is no solution involving fewer particles than the double-octet solution of unitary symmetry. The 5, 3 (SU_2) scheme, discussed in Sec. IVC, may lead to a solution of simplicity comparable to that of the double-octet scheme.

The third result is that deviations from degeneracy of the Okubo type are favored in the double-octet scheme.

The model is incomplete in several aspects. The various assumptions concerning the partial waves and configurations that are important have not been checked with detailed, dispersion-theoretic calculations. The criterion used for nondegenerate solutions is crude; its chief virtue is its simple applicability. Furthermore, no reason has been given why the particle multiplets should be nondegenerate. There is no compelling reason, other than simplicity, for the neglect of the baryon-antibaryon states. In fact, it is hoped that in more accurate bootstrap models the baryons will be necessary.

Thus, even if our basic assumption is right, i.e., that nature chooses the simplest self-consistent set of particles, the true consistency criteria may be quite different from those assumed here. The primary purpose of this paper is to demonstrate the falseness of the common assumption that if simple representations of one Lie group satisfy a particular bootstrap model, simple representations of any other Lie group must satisfy a similar model. The consistency criteria of Secs. III, IV, and V are examples of plausible criteria that distinguish between different group-representation schemes.

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Translational Inertial Spin Effect with Moving Particles

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Although the final conclusion of a preceding paper was incorrect, as we shall explain, the main point remains, and should entail the existence of *sui generis* recoil effects associated with nonzero values of curl $\boldsymbol{\sigma}$ ($\boldsymbol{\sigma}$ is the spin density). These should be observed by testing, not with solids as was previously proposed, but with the probability fluids associated with moving particles; this more refined type of experiment should be able to select, among the set of integrally equivalent energy-momentum tensors, the one describing locally the true or physical energy-momentum flux. In this paper it is shown, by an explicit calculation, that cylindrical type solutions of the extreme relativistic Dirac equation exist with no z dependence of the wave function (and thus no k_z component of the momentum) but still with a z component of the Dirac probability current; as this conclusion is reached with a t dependence of the wave function of strictly the form $\exp(-iWt/\hbar)$, there is no question of having to perform a Foldy-Wouthuysen transformation to extract the positive energy contribution (or equivalently, to use the Newton-Wigner position operator). The "transverse inertial spin effect" we predict is locally described by the flux of the Dirac current per time dt and surface $d\mathbf{s}$, and corresponds to the local transition probabilities between the dynamical state of the beam and a pointlike localization of the incident particles.

I. INTRODUCTION

In a preceding paper[1] it has been argued that the true, physical, energy-momentum tensor associated with a spin-$\frac{1}{2}$ wave is Tetrode's asymmetrical tensor

$$T^{ij}=-\tfrac{1}{2}c\hbar\bar{\psi}[\partial^i]\gamma^j\psi+ieA^i\bar{\psi}\gamma^j\psi \quad (1)$$

so that, according to the well-known[2] formula

$$\Theta^{ij}=T^{ij}-T^{ji}=-\partial_k\sigma^{ijk}=ic\epsilon^{ijkl}(\partial_l\sigma_k-\partial_k\sigma_l)\,, \quad (2)$$

where σ denotes Dirac's spin density

$$\sigma^{ijk}=ic\epsilon^{ijkl}\sigma_l=\tfrac{1}{2}c\hbar\bar{\psi}\gamma^{ijk}\psi\,, \quad (3)$$

the kinematical current lines and the energy-momentum lines may, under appropriate circumstances, be non-collinear. (Latin indexes run from 1 to 4; $x^4=ict$; $h=2\pi\hbar$, denotes Plank's constant, ϵ^{ijkl} Levi-Civita's indicator, γ^i the von Neumann matrices, $\bar{\psi}=\psi^\dagger\gamma^4$, $[\partial^i]$ the Gordon current operator, e the electron charge, A^i the electromagnetic potential; $\gamma^{ij\cdots}=\gamma^i\gamma^j\cdots$ if all indexes are different, 0 otherwise.)

The final conclusion of this preceding paper[1] was incorrect, as we shall explain later. However, the main point, which the above paragraph recalls, remains true; the present paper intends to show that by using as a test material the probability fluid associated with moving spin-$\frac{1}{2}$ particles rather than a solid, the recoil effect corresponding to the "transverse momentum"[1] should appear.

The test material, which is a beam of spin-$\frac{1}{2}$ particles, has the three following fundamental properties: (a) a

[1] O. Costa de Beauregard, Phys. Rev. **129**, 466 (1963); all the notations of this paper are retained here, except for Θ^{ij} which is taken in a different sense.
[2] H. Tetrode, Z. Physik **48**, 52 (1928).

velocity equal to or nearly equal to c, so that there are only two (longitudinal) spin, or helicity, states; (b) a pure energy state, with eigenvalue W, so that there is no question of having to perform a Foldy-Wouthuysen transformation in order to extract the positive-energy contribution; (c) no z dependence of the wave function ψ, so that there is certainly no momentum component in the (fixed) z direction.

We will prove by an explicit calculation that it is possible to (d) bend the beam around the z direction, that is, parallel to the xy plane as far as momentum (not necessarily velocity!) is concerned, in such a way that a pure (longitudinal) spin state is conserved; and (e) impose a radial distribution of the ψ wave amplitude such that the field of current and spin vectors (which are collinear) has a nonzero z component.

Finally, we will show that (f) these joint properties correspond precisely to the existence of a "transverse momentum" associated with a nonzero value of curl$\boldsymbol{\sigma}$, according to Tetrode's formula (2). The following remarks will be useful.

Let us define the "true" or "physical," the "pseudo-," and the "transverse" energy-momentum associated with an infinitesimal volume[3] $ic\epsilon^{ijkl}du_l=[dx^i dx^j dx^k]$ as, respectively,

$$dP^i=T^{ij}du_j\,, \tag{4}$$

$$dL^i=T^{ij}du_j\,, \tag{5}$$

$$dT^i=\Theta^{ij}du_j\,, \tag{6}$$

T and Θ being defined by (1) and (2); we will consider successively the cases where the 4-vector du_j is timelike and is spacelike.

A timelike du_j represents a volume element in the ordinary sense, and no generality will be lost in supposing that du_4 is the only nonzero component. Then, inserting in (4) and (5) the Tetrode expression (1), we obtain a "true" dP^i directed by the energy momentum operator $i\partial^i$, and a "pseudo" energy-momentum dL^i collinear with the Dirac current $i\bar{\psi}\gamma^i\psi$ [as ∂^4 will be the only differential operator present in (5) and the t dependence of ψ is, according to postulate (b), solely through a common factor $\exp(-iWt/\hbar)$]. Thus, in this case, dL^i may well be called the "longitudinal" energy-momentum. In view of the following, the prerelativistic form of formula (2)

$$\mathbf{T}=-\iiint \mathrm{curl}\boldsymbol{\sigma}du=\iint \boldsymbol{\sigma}\times d\mathbf{s} \tag{7}$$

should also be noted.

In the case where du_j is spacelike, no generality will be lost in supposing that the du_4 component is zero; the three other ones may be written as $du_\alpha=ds_\alpha dt\,(\alpha=1,2,3)$, so that formulas (4)–(6) now represent energy-momentum fluxes per time element through a surface element. In view of the following, we consider the case of a surface element orthogonal to the z axis; inserting the Tetrode expression (1) in (5) and remembering postulate (c) (no z dependence of the ψ wave), we find in this case a nil "pseudo" energy-momentum flux. But the "transverse," and thus the "total," energy-momentum flux will not be nonzero in general; using an abridged, but grammatically incorrect expression, we shall call c^2dT^4 and c^2dT^4/dt the "transverse energy" and "transverse power" fluxes; the formula

$$\bar{\omega}_t=c^2\iint \mathrm{curl}\boldsymbol{\sigma}\cdot d\mathbf{s}=c^2\oint \boldsymbol{\sigma}\cdot d\mathbf{l}, \tag{8}$$

(where $\bar{\omega}_t$ denotes the "transverse power") follows directly from Tetrode's formula (2).

We will show in Sec. III that the noncollinearity of the particles velocity and momentum found in Sec. II by a direct calculation is completely justifiable in terms of either the spacelike or the timelike integral considerations just stressed.

The problem of extending these conclusions to higher spin cases is an interesting one, but, as a few more difficult questions would be raised, we shall postpone consideration.

II. SPIN-½ PARTICLES IN A CIRCULAR ACCELERATOR (EXTREME RELATIVISTIC LIMIT)

We shall proceed to integrate the Dirac equations with a time- and z-independent, cylindrically symmetric, potential of the form

$$A_1=-a(r)\sin\theta\,,\quad A_2=a(r)\cos\theta\,,\quad A_3=0\,,\quad A_4=0\,, \tag{9}$$

generating the magnetic field

$$H_1=0\,,\quad H_2=0\,,\quad H_3=H_z=a(r)/r+a'(r)\,; \tag{10}$$

we use cylindrical coordinates r, θ, $z\,(r\geqslant 0)$.

In the extreme relativistic limit [postulate (a) above] the mass terms are negligible, and the four Dirac equations reduce to two pairs of equations implying the 2×2 Pauli matrices such that

$$\sigma_\mu\sigma_\nu=-\sigma_\nu\sigma_\mu=i\sigma_\rho\,,\quad \sigma_\mu{}^2=1\,,\quad \mu,\nu,\rho=1,2,3\,. \tag{11}$$

According to postulates (b) and (c), we will seek solutions of the form

$$\psi(r,\theta,t)=f(r)\exp[(i/\hbar)(C\theta-Wt)]\Lambda^{-1}(\theta)\varphi_0\,, \tag{12}$$

with C and W constant, and the (common) function $f(r)$ real; φ_0 is a constant 2-spinor, and $\Lambda^{-1}(\theta)$ a 2×2 matrix which will be specified later; W is clearly an energy eigenvalue, and C (due to the θ dependence through Λ^{-1}) is the mean value of the angular momentum around the z axis.

[3] To avoid confusion with the spin density, Schwinger's notation $d\sigma_i$ for the volume element is discarded.

According to (12) and (13),

$$\frac{\partial}{\partial x}+i\frac{e}{\hbar}A_x=\cos\theta\frac{\partial}{\partial r}-\sin\theta\left[\frac{1}{r}\frac{\partial}{\partial\theta}+i\frac{e}{\hbar}a(r)\right],$$
$$\frac{\partial}{\partial y}+i\frac{e}{\hbar}A_y=\sin\theta\frac{\partial}{\partial r}+\cos\theta\left[\frac{1}{r}\frac{\partial}{\partial\theta}+i\frac{e}{\hbar}a(r)\right], \quad (13)$$

the four Dirac equations may be written

$$\left\{(\sigma_1\cos\theta+\sigma_2\sin\theta)\frac{df}{dr}+(\sigma_2\cos\theta-\sigma_1\sin\theta)\left[\frac{1}{r}\frac{\partial}{\partial\theta}+\frac{i}{\hbar}\left(ea(r)+\frac{C}{r}\right)\right]f-\epsilon_1\frac{i}{\hbar}\frac{W}{c}f\right\}\Lambda^{-1}(\theta)\varphi_0=0; \quad (14)$$

it is well known that the double sign $\epsilon_1=\pm1$ allows jointly for the possibility of performing space inversions and describing antiparticles.[4]

Following postulate (d), we will seek solutions with the same cylindrical symmetry as the external potential (9); that is, we will identify $\Lambda(\theta)$ with the matrix

$$\Lambda^{\pm1}(\theta)=\cos\tfrac{1}{2}\theta\pm i\sigma_3\sin\tfrac{1}{2}\theta, \quad (15)$$

such that

$$\Lambda^{-1}\sigma_1\Lambda=\sigma_1\cos\theta+\sigma_2\sin\theta,$$
$$\Lambda^{-1}\sigma_2\Lambda=\sigma_2\cos\theta-\sigma_1\sin\theta. \quad (16)$$

$\Lambda(\partial/\partial\theta)\Lambda^{-1}=-\tfrac{1}{2}i\sigma_3$ is easily calculated, so that (14) can be rewritten as

$$\left\{\left(\frac{df}{dr}+\frac{f}{2r}\right)\sigma_1+\frac{i}{\hbar}\left[\left(ea(r)+\frac{C}{r}\right)\sigma_2-\epsilon_1\frac{W}{c}\right]f\right\}\varphi_0=0. \quad (17)$$

We now specify the representation of the σ's according to postulates (d), (e), and (f). When $\theta=0$, $\Lambda=\Lambda^{-1}=1$; we want φ_0 to be an eigenfunction of the spin projection on a unitary vector $u_1=0$, $u_2=\cos\alpha$, $u_3=\sin\alpha$; thus, we choose

$$\sigma_1=\sigma_{01},\quad \sigma_2=\sigma_{02}\cos\alpha+\sigma_{03}\sin\alpha,\quad \sigma_3=\sigma_{03}\cos\alpha-\sigma_{02}\sin\alpha, \quad (18)$$

with essentially $\alpha=$ const., the σ's being the standard set

$$\sigma_{01}=\begin{pmatrix}0&1\\1&0\end{pmatrix},\quad \sigma_{02}=\begin{pmatrix}1&0\\0&-1\end{pmatrix},\quad \sigma_{03}=\begin{pmatrix}0&i\\-i&0\end{pmatrix}; \quad (19)$$

and we take φ_0 as one of the two eigenfunctions

$$\varphi_0^+=\begin{pmatrix}u\\0\end{pmatrix},\quad \varphi_0^-=\begin{pmatrix}0\\u\end{pmatrix}, \quad (20)$$

with the normalization condition

$$u^*u=1. \quad (21)$$

[4] See, for instance, J. Hamilton, *The Theory of Elementary Particles* (Clarendon Press, Oxford, England, 1959), pp. 138, 139.

With solutions of the type (12), (18)–(21), the spin component tangent to the H helices of common slope α obeying the equations $r=$ const., $z=\alpha r(\theta-\theta_0)$, remains constant and equal to $\pm\tfrac{1}{2}\hbar$. In addition, the field trajectories of the current and spin-density vectors (which are collinear) are the H helices, as seen through (18), (19), and (20).

Finally we show that the integration of Eqs. (17) may be achieved along these lines. Inserting (18), (19), and (20) into (17) yields the two equations

$$ea(r)+C/r=\epsilon_1\epsilon_2 W/c\cos\alpha \quad (22)$$

and

$$\frac{df}{dr}+\left[\frac{1}{2r}-\epsilon_2\frac{\sin\alpha}{\hbar}\left(ea(r)+\frac{C}{r}\right)\right]f=0; \quad (23)$$

the double sign $\epsilon_2=\pm1$ is the same as in (20). Therefore, according to (22), the radial distribution of the vector potential (9) is not arbitrary, but depends on the *a priori* choice of the constants W, C, and α (energy and mean angular momentum of the particles, slope of their trajectories). The C/r term is merely a gauge term, ensuring that the kinetic energy momentum is a null 4-vector, as it has to be.

Inserting (22) in (23) yields

$$\frac{df}{dr}+\left[\frac{1}{2r}-\epsilon_1\frac{W\tan\alpha}{c\hbar}\right]f=0, \quad (24)$$

the integral of which is (B real)

$$f(r)=\frac{B}{\sqrt{r}}\exp\left(\frac{W\epsilon_1\tan\alpha}{c\hbar}r\right). \quad (25)$$

Thus, as was expected, we find a definite relation between the (common) slope α of the helical-current (and spin) lines, and the radial distribution of the waves amplitude $f(r)$; let us recall that, as the ψ wave has been taken as z-independent, there is no momentum component along the z axis.

There are essentially two classes of solutions according to the sign of $W\epsilon_1\tan\alpha$ (the intermediate case $\alpha=0$ being the one where no inertial spin effect exists, as the current lines are circles of axis z). Let us recall that W is taken as greater or less than 0 according as one deals with particles or antiparticles,[4] and restrict ourselves for brevity to the particle case: $W>0$.

We consider first the class of solutions $\epsilon_1\tan\alpha<0$, which are square integrable in the sense that

$$2\pi\int_0^{+\infty}f^2(r)rdr=-\frac{2\pi c\hbar}{W\epsilon_1\tan\alpha} \quad (26)$$

is finite. Then, according to the well known correspondence between energy and spin states in the two-component Fermion theory,[4] we see that the positive-

(negative-) helicity states are deviated downwards (upwards).

The class of solutions $\epsilon_1 \tan\alpha > 0$ is not square integrable, as it diverges in the upper limit $r \to +\infty$; consequently, when it is used, some kind of a cutoff should be introduced. These solutions are of physical interest since, going from $+\infty$ to $+\infty$ through a minimum, they are able to describe, in the minimum region, the behavior of a homogeneous thin beam injected tangentially in the field (10) at the right distance r. This can be worked out by simply solving (23) at the point $df/dr = 0$, which yields

$$\epsilon_1 \tan\alpha = c\hbar/2rW. \quad (27)$$

Thus, in this case, the positive- (negative-) helicity states will be deviated upwards (downwards).

III. CONNECTION BETWEEN THE PRECEDING FORMULAS AND THE GENERAL ONES IN THE INTRODUCTION

We will show now, by two different methods, that the connection existing between the transverse momentum of the particles and the nonzero value of curl$\boldsymbol{\sigma}$ is precisely the one described in the introduction.

First Method

The transverse momentum $\mathbf{T} = \iint \boldsymbol{\sigma} \times d\mathbf{s}$ contained inside a ring of radius r, thickness dr, and height dz is, according to formulas (18)–(25), parallel to the z axis with the value

$$dT = \pi\hbar \cos\alpha d(f^2 r) dz = 2\pi c^{-1} W \epsilon_1 \sin\alpha f^2 r dr dz; \quad (28)$$

inside the same volume, the probability of the presence of the Fermion (or, in a superquantized version, the mean number of Fermions) is

$$dn = 2\pi f^2 r dr dz, \quad (29)$$

so that the transverse momentum per Fermion is

$$T = dT/dn = c^{-1}\epsilon_1 W \sin\alpha. \quad (30)$$

But,

$$L = \epsilon_1 W/c \quad (31)$$

is the classical, longitudinal, momentum, so that finally[5]

$$T/L = \sin\alpha. \quad (32)$$

Incidentally, in the class of solutions $W\epsilon_1 \tan\alpha < 0$, formulas (28) and (29) are integrable over the whole plane $z = \text{const.}$, yielding, according to formula (25),

$$dn/dz = -\pi B^2 c\hbar / W\epsilon_1 \tan\alpha, \quad (33)$$

which is a normalization equation for the constant B, and

$$dT/dz = \pi B^2 \hbar. \quad (34)$$

The non-nullity of the latter expression, which is essentially due to the behavior of the wave function in the limit $r = 0$, shows that, in the problem under discussion, the various tensors $\alpha T^{ij} + \beta T^{ji}$, $\alpha + \beta = 1$, deduced from Tetrode's tensor (1) are *not* integrally equivalent; and that *the* correct one is unambiguously T^{ij}, which yields, by integration over the whole plane $z = \text{const.}$, $dP/dz = 0$. In particular, use of the classical symmetrized tensor $(\alpha = \beta = \frac{1}{2})$ would yield the wrong result $dP/dz = \frac{1}{2} dT/dz \neq 0$.

Second Method

According to formulas (8), (18)–(21), and (24), the "transverse power"[4] $d\bar{\omega}_t = c^2 d\oint \boldsymbol{\sigma} \cdot \delta\mathbf{l}$ through the ring comprised, in any plane $z = \text{const.}$, between two circles of radii r and $r + dr$ is

$$d\bar{\omega}_t = c^2 \epsilon_2 \cos\alpha \pi\hbar d(f^2 r) = 2\pi\epsilon_1\epsilon_2 \sin\alpha c W f^2 r dr. \quad (35)$$

This, according to what was said at the end of the introduction, must be precisely equal to the "true" or "physical" power running through the ring, i.e., the power transported by the deflected particles; this is $\epsilon_1 W$ times the particle flux through the ring, $2\pi c f^2 \varphi_0^\dagger \sigma_3 \varphi_0 r dr$, or, according to formulas (18) to (21),

$$d\bar{\omega} = 2\pi\epsilon_1\epsilon_2 \sin\alpha c W f^2 r dr. \quad (36)$$

By comparing (35) and (36) one verifies that $d\bar{\omega}_t = d\bar{\omega}$, and thus proves that the helical shape of the current lines, together with the absence of any momentum component along the z axis, is a direct consequence of the non-nullity of curl$\boldsymbol{\sigma}$, as explained in the introduction.

IV. DISCUSSION AND CONCLUSIONS

Physically speaking, the transverse dimensions of the accelerator are finite and this is obtained by a suitable choice of the external field. We are primarily interested in the effect of the radial limitation of the beam.

Calculating the integral (8) or (35) over a ring, normal to the z axis, limited by two circles outside the beam, one arbitrarily small and one arbitrarily large, will yield a zero value; this means that, globally, the transverse (z) component of the Dirac current induced by the existence of (nonzero) curl$\boldsymbol{\sigma}$ inside the beam, and that induced by the existence of a (strong) curl$\boldsymbol{\sigma}$ in the inner and outer sides of the beam, will compensate each other. In other words, one may conveniently distinguish an "interior" region of the beam, where the transverse deviation described in Secs. II and III will appear under appropriate radial distributions of the directing field and of the wave amplitude; and two sides of the beam, where transverse deviations (of opposite sign) will appear, which globally compensate the preceding one. Thus, if the experiment is conducted in such a way that only the total integral deviation is measured, no effect will be observed; and this *mutatis mutandis* is why a positive

[5] The presence of $\sin\alpha$ rather than $\tan\alpha$ is due to the fact that the integration all along a circular path has cut off the tangential projection of the particle's transverse momentum.

effect cannot be observed in the experiment described in our previous paper[1] (contrary to what we concluded there).

But, experimenting with a fluid as is described above, a positive effect should be observable, for it is then possible to test independently the probabilities of impacts of particles along the various stream lines of the Dirac current; that is, transitions are studied between the dynamical state of the beam and states corresponding to point locations of particles. This type of experiment, more refined than the one implying a global integration, should be able to distinguish, from among the various integrally equivalent[6] energy-momentum tensors, the one describing *locally* the true or physical energy-momentum flux.

The natural way to test experimentally the "transverse inertial spin effect" thus predicted would be to inject tangentially a monokinetic homogeneous beam of particles of energy W at the distance r defined by formula (27); thus, the two helicity states should be deviated in opposite z directions, as expressed by formula (27).

Of course, if one intended to use this procedure as a means for separating the two spin states of the particles, the callback in the z direction by the directing field should, at least, be partially removed; the corresponding difficult stability problems one would have to deal with are beyond the scope of the present study.

[6] The reason why the various energy-momentum tensors are integrally equivalent in the present physical circumstances and were not in the more schematic picture of Secs. II and III is of course due to the behavior of the ψ wave in the lower (eventually, upper) limits of the radius r.

Another important technical difficulty which we do not intend to discuss is associated with the imperfect vacuum inside the accelerator. Any Fermion colliding with the electron shell of an atom present along the track will be lost for testing the effect; the collapse of the wave packet associated with this position determination of the incident fermion will entail a corresponding narrowing of the subsequent probability current (as in the Wilson-chamber experiment); thus integrations all over the beam will be implied and, as explained above, a zero effect will follow as far as the subsequent path of these particles is considered. Finally, the effect we describe should be fully observable only on particles which undergo their first collision in the receiver.

Note added in proof. An explicit calculation has now been given in the case of a radial limitation of the beam[7]; the contents of the present paper are thus completely confirmed.

The effect has also been calculated in the case of a vertical limitation of the beam, obtained by a Fourier superposition of solutions analogous to those of the present paper (but with a phase exponent $C\theta+kz-Wt$).[8] The conclusion is that if the k distribution of amplitudes is a Gaussian one, centered on the value $k=0$ of the z-momentum component, the transformed Gaussian z distribution is centered on a value Z such that $\epsilon_1 Z/\sin\alpha = r\theta/\cos\alpha = ct$.

[7] Compt. Rend. **257**, 3327 (1963); the notations were defined in Compt. Rend. **256**, 4608 (1963).

[8] Compt. Rend. **258**, 1745 (1964). Formula (42) of that paper is obviously erroneous; it should be written with an m, and thus and f, essentially k-independent; this is a necessary condition for the validity of the conclusion that has been drawn.