Spectroscopy in the Nuclear $1f_{7/2}$ Shell*

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The intent of the work reported here has been to ascertain which properties of nuclei with $20 \leq Z, N \leq 28$ are consistent with wave functions composed of a Ca^{40} core plus Z-20 protons and N-20 neutrons in the $1f_{7/2}$ shell. We have attempted to determine the $1f_{7/2}$ nucleon-nucleon interaction from the observed levels of Sc^{42}, assuming that these can be interpreted as due to the configuration $(1f_{7/2})^2$. Following Talmi, we have then treated this interaction exactly, within the framework of our pure configuration assumption. We have found that many of the observed properties of these nuclei are reproduced by this model, in particular the systematics of low-lying nuclear spins, magnetic moments, ft values, and (p,d) and (d,p) spectroscopic factors. However, several nuclei have more low-lying states than the $(1f_{7/2})^n$ configuration can provide at the corresponding energies.

I. INTRODUCTION

THERE are two principal sources of uncertainty in nuclear shell-model calculations: (1) the number of configurations necessary to provide an adequate representation of the nuclear states under investigation, and (2) uncertainties in the effective residual interaction. Correspondingly, there are two general approaches to shell-model calculations, depending upon which of these uncertainties is regarded as more serious. One can assume a detailed residual interaction (shape, range, exchange character), and assume radial wave functions of the single-particle shell-model states. One can then calculate the interaction matrix elements between states of arbitrarily many configurations. This approach is beset by uncertainties of type (2), but is, in principle, free of those of type (1). On the other hand, one can make a very restrictive assumption about possible configurations. The interaction matrix elements are then determined by only a few numbers, which can themselves be determined from experiment. This approach minimizes uncertainties of type (2), but may be in error due to those of type (1).

This second approach,[1–4] whose application to nuclear physics has been vigorously pursued by Talmi and his co-workers, might be expected to have the greatest chance of success in regions of the periodic table in which the shell model exhibits well-separated single-particle levels. A good example is the $1f_{7/2}$ shell. In this paper we describe an application of the second approach to this shell. Thus we assume that certain levels of nuclei with $20 \leq Z, N \leq 28$ can be described by means of wave functions consisting of closed shells, plus Z-20 protons and N-20 neutrons in the $1f_{7/2}$ shell. Even this assumption admits great freedom in the choice of the wave functions. For example, in ${}_{22}Ti_{24}{}^{46}$ there are 23 orthogonal states of angular momentum 4 (and given z component), all within the pure $1f_{7/2}$ configuration. They correspond to possible angular momenta of the two $1f_{7/2}$ protons and the four $1f_{7/2}$ neutrons, consistent with a total angular momentum of 4. To determine which linear combinations of these represent real nuclei, we must calculate the matrix of the residual interaction with respect to these states and diagonalize it. If it is further assumed that the residual interaction is a two-body interaction, then these matrix elements are completely determined by the interaction energy differences between the eight states $(1f_{7/2})^2\, I=0, \cdots, 7$. These states presumably describe observed states in ${}_{21}Sc_{21}{}^{42}$, so that the desired energy differences can be extracted from the Sc^{42} spectrum. They can then be used for the calculation and subsequent diagonalization of the matrices. The resulting eigenvalues and eigenvectors enable us to predict energies and other properties (magnetic moments, ft values, spectroscopic factors) of the nuclear states. Our viewpoint is not a preconceived notion of the validity of the assumptions of a pure configuration and a two-body interaction. Rather, we seek to determine the consequences of these assumptions, to determine their compatibility with experiment, and to focus attention on the instances of sharp incompatibility. Having established phenomena which *cannot* be understood in terms of the pure $1f_{7/2}$ shell with two-body interactions, we hope to be able to present alternative explanations.

One test of the pure configuration assumption which can be applied without further calculation is a comparison of the nuclei $Z=20+a$, $N=20+b$ and $Z=28-b$, $N=28-a$, for example, ${}_{21}Sc_{23}{}^{44}$ and ${}_{25}Mn_{27}{}^{52}$. We refer to such nuclei as cross-conjugate pairs. It follows from well-known particle-hole theorems (see Sec. II) that cross-conjugate pairs should have identical spectra. Reference to the spectra of the pairs Sc^{44}-Mn^{52}, or Ti^{47}-V^{49}, shows that this expectation is not fulfilled, by amounts corresponding to shifts of several hundred keV per level. Thus it can already be said that at least some

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[1] R. D. Lawson and J. L. Uretsky, Phys. Rev. **106**, 1369 (1957); I. Talmi, in *Proceedings of the Rehovoth Conference on Nuclear Structure* (North-Holland Publishing Company, Amsterdam, 1958).
[2] I. Talmi and I. Unna, Ann. Rev. Nucl. Sci. **10**, 353 (1960).
[3] I. Talmi, Rev. Mod. Phys. **34**, 704 (1962).
[4] A. de-Shalit and I. Talmi, *Nuclear Shell Theory* (Academic Press Inc., New York, 1963).

of the nuclear states are perturbed to this extent by configurational admixtures.

Nuclei with only one type of fermion outside closed shells (the calcium isotopes and the $N=28$ isotones) have already been studied in this way.[1,3,11] We here extend the analysis to all nuclei with $20\leq Z,\ N\leq 28$, except those for which $23\leq Z,\ N\leq 25$.

II. THEORY

A. Notation and Particle-Hole Conventions

In the discussion below, the symbol j *always* signifies 7/2. The bracket notation $[\Psi^J\Phi^K]_M{}^L$ implies the vector coupling operation

$$[\Psi^J\Phi^K]_M{}^L\equiv\sum_{m_1,m_2}(JKm_1m_2|LM)\Psi_{m_1}{}^J\Phi_{m_2}{}^K,\quad (1)$$

where the coefficients in the sum (1) are the usual Wigner coefficients. We use Jahn's notation[5] for the normalized Racah coefficients,

$$\{[\Psi^J\Phi^K]^L\Theta^P\}_M{}^I=\sum_Q U(JKIP;LQ)\{\Psi^J[\Phi^K\Theta^P]^Q\}_M{}^I.\quad (2)$$

$\Psi_{12\cdots n}(j^nvIM)$ will imply the antisymmetric state of the j^n identical-particle configuration with seniority v and angular-momentum quantum numbers (I,M). The particle labels are subscripted. The fractional parentage coefficients (cfp) are defined as usual by

$$\Psi_{12\cdots n}(j^nvIM)=\sum_{v'I',v''I''}(j^{n-q}v'I';\,j^qv''I''\rrbracket j^nvI)\times[\Psi_{1\cdots(n-q)}(j^{n-q}v'I')\Psi_{(n-q+1)\cdots n}(j^qv''I'')]_M{}^I.\quad (3)$$

To define the hole state corresponding to $\Psi_{12\cdots n}(j^nvIM)$, we expand the latter in terms of normalized Slater determinants,

$$\Psi_{1\cdots n}(j^nvIM)=\sum_{m_1>m_2>\cdots>m_n}C^{vIM}(m_1m_2\cdots m_n)\times\frac{1}{(n!)^{1/2}}\det[m_1m_2\cdots m_n].\quad (4)$$

The $C^{vIM}(m_1m_2\cdots m_n)$ are constant coefficients. We then define the hole state $\Psi_{1\cdots(2j+1-n)}(j^{-n}vIM)$ by

$$\Psi_{1\cdots(2j+1-n)}(j^{-n}vIM)\equiv(-1)^{I(-j)}\sum_{m_1>m_2>\cdots>m_n}C^{vI-M}(m_1m_2\cdots m_n)\times\frac{1}{[(2j+1-n)!]^{1/2}}\det[m_1'm_2'\cdots m_n'],\quad (5)$$

where $(m_1',m_2',\cdots m_{2j+1-n}')$ is the set complementary to $(m_1m_2\cdots m_n)$, with $m_1'>m_2'\cdots>m_{2j+1-n}'$. The phase factor in (5) is $(-1)^I$ or $(-1)^{I-j}$ depending upon whether n is even or odd. Using the definition (5), one can show that if $V_q{}^k=\sum_i v_q{}^k(i)$ is a single-particle irreducible tensor operator, then

$$\langle\Psi(j^{-n}v_1I_1M_1)|\sum_{i=1}^{2j+1-n}v_q{}^k(i)|\Psi(j^{-n}v_2I_2M_2)\rangle=-(-1)^k\langle\Psi(j^nv_1I_1M_1)|\sum_{i=1}^{n}v_q{}^k(i)|\Psi(j^nv_2I_2M_2)\rangle,\quad k\neq0.\quad (6)$$

The $v_q{}^k$ (odd k) are infinitesimal generators of the group Sp$(2j+1)$. Thus (6) implies that the $\Psi(j^{-n}vIM)$ and $\Psi(j^nvIM)$ have the same transformation properties under Sp$(2j+1)$, so that $\Psi(j^{-n}vIM)$ is also a state of seniority v, total angular momentum I, and z-component M, as our notation has implied. It is also a consequence of (5) that a closed-shell wave function can be expanded as follows:

$$\begin{aligned}\Psi(j^{2j+1}00)&\equiv\frac{1}{[(2j+1)!]^{1/2}}\det[j\cdots-j]\\&=\left[\frac{n!(2j+1-n)!}{(2j+1)!}\right]^{1/2}\sum_{vI}(2I+1)^{1/2}[\Psi_{1\cdots n}(j^nvI)\Psi_{(n+1)\cdots2j+1}(j^{-n}vI)]_0{}^0\\&=\left[\frac{n!(2j+1-n)!}{(2j+1)!}\right]^{1/2}\sum_{\substack{vI\\v_1I_1,\,v_2I_2}}(2I+1)^{1/2}(j^{n-p}v_1I_1;\,j^pv_2I_2\rrbracket j^nvI)\\&\qquad\times\{[\Psi_{1\cdots n-p}(j^{n-p}v_1I_1)\Psi_{n-p+1\cdots n}(j^pv_2I_2)]^I\Psi_{n+1\cdots2j+1}(j^{-n}vI)\}_0{}^0\\&=\left[\frac{n!(2j+1-n)!}{(2j+1)!}\right]^{1/2}\sum_{\substack{vI\\v_1I_1,\,v_2I_2}}(2I+1)^{1/2}(j^{n-p}v_1I_1;\,j^pv_2I_2\rrbracket j^nvI)\\&\qquad\times\{\Psi_{1\cdots n-p}(j^{n-p}v_1I_1)[\Psi_{n-p+1\cdots n}(j^pv_2I_2)\Psi_{n+1\cdots2j+1}(j^{-n}vI)]^{I_1}\}_0{}^0\\&=\left[\frac{(n-p)!(2j+1-n+p)!}{(2j+1)!}\right]^{1/2}\sum_{v_1I_1}(2I_1+1)^{1/2}[\Psi_{1\cdots n-p}(j^{n-p}v_1I_1)\Psi_{n-p+1\cdots2j+1}(j^{-[n-p]}v_1I_1)]_0{}^0.\end{aligned}\quad (7)$$

[5] H. A. Jahn, Proc. Roy. Soc. (London) **A205**, 192 (1951).

Comparison of the coefficients of $\Psi_{1\cdots n-p}(j^{n-p}v_1I_1M_1)$ then yields

$$(j^pv_2I_2;\, j^{-n}vI\,]\!]\,j^{-[n-p]}v_1I_1)=\left(\frac{n!(2j+1-n)!}{(n-p)!(2j+1-n+p)!}\right)^{1/2}\left(\frac{2I+1}{2I_1+1}\right)^{1/2}(j^{n-p}v_1I_1;\, j^pv_2I_2\,]\!]\,j^nvI)$$

or equivalently

$$(j^{-[p+q]}vI;\, j^pv_2I_2\,]\!]\,j^{-q}v_1I_1)=(-1)^{p(p+q)+I+I_2-I_1}\left[\frac{(p+q)!(2j+1-p-q)!}{q!(2j+1-q)!}\,\frac{2I+1}{2I_1+1}\right]^{1/2}(j^qv_1I_1;\, j^pv_2I_2\,]\!]\,j^{p+q}vI). \quad (8)$$

This provides the amplitude for removing p particles from a state of q holes.

We will be mainly concerned with spherically symmetric two-particle interactions $W=\sum W_{ij}$. They can be expressed in the form

$$W=\textstyle\sum_k [V^kV^k]_0{}^0+T_0{}^0,$$

where $V_q{}^k$ and $T_0{}^0$ are single-particle operators. Then (6) implies that

$$\langle j^{-n}v_1I_1M_1|\sum_{1\leqslant i<j}^{2j+1-n} W_{ij}|\,j^{-n}v_2I_2M_2\rangle=\langle j^nv_1I_1M_1|\sum_{1\leqslant i<j}^{n} W_{ij}|\,j^nv_2I_2M_2\rangle+\delta_{v_1v_2}\delta_{I_1I_2}\delta_{M_1M_2}\times\text{const}, \quad (9)$$

so that corresponding hole-hole and particle-particle matrix elements are the same, apart from some multiple of the unit matrix. The [proton]-[neutron hole] interaction energies F_I:

$$F_I\equiv\langle[\Psi_{1\cdots 2j}(j^{-1}j)\psi_p(j)]_M{}^I|\sum_{i=1}^{2j} W_{ip}|[\Psi_{1\cdots 2j}(j^{-1}j)\psi_p(j)]_M{}^I\rangle \quad (10)$$

are related to the proton-neutron interaction energies E_J:

$$E_J\equiv\langle[\psi_n(j)\psi_p(j)]_M{}^J|W_{np}|[\psi_n(j)\psi_p(j)]_M{}^J\rangle \quad (11)$$

by a "Pandya relation"[6]:

$$F_I=\sum_J\left\{\frac{2J+1}{2j+1}-\left[\frac{2J+1}{2I+1}\right]^{1/2}U(jjjj;IJ)\right\}E_J, \quad (12)$$

which is easily derived with the help of (8).

B. Eigenvalues and Eigenvectors of the Scandium Isotopes and the $N=27$ Isotones

We use a representation composed of the states

$$[\Psi_{1\cdots n}(j^nvL)\psi_p(j)]_M{}^I. \quad (13)$$

The matrix of the neutron-neutron interaction is

$$\langle[\Psi_{1\cdots n}(j^nv_1L_1)\psi_p(j)]_M{}^I|\sum_{1\leqslant i<j}^{n} W_{ij}|[\Psi_{1\cdots n}(j^nv_2L_2)\psi_p(j)]_M{}^I\rangle$$

$$=\delta_{v_1v_2}\delta_{L_1L_2}\langle\Psi_{1\cdots n}(j^nv_1L_1)|\sum_{1\leqslant i<j}^{n} W_{ij}|\Psi_{1\cdots n}(j^nv_1L_1)\rangle$$

$$=\delta_{v_1v_2}\delta_{L_1L_2}\sum_{\text{even }J}\left[\frac{n(n-1)}{2}\sum_{vK}(j^{n-2}vK;\, j^2J\,]\!]\,j^nv_1L_1)^2\right]E_J. \quad (14)$$

The bracketed coefficients are given in Appendix I. The matrix of the neutron-proton interaction is

$$\langle[\Psi_{1\cdots n}(j^nv_1L_1)\psi_p(j)]_M{}^I|\sum_{i=1}^{n} W_{ip}|[\Psi_{1\cdots n}(j^nv_2L_2)\psi_p(j)]_M{}^I\rangle$$

$$=\sum_J\Big[n\sum_{vK}(j^{n-1}vK;\, j\,]\!]\,j^nv_1L_1)(j^{n-1}vK;\, j\,]\!]\,j^nv_2L_2)U(KjIj;L_1J)U(KjIj;L_2J)\Big]E_J. \quad (15)$$

[6] S. P. Pandya, Phys. Rev. **103**, 956 (1956); S. Goldstein and I. Talmi, Phys. Rev. **102**, 598 (1956).

The sum of (14) and (15) gives the complete energy matrix, which we diagonalize. The different eigenvectors with total angular momentum I will be distinguished by an index $a=1, 2, 3, \cdots$ (in order of increasing energy). The components of the normalized eigenvectors are labeled $D^{a,I}(v,L)$ so that

$$\Psi_M{}^{a,I}=\sum_{v,L} D^{a,I}(v,L)[\Psi_{1\cdots n}(j^n vL)\psi_p(j)]_M{}^I \tag{16}$$

is the corresponding eigenstate.

The values of E_J for odd J enter only into the neutron-proton interaction. E_J for even J enter into both neutron-proton and neutron-neutron interactions. We assume charge independence of the nucleon-nucleon force, so that the E_J for even J are the same in both (14) and (15). Then the energy eigenstates (16) will automatically be eigenstates of isotopic spin, unless there is an accidental energy degeneracy between states of different isotopic spin. Of course, the separate representation components (13) are generally not isotopic spin eigenstates.

The expressions (14) and (15) were used for $n=2$, 3, and 4. For $n=5$, 6 (and 7), we use a representation composed of the states

$$[\Psi_{1\cdots n}(j^{-r}vL)\psi_p(j)]_M{}^I \quad (r=8-n). \tag{17}$$

According to (9), the neutron-neutron interaction matrix elements are the same as those for the r-neutron case, apart from a multiple of the unit matrix. The proton-neutron interaction matrix elements are obtained by using the expression (15) with $n=r$, but replacing the E_J in (15) by the F_J given by (12).

The energy matrix calculated with respect to the particle states (13) is the same as that calculated with respect to the hole states

$$[\Psi_{1\cdots 8-n}(j^{-n}vL)\Psi_{1\cdots 7}(j^{-1}1j)]_M{}^I. \tag{13'}$$

The eigenvectors of these two matrices are therefore the same, so that

$$\Psi_M{}'^{aI}=\sum_{vL} D^{aI}(vL)[\Psi_{1\cdots 8-n}(j^{-n}vL)\Psi_{1\cdots 7}(j^{-1}1j)]_M{}^I \tag{16'}$$

would be an eigenstate of a nucleus with n proton holes and one neutron hole. In this way our scandium eigenvectors also provide us with eigenvectors for nuclei with $N=27$ and $20<Z<28$.

C. Eigenvalues and Eigenvectors of the Titanium Isotopes and $N=26$ Isotones

Here our representation is taken to be

$$[\Psi_{1,2}(j^2L_p)\Psi_{1,\ldots,n}(j^n v_n L_n)]_M{}^I. \tag{18}$$

The matrix elements of the proton-proton and neutron-neutron interactions are

$$\langle[\Psi_{1,2}(j^2L_p)\Psi_{1,\ldots,n}(j^n v_n L_n)]_M{}^I|W_{12}{}^{pp}+\sum_{1\leqslant i<j}^{n} W_{ij}{}^{nn}|[\Psi_{1,2}(j^2L_p{}')\Psi_{1,\ldots,n}(j^n v_n{}'L_n{}')]_M{}^I\rangle$$
$$=\delta_{L_pL_p{}'}\delta_{v_nv_n{}'}\delta_{L_nL_n{}'}\left\{E_{L_p}+\sum_{\text{even }J}\left[\frac{n(n-1)}{2}\sum_{vK}(j^{n-2}vK;\,j^2J\,]\!\}j^nv_nL_n)^2\right]E_J\right\}, \tag{19}$$

whereas for the proton-neutron interaction we get

$$\langle[\Psi_{12}(j^2L_p)\Psi_{1\cdots n}(j^nv_nL_n)]_M{}^I|\sum_{\substack{i=1,2\\ j=1,\ldots,n}} W_{ij}{}^{pn}|[\Psi_{12}(j^2L_p{}')\Psi_{1\cdots n}(j^nv_n{}'L_n{}')]_M{}^I\rangle$$
$$=(-1)^{L_n-L_n{}'}\sum_J\left[2n\sum_{\substack{vK\\A}}(j^{n-1}vK;\,j\,]\!\}j^nv_nL_n)(j^{n-1}vK;\,j\,]\!\}j^nv_n{}'L_n{}')U(jjIL_n;\,L_pA)U(jjIL_n{}';\,L_p{}'A)\times U(KjAj;\,L_nJ)U(KjAj;\,L_n{}'J)\right]E_J. \tag{20}$$

Here the components of the normalized eigenvectors are labeled $D^{a,I}(L_p,v_nL_n)$, so that

$$\Psi_M{}^{aI}=\sum_{L_p,v_nL_n} D^{a,I}(L_p,v_nL_n)[\Psi_{12}(j^2L_p)\Psi_{1\cdots n}(j^nv_nL_n]_M{}^I. \tag{21}$$

Equation (20) was used for $n\leq 4$. In the same fashion as in Sec. IIB, the [proton]-[neutron hole] representation was used for $n>4$, and the neutron-proton matrix elements were calculated by replacing the E_J of (20) by the

F_J of (13). The eigenstates of the $N=26$ isotones are given by

$$\Psi_M{}'^{aI}=\sum_{L_n,v_pL_p} D^{a,I}(L_n,v_pL_p)[\Psi_{1\cdots6}(j^{-2}L_n)\Psi_{1\cdots8-n}(j^{-n}v_pL_p)]\Psi^I. \tag{22}$$

The numerical evaluation of the expressions given above, and the subsequent diagonalization of the matrices, was performed with an IBM-7090 computer.

D. Spectroscopic Factors for Nucleon-Transfer Reactions

We use the customary definition for single-particle spectroscopic factors. For instance, if we have Z protons and N neutrons in the $1f_{7/2}$ shell, and if $\Psi_M{}^I(p_1\cdots p_Z;\, n_1\cdots n_N)$ and $\Phi_M{}^J(p_1\cdots p_Z;\, n_1\cdots n_{N-1})$ are wave functions of the states connected by the reaction then the spectroscopic factor for the transfer of an $1f_{7/2}$ neutron is

$$S=N[\langle\Psi_M{}^I(p_1\cdots p_Z;\, n_1\cdots n_N)|[\Phi^J(p_1\cdots p_Z;\, n_1\cdots n_{N-1})\psi^J(n_N)]_M{}^I\rangle]^2. \tag{23}$$

We illustrate the method of calculating (p,d) and (d,p) spectroscopic factors by considering the reactions $Sc^{45}(p,d)Sc^{44}$ and $Sc^{45}(d,p)Sc^{46}$.

If these nuclei consist of closed shells plus neutrons and a proton in the $1f_{7/2}$ shell, the differential cross section for the reaction $Sc^{45}(p,d)Sc^{44}$ can be written

$$d\sigma/d\Omega=\tfrac{3}{2}S\sigma_{f_{7/2}}(\theta) \qquad (p,d). \tag{24}$$

The quantity $\sigma_{f_{7/2}}(\theta)$ depends upon the theory of the reaction mechanism, which is described in the technical report of Bassel *et al.*,[7] or the monograph of Tobocman.[8]

We regard $\sigma_{f_{7/2}}(\theta)$ as a known function of its arguments, and use our wave functions to calculate S. To avoid confusion between the eigenvector components of Sc^{44} and Sc^{45}, we label the former by $C^{aI}(vL)$. As a result of the matrix diagonalization discussed in the previous sections, we have

$$\Psi_{123,p}(Sc^{44},aIM)=\sum_{vL} C^{aI}(vL)[\Psi_{123}(j^3vL)\psi_p(j)]_M{}^I,$$

$$\begin{aligned}\Psi_{1234,p}(Sc^{45},1jM)&=\sum_{v'L'} D^{1j}(v'L')[\Psi_{1234}(j^4v'L')\psi_p(j)]_M{}^j\\
&=\sum_{v'L',vL} D^{1j}(v'L')(j^3vL;j]\!]j^4v'L')[\{\Psi_{123}(j^3vL)\psi_4(j)\}^{L'}\psi_p(j)]_M{}^j\\
&=\sum_{v'L',vLI}(-1)^{j+L+L'-I}D^{1j}(v'L')(j^3vL;j]\!]j^4v'L')U(jLjj;IL')\{[\Psi_{123}(j^3vL)\psi_p(j)]^I\psi_4(j)\}_M{}^j\\
&=\sum_{a,I}\left[\sum_{v'L',vL}(-1)^{j+L+L'-I}D^{1j}(v'L')C^{aI}(vL)(j^3vL;j]\!]j^4v'L')\times U(jLjj;IL')\right][\Psi_{123,p}(Sc^{44},aI)\psi_4(j)]_M{}^j.\end{aligned}$$

Thus

$$S=4[\sum_{v'L',vL}(-1)^{j+L+L'-I}D^{1j}(v'L')C^{aI}(vL)(j^3vL;j]\!]j^4v'L')U(jLjj;IL')]^2. \tag{25}$$

The above refers to the pickup of a neutron. We can regard the stripping reaction as the pickup of a neutron hole. If we use in (24) coefficients D and C obtained with the j^{-4} and j^{-3} neutron wave functions, we are effectively calculating the spectroscopic factor for the pickup of a neutron hole from the ground state of Sc^{45}. This quantity is the $(2I+1)/8$ which appears in the relation[7]

$$d\sigma/d\Omega=[(2I+1)/8]S\sigma_{f_{7/2}}(\theta) \qquad (d,p). \tag{24'}$$

Proton spectroscopic factors are calculated in a similar manner.

E. Magnetic Moments

Using our assumption of the pure $1f_{7/2}$ configuration, we can write the z component of the magnetic moment operator as

$$\mu_z=g_pJ_z{}^p+g_nJ_z{}^N, \tag{26}$$

[7] R. H. Bassel, R. M. Drisko, and G. R. Satchler, Oak Ridge National Laboratory Technical Report ORNL-3240, 1962 (unpublished).

[8] W. Tobocman, *Theory of Direct Nuclear Reactions* (Oxford University Press, London, 1961).

where J_z^p and J_z^N are the z components of the proton and neutron angular-momentum operators, and g_p and g_n are the corresponding gyromagnetic ratios. The magnetic moment associated with the state (16) is then

$$\langle\Psi_I^{aI}|\mu_z|\Psi_I^{aI}\rangle=\frac{g_p+g_n}{2}I+\frac{g_p-g_n}{2}\sum_{vL}[D^{aI}(vL)]^2\frac{j(j+1)-L(L+1)}{I+1}, \tag{27}$$

whereas the corresponding expression for the state (21) is

$$\langle\Psi_I^{aI}|\mu_z|\Psi_I^{aI}\rangle=\frac{g_p+g_n}{2}I+\frac{g_p-g_n}{2}\sum_{L_p,v_nL_n}[D^{aI}(L_p,v_nL_n)]^2\frac{L_p(L_p+1)-L_n(L_n+1)}{I+1}. \tag{28}$$

F. Logft Values

The beta transitions we will discuss are allowed Gamow-Teller decays. It will be convenient to express their calculated logft values in terms of the Gamow-Teller decay $Sc^{42}{}_{I=7}\rightarrow Ca^{42}{}_{I=6}(\log ft=4.19)$[9]

$$[\log ft]_{\Psi_i\to\Psi_f}=4.19-\log\frac{\sum_{M_f,q}|\langle\Psi_f(I_fM_f)|\sum_\alpha\sigma_q(\alpha)t_+(\alpha)|\Psi_i(I_iM_i)\rangle|^2}{\sum_{M_f,q}|\langle\Psi(j^2,I=6,M_f)|\sum_\alpha\sigma_q(\alpha)t_+(\alpha)|\Psi(j^2,I=7,M_i)\rangle|^2}. \tag{29}$$

We must express our states (16) and (21) in the formalism of isotopic spin in order to evaluate matrix elements of t_+. If $\chi_+(\alpha)$ and $\chi_-(\alpha)$ represent neutron and proton states, respectively, then (16) and (21) become

$$\Psi_M{}^{aI}=\frac{1}{[n+1]^{1/2}}\sum_{vL}D^{aI}(vL)\mathcal{A}[\Psi_{\alpha\beta\cdots\gamma}(j^nvL)\psi_\delta(j)]_M{}^I\chi_+(\alpha)\chi_+(\beta)\cdots\chi_+(\gamma)\chi_-(\delta), \tag{16$''$}$$

$$\Psi_M{}^{aI}=\left[\frac{2}{(n+2)(n+1)}\right]^{1/2}\sum_{L_p,v_nL_n}D^{aI}(L_p,v_nL_n)\mathcal{A}[\Psi_{\alpha\beta}(j^2L_p)\Psi_{\gamma\cdots\delta}(j^nv_nL_n)]_M{}^I\chi_-(\alpha)\chi_-(\beta)\chi_+(\gamma)\cdots\chi_+(\delta). \tag{21$''$}$$

The symbol $\mathcal{A}$ implies antisymmetrization.

For the scandium-calcium decays, $\Psi_i(I_iM_i)$ of (29) is given by (16$''$), setting $a=1$ and I equal to the ground-state angular momentum. $\Psi_f(I_fM_f)$ of (29) will be simply

$$\Psi_f(I_fM_f)=\sum_{vL}D^{a_fI_f}(vL)[\Psi_{1\cdots n}(j^nvL)\psi_{n+1}(j)]_{M_f}{}^{I_f}\chi_+(1)\cdots\chi_+(n+1), \tag{30}$$

where the $D^{a_fI_f}(vL)$ are the components of an eigenvector with $T=T_z=\frac{1}{2}(n+1)$. Here it is not necessary to apply the operator $\mathcal{A}$ as (30) is automatically antisymmetric in all the particles. Then

$$\begin{aligned}\sum_{M_fq}&|\langle\Psi_f(I_fM_f)|\sum_\alpha\sigma_q(\alpha)t_+(\alpha)|\Psi_i(I_iM_i)\rangle|^2\\ &=\sum_{M_fq}|(n+1)\langle\Psi_f(I_fM_f)|\sigma_q(n+1)t_+(n+1)|\Psi_i(I_iM_i)\rangle|^2\\ &=(n+1)\sum_{M_fq}|\sum_{vL}D^{a_fI_f}(vL)D^{1I_i}(vL)\langle[\Psi_{1\cdots n}(j^nvL)\psi_{n+1}(j)]_{M_f}{}^{I_f}|\sigma_q(n+1)|[\Psi_{1\cdots n}(j^nvL)\psi_{n+1}(j)]_{M_i}{}^{I_i}\rangle|^2\\ &=(n+1)\frac{2I_f+1}{2I_i+1}\langle\psi(j)|[\sigma^1\psi^j]^j\rangle^2[\sum_{vL}D^{a_fI_f}(vL)D^{1I_i}(vL)U(1jI_fL;jI_i)]^2.\end{aligned} \tag{31}$$

For the reference Sc^{42} beta decay, $I_i=7$, $I_f=6$, $v=1$, $L=7/2$, $D^{a_fI_f}=D^{1I_i}=1$, so that (29) becomes

$$\begin{aligned}[\log ft]_{\Psi_i\to\Psi_f}&=4.19-\log\left[\frac{(n+1)(2I_f+1)[\sum_{vL}D^{a_fI_f}(vL)D^{1I_i}(vL)U(1jI_fL;jI_i)]^2}{(2I_i+1)2(13/15)U^2(1j6j;j7)}\right]\\ &=3.54-\log\left[(n+1)\frac{2I_f+1}{2I_i+1}[\sum_{vL}D^{a_fI_f}(vL)D^{1I_i}(vL)U(1jI_fL;jI_i)]^2\right]\end{aligned} \tag{32}$$

[9] P. C. Rogers and G. E. Gordon, Phys. Rev. **129**, 2653 (1963).

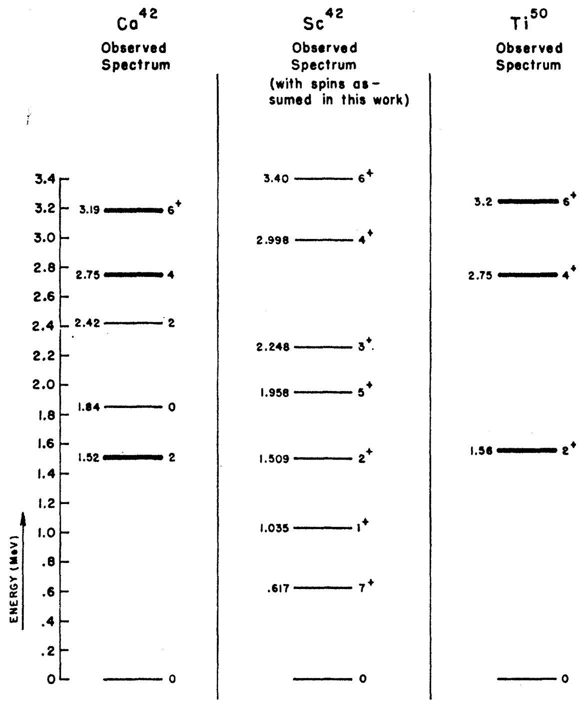

FIG. 1. The experimental spectra of $(f_{7/2})^2$ nuclei. The Sc^{42} energies are those determined by Sarma (Ref. 10). Spin assignments to levels in Sc^{42} are discussed in text.

expressing $\log ft$ in terms of the calculated eigenvector components. An analogous calculation is performed for the titanium decays.

G. Choice of Two-Body Energies

The two-body matrix elements of the residual interaction used for the matrix elements of Eqs. (14), (15), (19), and (20) were taken from the recent $Ca^{40}(He^3,p)Sc^{42}$ measurement.[10] The spectrum is shown in Fig. 1. The spins assigned to the energy levels have not been measured, but are inferred for the purposes of calculation. As far as possible, the criterion of a fit to the spectra of the many-particle isotopes is ignored in the spin assignments; the inferences are made on the basis of agreement with the assumption of isotopic spin conservation and consistency of the calculations with Coulomb energy differences between isotopic analog states.

The $T=1$ spectrum of two $f_{7/2}$ particles should be observable in three isotopes: Ca^{42}, Ti^{50}, and Sc^{42}. The $T=1$ levels will have only even spin, which means that in these three isotopes there should be identical 0, 2, 4, 6 level sequences, apart from Coulomb effects (see Sec. VI). The spectra are shown in Fig. 1; it is apparent that the expectation is approximately fulfilled. The levels in Sc^{42} deviate somewhat from the Ca^{42} and Ti^{50} spectrum, but the agreement is good enough to justify maintaining the assumption of isotopic spin conservation, and in fact it is on the basis of this agreement that the spin assignments are made. Neglecting this observed energy discrepancy in the input matrix elements, of course, sets an automatic limit of about 250 keV on the reliability of the calculated results. Using the observed values for the neutron-neutron and proton-proton residual interactions would not appreciably alter the results, and would destroy the isotopic spin symmetry, which proves to be useful.

The only ambiguity for even spin levels is the position of the 6^+ state in Sc^{42}. There are a number of experimentally observed states at about the right energy, with no positive indication which is the correct one. The choice of 3.4 MeV for the energy of the state resulted from the best agreement with the Coulomb energy arguments to be discussed in Sec. VI. Of the odd-spin levels, only the lowest has an unambiguous assignment. The 0.617-MeV level has been shown[9] to be isomeric and to decay by beta emission to the 6^+ state in Ca^{42}. It is undoubtedly the spin-7 level. The other three cannot be deduced from experiment, and must be inferred from other grounds. It was for this reason that the requirement of consistency with measured Coulomb energies was invoked.

The result of the analysis described in Sec. VI was to force the spin-1 level as low in energy as possible. It was for this reason that the 1.035-MeV state was assumed to be the proper spin-1 state, in spite of its weak excitation in the (He^3,p) experiments. The assignment of the other two spins was less critical, because of the similarity in energy between the remaining states, but the assignment shown in Fig. 1 gave the most consistent values for the Coulomb splitting.

III. DISCUSSION OF THE WAVE FUNCTIONS

The calculated ground-state eigenvectors are presented in Table AII, Appendix II. The first point to note is that the neutron-proton interaction is evidently strong enough to invalidate a simple description of the nuclear states as a single-proton state coupled to a single-neutron state. Thus the proton and neutron angular momenta are not separately constants of the motion. In particular we would not expect an "odd-group model" to be valid for the odd-odd nuclei. However, in certain cases we have found remarkable (though accidental) agreement between magnetic moments calculated with our and the odd-group wave functions. For the ground state of V^{50}, the odd-group model assumes the wave function

$$[\Psi(j^3L_p=7/2)\Psi(j^{-1}L_n=7/2)]_M{}^6, \qquad (33)$$

which implies the magnetic moment

$$\mu=3g_p+3g_n \quad \text{(odd group)}. \qquad (34)$$

Reference to Appendix II shows that our wave function has a 30% admixture of components other than (33),

[10] N. Sarma (private communication).

yet we find

$$\mu = 2.988 g_p + 3.012 g_n \quad \text{(our state)}. \tag{35}$$

In this example the magnetic moment is evidently very insensitive to (L_p, L_n) admixtures, illustrating the impossibility of inferring details of the wave function from isolated expectation values.

The wave functions of Ti^{44} (or Fe^{52}) can be divided into two classes, according to whether they are even or odd under the interchange of neutron and proton states. This is a consequence of the symmetry of the assumed interaction under this operation. Similarly, the wave functions of Ti^{48} are even or odd under the interchange of proton- and neutron-hole states. Apparently neither group of states is favored energetically over the other. The absence of coupling between the even and odd states has the consequence that pairs of very close eigenvalues can occur, for example $E=3.55$ (odd) and $E=3.485$ (even) are the energies of the two lowest $I=6$ states of Ti^{48} (see Fig. 14 later). Such nearly degenerate states might be expected to be especially sensitive to the effects of small admixtures of other configurations. A manifestation of the proton-(neutron hole) symmetry in Ti^{48} and Sc^{48} is that the Gamow-Teller β decay of Sc^{48} $(I=6)$ to the second $I=6$ level of Ti^{48} is absolutely forbidden according to our calculation.

An interesting consequence of Eqs. (19) and (20) is the equality

$$\langle [\Psi(j^2L)\Psi(j^2L)]_0{}^0 | \sum_{1\leqslant i<j}^{4} W_{ij} | [\Psi(j^2L')\Psi(j^2L')]_0{}^0 \rangle$$

$$= 2\langle [\Psi(j^2L)\psi(j)]_{m}{}^{j} | \sum_{1\leqslant i<j}^{3} W_{ij} | [\Psi(j^2L)\psi(j)]_{m}{}^{j} \rangle \tag{36}$$

for any W_{ij}. Since the $I=0$ matrix for Ti^{44} is double the $I=j=7/2$ matrix for Ti^{43}, it follows that these matrices have the same eigenvectors (as can be verified in Table AII), and eigenvalues differing by a factor of 2. Thus,

$$\Psi(\mathrm{Ti}^{44}, a=1, I=0)$$
$$= \sum_L D^{1,0}(L,L)[\Psi_{p_1p_2}(j^2L)\Psi_{n_1n_2}(j^2L)]_0{}^0,$$
$$= \sum_L D^{1,0}(L,L)[\{\Psi_{p_1p_2}(j^2L)\psi_{n_1}(j)\}^j\psi_{n_2}(j)]_0{}^0, \tag{37}$$

$$\Psi(\mathrm{Ti}^{43}, a=1, I=j)$$
$$= \sum_L D^{1,0}(L,L)\{\Psi_{p_1p_2}(j^2L)\psi_{n_1}(j)\}_m{}^j,$$

so that

$$\Psi_{p_1p_2n_1n_2}(\mathrm{Ti}^{44}, a=1, I=0)$$
$$= [\Psi_{p_1p_2n_1}(\mathrm{Ti}^{43}, a=1, I=j)\psi_{n_2}(j)]_0{}^0. \tag{38}$$

According to (23) this implies that the spectroscopic factor for the ground-state transition in the reaction $Ti^{44}(p,d)Ti^{43}$ is 2, and no strength remains for excited states. Equations similar to (36), (37), and (38) hold for the nuclei Ti^{48} and Ti^{49}. The corresponding physical consequence is that the reaction $Ti^{48}(d,p)Ti^{49}$ should populate only the lowest $I=j=7/2$ state of Ti^{49}, again with a spectroscopic factor of 2.

IV. COMPARISON WITH EXPERIMENT

A. The Spectra of Single-Closed-Shell Nuclei

The spectroscopy discussed in this paper has already been applied to single-closed-shell nuclei in the $f_{7/2}$ shell[1,11] and in fact it was the relative success of these analyses which spurred the extension of these techniques to other isotopes in the shell. Talmi used in his analysis the energies seen in Ca^{42} and Ti^{50} to calculate the spectra of the three- and four-particle spectra of nuclei containing only identical particles, i.e., the calcium isotopes, and Cr^{52}, V^{51}, and Mn^{53}. The use of the Ca^{42} energies gives a somewhat better over-all fit to these nuclei than do the energies used in the present work, although the two agree to approximately within the 250 keV mentioned above. The results of the present calculation, along with the pertinent experimental results, are shown[12] in Figs. 2 and 3.[13] The most obvious failing of the calculation is its inability to account for all the observed states. Such extra states are seen in many nuclei in the shell; in some cases the close-lying extra-configuration states appear to give rise to a considerable energy perturbation, as in the $3/2^-$ level in Ca^{43}, or the 2^+ level in Ca^{44}, while in other cases they show no effect, as is apparently true in Ca^{42}. Considering only those states accounted for by the theory, the over-all agreement is fairly good, and is particularly impressive in Cr^{52} and V^{51}. The predictions above 1 MeV in the odd-even nuclei are consistently high. This is to be expected since, simply from level densities, one would expect a more general mixing of configurations at high excitations, tending to depress observed energies. The energy agreement in Mn^{53} may be fortuitous. The spins of the excited levels have not been measured, but there are indications from (n,d) reaction experiments[14] on Fe^{54} of some $l=1$ mixing in the first excited state, which seems to imply a $3/2^-$ assignment for that level. Also, if Fe^{53} has ground-state spin $7/2^-$, a beta-decay branch to the 1.27-MeV state in Mn^{53} argues for a $9/2^-$ assignment.

The spectrum of the four-particle nuclei Ca^{44} and Cr^{52} have been discussed in detail elsewhere,[11] and only a

[11] I. Talmi, Phys. Rev. **126**, 1096 (1962).

[12] Unless otherwise noted, this and all succeeding data for the studies reported are taken from K. Way *et al.* in *Landolt-Bornstein, New Series, Energy Levels of Nuclei*, edited by A. M. Hellwege and K. H. Hellwege (Springer-Verlag, Berlin, 1961), Vol. 1.

[13] Data for V^{51} is taken from H. W. Kendall and I. Talmi, Phys. Rev. **128**, 792 (1962) and J. Vervier, Phys. Letters **5**, 79 (1963); J. E. Schwager, Phys. Rev. **121**, 569 (1961); for Ca^{45}, from Ref. 27.

[14] G. Bassani, L. Colli, E. Gadioli, and I. Iori, Nucl. Phys. **36**, 471 (1962).

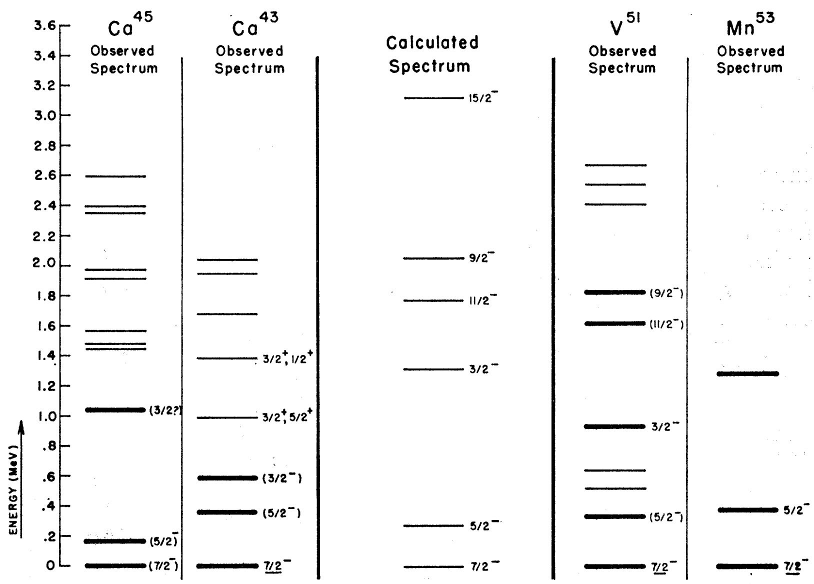

FIG. 2. Experimental energies of $(f_{7/2})^3$ and $(f_{7/2})^{-3}$ nuclei with one closed shell. The comparison with calculated values is complicated by the occurrence of extra-configurational levels. Levels assignable to the $(f_{7/2})^3$ or $(f_{7/2})^{-3}$ configuration are drawn heavier for ease of comparison.

few words need be said here. First, although the predicted energies for the first 4^+ states (at 2.369 MeV in Cr^{52}, and 2.28 MeV in Ca^{44})[15] are given reasonably well by the theory, there is some difficulty with the interpretation of these states in shell model terms. For configurations of identical particles, seniority is automatically a good quantum number, regardless of the form of the residual two-body interaction, and so the two 4^+ states in each of these two nuclei are characterized in the theory by a definite seniority. The 2.766 state in Cr^{52} is assigned seniority 2 and the 2.369 MeV, seniority 4. This assignment has been shown by Talmi to be consistent with the observed gamma-ray transitions, but it also implies that in (p,p') scattering and in the (p,α) reaction leading to levels in this nucleus, one of the two 4^+ levels will be excited preferentially. This, however, is not the case. In each of these experiments, the two spin 4^+ levels are excited with about the same probability.[16] Komoda[17] has recently shown this to be consistent with the effect of configuration mixing, which leaves the energies of the states almost unchanged, but which completely mixes the seniority wave functions.

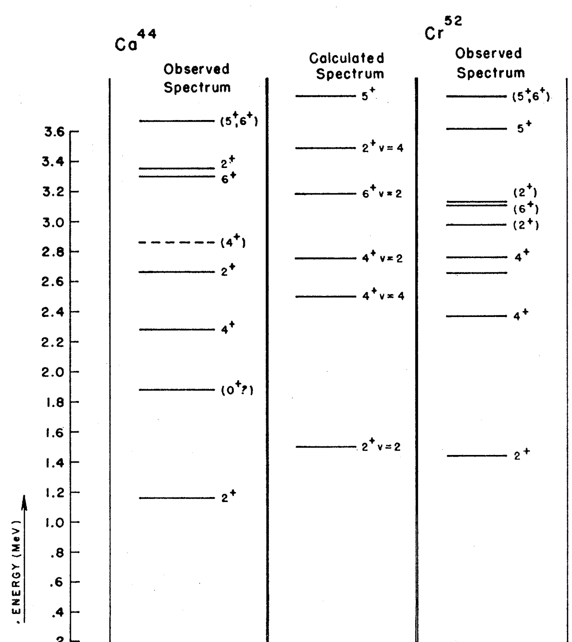

FIG. 3. Experimental energies of $(f_{7/2})^4$ nuclei with one closed shell. Data for Cr^{52} and Ca^{44} are taken from Ref. 15. Discussion of the levels with seniority values $v=4$ is found in text.

B. Odd-Even Nuclei with Neither Shell Closed

The calculation covers all the odd-even nuclei in the shell in which the odd configuration is a single particle or hole, or in which the even configuration is two particles or holes. Of the nuclei about which anything is known, this excludes only V^{47} and Cr^{49}. The nuclei with a single particle (hole) in the odd group are generally predicted to have a sizeable energy gap between ground- and first-excited states. In all such nuclei the spectra are expected to be quite similar, the complexity of the predicted spectra above the first excited state depending on the number of particles in the even shell. The $9/2^-$ and $11/2^-$ states are uniformly calculated to be rather low in energy, usually at or below the expected energy for the

[15] R. R. Wilson, A. A. Bartlett, J. J. Kraushaar, J. D. McCullen, and R. A. Ristinen, Phys. Rev. **125**, 1655 (1962); L. T. Dillman, J. J. Kraushaar, and J. D. McCullen, Nucl. Phys. **42**, 383 (1963).

[16] H. O. Funsten, N. R. Roberson, and E. Rost (to be published); B. F. Bayman, H. O. Funsten, N. R. Roberson, E. Rost, and R. Sherr, preliminary report by R. Sherr in *Proceedings of The Padua Conference on Direct Interactions and Nuclear Reaction Mechanisms* (Gordon and Breach, New York, 1963).

[17] T. Komoda (to be published).

$3/2^-$ and $5/2^-$ levels. This is quite different from the closed-shell prediction, in which the $5/2^-$ level is predicted to be very low lying.

When the odd group consists of three particles (holes) outside the closed shell, the predicted spectra have an extremely low-lying 5/2 state, akin to the three-particle spectra with one shell closed. It is not accurate to infer from this, however, that the even shell is inert and remains coupled to spin zero in the wave functions for the low-energy states. In Ti^{45}, for example, the amplitudes corresponding to the two protons coupled to spin 0 in the ground and first excited state wave functions are -0.6935 and 0.7261, respectively. And there are deviations from the predicted closed-shell spectra at higher excitations. Ti^{45} is predicted to have both $9/2^-$ and $11/2^-$ states below the first $3/2^-$ state, while the predicted Ca^{43} spectrum has these two above the $3/2^-$ state by 450 and 700 keV, respectively. In fact, the prediction for these nuclei are very comparable to the first class of unclosed-shell nuclei, except for the extreme lowering of the first $5/2^-$ state.

The theoretical predictions are less successful in the odd-even nuclei than in either the odd-odd or even-even. This is partly due to the mixing from the higher configurations, the effects of which are more pronounced in the odd-even isotopes; but partly it appears that there are other types of states, of uncertain origin, which are about the same energy as the $(f_{7/2})^n$ state.

Some of these are apparently positive parity states, analogous to the observed positive parity states of Ca^{43}. One explanation of such levels is that they are formed by the excitation of a $d_{3/2}$ hole, forming states $[d_{3/2}^{-1};(f_{7/2})^n I=0]$, where n is even. This is energetically reasonable. If this is the case, the intrusion of such states into the $(f_{7/2})^n$ spectrum will not perturb the rest of the spectrum.

The mixing of higher configurations $(p_{3/2},f_{5/2})$ in the odd-even spectra will have more serious results, particularly in view of the considerable energy gap predicted between ground and excited states. The next high single-particle level in these nuclei is expected to be the $p_{3/2}$, very close to 2 MeV above the 7/2 ground state (as seen in the Ca^{41} spectrum). The presence of this level should affect the energy of the first $3/2^-$ state (predicted at around 1.7 MeV in Cr^{51} and Sc^{45}), thus depressing the observed position of the state. The degree to which this effect occurs will depend on the relative position of the $(7/2)^n$ and $p_{3/2}$ levels. The best evidence indicates that the $f_{7/2}$-$p_{3/2}$ splitting decreases as the shell fills, which may lead to more serious mixing in the excited $3/2^-$ state in isotopes in the last part of the shell. This may be the explanation of the extremely low $3/2^-$ state seen in ${}_{23}V_{26}{}^{49}$.

The same mixing will affect the odd-odd and even-even nuclei, of course, but the effect on low-energy spectra is not so marked. In the odd-odd nuclei, most of the pertinent energy levels lie well below the 2 MeV $p_{3/2}$ state, and in the even-even nuclei, pairing energy arguments indicate that significant effects should not appear until about 3.5 MeV.

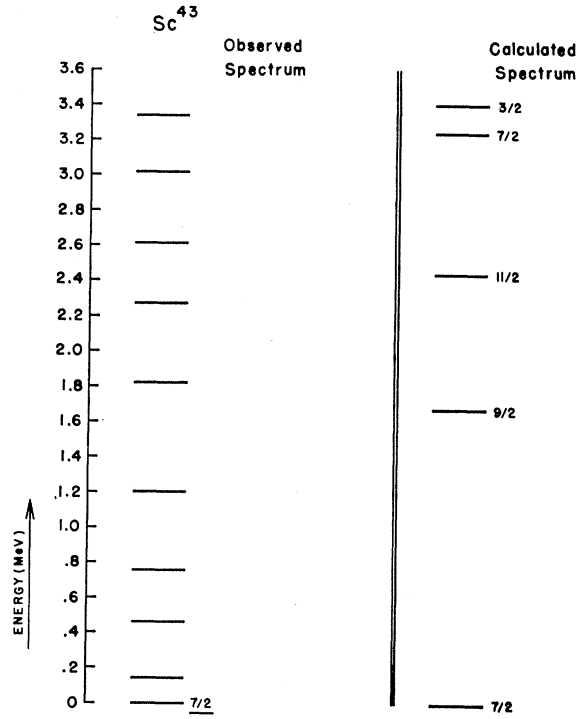

FIG. 4. Sc^{43} spectrum compared with calculation. Note the large number of apparent extra-configurational states. Energies of Sc^{43} taken from Ref. 18. The cross-conjugate nucleus to Sc^{43} is Fe^{53}, for which no data are available.

The low spin states in the odd-even nuclei will be most strongly affected by such mixing. At least at the beginning of the shell the high spin single-particle states, such as $f_{5/2}$ or $g_{9/2}$, lie considerably higher in energy, and will not significantly affect the states predicted here.

Sc^{43}

Little is known about the Sc^{43} spectrum. Several excited states are known through the (p,n) and (α,p) reactions[18] practically all of which are far too low to be consistent with the present calculation. (See Fig. 4.)

Sc^{45}

The pure $(f_{7/2})^5$ spectrum in Sc^{45} is predicted to have its first excited state at 1.47 MeV; (p,p') scattering experiments have located nine levels below this energy.[12] At least some of these are undoubtedly the positive parity levels mentioned above. Yntema[19] has evidence from (d,He^3) experiments of at least one very low-lying $l=2$ state, consistent with the discussion above. Above 1.5 MeV, any comparison of observed states to predicted

[18] G. J. McCallum, A. T. G. Ferguson, and G. S. Mani, Nucl. Phys. **17**, 116 (1960); N. O. Lassen and C. Larsen, Nucl. Phys. **43**, 183 (1963).
[19] J. L. Yntema (private communication).

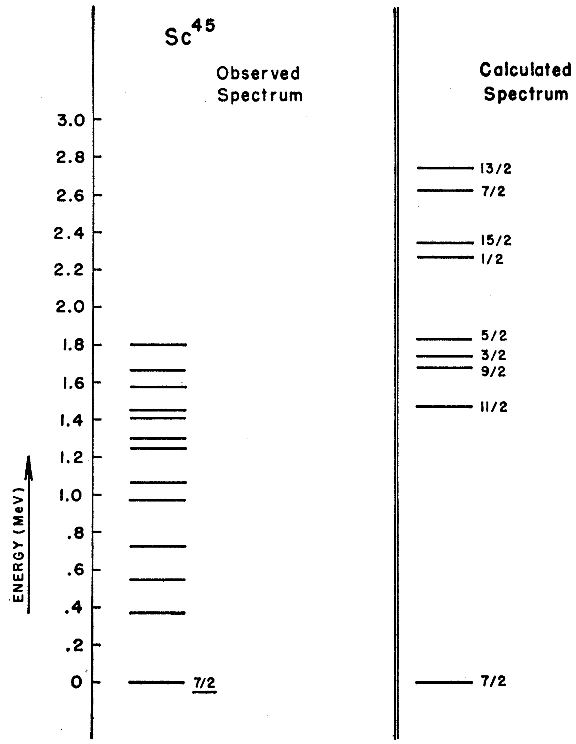

FIG. 5. Sc^{45} spectrum compared with calculation. Little further is known (see text).

states is hopeless at this time because of the density of observed states (see Fig. 5). Some of the states near 1.5 MeV may correspond to those calculated, but until at least parities are known, no comparison can be made. The ground-state wave function for Sc^{45} seems to be well represented. Evidence for this is the success of the wave function in predicting the magnetic moment of the nucleus, and in the results obtained in its use in beta-decay transition probabilities and stripping and pickup reactions to other nuclei.

Cr^{51}

The calculated spectrum agrees somewhat better with the observed spectrum of Cr^{51}, the cross conjugate of Sc^{45}. The anomalous low-energy states seen in Sc^{45} are missing here, and evidently have moved up in energy. Such an energy shift in Cr^{51} relative to the ground state lends credence to our assignment of them to another configuration, since the identity of spectra of cross-conjugate nuclei holds only for states with the $(f_{7/2})^n$ configuration. With the exception of the $3/2^-$ state at 0.76 MeV, the known levels of this nucleus are reasonably accurately reproduced by the calculation. The lowering of the $3/2^-$ level is apparently due to the $p_{3/2}$ mixing discussed above.

Thermal neutron capture and beta-decay studies have yielded information on the gamma-ray transitions between the excited levels in Cr^{51}; a summary of the information and an estimate of the relative transition probabilities has been given by Bauer *et al.*[20] Their conclusion is that the group of four levels between 1.17 and 1.57 MeV all decay to the ground state, but with a lower probability than that for cascade transitions between certain of the levels. There are three distinct cascade chains: one starting from the 1.57-MeV state, and proceeding through the 1.35-MeV state to the 0.76-MeV state, thence to the ground state (see Fig. 6). Competing with the first is the direct cascade from the 1.57-MeV level to that at 0.76 MeV. The third sequence starts from the 1.49-MeV level and cascades through the 1.17-MeV state, bypassing the 0.76-MeV level.

If the group of levels predicted by the theory between 1.47 and 1.82 MeV correspond to these experimental states, recognizing that the energy uncertainty does not permit the detailed level ordering in the group to be significant, the experimental transitions can be explained. The level at 0.76 MeV almost certainly is $3/2^-$. Hence, the $5/2^-$, $3/2^-$ pair in the group will decay by $M1$ transitions cascading through these levels, although a ground-state $E2$ transition will compete energetically. The other two members of the group are predicted to have higher spin, and will form a separate $M1$ cascade sequence to the ground state, again with a competing energetically favored $E2$. The spin ordering which seems most reasonable is shown in Fig. 6. There is a serious

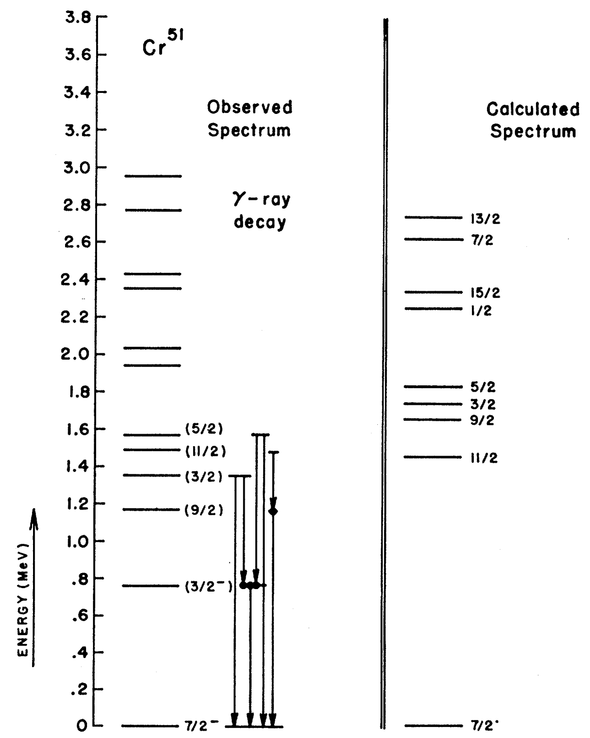

FIG. 6. Cr^{51} spectrum compared with calculation. The γ-ray summary is taken from Ref. 20. Spins are assigned to experimental levels in accord with discussion in text. This nucleus is cross conjugate with Sc^{45}.

[20] R. W. Bauer, J. D. Anderson, and L. J. Chrittensen, Phys. Rev. **130**, 312 (1963).

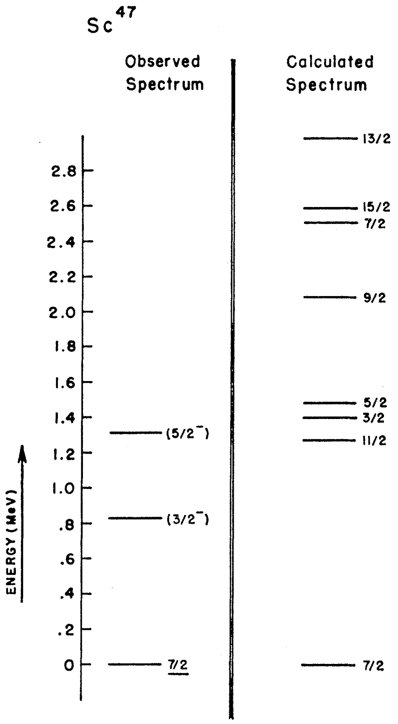

FIG. 7. Sc^{47} spectrum compared with calculation. The spin of the Sc^{47} ground state is measured to be 7/2 (Ref. b, Table I).

problem in this explanation, in that it is difficult to understand why a direct $M1$ transition from the presumed 5/2 level at 1.57 MeV to the ground state does not dominate the decay.

Because of the apparent $p_{3/2}$ mixing, one would expect that the 0.76-MeV state would be excited in a direct interaction such as (d,t) with an $l=1$ angular distribution. The level has indeed been seen,[21] but has been reported to have an angular distribution more consistent with $l=3$. This extremely hard to understand, since it implies that the spin of the state is 7/2 (or, remotely, 5/2), in serious disagreement with other more conclusive data.[20] Further investigation should clarify this point.

*Sc*47

Only two excited levels are known in the spectrum, with apparent spins $3/2^-$ and $5/2^-$. Their respective excitations are consistent with the theory, except that the $3/2^-$ seems to show the $p_{3/2}$ mixing effect discussed above. (See Fig. 7.[22])

*Ti*49

There are several puzzling features about the spectrum of this nucleus. It would appear that the calculations reproduce the observed levels remarkably well. There are several more levels observed around an excitation of 1.5–1.6 MeV than are consistent with the pure $f_{7/2}$ configuration, but other than that the agreement appears to be good. The trouble is that is is better than it should be. The observed state at 1.38 MeV has a rather large $l=1$ cross section in (d,p) stripping,[23] which, to be consistent with observations in the other odd-even nuclei, would lead us to expect another spin-3/2 state at a somewhat lower energy than that predicted. But this is not the case. Furthermore, the state at 1.72 MeV has a similarly large $l=1$ cross section, and has been assigned a spin 1/2. This is the only nucleus in the region in which an apparent $p_{1/2}$ level has been seen this low in energy. The admixture of $p_{1/2}$ amplitude to an $(f_{7/2})^n$ state of angular momentum 1/2 does not seem to be a satisfactory explanation; the lowest $(f_{7/2})^n$ state with $I=1/2$ is predicted to lie at 3.2 MeV, and to depress its energy to 1.72 MeV would require a very strong mixing interaction.

In spite of these confusing $l=1$ stripping observations, the $l=3$ states observed[24] in (p,d) reactions to Ti^{49}, although at much greater excitation (2.62 MeV) agrees almost exactly, both energetically and in relative cross section with the predicted 7/2 state. (See Fig. 8.)

It is interesting that the energy gap predicted by the theory is seen more clearly in Ti^{49} and Sc^{47} than in the lighter Sc^{43}, Sc^{45}, and perhaps Ti^{45}. This may be because Ti^{49} and Sc^{47} are close to the $Z=20$, $N=28$ shell closure, which, as has been pointed out before,[25] seems more tightly bound than the Ca^{40} closure.

*Ti*45

Ti^{45}, with three neutrons outside the closed shell, is predicted to have a spectrum with the ground and first excited states virtually degenerate in energy. This is in fact observed by high resolution (p,n) experiments. The ground state of this nucleus has been measured to have spin 7/2. Although the calculations predict a 5/2 ground state, the lowest 7/2 is predicted only 275 keV away. Except for this low-lying doublet, the calculated spectrum is rather like that for Cr^{51}. (See Fig. 9.)

Recent information obtained in the (p,d) reaction[24] has increased the total experimental knowledge of the spectrum, but has not appreciably clarified the situation. The only strongly excited state is the ground state (as is expected theoretically) and the strength of its excitation is consistent with the results on other titanium nuclei. All the other observed states show either flat or ambiguous angular distributions, so that identification of the configurations is not possible. Further, one state of the 0.23–0.33-MeV doublet appears to be isomeric,[26]

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[21] M. H. Macfarlane, B. J. Raz, J. L. Yntema, and B. Zeidman, Phys. Rev. **127**, 204 (1962).

[22] The position of the 0.83-MeV level is taken from P. C. Simms, N. Benczer-Koller, and C. S. Wu, Phys. Rev. **121**, 1169 (1961).

[23] L. H. Th. Rietjens, O. M. Bilaniuk, and M. H. Macfarlane, Phys. Rev. **120**, 527 (1960); J. L. Yntema, Phys. Rev. **131**, 811 (1963).

[24] E. Kashy and T. W. Conlon (private communication).

[25] R. H. Nussbaum, Rev. Mod. Phys. **28**, 423 (1956).

[26] A. M. Morozov, V. V. Remaev, and P. A. Yampol'ski, Zh. Eksperim. i Teor. Fiz. **39**, 973 (1960) [English transl.: Soviet Phys.—JETP **12**, 674 (1961)].

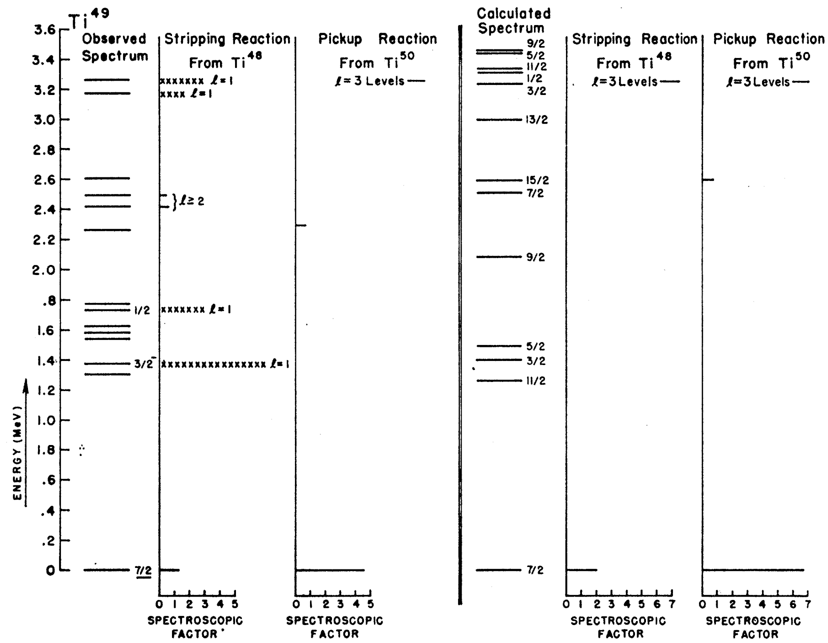

FIG. 8. Ti^{49} spectrum compared with calculation. This nucleus is cross-conjugate with Sc^{47}. Also compared are the observed relative cross sections for $Ti^{48}(d,p)Ti^{49}$ (Ref. 21) and preliminary value for absolute spectroscopic factors for $Ti^{50}(p,d)Ti^{49}$ (Ref. 24). Only the spectroscopic factor for the ground state is absolute, although the $l=1$ levels are approximately correct.

which is inexplicable in terms of the pure configuration, and probably is due to the positive parity of the state.

Ti^{47}

The drastic differences between the spectrum of Ti^{49}, where the first excited state lies 1.31 MeV above the ground state, and Ti^{47}, which has a low-energy doublet with a ground-state spin of 5/2, is one of the most striking experimental pecularities seen in the odd-even isotopes in this region, and is well reproduced by the theory, which predicts the ground- and first-excited states of Ti^{47} to be almost degenerate. Also worthy of note is the reasonably good agreement for the second excited state.[23,24] This is as low as the theory ever predicts the $3/2^-$ state, and Ti^{47} and its hole-hole conjugate V^{49} have the two lowest second-excited states in the shell. (See Fig. 10.)

It is apparent from (p,d) and (d,p) reactions[23,24,27] to levels in Ti^{47} that the lowering of the $5/2^-$ state is due primarily to the residual interactions between $f_{7/2}$ particles. In both reactions, only states of spin 7/2

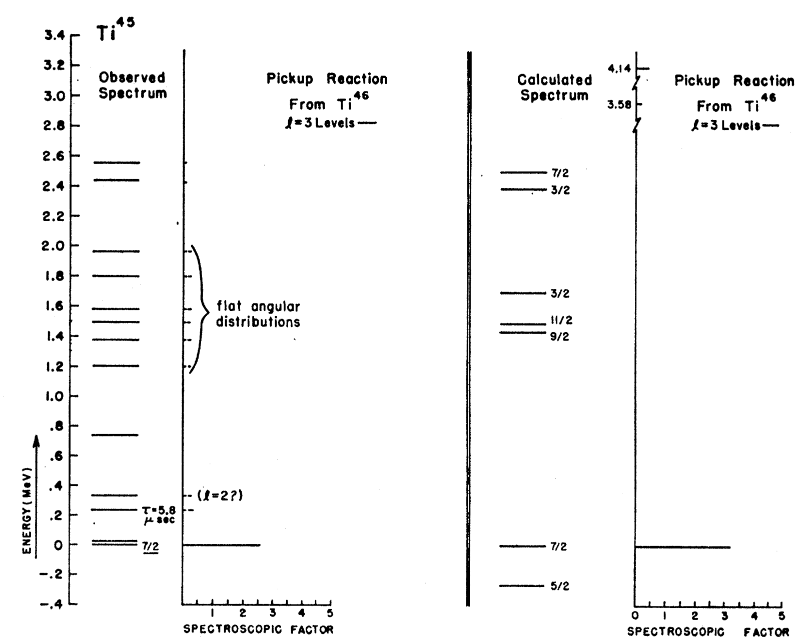

FIG. 9. Ti^{45} spectrum compared to calculation. Preliminary $Ti^{46}(p,d)$ data (Ref. 24) are also compared, and show no low-lying $l=3$ states besides the ground state. The 0.025-MeV level is probably the $5/2^-$ expected state. The cross-conjugate nucleus is Mn^{51}, for which no data are presently available.

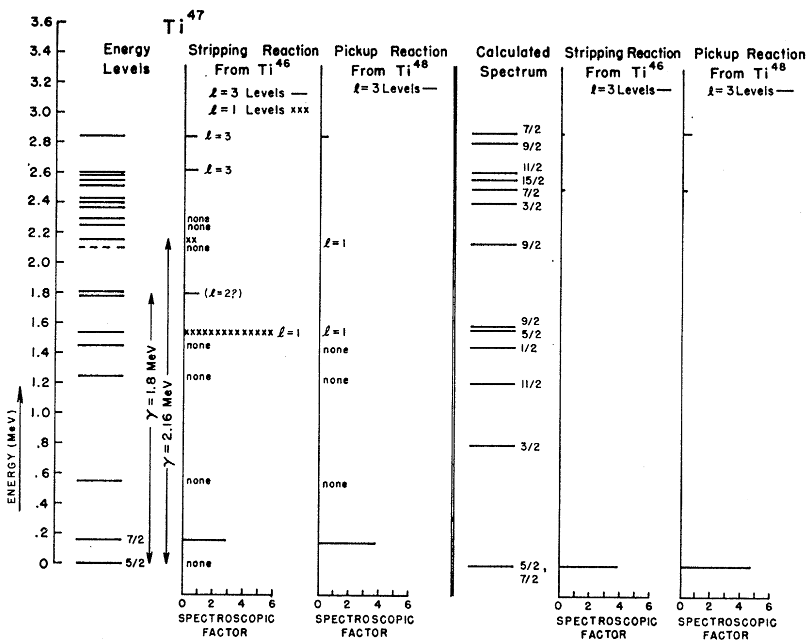

FIG. 10. Ti^{47} spectrum compared to calculation. Preliminary $Ti^{48}(p,d)Ti^{47}$ data (Ref. 23), and recent $Ti^{46}(d,p)$ results (Ref. 27) are also shown. The (d,p) ground state and (p,d) results are expressed in absolute spectroscopic factors (see Ref. 23).

should be excited in the present model, and indeed the 160-keV state is practically the only one seen below 1 MeV. Above this energy, $l=1$ states corresponding to the familiar $p_{3/2}$ level are seen. Apparently the usual mixing with the first $3/2^-$ level is inhibited because of its low position in the pure $(f_{7/2})^n$ scheme. At 2.60 and 2.84 MeV, two other $l=3$ levels are observed[27] in (d,p) with small cross section, remarkably close to the predicted 7/2 levels, but with somewhat larger cross section than expected.

V^{49}

As in its hole-hole conjugate Ti^{47}, the spectrum of V^{49} shows extremely low $5/2^-$ and $3/2^-$ states. The $3/2^-$ state is rather low to be consistent with the theory. This may be a depression due to configuration mixing. Only the energy of levels above these three low states are known, and the level density is such that no conclusions can be drawn. (See Fig. 11.)

Even-Even Nuclei

There are only three even-even nuclei in the shell about which anything is known, and for which neither shell is closed. One of these, Ti^{48}, is its own hole-hole conjugate, and is hence of more interest, because of the possible odd-even isotopic spin effects discussed in Sec. III.

In general, the prediction of the spectrum of an even-even nucleus is not a particularly impressive feat for a theory, assuming that the theory is sufficiently sophisticated to be able to generate an appreciable number of low-energy states. In all such calculations it is possible to fit many of the observed states, and success often hinges on assigning appropriate spins to unmeasured states, or on the prediction of a level which is not experimentally seen. The energy fits and spin assignments of these calculations are no exception to this general rule. Of more interest are the cross sections and predicted l values of stripping and pickup reactions; in this regard the present model has considerable success.

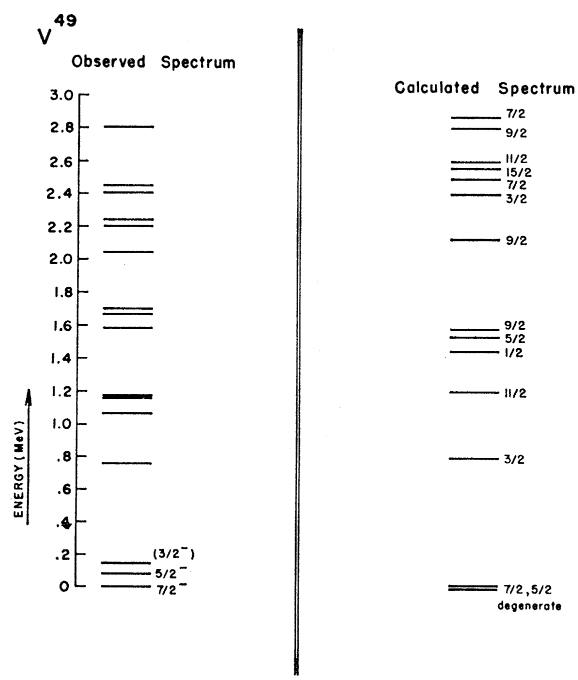

FIG. 11. V^{49} spectrum compared with calculation. Only energy levels are well known.

[27] J. Rapaport, MIT thesis (unpublished).

Ti^{46} *and Cr*^{50}

Until recently, the information on the levels of these two nuclei was too meager to make any sort of statement about the character of the states. Ti^{46} had been studied only through (p,p') reactions, and Cr^{50} states had been observed in the beta decay of Mn^{50}, but states were surely missing in both nuclei. Recently, however, (d,t) and (p,d) pickup reactions[28,24] have been employed with considerable success to study the excited states of Ti^{46}. The advantage of these reactions is that they provide considerably more information than merely the energy of the states, and actually measure the degree to which the final state is a parent of the target nucleus' ground state.

Low resolution (d,t) reactions[23] have given the initial information in these studies; more recently a series of high-resolution (p,d) experiments at Princeton[24] have been initiated to provide an exhaustive study of the low-excited states of Ti^{46}. A preliminary energy spectrum resulting from the latter is shown in Fig. 12, and the Cr^{50} is shown in Fig. 13.

Ti^{48}

The nucleus Ti^{48} is its own cross conjugate, and as such is predicted to display the odd-even classification of states discussed in Sec. III. The most interesting consequence of this feature of the Ti^{48} calculation is the prediction that there will occur two 6^+ states at about 3.4 MeV, separated by 150 keV. There is experimental evidence that this doubling actually occurs. The beta decay from the Sc^{48} ground state has been recently observed[29] to branch to two states 170 keV apart, with the highest energy state subsequently decaying by a cascade through a state with measured spin 6. This is what one would expect to see if the theory is correct; the $M1$ transition between the two predicted 6^+ states is allowed and will dominate the decay from the higher energy state at the expense of a slower $E2$ crossover to the 4^+ state at 2.29 MeV. (See Fig. 14.)

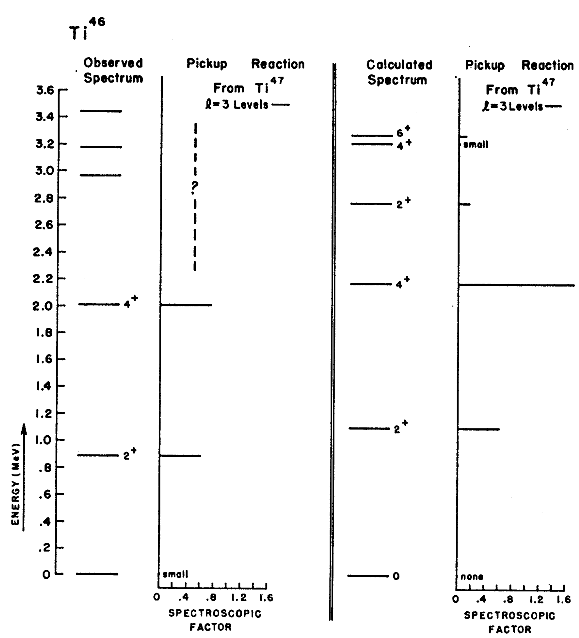

FIG. 12. Ti^{46} spectrum compared with calculation. Most data are from (p,p'), so that high spin levels may be missing. Preliminary $Ti^{47}(p,d)$ data (Ref. 24) are shown, expressed in absolute spectroscopic factors.

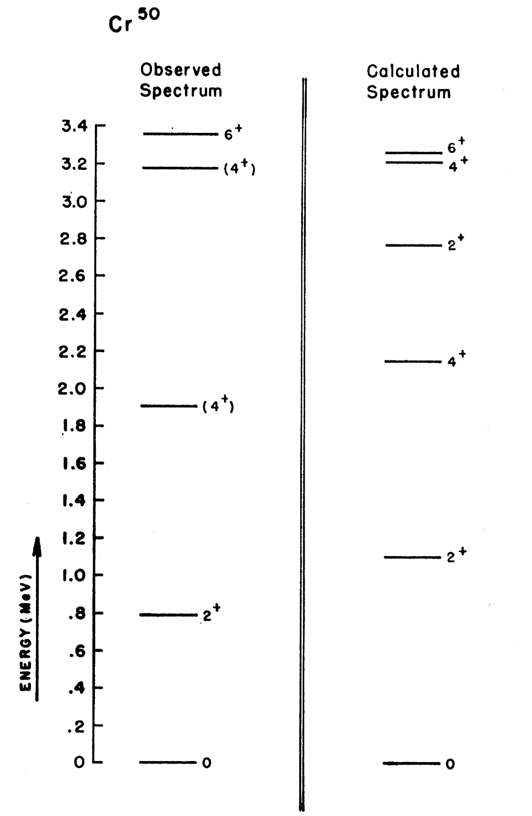

FIG. 13. Cr^{50} spectrum compared with calculation. Observed levels are populated in beta decay, so that high-lying low-spin levels may be missing.

The chief failure of the theory in the even-even isotopes is the predicted 3^+ level at 3.01 MeV in Ti^{48}, for which there is no experimental evidence. It may be that the actual 3^+ state lies higher in energy than is predicted, and the reason it has not been observed is that it is degenerate with some of the higher energy states. This explanation is unsatisfying, however, for it is hard to reconcile with the fact that if the state exists it should be populated in the beta decay from V^{48}, with a subsequent strong gamma ray to the 0.981-MeV 2^+ state.

The stripping and pickup reactions to levels in Ti^{48} are rapidly clarifying the nature of the excited states. The most detailed of these are the high resolution $Ti^{49}(p,d)Ti^{48}$ investigation of Kashy,[24] which show

[28] J. L. Yntema, Phys. Rev. **127**, 1659 (1962).

[29] M. Hillman, Phys. Rev. **129**, 2227 (1963).

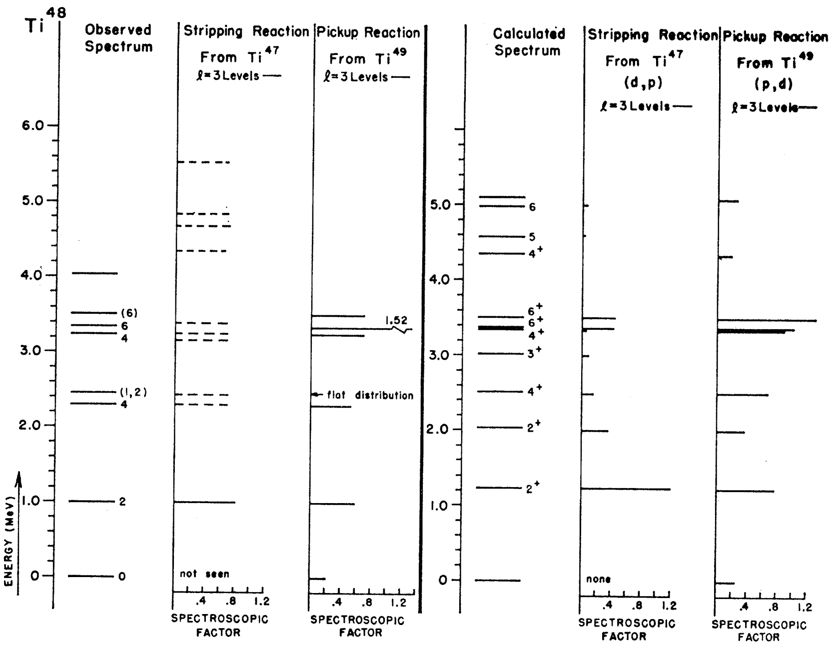

FIG. 14. Ti^{48} spectrum compared with calculation. (d,p) and (p,d) (Ref. 24) reaction data are also shown. (p,d) data are expressed in absolute spectroscopic factors. (d,t) reaction data to levels in this and other Ti nuclei are also available (Ref. 28), but the spectroscopic information is not as detailed. Three calculated levels between 3.0 and 5.0 MeV with negligible cross section have been omitted from the spectrum.

reasonable agreement with theoretical expectations up to about 4 MeV. The missing 3^+ state cannot be found by such experiments, however, because of its low expected cross section. There are a number of points of disagreement; the 2.42-MeV level should have an $l=3$ angular distribution, and instead is isotropic, and the two apparent 6^+ states, although qualitatively correctly described, are observed to have somewhat unbalanced spectroscopic factors. This may be an instance of the phenomenon discussed in Sec. III, namely, that if the energies predicted on the pure $f_{7/2}$ model are very close, the corresponding states are unstable with respect to small admixtures of other configurations.

Odd-Odd Nuclei

We are restricted in this calculation to odd-odd nuclei with one of the odd groups consisting of a single particle (hole) outside the closed shell. Thus, we cannot treat the interesting $N=Z$ cases V^{46} and Mn^{50}, but can study practically all the rest of the cases in the shell. As one might expect, the spectra of these isotopes are predicted to be fairly complicated, with many low-energy states. The general agreement turns out to be remarkably good.

Predictions of the spectra of odd-odd nuclei using the shell model are in general difficult to make because of the many uncertainties in the problem. The most common model used throughout the periodic table is that in which the neutrons and protons are treated as separate groups, with the low-energy states of the spectrum arising from a coupling of the groups via the neutron-proton interaction. When the interaction is weak, the neutron and proton angular momenta will be approximately good quantum numbers, and a general scheme proposed by Nordheim,[30] and extended by Brennan and Bernstein,[31] gives good results for the spins of the ground and possible isomeric states. But one would not expect this model to be applicable in the present studies and indeed the observed spins of these nuclei do not agree with the simple odd-group predictions.

There are a number of isomeric states observed in these nuclei. They seem to be of two types: Either they arise because of the close conjunction of two levels of the $f_{7/2}$ configuration, which have widely disparate spins, or they are due to levels of an apparently different configuration at about the same energy as the $f_{7/2}$ levels. We shall discuss the two types more fully for each individual nucleus.

Sc^{44}

The low-energy spectrum of Sc^{44} has recently been carefully studied using the $Sc^{45}(p,d)Sc^{44}$ reaction.[32] These results, coupled with previous spectroscopy experiments,[33] give a good picture of the spectrum up to 1.5 MeV. The agreement of the theory with present information is good (see Fig. 15). Notably missing from the theoretical predictions are the 70- and 140-keV states, but with these exceptions the correspondence between theory and experiment is of the order of the expected uncertainty in the calculated energies, and the relative cross sections in the (p,d) reaction are well reproduced.

[30] L. W. Nordheim, Rev. Mod. Phys. **23**, 322 (1951).
[31] M. H. Brennan and A. M. Bernstein, Phys. Rev. **120**, 927 (1960).
[32] E. Kashy, Phys. Rev. **134**, B378 (1964).
[33] J. K. Kliwer, J. J. Kraushaar, R. A. Ristinen, J. R. Keith, and A. A. Bartlett, Nucl. Phys. **50**, 328 (1963).

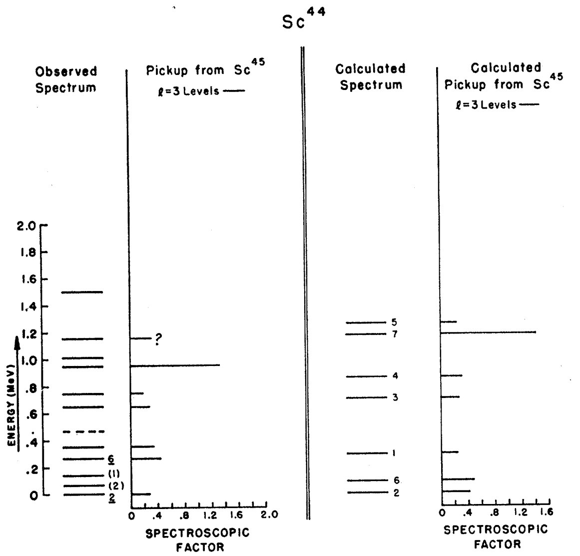

FIG. 15. Sc^{44} spectrum compared with calculation. The (p,d) reaction data are from Ref. 32 and are expressed in absolute spectroscopic factors. The first two excited states are both isomeric (Ref. 33).

It is interesting that the two low-energy states are not excited by the (p,d) experiments, indicating that their wave functions have little in common with the Sc^{45} ground state minus one neutron.

The experimental nature of these latter two states is somewhat peculiar. Their experimentally determined spins are either 1 or 2 (the best assignment is spin 2 to the 68-keV state, and 1 to the 140-keV state) and yet the gamma-ray decay from either of them to the spin-2 ground state is inhibited.[33] The 68-keV state has a lifetime of 0.18 μsec, and the 140-keV state decays exclusively by a cascade transition, rather than crossing over via an $M1$ transition to the ground state. This is a curious situation, because if either of these levels is spin 2, it is difficult to understand why sufficient mixing does not take place between it and the ground state to favor a completely allowed $M1$ transition. Thus, in Sc^{44}, there seem to be two separate isomeric phenomena; the 270-keV isomeric state is explainable in terms of the $(f_{7/2})^n$ configuration, but the 68- and 140-keV states apparently involve a more subtle inhibition in their transitions. The occurrence of the isomer in Ti^{45} may be another example of the same phenomenon.

Mn^{52}

The Mn^{52} spectrum is superficially similar to that for Sc^{44}; both have a low-energy 2–6 doublet, with a third state apparently of spin 1 nearby. In Mn^{52}, however, the order of the doublet is inverted. This would imply a displacement of the spin-2 state by about 650 keV relative to the spin-6 state in going from one nucleus to its conjugate. (See Fig. 16.)

Sc^{46}

The spectrum of Sc^{46} has been studied by several experimental techniques. One of the most accurate, and certainly the most fruitful in the spectroscopic analysis, has been the recent series of $Sc^{45}(d,p)$ experiments by Rapaport.[27] Of the 14 known levels below 1.2 MeV, these high-resolution experiments indicate that 9 show evidence for an $l=3$ angular distribution, characteristic of the $(f_{7/2})^n$ configuration. Of these 9 levels, the strongest is the state at 51 keV, which justifies its identification as the spin 6 level. (See Fig. 17.) The isomeric state at 140 keV is apparently analogous to the 140-keV level of Sc^{44}. It is not appreciably excited in the (d,p) reaction, and correspondingly, there is no theoretically predicted 1^+ state this low in energy. Other configurations are also needed to explain the 570–530-keV doublet marginally excited by (d,p).

V^{50}

The detailed information on this nucleus is not as exhaustive as for its Sc^{46} conjugate, but what is available indicates good agreement with the theory. The ground state has spin 6^+ and several low-lying levels which correspond well to prediction. Except for the inversion of the 4^+ and 6^+ levels, the correspondence to the Sc^{46} spectrum is good. The precision (p,d) studies to the levels have not yet been done, but low resolution studies by Legg and Rost[34] give good indications that relative cross sections in this reaction are also reproduced. The relative excitation of the ground state to the sum of the

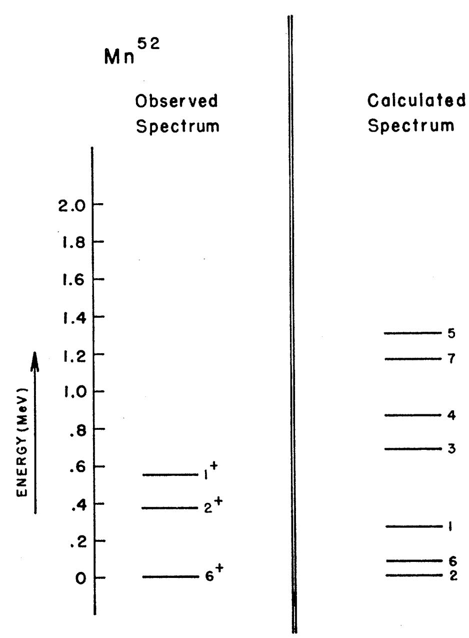

FIG. 16. Mn^{52} spectrum compared with calculation. This is the cross-conjugate of Sc^{44}. The magnetic moment of the 2^+ state is in violent disagreement with calculation (see text).

[34] J. C. Legg and E. Rost (to be published).

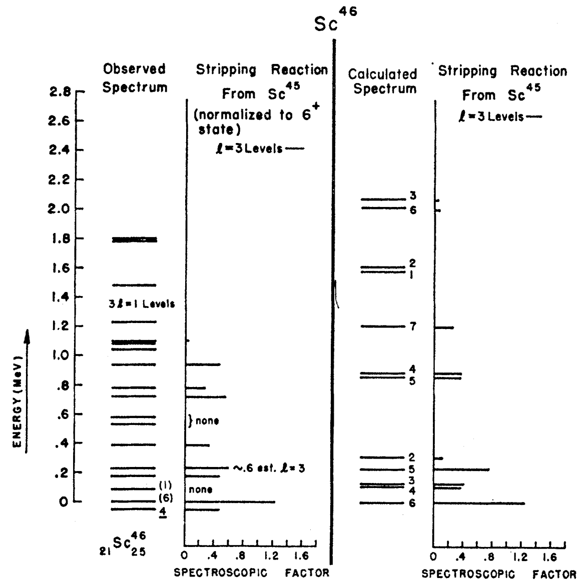

FIG. 17. Sc^{46} spectrum compared with calculation. The data are from Ref. 27 and are normalized to the calculated 6^+ state. The isomeric 1^+ level is not experimentally excited by the (d,p') reaction. A predicted 8^+ state at 1.95 MeV has been omitted, due to its low expected spectroscopic factor.

first four excited states agrees well with Legg's work, as does the strong excitation of the apparent 7^+ state at 0.87 MeV. (See Fig. 18.)

Sc^{48}

Since this nucleus is regarded here as consisting of one proton and one neutron hole, we predict a simple spectrum consisting of levels with $I=0$ to 7, at energies given by the F_I of Eq. (10). There again are several too many states seen in the spectrum of this nucleus; the lowest five states, however, seem to belong to the $f_{7/2}$ configuration, and the energy agreement is excellent. The measured ground state of this nucleus is 6^+; not enough further information is available to comment on the spectrum. (See Fig. 19.)

Pickup Reactions in the Ti Isotopes

In the preceding discussion and in the corresponding figures, mention has been made of stripping and pickup reaction cross sections. To make clear comparisons between theory and experiment for these studies is often difficult, because in some cases the total absolute cross section is not available, or the experimental energy resolution is not adequate to insure that single states of the spectrum are excited. The former problem has become more generally appreciated because the apparent validity of the distorted-wave Born approximation now enables one to extract meaningful absolute spectroscopic factors, by using σ_{DW} calculated from it in Eq. (24).

The (p,d) experiments of Kashy and Conlon[24] suffer from neither of these problems and have been quoted extensively above. The agreement of the calculations with their results for a given nucleus is impressive, but even more impressive is the agreement from nucleus to nucleus. The data of Kashy and Conlon gives absolute numbers, assuming the distorted wave calculation to be exact; no normalization to observed cross sections is necessary. Thus, in $Ti^{46}(p,d)Ti^{45}$ the ratio of their observed cross section to the distorted-wave prediction gives for S a value 2.75, where our theory predicts 3.10, and the simplest single-particle view of $1f_{7/2}$ shell wave functions gives 4. But the ratios of the experimental spectroscopic factors to those predicted by this theory for reactions leading to the 7/2 states in Ti^{45}, Ti^{47}, and Ti^{49}, are 0.70, 0.84, and 0.86, respectively. Thus, the theory, although uniformly high agrees from nucleus to nucleus within 20%.

V. MAGNETIC MOMENTS AND BETA-DECAY MATRIX ELEMENTS

The magnetic moments and beta-decay matrix elements calculated with these wave functions have been reported previously.[35] No changes have been made from those values in the entries in Table I, but several new calculated values have been added. The general rationale for choosing the gyromagnetic ratios g_p and g_n, and the Gamow-Teller matrix element for the single-particle decay may be found in Ref. 35.

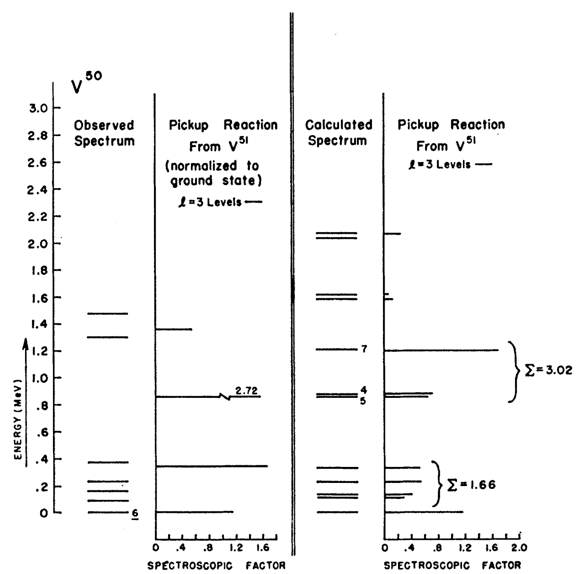

FIG. 18. V^{50} spectrum compared with calculation. There is some evidence the 0.225-MeV state may be 4^+ (Ref. 12). Also shown are the results of low resolution (p,d) spectroscopy of V^{51} (Ref. 32). If sums are taken over unresolved levels, the spectroscopic factors agree reasonably well. Higher resolution studies indicate that the 0.34-MeV line shown above is a compound of three levels, and the 0.87-MeV line a compound of two, further supporting the theory [C. A. Whitten and E. Kashy (private communication)].

[35] B. F. Bayman, J. D. McCullen, and L. Zamick, Phys. Rev. Letters **11**, 215 (1963).

TABLE I. Log ft values.

Nuclear decay	Initial spin → Final spin	Experiment[a]	Calculated	Experiment[a] cross-conjugate	Nuclear decay cross-conjugate	Initial spin → Final spin
$Sc^{43} \rightarrow Ca^{43}$	$7/2 \rightarrow 7/2$	4.9	4.58	5.4	$Fe^{53} \rightarrow Mn^{53}$	$7/2 \rightarrow 7/2$
	$7/2 \rightarrow 5/2$	5.0	4.45	5.2		$7/2 \rightarrow 5/2$
$Sc^{44} \rightarrow Ca^{44}$	$2 \rightarrow 2$	5.3	5.11	5.5	$Mn^{52} \rightarrow Cr^{52}$	$2 \rightarrow 2$
	$6 \rightarrow 6$	5.8[b]	5.40	5.5		$6 \rightarrow 6$
$Ca^{45} \rightarrow Sc^{45}$	$7/2 \rightarrow 7/2$	6.0	5.28	5.4	$Cr^{51} \rightarrow V^{51}$	$7/2 \rightarrow 7/2$
$Ti^{45} \rightarrow Sc^{45}$	$7/2 \rightarrow 7/2$	4.6	4.55	5.1	$Mn^{51} \rightarrow Cr^{51}$	$7/2 \rightarrow 7/2$
$Sc^{46} \rightarrow Ti^{46}$	$4 \rightarrow 4$	6.2	5.94	...	...	...
$Ca^{47} \rightarrow Sc^{47}$	$7/2 \rightarrow 7/2$	8.5	5.85	5.7	$Sc^{49} \rightarrow Ti^{49}$	$7/2 \rightarrow 7/2$
	$7/2 \rightarrow 5/2$	6.0	4.99	...		
$Sc^{47} \rightarrow Ti^{47}$	$7/2 \rightarrow 7/2$	5.3	5.41	6.2	$V^{49} \rightarrow Ti^{49}$	$7/2 \rightarrow 7/2$
	$7/2 \rightarrow 5/2$	6.1	5.62	...	...	...
$Sc^{48} \rightarrow Ti^{48}$	$6 \rightarrow 6$	5.5	5.45	...	...	...
	$6 \rightarrow 6$	6.4	Forbidden	...	...	...

[a] Taken from Landolt-Bornstein, *Energy Levels of Nuclei* (Springer-Verlag, Berlin 1961), Vol. I.
[b] See Ref. 15.

As with the rest of the calculations reported, the magnetic moments and log ft values showed remarkably little variation in their general features as the input Sc^{42} energies were varied over about 300 keV. As a result of such energy variation, the magnetic moments were seen to vary over an average of 0.2 nm, and the log ft values about 0.2. Although the highly forbidden transitions, such as $Ca^{47} \rightarrow Sc^{47}$, varied more rapidly, the general order of forbiddenness remained about the same. Several general features were remarkably insensitive to energy variation, in particular the high magnetic moments of the odd-odd scandium isotopes, and the increase in log ft value in the scandiums as the number of neutrons was increased. This latter effect may not be completely explainable in terms of $(f_{7/2})^n$ configurations, but clearly a large part of it is.

The worst agreement of any of the calculated values is in the magnetic moment of the $I=2$ state in Mn^{52}. This moment is completely inexplicable in terms of the $(f_{7/2})^n$ configuration. For any reasonable choice of g_p and g_n, the sum g_p+g_n will be very close to 1.1 nm, and for such values, the moments of the Sc^{44} and Mn^{52} spin-2 states cannot be simultaneously fit.[36] This is apparently a mixing effect, although it is hard to understand why it should show up so strongly in only this state.

One other disturbing feature is the good agreement of the neutron gyromagnetic ratio ($g_n=-0.39$ nm) with the measured moment of Ca^{43}, rather than Ca^{41}. In a similar calculation to that done here, Talmi and Unna[2] have calculated the "best" values for $g_n(f_{7/2})$ and $g_p(d_{3/2})$ for the potassium isotopes and find a value of g_n which is virtually identical to the experimental g_n of Ca^{41}.[37] We have no good explanation for this discrepancy.

A comparison of our results with the work of Lawson[38] shows the two approaches meet with about the same degree of success. Not surprisingly, when the moments predicted in each scheme are the same, the wave functions strongly overlap. There are several cases, however, in which Lawson's predictions are quite different from the present ones; one of these, Ti^{45}, is now being measured[39] and may help to resolve the question of relative success.

[36] D. L. Harris and J. D. McCullen, Phys. Rev. **132**, 310 (1963).
[37] E. Brun, J. J. Kraushaar, W. L. Pierce, and W. J. Veigele, Phys. Rev. Letters **9**, 166 (1962).
[38] R. D. Lawson, Phys. Rev. **124**, 1500 (1961).

VI. SCANDIUM-CALCIUM BINDING ENERGY DIFFERENCES

The ground-state energies of Sc^{41} and Ca^{41} differ by 7.279 MeV.[40] These are isobaric analog states, and the energy difference must be ascribed to the Coulomb interaction between the $1f_{7/2}$ proton in Sc^{41} and the

TABLE II. Magnetic moments ($g_p=+1.499$ nm $g_n=-0.39$ nm).

	Nucleus	Experiment[a]	Calculated
Odd neutron	Ca^{41}	-1.5946	-1.35
	Ca^{43}	-1.317	-1.35
	Ti^{45}	...	-0.38
	Ti^{47}	-0.7881	-0.73
	Ti^{49}	-1.1036	-1.14
Odd proton	Sc^{43}	4.46 ± 0.1[b]	$+4.53$
	Sc^{45}	4.7563	$+4.75$
	Sc^{47}	5.31 ± 0.06[c]	$+5.04$
	V^{49}	4.46	$+4.68$
	V^{51}	5.148	$+5.25$
	Mn^{51}	...	$+4.28$
	Mn^{53}	5.050	$+5.25$
Odd odd	Sc^{44} $I=2$	2.56 ± 0.04	$+2.21$
	$I=6$	3.96 ± 0.15	$+3.62$
	Sc^{46} $I=4$	3.03	$+3.23$
	V^{50} $I=6$	3.3466	$+3.32$
	Mn^{52} $I=6$	3.08	$+3.05$
	$I=2$	1.02	$+0.01$

[a] Unless otherwise noted, values are taken from compilation of I. Lindgren (to be published).
[b] R. G. Cornwell and J. D. McCullen (unpublished).
[c] W. Happer and J. D. McCullen (unpublished).

[39] R. G. Cornwell (private communication).
[40] This and all further binding energy information, unless otherwise noted, is taken from L. A. Konig, J. H. W. Mattauch, and A. H. Wapstra, Nucl. Phys. **31**, 18 (1962).

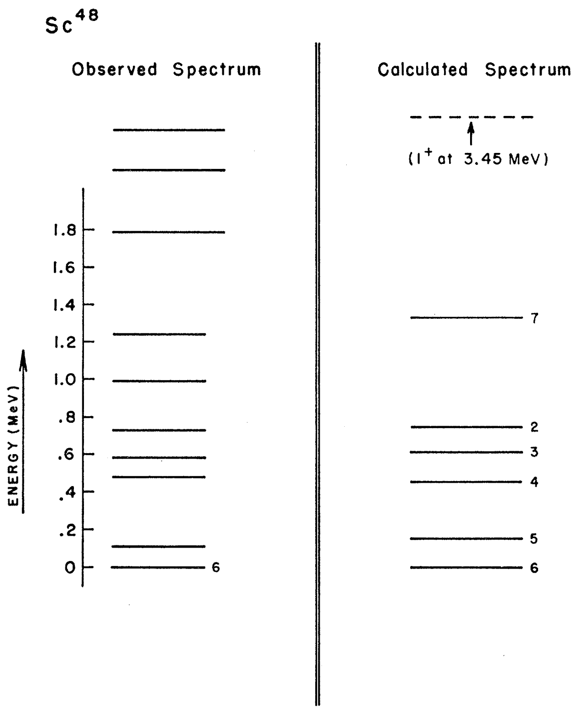

FIG. 19. Sc^{48} spectrum compared with calculation. Only the 6^+ ground state has known spin. The energy agreement is remarkable.

20 protons in the Ca^{40} core. If we assume that the 21st proton in all the scandium isotopes remains in the $1f_{7/2}$ shell, then the energy difference between all scandium-calcium isobaric analog states should equal 7.279 MeV. Thus, if we subtract from 7.279 MeV the observed energy difference between the ground states of, say, Sc^{43} and Ca^{43}, we will have an experimental determination of the excitation energy in the Sc^{43} spectrum of the state analogous to the ground state of Ca^{43}.

Table III shows the excitation energies calculated in this way, together with our theoretical predictions for the position in scandium of these calcium-analog states. The agreement between theory and observation is of the same order as that encountered in discussing the low-lying levels.

TABLE III. Excitation energies in scandium isotopes of analogs of calcium ground states.

A	B.E.(Ca)-B.E.(Sc)	Analog ex'n energy	Calculated	Exp-calc
41	7.279	0	0	⋯
42	7.194	0.085	0	0.085
43	3.003	4.276	4.17	0.11
44	4.431	2.848	2.81	0.04
45	0.530	6.749	6.48	0.27
46	2.165	5.114	4.64	0.47
47	−1.182	8.461	8.40	0.06
48	0.660	6.619	6.44	0.18

VII. WAVE FUNCTIONS AND SPECTRA OBTAINED BY USING OTHER POTENTIALS

In this work, the problem of what to take for the interaction of nucleons in an open shell has been resolved by taking the two-body matrix elements from the experimental data, in particular, from the Ca^{42} and Sc^{42} spectra. In many similar shell-model calculations, specific forces are assumed and two of the most common simple interactions used are the pairing interaction and the quadrupole interaction. These have been used either separately or in combination. In this section the spectra and ground-state wave functions of these two interactions are compared with the corresponding quantities obtained using the "experimental interaction."

The pairing force is defined in terms of its matrix elements as follows:

$$E_0=0$$
$$E_I=G(>0)\,,\quad I=1,\,2,\,\cdots,\,2j.$$

It acts between all the nucleons. The eigenvectors of this force define the coupled seniority scheme, introduced by Flowers.[41] The spectrum of this force can be written

$$E(A,s,T,t)=\frac{-G}{2j+1}\left\{\left[A\left(j+2-\frac{A}{4}\right)-T(T+1)\right]-\left[s\left(j+2-\frac{s}{4}\right)-t(t+1)\right]\right\}+G\frac{A(A-1)}{2},$$

where A is the number of $1f_{7/2}$ nucleons, s is the (coupled) seniority, T the isotopic spin, and t the "reduced isotopic spin." Flowers lists the angular momenta associated with each set (A,s,T,t) and, except for the ground states, there are usually several. They all have the energy $E(A,s,T,t)$. Thus, the spectrum of the pairing force is characterized by high degeneracies. For example, the first excited state in Ti^{46} is sixfold degenerate ($I=2$, 4, 6, each occurring twice), in contrast to the observed $I=2$ first excited state which is well isolated above and below.

For identical nucleons in the $f_{7/2}$ shell, any two-particle interaction leads to the (identical-particle) seniority wave functions. However, there is no such theorem valid when both neutrons and protons are present. Table IV shows overlaps between some ground-state coupled seniority wave functions and the ground-state wave functions calculated with the interaction of Fig. 1. The overlaps range from about 52 to 93%, indicating that coupled seniority is appreciably violated by our wave functions.

The quadrupole interaction $W_{ij}\propto[Y^2(\hat{r}_i)Y^2(\hat{r}_j)]_0^0$ leads to two-body matrix elements, E^I, which are proportional to $U(jI2j;jj)$. The actual numbers used

[41] B. H. Flowers, Proc. Roy. Soc. (London) A212, 248 (1952).

TABLE IV. Overlap integrals for various wave functions in the $f_{7/2}$ shell.

Nucleus	J	Pairing M. B. Z.[a] (%)	Quadrupole M. B. Z.[a] (%)	Lawson (+) M. B. Z.[a] (%)	Lawson (−) M. B. Z.[a] (%)	Lawson (−) quadrupole (%)
Sc^{43}	7/2	87.9	99.3	97.6	98.7	99.9
Sc^{44}	2	87.3	97.4	94.0	95.5	99.5
Sc^{45}	7/2	86.5	97.6	93.6	92.4	97.3
Sc^{46}	4	52.1	98.3	94.8	89.2	92.8
Sc^{47}	7/2	93.0	97.0	94.7	94.7	99.7
Ti^{43}	7/2	87.9	99.3	97.6	98.7	99.9
Ti^{44}	0	87.9	99.3	97.6	98.7	99.9
Ti^{45}	7/2	83.8	98.0	85.6	93.2	98.2
Ti^{46}	0	86.7	97.7	93.1	92.7	98.0
Ti^{47}	5/2	···	98.7	92.5	85.4	90.5
Ti^{48}	0	93.0	97.0	94.7	94.7	99.7
Ti^{49}	7/2	93.0	97.0	94.7	94.7	99.7

[a] M. B. Z. refers to wave functions of this work.

were

$$0,\ 0.4096,\ 1.1471,\ 2.0483,\ 2.8677,\ 3.2774,\ 2.8677,\ 1.1471,$$

corresponding to $I=0, 1, 2, \cdots, 7$. The spectrum of the quadrupole force is listed in Table V. Although there are some deficiencies, it is quite clear that the quadrupole force spectrum agrees fairly well with experiment. The major deficiencies are that the predicted ground state spins of Sc^{44}, Sc^{46}, Sc^{48}, Ti^{45} are not correct. The quadrupole force predicts spins 1, 3, 5, and 5/2 rather than 2, 4, 6, and 7/2. The tendency to get an odd spin for the ground states of Sc^{44}, Sc^{46}, and Sc^{48} is undoubtedly due to the low-lying spin-1 state in the two-body interaction.

The overlap of the ground-state wave functions using the quadrupole force and the experimental force are shown in Table IV. They are quite high, ranging from 97.0% to 99.3%. This indicates that a quadrupole force can furnish ground-state wave functions about as well as the interaction obtained from experiment.

TABLE V. Spectrum of a quadrupole force.

Nucleus	Energy	Spin	Nucleus	Energy	Spin
Sc^{43}	0	7/2	Ti^{45}	0	5/2
	1.48	9/2		0.636	7/2
	2.70	11/2		1.979	9/2
	2.82	5/2		2.359	11/2
	3.01	5/2		3.178	3/2
Sc^{44}	0	1	Ti^{46}	0	0
	0.237	2		0.863	2
	0.287	6		1.930	4
	0.921	3		3.306	6
	1.520	4		3.351	4
				3.515	2
Sc^{45}	0	7/2		4.136	5
	1.26	9/2			
	1.36	11/2	Ti^{47}	0	5/2
	2.03	1/2		0.153	7/2
	2.04	3/2		0.489	3/2
				1.280	11/2
Sc^{46}	0	3		1.690	9/2
	0.135	4			
	0.173	5	Ti^{48}	0	0
	0.201	6		1.025	2
	0.573	2		2.228	2
	1.259	5		2.242	4
				3.059	3
Sc^{47}	0	7/2		3.122	6
	0.996	3/2		3.758	6
	1.204	11/2		3.943	4
	1.944	5/2			
			Ti^{49}	0	7/2
Sc^{48}	0	5		0.996	3/2
	0.410	6, 4		1.205	11/2
	1.230	3		1.944	5/2
	2.130	2, 7		2.232	9/2
	2.867	1		2.649	13/2
	3.277	0		2.725	15/2

A. Lawson Wave Functions

A different procedure for obtaining wave functions in the $f_{7/2}$ shell, one which has the virtue of being considerably simpler than the straightforward matrix diagonalization procedure, has been carried out by Lawson.[38,42] The basic assumption made is that the single-particle potential in which the particles move is deformed. The deformation splits the degenerate $f_{7/2}$ level into 4 doubly-degenerate levels which can be labeled by the Z-quantum number m. One first obtains an intrinsic wave function, in the manner of Nilsson,[43] by putting the particles in the lowest levels, For a positive deformation the order of the levels is in the order of increasing m; for a negative deformation the order of the levels is in the order decreasing m. From this intrinsic wave function a state of definite angular momentum is projected out by means of the Hill-Wheeler integral

$$\Psi_{MK}{}^{I}(x)=(2I+1)C_K{}^I\int dRD_{MK}{}^I(R)\chi_K(Rx)\,,$$

where R is the rotation matrix and $C_K{}^I$ is a normalization coefficient, and K is the component of angular momentum of the particles along the intrinsic axis. The wave function $\Psi_{MK}{}^I$ obtained by the above procedure is then the assumed ground-state wave function of the

[42] R. D. Lawson and B. Zeidman, Phys. Rev. **128**, 821 (1962).
[43] S. G. Nilsson, Kgl. Danske Videnskab. Selskab, Mat. Fys. Medd. **29**, No. 16 (1955).

nucleus. The remaining details of the calculation are present in Ref. 41.

Comparing the procedure of Lawson and the present work, we see that whereas we start with a spherical nucleus and require a two-body interaction between the particles to remove the degeneracy of the wave functions, Lawson starts with a different single-particle well and then does not require any residual interaction (except for odd-odd nuclei) in order to obtain the nuclear wave functions. The overlap integrals listed in Table IV indicate that Lawson's wave functions for the ground states closely approximate those that diagonalize the two-particle interaction we have used. Lawson's wave functions with negative deformation (near the beginning of the shell) show the expected strong overlap with eigenstates of the pairing force.

VIII. DISCUSSION

In this paper we have investigated the consequences of a nuclear model in which the extra-core nucleons remain within the $1f_{7/2}$ shell, so that the nuclear spectra are due entirely to the splitting of states of the pure configuration by a residual nucleon-nucleon interaction. We have seen that many of the properties of nuclei with $20 \leq Z,\ N \leq 28$ are entirely consistent with this model. Some of the features of the spectra are straightforward, such as the regularity with which the ground-state spin of the odd-even nuclei is 7/2, and some have been suspected for a long time to be entirely due to the residual interaction between $1f_{7/2}$ particles, such as the low-lying 5/2 state in systems with three identical nucleons. The reproduction of these features by the present calculation is encouraging, but hardly surprising.

There are several results of the calculation which are more remarkable. One is the good agreement with the pattern of the ground-state spins of the odd-odd nuclei. In the Sc^{44}-Mn^{52} cross-conjugate pair, the experimental evidence requires a low 2^+–6^+ doublet, while the Sc^{46}-V^{50} pair requires low-lying 4^+ and 6^+ states. The calculation predicts for Sc^{44} a low doublet with the correct spins, with the lowest 4^+ at 0.8 MeV, while for Sc^{46} it predicts the 4^+ practically degenerate with the ground state.

Other trends in closely related nuclei are equally well reproduced. The increase in magnetic moment and β-decay $\log ft$ value as neutrons are added to the odd-even scandium isotopes is shown to be simply understandable in terms of pure configuration, as is the general lowering of the first 2^+ state in even-even nuclei as the middle of the shell is reached. Further, the calculation gives a natural explanation to the apparent existence of two 6^+ states in Ti^{48} which are only 150 keV apart.

There is also striking agreement between $l=3(d,p)$ and (p,d) spectroscopic factors calculated here and extracted from experiment by means of the distorted wave Born approximation. The successful description of low 7/2 states of the odd-even nuclei is not surprising, but the general agreement of excited states, particularly in even-even nuclei, is somewhat more so. In forthcoming publications it will be shown that the wave functions we have calculated also account for many of the systematics of the (p,t) and (p,α) reactions, although they suffer from additional uncertainties due to the more complicated nature of the reaction. Magnetic moments, ft values, and binding energies are, with some striking exceptions, also compatible with the simple configurational assignment.

All these features are calculated from a two-body residual interaction taken as directly as possible from the experimental levels in Sc^{42}. Hence, the *detailed* application of the results of this paper to future investigations hinge on the essential correctness of the energy and spin assignments made for that spectrum. But the general conclusions are not so strictly tied to those experimental results, because within reasonable limits variations in the two-body energies do not materially affect the results. This can be clearly seen by considering the wave functions generated from the $P_2(\cos\theta)$ force described in Sec. VII; although rather different in the specific input energies it uses, such a force generates wave functions with remarkably complete overlap with ours.

Of course these statements do not constitute proof that the real nuclear states are constructed entirely from extra-core nucleons in the $1f_{7/2}$ shell. To make quantitative estimates of the configuration purity of the wave functions, one would have to study how configurational admixtures affect the agreement achieved here between theory and experiment. It remains to be seen whether such admixtures are best described in terms of the effect of an assumed two-body interaction, or in terms of wave functions projected from products of Nilsson-type orbitals.

In a sense we learn most from the few situations in which there is hopeless disagreement between experiment and the pure-configuration predictions. The most striking example is the occurrence of many states of Sc^{45} below 1.3 MeV. In general there are too many low-lying states in all nuclei to be accounted for by our scheme. Such conflicts clearly indicate the participation of other configurations in some of the low-lying states of the nuclei we are studying. In fact, one of the puzzling features of the $1f_{7/2}$ shell is the frequent proximity of two states of the same spin and parity, one of which is a model example of pure $1f_{7/2}$ behavior, and the other a clear product of configurational admixture. Perhaps the most valuable information revealed by a calculation such as this one is where one can expect to find states predominantly due to the $(1f_{7/2})$ configuration, and how strongly such states will be excited in various reactions.

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When this manuscript was in the final stages of preparation, we received from Dr. J. Ginocchio and Dr. B. French a preprint of their work on the energy levels and binding energies of $1f_{7/2}$ shell nuclei. Their calculated spectra are the same as ours, as they have also confined themselves to the $1f_{7/2}$ shell and have used the two-body matrix elements that we inferred from the $A=42$ system (Fig. 1).

APPENDIX I

For $n=3$ we have

$$\langle\psi(j^3IM)|\sum_{1\leqslant i<j}^{3}W_{ij}|\psi(j^3IM)\rangle=\sum_L 3(j;j^2L]\!]j^3I)^2E_L.$$

These three-to-two fractional parentage coefficients are available in the tables of Edmonds and Flowers.[42] For $n=4$,

$$\langle\psi(j^4vIM)|\sum_{1\leqslant i<j}^{4}W_{ij}|\psi(j^4vIM)\rangle=\sum_L[6\sum_K(j^2K;j^2L]\!]j^4vI)^2]E_L.$$

These coefficients are given in Table AI.

TABLE AI. $6\sum_K(j^2K;j^2L]\!]j^4vI)^2$.

vI \ L	0	2	4	6
00	3/2	5/6	3/2	13/6
22	1/2	11/6	3/2	13/6
24	1/2	5/6	5/2	13/6
26	1/2	5/6	3/2	19/6
42	0	1	42/11	13/11
44	0	7/3	1	8/3
45	0	8/7	192/77	26/11
48	0	10/21	129/77	127/33

APPENDIX II. THE GROUND-STATE WAVE FUNCTIONS

In Table AII we list eigenvectors corresponding to the ground states. A tabulation of all the eigenvectors is

TABLE AII. Ground-state eigenvectors.

$Sc^{43}(Fe^{53})$

	L_2	D
$I=7/2$	0	0.7608
$E=0.00$	2	0.6090
$T=1/2$	4	0.2093
	6	0.0812

$Sc^{44}(Mn^{52})$

	L_3	D
$I=2$	3/2	−0.1439
$E=0.00$	5/2	0.6850
$T=1$	7/2	0.7126
	9/2	−0.0197
	11/2	0.0424
$I=6$	5/2	0.7257
$E=0.09$	7/2	−0.6199
$T=1$	9/2	−0.1744
	11/2	0.2335
	15/2	−0.0649

$Sc^{45}(Cr^{51})$

	L_4	D
$I=7/2$	0	0.8210
$E=0.00$	2	0.5434
$T=3/2$	4	0.0846
	6	−0.0130
	2′	0.0428
	4′	−0.1462
	5	−0.0121

$Sc^{46}(V^{50})$

	L_{-3}	D
$I=4$	3/2	0.1739
$E=0.11$	5/2	0.8498
$T=2$	7/2	−0.4912
	9/2	0.0749
	11/2	−0.0200
	15/2	−0.0179
$I=6$	5/2	0.4715
$E=0.00$	7/2	0.8397
$T=2$	9/2	−0.0776
	11/2	0.2580
	15/2	0.0013

$Sc^{47}(Ti^{49})$

	L_{-2}	D
$I=7/2$	0	+0.9136
$E=0.00$	2	−0.4058
$T=5/2$	4	−0.0196
	6	0.0146

$Ti^{43}(Co^{53})$

	L_2	L_1	D
$I=7/2$	0	7/2	−0.7608
$E=0.00$	2	7/2	−0.6090
$T=1/2$	4	7/2	−0.2093
	6	7/2	−0.0812

$Ti^{44}(Fe^{52})$

	L_2	L_2	D
$I=0$	0	0	−0.7608
$E=0$	2	2	−0.6090
$T=0$	4	4	−0.2093
	6	6	−0.0812

$Ti^{45}(Mn^{51})$

	L_2	L_3	D
$I=5/2$	0	5/2	−0.6935
$E=0$	2	3/2	−0.0610
$T=1/2$	2	5/2	−0.4359
	2	7/2	−0.5031
	2	9/2	0.1545
	4	3/2	−0.0233
	4	5/2	0.0074
	4	7/2	−0.1499
	4	9/2	0.0615
	4	11/2	0.1322
	6	7/2	0.0560
	6	9/2	0.0125
	6	11/2	0.0091
	6	15/2	−0.0260

$Ti^{45}(Mn^{51})$

	L_2	L_3	D
$I=7/2$	0	7/2	0.7261
$E=0.28$	2	3/2	0.1198
$T=1/2$	2	5/2	−0.5354
	2	7/2	−0.0175
	2	9/2	−0.1753
	2	11/2	−0.2587
	4	3/2	−0.1058
	4	5/2	−0.1354
	4	7/2	−0.1319
	4	9/2	0.1052
	4	11/2	0.0162
	4	15/2	0.0662
	6	5/2	−0.0068
	6	7/2	−0.0807
	6	9/2	−0.0224
	6	11/2	0.0590
	6	15/2	−0.0254

$Ti^{46}(Cr^{50})$

	L_2	L_4	D
$I=0$	0	0	0.8224
$E=0.00$	2	2	0.5420
$T=1$	2	2′	0.0563
	4	4	0.0861
	4	4′	−0.1383
	6	6	−0.0127

$Ti^{47}(V^{49})$

	L_2	L_{-3}	D
$I=5/2$	0	5/2	0.8142
$E=0.001$	2	3/2	−0.2153
$T=3/2$	2	5/2	−0.2690
	2	7/2	−0.4336
	2	9/2	0.1047
	4	3/2	−0.0396
	4	5/2	0.0312
	4	7/2	0.0864
	4	9/2	−0.0782
	4	11/2	−0.0539
	6	7/2	0.0152
	6	9/2	0.0144
	6	11/2	0.0070
	6	15/2	−0.0019

	L_2	L_{-3}	D
$I=7/2$	0	7/2	0.8121
$E=0$	2	3/2	−0.1811
$T=3/2$	2	5/2	0.3856
	2	7/2	−0.2459
	2	9/2	0.0756
	2	11/2	0.2424
	4	3/2	0.0008
	4	5/2	−0.1223
	4	7/2	−0.0985
	4	9/2	−0.0473
	4	11/2	−0.0806
	4	15/2	0.0029
	6	5/2	0.0195
	6	7/2	0.0092
	6	9/2	−0.0157
	6	11/2	−0.0010
	6	15/2	−0.0030

Ti^{48}

	L_2	L_{-2}	D
$I=0$	0	0	−0.9136
$E=0$	2	2	0.4058
$T=2$	4	4	0.0196
	6	6	−0.0146

$Ti^{49}(Sc^{47})$

	L_2	L_{-1}	D
$I=7/2$	0	7/2	−0.9136
$E=0$	2	7/2	0.4058
$T=5/2$	4	7/2	0.0196
	6	7/2	−0.0146

contained in the Princeton University Technical Report NYO-9891.[45] The use of Table AII to construct a wave function is best illustrated by an example. Suppose we want the wave function of the ground state of Ti^{45}. Here are two $f_{7/2}$ protons, and three $f_{7/2}$ neutrons. The symbol $L(\pm 2)$ refers to the group containing two particles or holes, hence in this case to the two protons. Similarly the symbol $L(\pm 3)$ refers here to the three neutrons. Reference to Table AII then gives

$$\Psi(Ti^{45}, a=1, I=7/2, m) = 0.7261[\psi(j^2 0)\psi(j^3 7/2)]_m{}^{7/2} + 0.1198[\psi(j^2 2)\psi(j^3 3/2)]_m{}^{7/2} - 0.5354[\psi(j^2 2)\psi(j^3 5/2)]_m{}^{7/2} - \cdots \text{ etc.}$$

These eigenvector components also furnish us with a wave function for Mn^{51}, which we regard as three proton holes and two neutrons holes. Thus, here $L(\pm 3)$ refers to the protons, and $L(\pm 2)$ to the neutrons.

$$\Psi(\mathrm{Mn}^{51}, a=1, I=7/2, m) = 0.7261[\psi(j^{-2} 0)\psi(j^{-3} 7/2]_m{}^{7/2} + 0.1198[\psi(j^{-2} 2)\psi(j^{-3} 3/2)]_m{}^{7/2} - 0.5354[\psi(j^{-2} 2)\psi(j^{-3} 5/2)]_m{}^{7/2} - \cdots \text{ etc.}$$

The states $\psi(j^3 IM)$ and $\psi(j^4 vIM)$ are defined by the fractional parentage coefficients $(j^2L; j]\!]j^3I)$ and $(j^3L; j]\!]j^4vI)$. These are taken from the tables of Edmonds and Flowers,[44] except for the signs of $(7/2^2 6; 7/2]\!]7/2^3 15/2)$ and $(7/2^2 L; 7/2]\!]7/2^3 7/2)$, $L=0$, 2, 4, 6, where we believe the signs given by these authors are incorrect.[46] The hole states $\psi(j^{-4})$, $\psi(j^{-3})$, $\psi(j^{-2})$, $\psi(j^{-1})$ are defined by (5) or (8).

The isotopic spins of the states are obtained by modifying the two-particle interaction as given in Fig. 1 by the addition of 1 MeV to the levels of odd angular momentum (isotopic spins zero). This is equivalent to the addition of the operator $\frac{1}{4}-\mathbf{t}_1\cdot\mathbf{t}_2$, where $\mathbf{t}_i$ are the isotopic spin operators for particle i. Summing this over all the A nucleons gives

$$\sum_{1=i<j}^{A}(\tfrac{1}{4}-\mathbf{t}_i\cdot\mathbf{t}_j)=\tfrac{1}{8}A(A-1) -\tfrac{1}{2}[(\sum \mathbf{t}_i)\cdot(\sum \mathbf{t}_j)-\sum \mathbf{t}_i{}^2], \quad (33)$$

whose expectation value in a state of A nucleons with isotopic spin T is

$$\tfrac{1}{8}A(A+2)-\tfrac{1}{2}T(T+1). \quad (34)$$

Since (33) is a scalar in isotopic spin space, its addition has no effect upon the wave function, but merely shifts the energy levels by an amount (34). This shift is used to ascertain the isotopic spin of each eigenvector.

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[44] A. R. Edmonds and B. H. Flowers, Proc. Roy. Soc. (London) **A214**, 515 (1952).

[45] Copies of this report may be obtained by writing to the authors at Palmer Physical Laboratory, Post Office Box 708, Princeton, New Jersey.

[46] If we would omit this over-all change of sign in $\psi(j^3 I=7/2M)$ we would have to change the signs of all the coefficients, $(j^3 7/2; j]\!]j^4vI)$.