''}\mathfrak{C}(j'j''\omega rP)\mathfrak{M}(j'j''\omega P)\times\langle n_i'l_i'|\,\varphi^2(r_i)\,|n_i''l_i''\rangle, \quad \text{(A6b)}$$

$$\mathfrak{C}=c_{j_1'\omega P1}{}^{(rP1)}c_{j_1''\omega_1}{}^{(r1)}c_{j_2'\omega P_2}{}^{(rP2)}c_{j_2''\omega_2}{}^{(r_2)}\cdots\times c_{j_n'\omega P_n}{}^{(rP_n)}c_{j_n''\omega_n}{}^{(r_n)}, \quad \text{(A6c)}$$

$$\mathfrak{M}=\langle j_1'\omega_{P1},\cdots,j_n'\omega_{Pn}|\,\mathbf{D}^\dagger(\theta')\,\mathbf{D}(\theta'')\times|\,j_1''\omega_1,\cdots,j_n''\omega_n\rangle. \quad \text{(A6d)}$$

The string of Kronecker delta's in the isospin quantum numbers in (6a) results from the isospin integrations. Denoting this henceforth by $\Delta_i(P)$, we note that this restricts the possible permutations to those of the form $P=ST$, where S is a permutation acting just among the n_π sets of proton quantum numbers and T is a permutation acting among the $n_\nu=n-n_\pi$ sets of neutron quantum numbers. The operator $\frac{1}{2}(1+\tau_3{}^{(i)})$ acts as a projection operator and requires the ith nucleon to be a proton. In (A6c) we have used the fact that the Nilsson coefficients are real.

Recalling properties (11) and (12) of the rotation operator $\mathbf{D}(\theta)$, (A6d) becomes

$$\begin{aligned}\mathfrak{M}=&\sum_{m_1'}\cdots\sum_{m_n'}\sum_{m_1''}\cdots\sum_{m_n''}D_{m_1'\omega P1}{}^{j1'}(\theta')\cdots\\&\times D_{m_n'\omega P_n}{}^{jn'}(\theta'')D_{m_1''\omega_1}{}^{j1''*}(\theta'')\cdots D_{m_n''\omega_n}{}^{jn''*}(\theta'')\\&\times\langle j_1'm_1',\cdots,j_n'm_n'|\,j_1''m_1'',\cdots,j_n''m_n''\rangle\\=&\,\delta_{j1'j1''}\cdots\delta_{jn'jn''}\sum_{m_1}\cdots\sum_{m_n}D_{m_1\omega P1}{}^{j1}(\theta')\cdots\\&\times D_{m_n\omega_{P_n}}{}^{jn}(\theta')D_{m_1\omega_1}{}^{j1*}(\theta')\cdots D_{m_n\omega_n}{}^{jn*}(\theta''). \quad \text{(A7)}\end{aligned}$$

The orthogonality between the primed and double-

primed quantities throws away half the summations in (A6b) and (A7), as well as reducing the radial integral in (A6b) to the $\mathcal{R}_{l_i}$ given in the text.

At this point we can carry out the Hill-Wheeler integrations indicated in (A1). These can in fact be done independently, as the dependence on θ' and θ'' in (A7) has been separated into two factors, each being a string of n D functions. We get an additional D function from (A1). We treat the integral,

$$\mathcal{I}=\int d\theta D_{m_1\omega_1}{}^{j_1}(\theta)D_{m_2\omega_2}{}^{j_2}(\theta)\cdots \times D_{m_n\omega_n}{}^{j_n}(\theta)D_{MK}{}^{I*}(\theta), \quad \text{(A8)}$$

in the following way. We first note that the product of two D functions can be expressed as a linear combination of D functions, the so-called Clebsch-Gordan series.[26]

$$D_{m_1\omega_1}{}^{j_1}(\theta)D_{m_2\omega_2}{}^{j_2}(\theta)=\sum_{\mathcal{J}}(j_1j_2m_1m_2|\,\mathcal{J}\mathfrak{M}) \times(j_1j_2\omega_1\omega_2|\,\mathcal{J}\Omega)D_{\mathfrak{M}\Omega}{}^{\mathcal{J}}(\theta). \quad \text{(A9)}$$

We go through this coupling process $n-1$ times, proceeding from the left, to obtain

$$\mathcal{I}=\sum_{\mathcal{J}_1}\cdots\sum_{\mathcal{J}_{n-1}}\mathcal{K}'(j\mathcal{J}\omega)\mathfrak{m}'(j\mathcal{J}m) \times\int d\theta D_{\mathfrak{M}_{n-1}\Omega_{n-1}}{}^{\mathcal{J}_{n-1}}(\theta)D_{MK}{}^{I*}(\theta), \quad \text{(A10a)}$$

$$\mathcal{K}'=(j_1j_2\omega_1\omega_2|\,\mathcal{J}_1\Omega_1)(\mathcal{J}_1j_3\Omega_1\omega_3|\,\mathcal{J}_2\Omega_2)\cdots \times(\mathcal{J}_{n-2}j_n\Omega_{n-2}\omega_n|\,\mathcal{J}_{n-1}\Omega_{n-1}), \quad \text{(A10b)}$$

$$\mathfrak{m}'=(j_1j_2m_1m_2|\,\mathcal{J}_1\mathfrak{M}_1)(\mathcal{J}_1j_3\mathfrak{M}_1m_3|\,\mathcal{J}_2\mathfrak{M}_2)\cdots \times(\mathcal{J}_{n-2}j_n\mathfrak{M}_{n-2}m_n|\,\mathcal{J}_{n-1}\mathfrak{M}_{n-1}). \quad \text{(A10c)}$$

At this point we use the orthogonality of the D functions[26]

$$\int D_{MK}{}^{I*}(\theta)D_{\mathfrak{M}_{n-1}\Omega_{n-1}}{}^{\mathcal{J}_{n-1}}(\theta)d\theta =\frac{8\pi^2}{2I+1}\delta_{I\mathcal{J}_{n-1}}\delta_{M\mathfrak{M}_{n-1}}\delta_{K\Omega_{n-1}}, \quad \text{(A11)}$$

so that

$$\mathcal{I}=\frac{8\pi^2}{2I+1}\sum_{\mathcal{J}_1}\cdots\sum_{\mathcal{J}_{n-2}}\mathcal{K}(j\mathcal{J}\omega)\mathfrak{m}(j\mathcal{J}m), \quad \text{(A12a)}$$

$$\mathcal{K}=(j_1j_2\omega_1\omega_2|\,\mathcal{J}_1\Omega_1)\cdots(\mathcal{J}_{n-2}j_n\Omega_{n-2}\omega_n|\,IK), \quad \text{(A12b)}$$

$$\mathfrak{m}=(j_1j_2m_1m_2|\,\mathcal{J}_1\mathfrak{M}_1)\cdots(\mathcal{J}_{n-2}j_n\mathfrak{M}_{n-2}m_n|\,IM). \quad \text{(A12c)}$$

There are two such integrations, one for θ' and the other for θ''. Since we end up with sums over Clebsch-Gordan coefficients, which are real, the fact that the θ'' integral is the complex conjugate of (A8) does not bother us. Also, since the "parent momenta" $\mathcal{J}_i$ have no physical meaning, i.e., are not quantum numbers of the wave function, we can couple the D functions in (A8) in any order. The choice which leads to (A12) is merely one of convenience.

The θ' integration has $\mathcal{K}$ depend on the permutation P, as this is what orders the ω's. Both $\mathcal{K}$'s from the two integrations are independent of the m's. Thus, dropping an over-all factor of $[8\pi^2/(2I+1)]^2$ and reordering the $\mathcal{J}$ and the m summations, (A1) becomes

$$L^{(1)}=a\sum_{i=1}^{n}\sum_{P}\epsilon_P\Delta_i(P)\times j\text{ sums}/K, \quad \text{(A13a)}$$

$$j\text{ sums}=\sum_{j_1}\cdots\sum_{j_n}\mathfrak{C}(j\omega rP)\mathcal{R}_{l_i}\times\mathcal{J}\text{ sums}, \quad \text{(A13b)}$$

$$\mathcal{J}\text{ sums}=\sum_{\mathcal{J}_1'}\cdots\sum_{\mathcal{J}_{n-2}'}\sum_{\mathcal{J}_1''}\cdots\sum_{\mathcal{J}_{n-2}''}\mathcal{K}(j\mathcal{J}'\omega P) \times\mathcal{K}(j\mathcal{J}''\omega E)\times m\text{ sums}, \quad \text{(A13c)}$$

$$m\text{ sums}=\sum_{m_1}\cdots\sum_{m_n}\mathfrak{m}(j\mathcal{J}'m)\mathfrak{m}(j\mathcal{J}''m). \quad \text{(A13d)}$$

We now finally get the result given in the text when we recognize that the m sums collapse into $\delta_{\mathcal{J}_1'\mathcal{J}_1''}\cdots \times\delta_{\mathcal{J}_{n-2}'\mathcal{J}_{n-2}''}$. This comes about from the orthogonality of the Cebsch-Gordan coefficients appearing in the $\mathfrak{m}$'s. Although not really necessary in this case, it is best to unravel the sums proceeding from right to left. In the more complicated two-particle cases, this then gives rise to the 6-j symbols appearing in (40).