# Nonrenormalizability and the Short-Range Force in Some Field-Theoretic Models*

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Bethe-Salpeter equations for the scattering Green's functions are discussed in some nonrenormalizable models. The models involve the multiple exchanges of pairs of Dirac particles, coupled to the scattering particles by a four-Fermi interaction. The Green's function is constructed from a Bethe-Salpeter scattering wave function. For certain forms of coupling the forces are repulsive at short distances and analogous to potentials in a nonrelativistic scattering problem which behave as $r^{-6}$ at the origin. In these cases there is a unique, well defined, and Fourier transformable solution for the Green's function in space-time; and the scattering amplitude exists. A class of terms corresponding to delta-function potentials may be included in the interaction kernel without changing the solution. In a case with zero-mass particles and zero total energy an exact solution for the scattering amplitude is obtained.

## I. INTRODUCTION

IN this paper we discuss the solution of some nonrenormalizable Bethe-Salpeter equations. These are integral equations for two-body scattering Green's functions which arise in an approximation to a nonrenormalizable field theory. Even if we manage to find honest solutions it will be unclear what their significance is to the general problem of unrenormalizable field theories, since we are dealing here with just one of the Green's functions of the theory, and only in an approximation. Nonetheless our models, if expanded in perturbation theory, give rise to all of the pathologies of a nonrenormalizable theory. That is, their perturbation developments require an infinite number of subtractions. There is thus some interest in seeing whether there exist well defined nonperturbative solutions.[1]

Our equation is an integral equation for the scattering of two Dirac particles, where the interaction term comes from four-Fermi interactions. The approximation is the ladder approximation, where the exchanged object is a Fermion bubble and the graphs generated by an (incorrect) iteration procedure are those of Fig. 1. We shall consider several different forms of coupling.

The main point of our work is that in some of the cases to be considered the integral equation for the scattering Green's function has a perfectly well defined and unique solution. No regulators, cutoffs, or arbitrary procedures of any kind are involved in defining this solution. The solution for the Green's function in space-time is Fourier transformable; the scattering amplitude exists. No concept of a sum of graphs is involved. For these well-defined cases in general we can prove only the existence of a solution. The only cases we have managed to solve exactly involve the scattering of mass-zero particles by exchanges of mass-zero particles.

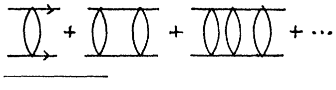

FIG. 1. The graphs which would be obtained if we (incorrectly) iterated the Bethe-Salpeter equation.

There is a way of gaining some qualitative insight into the nature of our solutions, in terms of singularities of forces at short distances. Our nonrenormalizable equation is quite similar in structure to a Schrödinger equation in a potential which has a $r^{-6}$ singularity at the origin. Our well-defined cases correspond to a repulsive short-range force. In the corresponding potential-scattering problems the scattering amplitude is well defined, but the integrals in the perturbation expansion diverge more and more strongly as the order of perturbation theory is increased. Predazzi and Regge have given a systematic development of scattering formalism for such potentials[2] and it is our aim to develop the relativistic analog to their work.

The great restriction in our work will be to zero-total four momentum. We shall consider the amplitude for scattering two particles of equal mass, analytically continued to zero-total energy in the center-of-mass system. From analogy to the potential-scattering problem we may anticipate, however, that certain interesting properties of the scattering amplitude are independent of the total energy, following only from the behavior of the wave function at the origin. For our short-range repulsions the behaviors near the origin are independent of the energy.

It has been noted that in a number of renormalizable models the location of a branch point in the angular-momentum plane depends only on behavior near the origin.[3] In our nonrenormalizable models there will be a branch point in the coupling-constant plane, at $G=0$, the nature of which is probably energy-independent.

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[1] There have been a number of schemes for defining a result in nonrenormalizable models: R. Arnowitt and S. Deser, Phys. Rev. **100**, 349 (1955); L. N. Cooper, *ibid.* **100**, 362 (1955); T. D. Lee and C. N. Yang, *ibid.* **119**, 1410 (1960); G. Feinberg and A. Pais, *ibid.* **131**, 2724 (1963).

[2] E. Predazzi and T. Regge, Nuovo Cimento **24**, 518 (1962).

[3] Fixed branch points in the $l$ plane for the Goldstein amplitude (see Ref. 7) are implicit in the discussion by G. C. Wick [Phys. Rev. **96**, 1124 (1952)]. For boson theories ($\lambda\varphi^4$ and vector-meson exchange) they are discussed by R. F. Sawyer [Phys. Rev. **131**, 1384 (1963)] and by Baker and Muzinich (to be published); for Fermion theories by A. R. Swift and B. W. Lee [Phys. Rev. **131**, 1857 (1963)]. The general connection between such branch points and the solution near the origin is implicit in Refs. 11. The most complete treatments are by G. Domokos and P. Suranyi, Nucl. Phys. (to be published); and by G. Cosenza, L. Sertorio, M. Toller (to be published).

Another reason for interest in the zero four-momentum case is the fact that it corresponds to zero momentum transfer for the crossed process. Our models, therefore, give some predictions for forward scattering in a physical region.

## II. BETHE-SALPETER SCATTERING FORMALISM

We consider the scattering process, $A+\bar{A}\rightarrow A+\bar{A}$, where $A$ is a Dirac particle of mass $m$. Let $p_1$, $p_2$ be the incident four momenta of $A$ and $\bar{A}$, respectively; $p_1'$, $p_2'$ the final momenta. We define

$$W=p_1+p_2=p_1'+p_2',$$
$$p=\tfrac{1}{2}(p_1-p_2)\quad p'=\tfrac{1}{2}(p_1'-p_2'). \tag{1}$$

The $T$-matrix element for scattering may be written in the form,

$$T(p',p,W)=M\bar{U}_\gamma{}^c(\tfrac{1}{2}W+p')U_\alpha{}^a(\tfrac{1}{2}W+p)\bar{U}_\beta{}^b(\tfrac{1}{2}W+p)\times U_\delta{}^d(-\tfrac{1}{2}W+p')T_{\alpha\beta,\gamma\delta}(p',p,W), \tag{2}$$

where

$$T_{\alpha\beta,\gamma\delta}(p',p,W)=-i\int d^4x d^4x' e^{-ip'\cdot x'+ip\cdot x}T_{\alpha\beta,\gamma\delta}(x',x,W). \tag{3}$$

Here $M$ is the usual product of $2\pi$'s and $(M/E)^{1/2}$'s. The labels $a$ and $c$ distinguish the initial and final spin states for the particles; $b$ and $d$ for the antiparticles. We are describing the antiparticles as particles with reversed four momenta. The object we shall study is the scattering operator in relative space-time coordinates for zero-total four momentum.

$$T_{\alpha\beta,\gamma\delta}(x',x)\equiv T_{\alpha\beta,\gamma\delta}(x',x,W_\mu=0). \tag{4}$$

The Bethe-Salpeter equation in the ladder approximation for our models involving a Fermion bubble exchange is of the form,

$$T_{\alpha\beta,\gamma\delta}(x',x)=V_{\alpha\beta,\gamma\delta}(x)\delta^4(x-x')+\int d^4x_1d^4x_2T_{\alpha\beta,\gamma'\delta'}(x',x_1)\times S_{\gamma'\gamma''}{}^F(x_1-x_2)S_{\delta'',\delta'}{}^F(x_2-x)V_{\gamma''\delta'',\gamma\delta}(x). \tag{5}$$

Here the interaction function $V(x)$ depends on the nature of the coupling chosen. Various choices are discussed in Sec. III. The details of the reduction to relative coordinates and the derivation of Eq. (5) are in the Appendix.

It should be noted that Eqs. (3) and (5) together already involve a continuation out of the physical region if the scattering particles have nonzero mass. For the case of $W_\mu=0$ the mass shell is defined by $p^2=-m^2$, $p'^2=-m^2$, $p_0=p_0'=0$; i.e., the relative momenta $p$ and $p'$ are purely spatial and imaginary. From the later developments, it will be clear that the Fourier transform (3) still correctly defines the analytically continued scattering amplitude provided the minimum exchanged mass is greater than or equal to $2m$, where $m$ is the mass of the scattering particle $A$.

Whether or not this inequality is satisfied, a general procedure is to begin with the four momentum of each external particle space-like.

$$p^2=p'^2=p_1{}^2=p_2{}^2=p_1'^2=p_2'^2=\lambda^2\quad(\text{for}\quad W_\mu=0),$$
$$p_0=p_0'=0.$$

In this case the relative momenta $p$ and $p'$ are real and the Fourier transformation (3) is well defined. The continuation to the mass shell $\lambda^2=-m^2$ is to be deferred until the end of the calculation.

We shall claim a superiority of Eq. (5) over other formulations of the scattering problem, for example sums of graphs or integral equations in momentum space. Therefore, an approach to the ladder approximation which avoids perturbation theory is required, for example the Green's function approach of Schwinger.[4] The reduction of the Green's function equation to our Eq. (5) for the scattering operator is given in the Appendix.

In our four-Fermi coupling models the interaction function, $V(x)$, leaving aside spinor complications, will be of the general form $(S_F(x))^2$. This is well defined except on the light cone, but the Fourier transform does not exist, so that the conversion of Eq. (5) to momentum space is not possible. One might object that also in space-time Eq. (5) has no meaning because of the lack of definition of $V(x)$ on the light cone. However, we shall find certain favorable cases in which this lack of definition has no effect on the solutions.

Next we restrict our considerations to a single invariant amplitude.[5] We define

$$T_1(x',x)=\tfrac{1}{4}(\gamma_5)_{\beta\alpha}(\gamma_5)_{\gamma\delta}T_{\alpha\beta,\gamma\delta}(x',x). \tag{6}$$

Roughly speaking, $T_1(x',x)$ is an amplitude for singlet $\bar{A}A$ scattering.

The interaction functions $V_{\alpha\beta,\gamma\delta}(x)$, which arise in our models, all have the following property,

$$(\gamma_5)_{\gamma\delta}V_{\alpha\beta,\gamma\delta}(x)=(\gamma_5)_{\alpha\beta}V(x). \tag{7}$$

Restricting to this class of interactions, Eq. (5) leads directly to an integral equation for the amplitude $T_1(x',x)$,[6]

$$T_1(x',x)=-V(x)\delta^4(x-x')-i\times\int d^4x''T_1(x',x'')\Delta_F(x''-x)V(x). \tag{8}$$

This equation is the starting point for all that follows. It is closely related to Goldstein's differential equation

[4] J. Schwinger, Proc. Natl. Acad. Sci. U.S. **37**, 452 (1951).

[5] Much of the apparatus for treating the other amplitudes is contained in a paper by W. Kummer (to be published).

[6] Using $\int dx''S_F(x-x'')\gamma_5S_F(x''-x')=-i\gamma_5\Delta_F(x-x')$.

for a bound-state wave function.[7] Differential equations are not sufficient for our purpose; the boundary conditions implied by Eq. (8) will be very important.

We now assume that Eq. (8) can be transformed into an equation in a four-dimensional Euclidean space. This assumption is discussed in the Appendix. The standard objections to using the Euclidean space for a scattering problem do not apply since we are below threshold for real scattering. It is found that there is no difficulty in constructing the scattering amplitude from the Euclidean Green's function, for the case of $W_\mu=0$, if the individual four momenta of the scattering particles are taken to be space-like.

The Euclidean $T$ operator is defined as[8]

$$T^{(e)}(x',x)=iT_1(\mathbf{x}',x_0'e^{-(i\pi/2)};\,\mathbf{x},\,x_0e^{-(i\pi/2)}). \qquad (9)$$

Since we shall deal always with the case $p_0=p_0'=0$, we can rotate the time contours clockwise in the integrals in (3) to obtain,

$$T_1(p',p)=\int d^4x d^4x' e^{-i(p'\cdot x')+ip\cdot x}T^{(e)}(x',x). \qquad (10)$$

The Euclidean $T$ operator obeys the equation,

$$T^{(e)}(x',x)=V(x)\delta^4(x-x')+\int d^4x'' T_1^{(e)}(x',x'')\Delta_F(x''-x)V(x), \qquad (11)$$

where the Euclidean space is to be understood throughout. $T^{(e)}(x',x)$ and $T_1(p',p)$ are now to be expanded in the four-dimensional partial-wave series,[9]

$$T^{(e)}(x',x)=\sum_{n=0}^{\infty}\frac{n+1}{2\pi^2}C_n\left(\frac{x_\mu x_\mu'}{rr'}\right)T^{(n)}(r',r), \qquad (12)$$

$$T_1(p',p)=\sum_{n=0}^{\infty}\frac{(n+1)}{2\pi^2}C_n\left(\frac{p_\mu' p_\mu}{qq'}\right)T^{(n)}(q',q), \qquad (13)$$

where $q=(p_\mu^2)^{1/2}$, $r=(x_\mu^2)^{1/2}$. The $C_n(x)$ are Tschebyscheff polynomials (see Appendix).[10]

We note the relation

$$T^{(n)}(q',q)=\frac{16\pi^2}{qq'}\int_0^\infty drdr'r^2r'^2 \times J_{n+1}(q'r')T^{(n)}(r',r)J_{n+1}(qr). \qquad (14)$$

The equation for $T^{(n)}(r'r)$ follows from Eq. (11),

$$T^{(n)}(r',r)=\frac{V(r)}{r^3}\delta(r-r')+\int_0^\infty dr''(r'')^3T^{(n)}(r',r'')G_n(r'',r)V(r), \qquad (15)$$

where

$$G_n(r_1,r_2)=\frac{I_{n+1}(mr_<)K_{n+1}(mr_>)}{r_1r_2}, \qquad (16)$$

$$r_<=\min(r_1,r_2) \quad r_>=\max(r_1,r_2).$$

$I$ and $K$ are modified Bessel functions.

From the integral equation (15) and the equation for the scattering amplitude (14) we can construct the scattering amplitude in terms of a scattering wave function $\psi(q,r)$,

$$T^{(n)}(q',q)=\frac{+16\pi^2}{q'q}\int_0^\infty r^2drJ_{n+1}(q'r)V(r)\psi_n(q,r). \qquad (17)$$

The wave function $\psi_n(q,r)$ is given by the solution to the integral equation,

$$\psi_n(q,r)=\frac{J_{n+1}(qr)}{r}+\int_0^\infty dr'(r')^3 \times G_n(r,r')V(r')\psi_n(q,r'). \qquad (18)$$

The mass shell is defined by $q$, $q'=im$. We now go onto the mass shell for the initial particles, $q=im$, keeping the four momenta of the final particles space-like for the moment ($q'$real). From (18) and (16) follows the behavior of $\psi_n(im,r)$ at infinity,

$$\psi_n(im,r)\underset{r\to\infty}{\longrightarrow}\frac{e^{mr}}{r^{3/2}}\left(\frac{\pi}{2m}\right)^{1/2}(i)^{n+1}. \qquad (19)$$

The potential $V(r)$ behaves like $e^{-\mu_0 r}$ at infinity, where $\mu_0$ is the minimum mass exchanged. When $m<\mu_0$ there is no problem in going onto the mass shell for the incident state directly in Eq. (17). To approach the mass shell for both the initial and final particles without worrying about continuations requires $2m<\mu_0$.

Setting $q=im$ in Eq. (18) gives an integral equation which is transformable into the differential Bethe-Salpeter equation for the scattering wave function[11] $\psi_n(\mathrm{Im},\mathrm{Re})$

$$(\nabla_r^2-m^2)\psi_n(im,r)=V(r)\psi_n(im,r), \qquad (20)$$

where

$$\nabla_r^2=\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}-\frac{n(n+2)}{r^2}. \qquad (21)$$

[7] J. S. Goldstein, Phys. Rev. **91**, 1516 (1953).

[8] J. Schwinger, Proc. Natl. Acad. Sci., U.S. **44**, 617 (1958).

[9] The first use, known to us, of the expansion in hyperspherical harmonics in a scattering problem was by M. Gourdin, thesis, University of Paris, 1959 (unpublished). The expansion (13) for a zero-energy problem, was used by M. Baker and I. Muzenich (to be published) who attribute it to J. D. Bjorken.

[10] See, e.g., *Higher Transcendental Function*, edited by A. Erdelyi (McGraw-Hill Book Company, Inc., New York 1955), Vol. 2 Chapters 10 and 11.

[11] Related radial Bethe-Salpeter equations in renormalizable models have been considered recently in connection with the bound-state problem by A. Bastai, L. Bertocchi, S. Fubini, G. Furlan, and M. Tonin (to be published), and by G. Domokos and P. Suranyi, Nucl. Phys. (to be published). Also see S. Fubini, in Report at the Stanford Conference on Nuclear Structure, 1963 (unpublished).

This equation is to be supplemented with a boundary condition at the origin which we can determine only by knowing the potential $V$ and a boundary condition at infinity, Eq. (19) (really a normalization condition, analogous to the normalization of the incident flux in ordinary scattering theory).

The problem has now been formulated: to solve Eq. (20) with the boundary condition (19) at infinity and with behavior at the origin such that the integrals in Eqs. (14), (15), and (18) converge. We now look for models in which this problem has a unique solution.

## III. THE INTERACTION FUNCTIONS

Our Bethe-Salpeter kernels are those associated with the graph of Fig. 2. We take the interaction Lagrangian,

$$\mathcal{L}_I=\tfrac{1}{2}\lambda(\bar{\psi}_A\Gamma^a\psi_A)(\bar{\psi}_B\Gamma_a\psi_B) \tag{22}$$

and consider S, P, V, or A couplings.

The potential $V(x)$ defined by Eqs. (7) and (5) is given by

$$V(x)=-\frac{\lambda^2}{16}(\mathrm{Tr}\Gamma^a\gamma_5\Gamma^b\gamma_5)\mathrm{Tr}\{[(i\gamma^\mu\partial_\mu-\mu)\Delta_F(x)]\Gamma_a \times[(-i\gamma^\mu\partial_\mu-\mu)\Delta_F(x)]\Gamma_b\}\,, \tag{23}$$

where $\mu$ is the mass of $B$ and $\Delta_F(x)$ is now the Feynman propagator with mass $\mu$.

For the four types of coupling we shall consider the potentials defined by Eq. (23) come out to be (with Euclidean metric now understood)

(a) Scalar $V(r)=\lambda^2[-(\partial_\mu\Delta_F(x))^2+m^2\Delta_F{}^2(x)]$, (24a)

(b) Pseudoscalar
$$V(r)=\lambda^2[(\partial_\mu\Delta_F(x))^2+\mu^2\Delta_F{}^2(x)]\,, \tag{24b}$$

(c) Vector $V(r)=\lambda^2[-2(\partial_\mu\Delta_F(x))^2-4\mu^2\Delta_F{}^2(x)]$, (24c)

(d) Axial vector
$$V(r)=\lambda^2[2(\partial_\mu\Delta_F(x))^2-4\mu^2\Delta_F{}^2(x)]\,. \tag{24d}$$

Leaving out the $\delta$ functions at the origin we have

$$\Delta_F(r)=(\mu/4\pi^2r)K_1(\mu r)\,. \tag{25}$$

We see that in each case $V(r)$ behaves near the origin as $r^{-6}$.

$$\lim_{r\to 0}V(r)=4G/r^6\,,$$

where

$$\begin{array}{lll}(\mathrm{a}) & (\mathrm{S}) & G=-\lambda^2/4\pi^2\,,\\ (\mathrm{b}) & (\mathrm{P}) & G=\lambda^2/4\pi^2\,,\\ (\mathrm{c}) & (\mathrm{V}) & G=-\lambda^2/2\pi^2\,,\\ (\mathrm{d}) & (\mathrm{A}) & G=\lambda^2/2\pi^2\,.\end{array} \tag{26}$$

These singularities in the cases S and V are attractive, in the cases P and A repulsive, if we make an analogy of Eq. (20) to the Schrödinger equation.

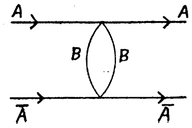


FIG. 2. The Bethe-Salpeter kernel.

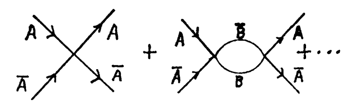


FIG. 3. "Local" kernels. They do not change the solution for the repulsive cases and are to be discarded.

## IV. SOLUTIONS NEAR THE ORIGIN

The solutions to Eq. (20) near the origin for the cases a and c in which the $r^{-6}$ term in $V(r)$ is attractive are of the form.

$$\psi_n(im,r)\to\mathrm{const}\times\exp\left[\pm\frac{i\sqrt{|G|}}{r^2}\right],\quad r\to 0\,. \tag{27}$$

In the repulsive cases, b and d, the solutions are of the form

$$\psi_n(im,r)\to\mathrm{const}\times\exp\left[\pm\frac{\sqrt{G}}{r^2}\right],\quad r\to 0\,. \tag{28}$$

In the repulsive case it is clear that the regular solution, that is the solution which vanishes exponentially at the origin, must be chosen in order that the integrals in the integral equation (18) exist. This integrability is the only boundary condition to be met, the condition at infinity, Eq. (19), being a normalization convention which can be satisfied except in the exceptional case of a bound state.

Therefore we find a unique and well defined solution to the integral equations (18) and (16) in the case of a repulsive short-range force.

The term $\delta(r^2)$, which was omitted from the Feynman propagator in Eq. (25) cannot have any effect on the scattering in the repulsive cases, in which the wave function vanishes at the origin faster than any power of $r$. Likewise certain additional terms which correspond to $\delta$-function potentials may be added to our original Bethe-Salpeter kernel without changing the scattering. Figure 3 gives the diagrammatic representation of two such "local" kernels.

In the case of the attractive short-range force, Eq. (27), the solution is apparently not well defined. The boundary condition at $r=0$ is ambiguous since both behaviors of (27) at the origin are equally regular. Also there is no reason to discard the $\delta$-function terms at the origin in this case. These terms are completely undefined since the potentials (24) involve products of singular functions.

There is a possibility of treating this case by supplementing the differential equation (20) with a boundary condition at the origin. The field-theoretic analog of the procedure of Case[12] for the treatment of scattering by singular potentials would be to impose the boundary condition,

$$\psi_n(im,r)\underset{r\to 0}{\longrightarrow}\mathrm{const}\,\cos((|G|/r^2)+\varphi)\,. \tag{29}$$

[12] K. M. Case, Phys. Rev. **80**, 797 (1950).

Here $\varphi$ is an additional parameter necessary to define the theory. Its physical significance is unclear. By analogy with Ref. 12, $\varphi$ should be independent of the total energy. But it is unclear whether one should pick a single $\varphi$ for all angular-momentum states $n$, or a different $\varphi(n)$ for each value of $n$. The latter choice would involve the infinite number of parameters one expects in a nonrenormalizable theory. Choosing a universal $\varphi$, however, might give a well-defined theory; we could even hope that choosing $\varphi$ was equivalent to giving a definition to the product of singular functions which occurs in the potential.

Though all our considerations have been for the case of zero-total energy we may anticipate that the behaviors at the origin (27) and (28) are independent of the total energy. The difficulties in carrying through an analysis which takes correct account of the most singular force and treats the energy as a perturbation (in analogy to the work of Predazzi and Regge) are connected with the additional variables which enter the problem when $W_\mu \neq 0$.[13] We have found no simple way to approach this problem.

## V. AN EXACT SOLUTION

We consider the cases of pseudoscalar or axial vector coupling, (24b,d), with zero-mass particles exchanged. For $\mu=0$ Eq. (24b) becomes

$$V(r)=4G/r^6. \tag{30}$$

We also take zero mass for the scattering particles, $m=0$. The differential equation for $\psi_n(0,r)\equiv\psi_n(r)$, Eq. (20), becomes

$$\left(\frac{d^2}{dr^2}+\frac{3}{r}\frac{d}{dr}-\frac{n(n+2)}{r^2}\right)\psi_n(r)=\frac{4G}{r^6}\psi_n(r). \tag{31}$$

The regular solution of (32) is,

$$\psi_n(r)=(A/r)K_{(n+1)/2}(\sqrt{G}/r^2). \tag{32}$$

Because all masses are zero we may no longer use the boundary condition (19) to determine the constant $A$ in (32). We determine the normalization instead by substituting the potential (30) into the integral Eq. (18) and taking the limit $q\to 0$, obtaining,

$$\psi_n(r)=\frac{(2)^{-n-1}q^{n+1}}{(2+n)}r^n -\frac{G}{n+1}r^{-n-2}\int_0^r dr'(r')^{n-3}\psi(r') -\frac{G}{n+1}r^n\int_r^\infty dr'(r')^{-n-5}\psi(r'). \tag{33}$$

Here we have retained the leading term for small $q$. We shall set $q=0$ later. By direct substitution we verify the solution of (33) to be given by Eq. (32) with

$$A=(G)^{n/4+1/4}q^{n+1}2^{-3/2n-3/2}[\Gamma(1+n)\Gamma(\tfrac{1}{2}\mu+\tfrac{3}{2})]^{-1}. \tag{34}$$

The mass-shell $T$-matrix element is given by Eq. (17) in the limit in which $q$ and $q'$ approach zero. For the case $n=0$ the powers of $q$, $q'$ disappear completely and we obtain

$$T^{(0)}{}_{(0,0)}=\frac{(2)^{5/2}\pi^2}{\Gamma(\frac{3}{2})}(G)^{5/4}\int_0^\infty dr r^{-4}K_{1/2}\left(\frac{\sqrt{G}}{r^2}\right)=\tfrac{1}{4}\pi^2\sqrt{G}. \tag{35}$$

All the remaining terms in (13) vanish in the limit $q, q'\to 0$. We thus obtain a closed form for the complete $T$-matrix element of (10).

$$T_1(0,0)=\tfrac{1}{8}\sqrt{G}=\lambda/16\pi. \tag{36}$$

Since the total energy and all masses are zero for this solution, $\lambda$ is in fact the only quantity of dimension (length)$^2$ remaining in our theory. With all particles of zero mass, however, infrared divergences might have been anticipated. Thus we expect a result of the form

$$\lim_{q\to0} T_1(q,q)=\lim_{q\to0}\lambda f(\lambda q), \tag{37}$$

in which the limit might be expected to be zero or infinity. What is worthy of note here is that the limit is finite.

## VI. DISCUSSION

Since we have dealt only with the very special case of a single invariant amplitude, at zero total energy, and in the ladder approximation, we must ask if any of the results are of more general significance.

The interesting property of our model is that it contains all of the pathologies of a nonrenormalizable field theory, in a perturbation expansion. Yet it allows a well defined and unique solution, provided that the problem is formulated from the Green's function equations in space-time. The integral equation itself cannot be written in momentum space, although the solution is Fourier transformable so that the scattering amplitude exists.

There are two properties of the "bubble exchange" potential (interaction kernel) which create the difficulty in the perturbation expansion, and it is important for our purposes to consider these properties separately. The potential near the origin in the Euclidean space (i.e., near the light cone) consists of an analytic part with an $r^{-6}$ singularity at the origin plus a part with an undefined product of $\delta$ functions. However, we escape the problem of defining this product in the cases in which the coefficient of the $r^{-6}$ singularity corresponds to a repulsion. In this case the Bethe-Salpeter scattering wave function approaches zero so strongly at the origin

[13] A possible method is outlined by G. Domokos and P. Suranyi (unpublished).

that no products of $\delta$ functions and their derivatives in the potential can have an effect on the scattering. An additional dividend in these cases is that we can discard a class of contributions corresponding to "local" graphs (Fig. 3) from the interaction kernel.

Our first hope for a more complete theory is thus that a short-range repulsion may act between every pair of particles. If this repulsion is sufficiently singular we may expect that every two-body Bethe-Salpeter scattering wave function goes at small distances as $\exp(-\lambda r^{-a})$ where $a>0$, and that the solutions to all integral equations in space-time are unique, well defined, and Fourier transformable.

But it is quite likely that no model can be found with a repulsion in every state (e.g., for $AA$ scattering as well as $A\bar{A}$ scattering). It must then be asked if we can replace the effect of the undefined terms at $r=0$ by a boundary condition, analogous to that of Case in the case of a too-singular attractive potential. This procedure would introduce new parameters, but possibly only a finite number of them.

## APPENDIX A

We begin from the integral equation for the Green's function,

$$G_{12}(x_1,x_2;x_1',x_2')=S_1^F(x_1-x_1')S_2^F(x_2-x_2') + \int G_{12}(x_1,x_2;x_1''',x_2''')V_{12}(x_1''',x_2''',x_1'',x_2'') \times S^F{}_{(1)}(x_1''-x_1')S^F{}_{(2)}(x_2''-x_2')dx_1'''dx_2'''dx_1''dx_2'', \quad \text{(A1)}$$

where spinor indices have been suppressed but are implied in the subscripts 1 and 2.

Abbreviating this equation as

$$G_{1,2}=S_1S_2+G_{12}V_{12}S_1S_2, \quad \text{(A2)}$$

we define $T$ of Eq. (5) by

$$G_{12}=S_1S_2+S_1S_2TS_1S_2. \quad \text{(A3)}$$

The integral equation for $T$ follows,

$$T_{12}=V_{12}+T_{12}S_1S_2V_{12}. \quad \text{(A4)}$$

In $T_{12}(x_1,x_2;x_1',x_2')$ we introduce relative coordinates

$$x=x_1-x_2, \quad x'=x_1'-x_2', \quad y=x_1+x_2-x_1'-x_2'. \quad \text{(A5)}$$

We define

$$T(x,x';W)=\int e^{(iW/2)\cdot y}T(x,x',y)d^4y. \quad \text{(A6)}$$

For the particular case $W_\mu=0$ and

$$V(x_1,x_2,x_1',x_2')=\delta^4(x_1-x_2')\delta^4(x_2-x_2')V(x_1-x_2), \quad \text{(A7)}$$

the integral equation for $T(x,x')\equiv T(x,x',0)$ reduces to (5).

According to Ref. 8 Eq. (A1) can directly be transformed to an equation of the same form in a Euclidean space. If we accept this result the only question remaining is whether the time integrals for the removal of the center-of-mass motion (A6) and in the expression for the scattering amplitude in momentum space (3) may be rotated to the imaginary axis. As mentioned before there is no problem in rotating the contours in (3) provided that the relative energies $p_0$ and $p_0'$ are zero The time integral in (A6) however may be rotated (clockwise by $\frac{1}{2}\pi$) only in the region, $W_0<2m$, because of the convergence problem at $y_0=\infty$. In our case, $W_\mu=0$ there is no difficulty.

## APPENDIX B

We list some properties of the hyperspherical harmonics which have been used (see Ref. 10)

$$d\Omega_k=\sin\beta(d\cos\beta)(d\cos\theta)(d\varphi), \quad \text{(B1)}$$

$$C_n(\cos\beta)=\sin(n+1)\beta/\sin\beta, \quad \text{(B2)}$$

$$\frac{1}{2\pi^2}\int d\Omega_k C_n(\cos\beta_{ik})C_n(\cos\beta_{kf})=\frac{\delta_{nm}}{n+1}C_n(\cos\beta_{if}), \quad \text{(B3)}$$

$$\delta^4(p-k)=\frac{\delta(|p|-|k|)}{k^3}\sum_{n=0}^{\infty}\frac{n+1}{(2\pi)^2}C_n(\cos\beta_{p,k}), \quad \text{(B4)}$$

$$e^{ip_\mu x_\mu}=\sum_{n=0}^{\infty}2(n+1)(i)^n\frac{J_{n+1}(pr)}{pr}C_n(\cos\beta_{p,x}). \quad \text{(B5)}$$

For $x_4=ix_0$, and $x_4'=ix_0'$ real (i.e., for the Euclidean case)

$$\Delta_F(x-x')=\sum_{n=0}^{\infty}\frac{n+1}{2\pi^2}C_n(\cos\beta)\frac{I_{n+1}(mr_<)K_{n+1}(mr_>)}{rr'}, \quad \text{(B6)}$$

$$r=(x_\mu^2)^{1/2}; \quad r_<=\min r, r'; \quad r_>=\max r, r'.$$