Answer to Fock Concerning the Time Energy Indeterminacy Relation*

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An answer is given to a recent criticizm by Fock, concerning our paper "Time in the Quantum Theory and the Uncertainty Relation for Time and Energy." It is proved that Fock's criticizm is wrong, and that our previous conclusion that energy can be measured in an arbitrarily short period of time is valid.

IN a recent article[1] Fock has criticized one of our previous papers.[2] In that paper we attempted to show that contrary to a widespread view,[3] it is possible to measure the energy of a physical system within an arbitrarily short time interval. We first emphasized that the time in question is a dynamical variable belonging to the measuring apparatus and therefore commutes with the energy of the system. Hence no reciprocal limitations in the mutual definability of these quantities should be expected. Nevertheless, since analysis of apparently illustrative examples of energy measurements (e.g., by collision) had seemed to indicate the opposite, we proceeded to show that the arrangements considered in these examples did not exhaust the measuring possibilities. By analyzing the problem of energy measurements along the lines of von Neumann[4] we arrived at the conclusion that measurements of energy in arbitrarily short intervals of time are indeed possible. However, as indicated by the mathematics, the execution of such measurements would require an interaction of a type different from those commonly considered (such as in a one-collision experiment). Using this information, we then described an experimental setup that introduces the proper interaction for this measurement.[4] As expected, this setup permits us to measure the energy of the system in as short a time interval as we choose.

Fock's criticism consists of two steps. He first raises objection to the Hamiltonian which we used in our mathematical considerations, namely,

$$H=p_x^2/2m+p_y^2/2M+yp_xg(t) \tag{1}$$

[where p_x and m are the momentum and the mass of the observed particle, p_y and M are the corresponding quantities of the test body and y is its position. The function $g(t)$ measures the strength of the interaction and differs from zero only during a short interval of time, when it equals a constant].

As Fock points out, this Hamiltonian describes an interaction between the particle under consideration and a field g which is switched on instantaneously at a certain time t and is similarly switched off at a later time $t+\Delta t$. Fock claims that we commit the logical error of "begging the question," since our procedure amounts to using this field as a classical entity, i.e., we neglect the time-energy uncertainties associated with it. For example, we neglect right from the outset the field quanta of infinite energy which are created when g is switched on and off instantaneously.

While it is quite correct that such infinite uncertainties in the energy of the field are created, this would lead to no contradiction with our basic argument unless one assumes further (as Fock does) that such quanta of the field by necessity produce a corresponding (infinite) uncertainty in the energy of the particle itself. This, however, is an erroneous assumption, as will be shown below, and thus Fock's first argument will prove to be invalid.

In his second argument Fock uses a "corrected" $g(t)$, which is switched on and off smoothly [during a time of the order Δt of the duration of $g(t)$ itself]. He then proposes to show that with a thus modified Hamiltonian one arrives at conclusions opposite to ours, namely that the energy cannot be measured in arbitrarily short times. His argument is based on his calculation showing that the kinetic energy of the observed particle is changed by an amount which is uncertain to the order $\Delta E\geq h/\Delta t$. In fact, this point was emphasized in our paper.[5] But, as we pointed out, this uncertainty in the kinetic energy is produced during the *first stage* of the measurement process; the measurement which we considered possessed, however, as we explained in detail that paper, also a *second stage*, during which another shift in the kinetic energy takes place, which exactly cancels the first uncertainty. Thus, when the interaction is over, the final kinetic energy again equals the initial $p_x^2/2m$, and, since p_x has been measured accurately, we are left with no uncertainty in the energy.[6] Since the

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[1] V. Fock, Zh. Eksperim. i Teor. Fiz. **42**, 1135 (1962) [English transl.: Soviet Phys.—JETP **15**, 784 (1962)].

[2] Y. Aharonov and D. Bohm, Phys. Rev. **122**, 1649 (1961).

[3] See, for example, V. Fock and N. Krylov, J. Phys. (USSR) **11**, 112 (1947).

[4] See Sec. 4 in Ref. 2.

[5] Reference 2, p. 1658.

[6] One might wonder whether it is a general condition that the kinetic energy must become uncertain in the intermediate stage of the measurement or whether this is just a characteristic of the examples discussed by us. The answer is that it must be a general property of every kinetic energy measurement. This is connected with the fact that measurement of the kinetic energy is equivalent to measurement of velocity. Since the velocity is not a canonical variable, it is impossible to add an interaction term to the Hamiltonian which is proportional to the velocity. This point as well as the general problem of measurement of noncanonical variables will be discussed more fully in a future paper written by one of us (Y. Aharonov), Gideon Carmi, and Aage Petersen.

duration of $g(t)$ (which includes both stages of the measurement process) can be made arbitrarily short, we thus had achieved our object, of measuring the energy in an arbitrarily short time interval, without introducing any *over-all* change in the energy.

We now return to Fock's first argument, although, as should be clear from the previous discussion, our basic assertion about the measurability of the energy in arbitrarily short times, was established by answering Fock's second argument alone. We emphasize again that the validity of Fock's first argument depends on the question whether or not the occurrence of uncertainties in the energy of the "field" $g(t)$ *by necessity* also introduces equal uncertainties in the energy of the observed particle. The simple example which follows will show that this is not at all the case. Rather, the example indicates that the transfer of uncertainty to the particle can be arbitrarily small; the reason being that this transfer is governed by momentum conservation (and, in general, by some suitable conservation law other than that of energy). Thus, there is no *a priori* argument against the use of our Hamiltonian (1). We will then proceed to show directly that the Hamiltonian (1) indeed describes a well-defined physical situation.

As an example of an interaction process in which the uncertainty of the energy exchange between the constituents is arbitrarily small, consider a collision between two particles, one of which is light (mass m) and the other very heavy (mass M). If the light particle has an initial velocity V and a latitude of position Δ_x, it is possible to fix the time of collision to any accuracy, $\Delta t = \Delta x/V$, by making V arbitrarily large. (We assume that the uncertainty in the position of the second particle is of the same order as Δ_x, but if its mass is large enough, its velocity may be made arbitrarily small.) To see that the energy exchange can be controlled to an arbitrary accuracy, it is enough to observe that the maximum energy transfer will be $(mV)^2/2M$, which approaches zero when M goes to infinity. Thus, it is clear that the interaction term in the Hamiltonian describing these two systems will be effectively different from zero for an arbitrarily short time and still the energy exchange will be uncertain to an amount far smaller than h divided by the time of interaction. Hence, if we would have multiplied the interaction term of this Hamiltonian by an explicit time-dependent function of the form of $g(t)$ in (1), we would have made no essential change provided the period in which $g(t)$ is different from zero is larger than the uncertainty in the time of collision.

It should now be clear how we can derive the Hamiltonian (1) from a Hamiltonian which is not explicitly time-dependent and which therefore satisfies even Fock's demands. The equivalent will be

$$H=\frac{p_x^2}{2m}+\frac{p_y^2}{2M}+\frac{p_z^2}{2M'}+yp_xg(z)\,, \qquad (2)$$

where z, p_z and M' are the coordinate, momentum, and mass of an extremely heavy particle. (The z degree of freedom serves to introduce a dynamical time in the Hamiltonian.)

If the mass of z is large enough we may, as is well known, approximate the z dependence of the total wave function by a term $\delta\ (z-V_zt)$ where V_z is the velocity of z which may be taken as a constant equal to 1. In this approximation all the results derived from Hamiltonian (2) will be exactly equivalent to those derived from Hamiltonian (1) [since $g(z)$ is equal to $g(t)$]. We thus conclude that every explicitly time-dependent Hamiltonian may be approached with an arbitrary accuracy, and that no extra consistency limitations can be imposed.

In summary, both objections raised by Fock are untenable, and the conclusion of our previous paper, viz., that reproducible energy measurements can be performed in arbitrarily short periods of time, remains valid.

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