Analysis of the 142-MeV Proton-Proton Scattering Data*

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After some amount of data selection, modified phase-shift analyses were made on the 137.5–147-MeV proton-proton scattering data. Standard deviations were obtained for phase shifts of the Stapp-Ypsilantis-Metropolis solution type 1. As previously reported by Perring, solution 2 was found to be very improbable. The pion-nucleon coupling constant was not well defined by the data at this energy. The Amati-Leader-Vitale phases were examined, and several discrepancies noted. Further experiments are suggested.

I. INTRODUCTION

EXPERIMENTS in the neighborhood of 142 MeV have produced more proton-proton scattering data than at any other energy. Included are single-, double-, and triple-scattering measurements. In addition, this is the only energy at which comprehensive sets of experiments have been performed at two different laboratories.

Perring[1,2] has made an analysis of some of the data, using the "modified phase shift analysis" originated by Moravcsik.[3] The higher angular-momentum (L) phases were assumed to be exactly one-pion-exchange (OPE), while the lower L phases were adjusted so as to produce a least-squares fit to the data. Perring did not compute the standard deviations for the final low-L phases he obtained, nor did he investigate the effects of his data selection. Those points are among the main items to be examined here.

II. DATA USED AND TREATMENT

All of the 168 data considered for this analysis are shown in Table I. Perring used 71 of these, apparently because of limitations in his computer program. His selection consisted of all of the Harwell data, plus the Harvard triple-scattering parameters D, R, and A. He did not use the 147-MeV Harvard cross section, which has a shape noticeably different from that found by the Harwell group at 142 MeV.[4] Perring assumed that the unusual Harwell shape could be accounted for by small changes in the phases as the energy is changed. This assumption is examined in Fig. 1, which shows the ratio $\sigma(45°)/\sigma(90°)$ as a function of energy. The crosses denote values from smoothing cross sections. One of us (P. S.) and Yoder have made several preliminary energy-dependent modified phase-shift analysis,[4] using various parameterizations and data combinations. The cross section ratio produced thereby was always a very slowly varying function of energy. Thus, there appears to be a real conflict in the cross section shapes, with the weight of the evidence from nearby energies quite strongly in favor of the Harvard results (see Fig. 1).[5] Next, Perring's phases were compared to those obtained at higher energies by one of us.[6] Perring's $^3F_2$ and $^1G_4$ phases were rather out of line, while analyses using the Harvard data alone were not. Finally, modified phase analyses were made, with 11 free (searched upon) phases and with OPE for the contributions of higher L phases. With all 168 data, the Harwell cross section points contributed an average of 4.1 per datum to χ^2, compared to 1.1 for the Harvard cross section data. Considering all of the above evidence, it was decided that the Harwell cross section would not be included in most of the work to be reported here.

With the remaining 132 data, the analysis was remade. The χ^2 ratio, χ^2 divided by its expected value, dropped from 2.05 to 1.17. It was noticed that the 142-MeV Harwell polarization set still yielded an average χ^2 of 2.0 per datum, little change from its 2.1 per datum in the 168 data analysis. The Harwell $s(\theta)=P(\theta)/(\sin\theta\cos\theta)$ varies rapidly with angle in the region 65–90°; the Harvard measurements, and all

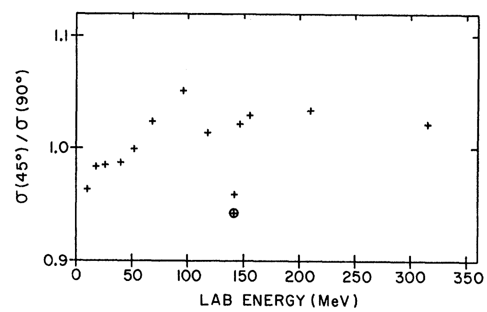

FIG. 1. The experimental ratios $\sigma(45°)/\sigma(90°)$ for proton-proton scattering at several energies. The circled 142-MeV point is the prediction of OPE(11) (see text, Sec. IV), when the Harwell cross section was substituted for the Harvard cross section and χ^2 reminimized.[5]

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[1] J. K. Perring, Nucl. Phys. **30**, 424 (1962).

[2] J. K. Perring, Nucl. Phys. **42**, 306 (1963).

[3] M. H. MacGregor, M. J. Moravcsik, and H. P. Stapp, Phys. Rev. **116**, 1248 (1959), and previous publications cited therein.

[4] P. Signell and N. R. Yoder, Phys. Rev. **134**, B100 (1964).

[5] The references for the points are: 9.69 MeV Minnesota, Phys. Rev. **116**, 989 (1959); 18.2 MeV Princeton, Phys. Rev. **95**, 1226 (1954); 25.63 MeV Minnesota, Phys. Rev. **118**, 1080 (1960); 39.4 MeV Minnesota, Phys. Rev. **111**, 212 (1958); 51.8 MeV Tokyo, INSJ-45, 1961 (unpublished); 68.3 MeV Minnesota, Phys. Rev. **119**, 313 (1960); 95, 118, 147 MeV Harvard, Ann. Phys. **5**, 299 (1958); 142 MeV Harwell, Nucl. Phys. **16**, 320 (1960); 156 MeV Orsay, J. Phys. Rad. **22**, 628 (1961); 213 MeV Rochester, thesis of A. Konradi, 1961 (unpublished); 315 MeV Berkeley, Phys. Rev. **105**, 288, Table II (1957).

[6] P. Signell, Phys. Rev. **133**, B982 (1964).

TABLE I. Data used. N_σ indicates (absolute) normalization for the relative σ's following it. Likewise, N_P indicates the normalization for the relative P's which follow it.

Experimental energy (MeV)	C. M. angle (degrees)	Type	Parameter	Error	Reference, remarks
137.5	43.0	R'	0.562	0.052	a
	52.5		0.472	0.054	
	62.0		0.376	0.068	
	72.5		0.238	0.084	
	82.1		0.251	0.121	
139.0	31.1	A	−0.368	0.032	b
	41.4		−0.344	0.031	
	51.7		−0.311	0.035	
	61.9		−0.231	0.046	
	72.0		−0.187	0.056	
	82.1		−0.099	0.079	
140.5	31.1	R	−0.252	0.030	c
	41.1		−0.227	0.028	
	51.7		−0.271	0.035	
	61.9		−0.146	0.037	
	72.0		−0.151	0.055	
	82.1		−0.047	0.080	
142.0	12.0	$\sigma_{\text{int.}}$	24.2	0.3	d interpolated
142.0		N_σ	1.00	0.045	e used only in preliminary analyses
	5.19	σ	69.24	2.60	
	6.23		28.77	0.59	
	7.27		12.96	0.43	
	8.30		7.75	0.28	
	8.82		5.76	0.30	
	9.34		4.91	0.25	
	10.38		4.37	0.21	
		N_σ	1.00	0.045	e used only in preliminary analyses
	10.38	σ	4.32	0.14	
	12.46		3.63	0.13	
	14.53		3.77	0.14	
	16.61		4.10	0.14	
	20.76		3.74	0.13	
	25.95		3.83	0.16	
	31.06		3.60	0.16	
		N_σ	1.00	0.045	e used only in preliminary analyses
	10.38	σ	4.27	0.04	
	12.46		3.34	0.06	
	14.53		3.28	0.05	
	16.61		3.39	0.05	
	20.76		3.63	0.05	
	25.95		3.62	0.06	
	31.06		3.62	0.06	
	37.20		3.95	0.06	
	41.34		3.66	0.05	
		N_σ	1.00	0.045	e used only in preliminary analyses
	20.76	σ	3.44	0.07	
	25.95		3.77	0.07	
	31.06		3.90	0.07	
	41.34		3.89	0.07	
	51.62		3.96	0.07	
	61.84		3.90	0.07	
	71.98		4.00	0.07	
	82.06		4.02	0.07	
	90.00		4.08	0.07	
142.0		N_P	1.00	0.02	e used only in preliminary analyses
	5.19	P	−0.037	0.034	
	6.23		−0.027	0.009	
	8.30		0.031	0.024	
	9.34		0.089	0.023	
	10.38		0.153	0.035	
	10.38		0.107	0.021	
	12.46		0.130	0.033	
	14.53		0.180	0.031	
	16.61		0.155	0.028	
	20.76		0.189	0.009	
	24.80		0.216	0.037	
	25.95		0.225	0.011	
	31.06		0.241	0.010	
	37.20		0.283	0.030	
	41.34		0.238	0.010	
	45.45		0.242	0.005	
	49.55		0.240	0.004	
	51.62		0.229	0.006	
	53.65		0.213	0.004	
	57.70		0.205	0.006	
	59.75		0.197	0.005	
	61.84		0.183	0.005	
	65.90		0.170	0.005	
	69.95		0.141	0.005	
	71.98		0.118	0.005	
	74.05		0.097	0.006	
	78.05		0.054	0.007	
	82.06		0.060	0.009	
142.0	12.45	D	−0.262	0.063	f
	20.76		−0.008	0.038	
	31.06		0.137	0.033	
	41.34		0.156	0.031	
	51.62		0.178	0.033	
	61.48		0.076	0.031	
	71.98		0.147	0.070	
	82.06		0.286	0.099	
142.0	24.0	R	−0.224	0.051	g
	32.7		−0.203	0.051	
	45.7		−0.178	0.031	
	54.4		−0.212	0.042	
	67.2		−0.213	0.040	
	76.1		−0.147	0.063	
	84.0		−0.142	0.136	
	90.0		0.110	0.131	
143.0	31.1	D	0.082	0.077	h
	41.4		0.162	0.040	
	51.7		0.110	0.050	
	61.9		0.045	0.060	
	72.0		0.019	0.100	
	82.1		−0.037	0.133	
	92.2		−0.027	0.170	
143.0	31.1	A	−0.408	0.032	i
	41.4		−0.377	0.037	
	51.7		−0.342	0.050	
	61.9		−0.355	0.075	
	72.0		−0.198	0.079	
	82.1		0.022	0.154	
147.0		N_σ	1.00		j
	12.4	σ	3.79	0.10	
	14.5		3.88	0.10	
	16.6		4.02	0.10	
	18.7		4.03	0.10	
	20.7		4.15	0.10	
	22.8		4.14	0.11	
	24.9		4.26	0.11	
	31.1		4.22	0.11	
		N_σ	1.00		j
	20.7	σ	4.17	0.08	
	25.9		4.29	0.08	
	31.1		4.39	0.08	
	36.3		4.31	0.08	
	41.4		4.21	0.04	
	46.5		4.21	0.04	
	51.7		4.16	0.04	

TABLE I (*Continued*).

Experimental energy (MeV)	C. M. angle (degrees)	Type	Parameter	Error	Reference, remarks
	56.8		4.14	0.04	
	61.9		4.12	0.04	
	67.0		4.12	0.04	
	68.0		4.09	0.05	
	72.0		4.07	0.04	
	72.9		4.14	0.05	
	77.1		4.06	0.05	
	77.9		4.12	0.05	
	82.1		4.07	0.05	
	82.9		4.13	0.05	
	87.2		4.11	0.05	
	87.8		4.12	0.05	
147.0		N_P	1.00	0.03	j
	6.20	P	−0.004	0.014	
	8.34		0.045	0.014	
	10.4		0.103	0.014	
	12.4		0.126	0.011	
	14.5		0.155	0.014	
	16.6		0.180	0.010	
	18.7		0.193	0.015	
	20.7		0.198	0.009	
	22.8		0.183	0.015	
	24.9		0.227	0.014	
	25.9		0.203	0.011	
	31.1		0.228	0.009	
	36.3		0.247	0.011	
	41.4		0.239	0.006	
	46.5		0.233	0.006	
	51.7		0.229	0.006	
	56.8		0.205	0.006	
	61.9		0.171	0.006	
	67.0		0.154	0.006	
	68.0		0.144	0.009	
	72.0		0.131	0.006	
	72.9		0.109	0.008	
	77.1		0.098	0.006	
	77.9		0.068	0.008	
	82.1		0.052	0.008	
	82.9		0.041	0.007	
	87.2		0.030	0.008	
	87.8		0.006	0.009	

[a] S. Hee and R. Wilson, Harvard Cyclotron Report, May 1963 (to be published).
[b] S. Hee and E. H. Thorndike, Phys. Rev. **132**, 744 (1963).
[c] E. Thorndike, J. Le Francois, and R. Wilson, Phys. Rev. **120**, 1819 (1960).
[d] J. N. Palmieri (private communication).
[e] A. E. Taylor, E. Wood, and L. Bird, Nucl. Phys. **16**, 320 (1960).
[f] C. F. Hwang, T. R. Ophel, E. H. Thorndike, and R. Wilson, Phys. Rev. **119**, 352 (1960).
[g] L. Bird, D. N. Edwards, B. Rose, A. E. Taylor, and E. Wood, Phys. Rev. Letters **4**, 312 (1960).
[h] L. Bird, P. Christmas, A. E. Taylor, and E. Wood, Nucl. Phys. **27**, 586 (1961).
[i] O. N. Jarvis, B. Rose, J. P. Scanlon, and E. Wood, Nucl. Phys. **42**, 294 (1963).
[j] J. N. Palmieri, A. M. Cormack, N. F. Ramsey, and R. Wilson, Ann. Phys. **5**, 299 (1958). All N_σ's in this reference have been withdrawn (private communication from R. Wilson). Small angle points were omitted because of possible multiple-scattering corrections (private communication from A. M. Cormack).

analyses and models, yield an $s(\theta)$ which varies only slowly with angle in the aforementioned range. The high χ^2 of the points is thus an indication of an incompatibility of the Harwell $s(\theta)$ with this kind of analysis and with the Harvard $s(\theta)$. When the Harwell polarization data were removed, the remaining 103 data yielded a χ^2 ratio of 0.86. Close examination showed no data group with an abnormally high contribution to χ^2, and

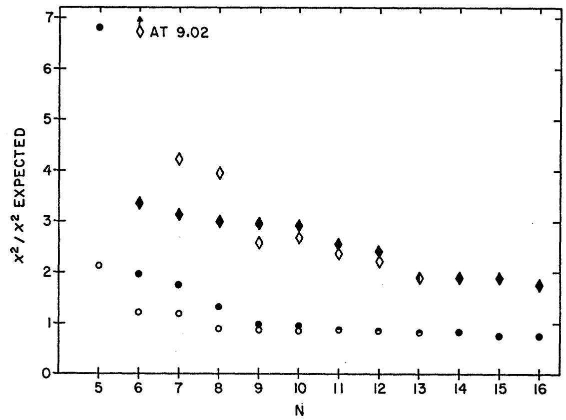

FIG. 2. The ratio of χ^2 to its expected value for various numbers N of free (searched-upon) low-angular momentum phases. Open circles (○) indicate ALV(N), solution 1; darkened circles (●), OPE(N). Open diamonds (◇) denote ALV(N), solution 2; darkened diamonds (♦), OPE(N) for solution 2. Half-darkening indicates coinciding points.

in fact no contributions higher than what one would expect from normal fluctuations. This set was then adopted as standard for the rest of the work.

All of the data were treated as though they had been measured at 142 MeV, this figure being a compromise among the actual experimental energies. As discussed above, the *shapes* of the angular distributions seem to change rather slowly with energy, so there should not be much error in taking the Harvard *relative* values to be at a slightly different energy than the measured one. The triple-scattering parameters were less accurately measured, so could probably tolerate the small energy shifting of several of them. In addition, energy-dependent analyses of the type mentioned above were made with the data taken first at the experimental energies and then at the single energy. The differences in the phase parameters were slight compared to their standard deviations.

The cross section and polarization relative angular distributions were treated in the same fashion as by Perring,[2] except that normalization of the experimental standard deviation was properly included here.[7]

III. METHOD AND NOTATION

The procedures for minimizing χ^2 and for computing the standard deviations on the low-L phases were as in a previously reported analysis at 52 MeV.[6] Explanations of notation will also be found there. Briefly, OPE(N) indicates that N low-L phases were searched upon, the other phases being fixed at their OPE values. ALV(N) indicates that those phases which were not searched upon, and had $L\leq 5$, were fixed at their Amati-Leader-Vitale[8] theoretical values. In both cases, the

[7] P. Signell, N. R. Yoder, and N. M. Miskovsky, Phys. Rev. **133**, B149 (1964).
[8] D. Amati, E. Leader, and B. Vitale, Phys. Rev. **130**, 750 (1963).

TABLE II. Results of modified phase analyses on the 103-piece data set, using OPE for the higher L contributions. Nuclear bar, in degrees.

N	χ^2	χ^2 ratio	$^1S_0$	$^3P_0$	$^3P_1$	$^3P_2$	ϵ_2
6	189.7	1.96	16.69±0.73	5.89±0.54	−16.48±0.24	14.15±0.19	−2.95±0.14
7	170.3	1.77	15.99±0.72	5.76±0.52	−16.78±0.25	14.05±0.18	−3.04±0.13
8	124.0	1.31	16.45±0.63	5.88±0.46	−16.91±0.21	14.02±0.16	−2.75±0.12
9	91.4	0.97	17.30±0.61	7.43±0.49	−16.88±0.18	14.03±0.14	−2.45±0.12
10	85.8	0.92	16.55±0.66	6.67±0.58	−17.28±0.24	13.89±0.15	−2.62±0.13
11	79.2	0.86	16.52±0.64	6.26±0.59	−17.35±0.23	13.88±0.15	−2.63±0.13
12	76.8	0.84	16.93±0.73	6.39±0.69	−17.21±0.29	13.95±0.17	−2.54±0.16
13	74.5	0.83	17.05±0.78	6.60±0.76	−17.08±0.34	14.01±0.18	−2.53±0.17
14	73.5	0.83	17.06±0.76	6.35±0.76	−17.04±0.32	14.01±0.17	−2.56±0.15
15	67.5	0.77	16.76±0.75	6.74±0.74	−17.00±0.31	14.13±0.17	−2.45±0.16
16	66.0	0.76	16.77±0.73	6.45±0.70	−17.04±0.28	14.11±0.16	−2.39±0.15
11[a]	322.0	2.05	16.00±0.94	6.17±0.97	−17.33±0.37	13.80±0.22	−2.46±0.19
11[b]	141.6	1.17	16.15±0.72	6.20±0.70	−17.52±0.26	13.99±0.15	−2.50±0.12
ALV(10)	78.7	0.85	16.54±0.64	6.39±0.57	−17.32±0.23	13.89±0.15	−2.63±0.13
11[c]	228.4	2.48	−8.12±0.87	−28.93±1.06	2.76±0.38	14.32±0.30	−4.26±0.27

N	$^1D_2$	$^3F_2$	$^3F_3$	$^3F_4$	ϵ_4	$^1G_4$	$^3H_4$
6	5.61±0.19						
7	5.82±0.19	0.94±0.15					
8	5.92±0.16	0.22±0.17	−1.45±0.19				
9	5.37±0.18	−0.43±0.18	−1.31±0.17		−0.50±0.06		
10	5.25±0.18	0.25±0.35	−1.69±0.22	0.85±0.19	−0.51±0.06		
11	4.99±0.19	0.51±0.36	−1.78±0.21	1.05±0.20	−0.57±0.06	0.83±0.11	
12	4.94±0.20	0.14±0.47	−1.75±0.24	0.79±0.31	−0.64±0.07	0.86±0.12	
13	5.05±0.22	0.02±0.53	−1.72±0.26	0.74±0.34	−0.59±0.07	0.83±0.13	
14	5.20±0.23	0.10±0.53	−1.68±0.25	0.75±0.33	−0.58±0.08	0.84±0.12	−0.02±0.21
15	5.23±0.23	0.17±0.50	−1.66±0.25	0.80±0.31	−0.54±0.08	0.64±0.14	0.09±0.20
16	5.36±0.24	0.30±0.48	−1.62±0.24	0.81±0.29	−0.49±0.08	0.57±0.14	−0.14±0.24
11[a]	4.86±0.30	−0.07±0.51	−1.68±0.35	0.71±0.30	−0.50±0.09	0.69±0.15	
11[b]	4.91±0.22	0.39±0.40	−1.68±0.23	1.10±0.22	−0.52±0.06	0.84±0.13	
ALV(10)	5.02±0.17	0.41±0.35	−1.80±0.21	1.01±0.18	−0.55±0.05		
11[c]	3.74±0.32	4.02±0.29	−3.51±0.18	1.62±0.15	−1.70±0.16	0.25±0.20	

N	$^3H_5$	$^3H_6$	ϵ_6	$^1I_6$
6				
7				
8				
9				
10				
11				
12		0.21±0.07		
13	−0.39±0.11	0.22±0.07		
14	−0.23±0.19	0.08±0.16		
15	−0.31±0.18	0.21±0.16		0.39±0.08
16	−0.21±0.19	0.09±0.17	−0.15±0.05	0.31±0.09
11[a]				
11[b]				
ALV(10)				
11[c]				

[a] 168-piece data set. [b] 132-piece data set. [c] Solution 2, 103-piece data set.

highest L phases were represented by their OPE contributions with $g^2=14.4$, $\mu=135.1$ MeV. Perring's analyses would here be labeled OPE(9) and OPE(12).

IV. RESULTS OF THE MODIFIED PHASE ANALYSES

A minimum in the χ^2 surface was reached for each of the analyses reported in Table II. The ratio of χ^2 to its expected value is plotted versus the number, N, of searched-upon phases, in Fig. 2. The dark circles are the OPE points, the open circles denote ALV. The order in which the phases were released was arranged in an attempt to reach a minimum as soon as possible, and to make the slope of the points monotonically increasing. However, no phase was put before another which was two angular momentum units below it. For example, an H phase was never released before an F phase, even if it was desirable on statistical grounds. This is nothing more than a centrifugal barrier argument.

Using the χ^2 test, one finds in Table II that the OPE($N\geq 9$) and ALV($N\geq 8$) runs all resulted in about the same χ^2 probability.[3] The F-test probabilities[3] never reached a usefully high (at least $\frac{1}{3}$) value, so are not shown. Strictly statistically then, one should go to a higher number of search parameters in order to increase the F-test probabilities. Note, however, the behavior of the $^1I_6$ phase in Table II. When released, it went

TABLE III. Comparison of phase shifts from various models (see text). Values are nuclear bar, in degrees.

N	$^1S_0$	$^3P_0$	$^3P_1$	$^3P_2$	ϵ_2	$^1D_2$	$^3F_2$	$^3F_3$	$^3F_4$	ϵ_4	$^1G_4$	$^3H_4$	$^3H_5$	$^3H_6$
Perring(12)	16.0	6.8	−17.1	14.1	−2.7	5.7	−0.3	−1.1	0.2	−0.5	0.5	0.2		
YLAM	13.29	4.01	−17.19	14.21	−2.69	5.67	0.55	−2.55	0.40	−0.85				
YRB1	16.04	8.94	−17.19	12.95	−2.83	5.21	0.23	−1.36	0.32	−0.52	0.50			
SBM	19.90	7.13	−16.90	14.40	−3.00	4.79	1.13	−2.33	0.53	−0.84	0.61	0.19	−0.53	0.08
HJ	16.84	4.66	−16.10	14.43	−2.75	5.42	1.26	−2.04	0.63	−0.82	0.62	0.25	−0.47	0.05
Yale	12.91	4.95	−17.52	14.47	−2.95	6.15	0.98	−2.62	0.73	−1.02	0.74	0.26	−0.61	0.13
SW	12.7	6.30	−15.0	14.36	−3.04	5.64	0.95	−2.08	1.05	−0.90	0.67	0.21		
OPE					−4.26	2.03	1.44	−2.59	0.40	−0.85	0.56	0.21	−0.56	0.08
ALV						5.47	0.19	−2.05	1.39	−0.25	0.79	0.23	−0.52	0.12
OPE(11)	16.46	6.22	−17.38	13.88	−2.61	4.98	0.52	−1.78	1.07	−0.57	0.84			
	(±0.60)	(±0.59)	(±0.23)	(±0.15)	(±0.12)	(±0.11)	(±0.36)	(±0.20)	(±0.20)	(±0.06)	(±0.10)			

immediately to a value three standard deviations away from its OPE value of 0.17°. The χ^2 dropped correspondingly, making the F-test probability much less than one percent for OPE being correct for $^1I_6$. Yet, current potential[9] models yield $^1I_6$ phases which differ from the OPE value by at most 0.02° at 142 MeV, even though some of the models contain strong, but short range, quadratic spin-orbit interactions. *We regard as unphysical the large departure of* $^1I_6$ *from the* OPE *and*

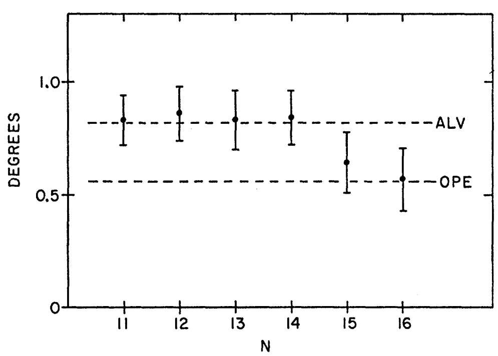

FIG. 3. The predicted $^1G_4$ phase shift for OPE(N) versus N.

model values. Figure 3 shows the behavior of the $^1G_4$ phase as the number of search phases is increased. The statistically preferred value is at OPE, but this is regarded as spurious in light of the above discussion.

Comparison of Tables II and III shows that the $^3H_6$ phase in OPE(12) is outside the range of the model values. As in the case of the $^1I_6$, we regard this as probably unphysical. Here, however, the model values are rather dispersed. We have chosen to take the $^3H_6$ OPE value as a median. Among the OPE runs, then, we favor OPE(11). Figures 4–9 display the predictions of OPE(11) for the various kinds of data.

Using the ALV higher L phases, the 14 lowest L phases were again arranged in such an order that the χ^2 ratio was made as small as possible at each number of search parameters. Since the ALV $^3H_4$ and $^3H_5$ phases are almost exactly at the OPE values (see Table III), the ALV($N\geq 13$) are the same as OPE($N\geq 13$) and consequently are not shown in Table II. From this equality and the criterion used for the OPE series, one should select ALV(11). Note, however, that the latter is almost identical to ALV(10), because of the ALV $^1G_4$ being at the desired value. Finally, then, $N=10$ is taken as the preferred ALV run.

V. OTHER MODELS

There are a number of two-nucleon models which one can use to make predictions for the 142-MeV data. Table III compares the phase shifts from the Perring[2] modified analysis, the Yale phase-shift representations[10] YLAM and YRB1, the Saylor-Bryan-Marshak[11] (SBM) boundary-condition-plus-potential model, the Hamada-

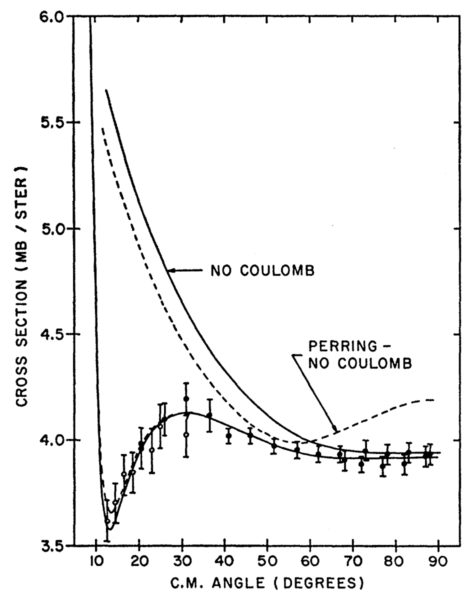

FIG. 4. The Harvard cross section data, renormalized by OPE(11) values N_σ(small angles) = 0.954, N_σ(large angles) = 0.955. The open circles (○) are from the small angle telescope, the darkened (●) from the large angle telescope. The OPE(11) prediction with the Coulomb amplitude is the solid line; the dashed line denotes OPE(11) with the Garren amplitude. The OPE(11) and Perring predictions with $e^2=0$ are also shown for comparison.

[9] The $^1I_6$ phases were computed from the parameters of the models listed in Table III.

[10] G. Breit, M. H. Hull, Jr., K. E. Lassila, and K. D. Pyatt, Jr., Phys. Rev. **120**, 2227 (1960).

[11] D. P. Saylor, R. A. Bryan, and R. E. Marshak, Phys. Rev. Letters **5**, 266 (1960), and corrections in P. Signell and N. R. Yoder, Phys. Rev. **132**, 1707 (1963).

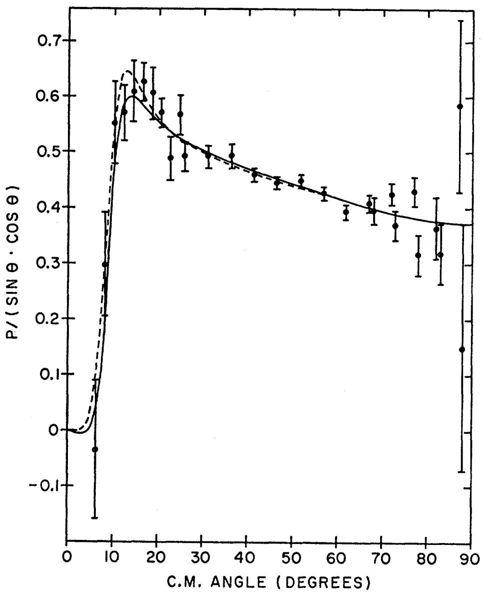

FIG. 5. The Harvard $s(\theta)=P(\theta)/(\sin\theta\cos\theta)$ data, renormalized by the OPE(11) value $N_P=0.952$. The solid line is the OPE(11) prediction using the Coulomb amplitude; the dashed line, using the Garren amplitude.

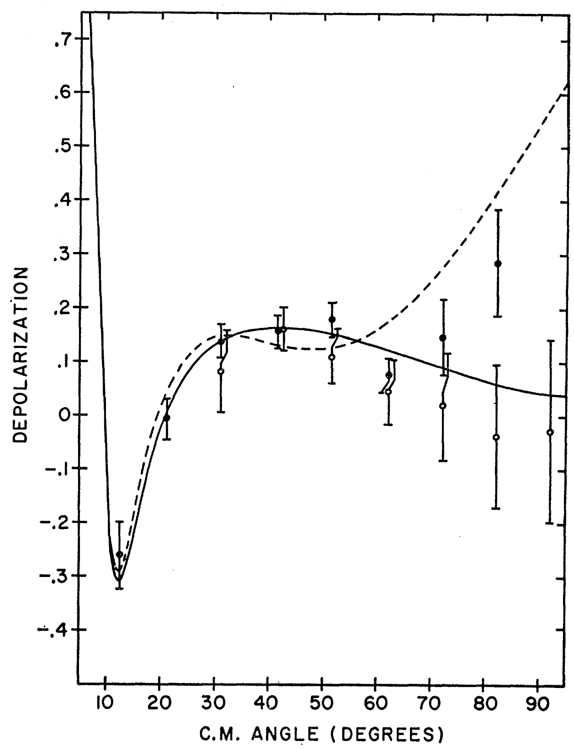

FIG. 6. The Harvard (●) and Harwell (○) $D(\theta)$ data. The solid line is OPE(11) for solution 1; the dashed line, solution 2.

Johnston[12] (HJ) potential, the Yale[13] potential type model, and the Scotti-Wong[14] (SW) resonant-boson-exchange model. The phases for the models were obtained as in a previous communication.[6]

The χ^2 fit to the 103 data for the various models is shown in Table IV. Release of the $^1S_0$ phase from its model value had little effect on any model fit except for that of YRB1, where the χ^2 dropped from 529 to a minimized 337. This would seem to indicate a possible defect in the YRB1 parameterization of the $^1S_0$ phase shift.

The Perring phases produced a χ^2 larger than for the SBM, HJ, and Yale models. Over $\frac{2}{3}$ of the Perring χ^2 was contributed by the Harvard cross section, again indicating the incompatibility of the Harvard and Harwell cross section shapes.

TABLE IV. Goodness-of-fit of various models to the 142-MeV data.

Model	χ^2	χ^2/χ^2 [OPE(11)]	Remarks
Perring(12)	303	3.8	σ has Harwell shape
YLAM	241	3.0	σ(20–40°) too small, R too positive
YRB1	523	6.1	N_P too small, σ has Harwell shape, D too negative
SBM	238	3.0	N_P too small, $\sigma_{\text{int.}}$ too large
HJ	209	2.6	N_P too small, D(60°) too positive
Yale	234	3.0	$\sigma_{\text{int.}}$ too large, R(50–70°) too positive
SW	498	6.3	N_P too small, σ(45°) too large, R too positive, $\sigma_{\text{int.}}$ too small
ALV(10)	78.7	0.99	

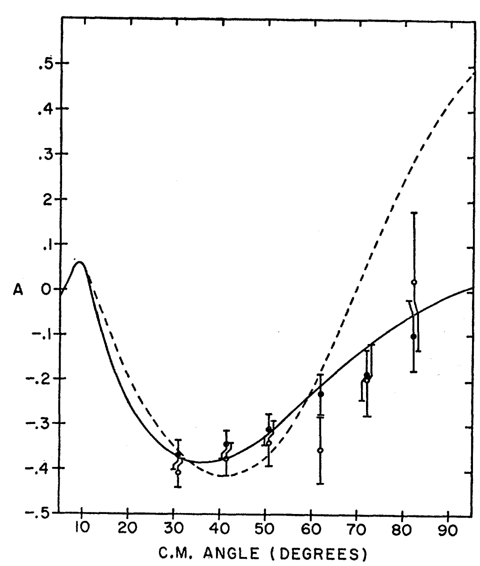

FIG. 7. The Harvard (●) and Harwell (○) $A(\theta)$ data. The solid line is OPE(11) for solution 1; the dashed line, solution 2.

[12] T. Hamada and I. D. Johnston, Nucl. Phys. **34**, 382 (1962).
[13] K. E. Lassila, M. H. Hull, Jr., H. M. Ruppel, F. A. McDonald, and G. Breit, Phys. Rev. **126**, 881 (1962).
[14] A. Scotti and D. Y. Wong, Phys. Rev. Letters **10**, 142 (1963).

TABLE V. Fractional *increases* in the phase-shift standard deviations, for OPE(11), resulting from the removal of various data groups. N_P, σ(small angle), and R' are not shown: Their removal produced only negligible effects.

Data removed	$^1S_0$	$^3P_0$	$^3P_1$	$^3P_2$	$^1D_2$	ϵ_2	$^3F_2$	$^3F_3$	$^3F_4$	$^1G_4$	ϵ_4
$\sigma_{\text{int.}}$	0.48	−0.07	3.34	2.01	0.29	1.34	0.26	0.50	0.07	0.17	0.05
σ(large angle)	0.10	0.17	0.15	0.19	0.24	0.49	0.23	0.29	0.72	0.47	0.57
P	0.18	−0.03	0.44	0.19	0.25	0.38	0.17	0.29	0.66	0.15	0.17
D	0.21	1.22	0.04	0.01	0.17	0.33	0.55	0.33	0.25	0.04	0.38
R	0.65	1.14	1.34	1.83	0.85	0.66	0.23	0.16	0.59	0.69	0.09
A	−0.04	0.17	0.18	0.05	0.13	0.61	0.45	1.02	0.29	0.08	0.16

VI. CONSISTENCY CHECKS ON THE ALV CALCULATION

The published ALV $^3F_2$ and ϵ_4 phases have unusual energy dependences, in that they do not become OPE at low energies. The calculations necessary to check

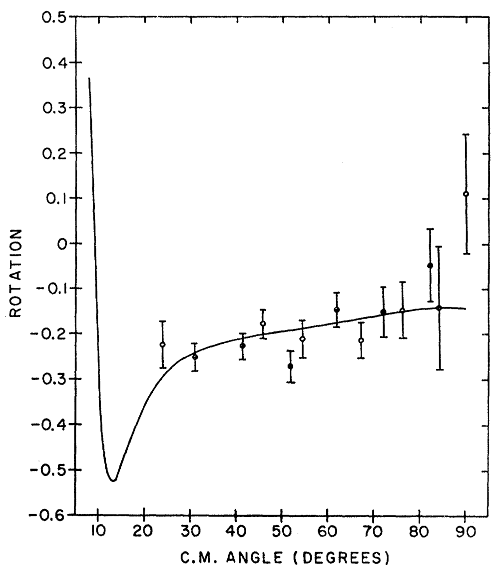

FIG. 8. The Harvard (●) and Harwell (○) R(θ) data. The curve is OPE(11).

these doubtful ALV predictions are being undertaken, but are quite lengthy. However, some simple calculations using the published ALV numbers provide a check of the $^3F_2$ and ϵ_4 phases.

Using the T_{ij}'s and δ's from Figs. 1 and 2 of ALV,[4] one can evaluate their $^3F_2$ phase by the approximate relations[15]

$$10\delta_{32} = -18\delta_{34} - 14\delta_{33} + k(2T_{11} + T_{00})$$
$$(20/3)\delta_{32} = 9\delta_{34} - (7/3)\delta_{33} - (4k/\sqrt{2})(T_{10} - T_{01}),$$

which are of sufficient accuracy for the small phases involved. The actual numbers which were used in the calculation were taken from the full page graphs in the preprint of the ALV paper. The resulting $^3F_2$ phases from the two equations are 0.64 and 1.25°, which straddle the model values but are far from the 0.2° displayed by ALV in their Fig. 2. A similar procedure can be used for ϵ_4, with[14]

$$5^{1/2}\epsilon_4 = -3\delta_{56} - \tfrac{5}{2}\delta_{54} - \tfrac{1}{2}(42)^{1/2}\epsilon_6 + (k/4)T_{00}$$

and the OPE value of −0.225° for ϵ_6. The result is $\epsilon_4 = -0.79°$, close to the OPE value of −0.85° but far from the ALV Fig. 2 value of −0.23°.

One concludes that: (a) there seems to be an inconsistency in the published ALV $T_{11}(3)$, $T_{00}(3)$, $T_{10}(3) - T_{01}(3)$, δ_{34}, and δ_{33}, and (b) the $^3F_2$ and ϵ_4 phases from an ALV-type calculation are probably close to the values from other models at 142 MeV.

VII. SENSITIVITY OF PHASES TO DATA SUBGROUPS

A measure of the sensitivity of the phases to a data subgroup can be obtained by removing that group from the data set. After again minimizing χ^2, one can compare the resulting phase-shift standard deviations to their values when the data group was included. Such analysis have been performed for each of the kinds of data in the full 103-piece data set. The resulting fractional increases in the phase-shift standard deviations are shown in Table V. It is apparent that most of the phases are most sensitive to $\sigma_{\text{int.}}$, D, and R. It would

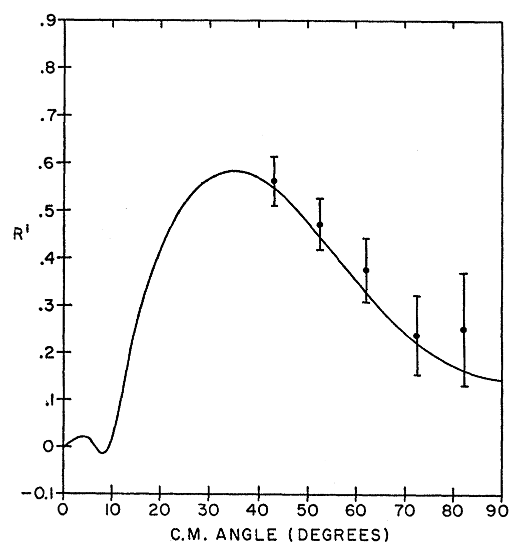

FIG. 9. The Harvard $R'(\theta)$ data. The curve is OPE(11).

[15] H. P. Stapp, T. J. Ypsilantis, and N. Metropolis, Phys. Rev. **105**, 302 (1957).

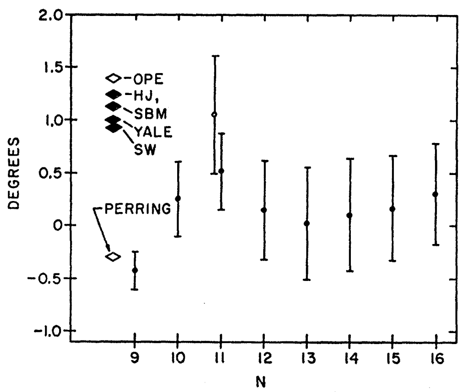

FIG. 10. The $^3F_2$ phase shift for OPE(N) versus N, the number of searched-upon phases. The open circle is the predicted value with all of the D data removed from the data set. The values are from Table III.

appear that R' makes lettle contribution to the analysis. However, the removal of R' from the data set does cause the *value* of the $^1S_0$ phase to decrease by 60% of its standard deviation. The other phases are much less affected.

Examination of Fig. 10 and Table III shows that the $^3F_2$ phase is rather ambiguously given by the analyses, but is probably lower than any of the model values. Perring's negative $^3F_2$ for twelve-search phases seems to indicate some sensitivity of the $^3F_2$ phase to data selection. Table V shows that the $^3F_2$ is most sensitive to D and A. A series of OPE(11)-type runs were made, but with the $^3F_2$ fixed at a sequence of values from zero to one degree. The data predictions showed little variation for any kind of data except D. The latter

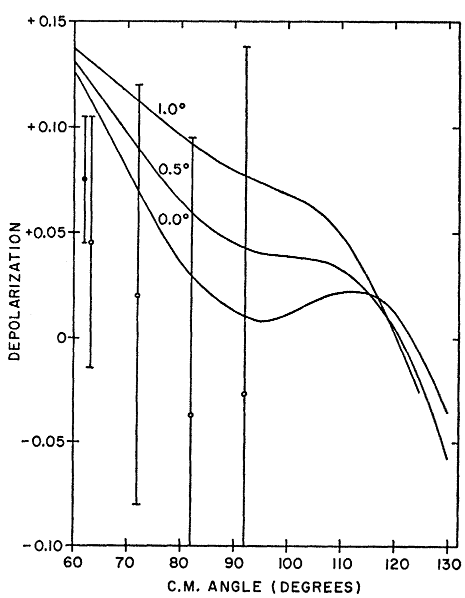

FIG. 11. The high angle portion of the $D(\theta)$ in Fig. 6, expanded and with curves from OPE(11)-type minimizations of χ^2 but for fixed values of the $^3F_2$ phase.

and A were examined out to 130°, but only D, shown in Fig. 11, was found to vary appreciably. Runs were made with first only the Harvard, then only the Harwell $D(\theta \geq 70°)$ removed, thus removing the discrepancy in the data (see Fig. 6). The predicted $^3F_2$ phases and the D curves were not very different than for the full data set curve in Fig. 6. Removal of all of the D data left the *predicted* D curve unchanged up to 50°, but made it almost constant from there out to 90°. The predicted $^3F_2$ for that case was 1.0°, shown as an open circle in Fig. 10.

It is apparent in Figs. 6 and 11, and from the above discussion, that the Harvard $D(62°)$ is low and that the Harvard $D(72°)$ and $D(82°)$ are high. In order to determine the $^3F_2$ phase more accurately, it would seem desirable to have D(60–100°) remeasured; preferably to uniform accuracy.

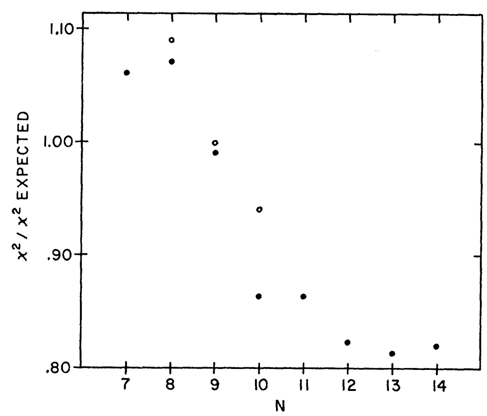

FIG. 12. χ^2 ratio versus number of searched-upon phases, N, with the pion-nucleon coupling constant g^2 also a free parameter. The darkened circles represent the release of the phase shifts from their OPE values in the order: $^3F_2$, ϵ_4, $^1G_4$, $^3F_4$, $^3F_3$, $^3H_6$, $^3H_4$, $^3H_5$. The open circles represent the order: $^3F_3$, ϵ_4, $^3F_4$.

VIII. THE PION-NUCLEON COUPLING CONSTANT

Figures 12 and 13 show the effects of including the pion-nucleon coupling constant g^2 along with the low-L phases as a free, searched-upon, parameter. Two different phase-shift orderings are shown. The $N=9$ case is of particular interest; the two χ^2 probabilities are reasonable and almost identical, but the predicted g^2 values are very different. One can probably only conclude that the predicted g^2 is somewhere between five and seventeen, with a slight preference for the range $g^2=10\pm3$. In order to obtain phase-shift values it would seem best, for the present, to fix g^2 at its pion-nucleon value.

IX. SOLUTION 2

The solution to the least-squares fitting problem, used in the previous sections, is of the type labeled No. 1 by Stapp, Ypsilantis, and Metropolis[15] (SYM). Of the six main solutions found by SYM at 310 MeV,

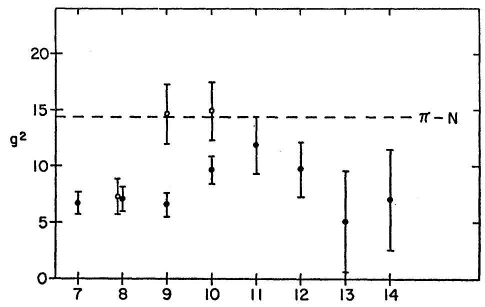

FIG. 13. The values of g^2 and standard deviations for the χ^2 ratios shown in Fig. 12.

the subsequent *modified* phase analysis[3] at 310 MeV left only Nos. 1 and 2. One of Perring's most interesting results was his very large value of χ^2, at 142 MeV, for the solution of the second type. Solution 2 was also examined here: The results are shown as diamonds in Fig. 2. The χ^2 probability for solution 2 is considerably less than one percent. A definite value cannot be given, since probability estimates cannot be reliably made for much less than the inverse of the number of degrees of freedom (number of data minus number of searched-upon parameters). Figure 2 shows clearly that the elimination of solution 2 at 142 MeV does not depend on taking a particular number of search parameters.

The main experimental data which forced the χ^2 ratio to a high value for solution 2 were A and D. When A was removed from the data set, the solution 2, χ^2 ratio, fell from 2.48 to 1.63. Removal of D alone caused a drop to 1.08, and removal of both D and A resulted in a ratio of 0.74. The poor fit of solution 2 to D and A is shown in Figs. 6 and 7: The curves are generally similar to those displayed by Perring.

It is possible that better measurement of D(60–100°) might decide even more strongly in favor of solution 1; this provides another reason for further measurements.

X. THE GARREN AMPLITUDE

Garren[16] has constructed a relativistic electromagnetic amplitude which includes charge-charge, charge-moment, and moment-moment interactions. Palmieri *et al.*,[17] in reporting the polarization data used in the present analyses, examined the ratio of the measured 147 polarization to the polarization arising from "purely nuclear phenomena." For the latter, they used one of Garren's sets of $L=0,1$ 213-MeV phase shifts. The result was good qualitative agreement with the 147-MeV measurements.

Figures 4 and 5 show the effect, on the OPE(11)-predicted σ and P, of including the Garren amplitude and reminimizing χ^2. The differences from the Coulomb amplitude OPE(11) curves are seen to be quite small. Other runs were made, with $N=9$ to 14 in OPE(N). The Garren amplitude produced slightly better fits for $N<12$ and slightly poorer fits for $N>12$.

XI. SUMMARY

With some selection, a 103-piece data set was obtained which gave reasonable average contributions to χ^2 from the various kinds of data. An independent measurement of $\sigma(45°)/\sigma(90°)$ at 142 MeV would be helpful in resolving the discrepancy in this quantity between the Harvard and Harwell measurements.

Modified phase analyses were performed on the 103-piece data set, resulting in predictions for phase shifts and their standard deviations. Using the one-pion-exchange contributions for higher angular-momentum states, statistical and physical arguments led to a preference for eleven or more searched-upon low-L phases, with a slight preference for the lower limit of eleven. Use of the Amati-Leader-Vitale higher L phases allowed a reduction in the required number of free phases to ten, owing to the ALV $^1G_4$ being at the desired value. Some discrepancies were noted in the ALV calculations.

The χ^2 fit of several potential-type models and phase-shift representations were computed for this data set. All gave values of χ^2 at least several times that of the modified phase analysis, but for differing reasons among the various models.

The $^3F_2$ phase shift from the modified phase analyses was found to be somewhat lower than that of the models, due mainly to the measured D(60–90°). In view of the dissimilar high-angle trends between Harwell and Harvard, and the apparently low D(62°) data, it is suggested that the D(60–100°) be reexamined experimentally.

The pion-nucleon coupling constant was found to be rather ambiguously given by this data set, providing little evidence for the one-pion-exchange mechanism.

Replacement of the Coulomb amplitude by the Garren amplitude produced little change in the prediction and χ^2.

The SYM type-2 solution was found to have a χ^2 probability of much less than one percent. For a small number of free phases, the use of ALV higher L phases lowered χ^2 from its OPE value for solution 1, and raised it for solution 2, as a good theory should.

ACKNOWLEDGMENTS

We wish to thank R. Wilson, J. Palmieri, and A. M. Cormack for very helpful discussions of the data.

The computer program used was developed by one of us (P. S.) and N. R. Yoder, to whom we are indebted. The program was run in the Computation Center of the Pennsylvania State University and in the A. E. C. Computation Center at New York University.

[16] A. Garren, Phys. Rev. **101**, 419 (1956).
[17] Reference j of Table I.