PHYSICAL REVIEW VOLUME 134, NUMBER 6B 22 JUNE 1964

Theory of Low-Energy π-π Scattering*

KYUNGSIK KANG†
Department of Physics, Indiana University, Bloomington, Indiana
(Received 9 January 1964)

We make a self-contained examination of the π-π scattering problem on the basis of the S-matrix conjecture, using analyticity, elastic unitarity, and crossing symmetry. The crossing-symmetry relations are derived without using the double-dispersion relations. The nearest singularity hypothesis is adopted. Inelastic effects and distant left-hand singularities are crudely incorporated by means of polynomials. Nearby left-hand singularities are chosen in such a way the functional forms are correct near the branch point with unknown parameters. The amplitude so constructed satisfies elastic unitarity exactly in one channel; the parameters are adjusted through self-consistency conditions derived from crossing symmetry. By use of the inverse amplitude, we are able to construct simple amplitudes and find solutions by hand calculation. A p-wave resonance is generated for negative values of λ with $|\lambda|<0.39$. Solutions exhibit the development of $I=0$ s-wave bound states for negative values of λ with $|\lambda|\geq 0.44$ and of $I=1$ p-wave bound states for negative values of λ with $|\lambda|\geq 0.39$. We find the range of λ, $-0.3\lesssim\lambda<0.1$, to be self-consistent.

WE wish to make a limited examination of the π-π scattering problem which we hope will shed some light on the conjectures about analyticity and crossing symmetry made by Chew and Mandelstam. Elastic π-π scattering provides the simplest physical application of the S-matrix approach.[1] We have attempted to make this article self-contained (we do not attempt, however, an axiomatic approach), and we hope it will be read by many who have not yet studied π-π scattering.

In elastic π-π scattering, there are three physical channels. The amplitude thus describes the same process in each of the three physical regions of the complex plane. The basic conjecture is that the amplitude can be continued analytically from one physical region to another, thus very strong conditions are imposed. These crossing-symmetry conditions are especially strong where the three physical regions are close to each other. The crossing-symmetry relations are derived without using the double-dispersion relations.

There have been two approaches to practical calculations: Some emphasize the description of the amplitude in remote regions of the complex plane, others stress detailed satisfaction of the conditions in a limited region. (Note that neither of them explicitly examines inelastic processes.) We adopt the latter approach. We wish to construct an elastic π-π amplitude satisfying the conditions of analyticity, elastic unitarity, and crossing symmetry in a limited region of the complex plane. It is hoped that the information from the nearest singularities leads to estimation of the most important aspects of the processes.

A simplified description of our procedure is as follows: We write down an amplitude which manifestly satisfies the analyticity and unitarity requirement in a single channel in the low-energy region. By continuation and crossing symmetry, we then obtain self-consistency conditions which we try to satisfy by adjusting parameters of the original amplitude. Our amplitude, as is generally the case, will thus satisfy elastic unitarity exactly and crossing symmetry only approximately. By use of the inverse amplitude, we keep the problem simple enough so that we are able to construct nontrivial amplitudes and find solutions by hand calculations.

I. INTRODUCTION

It is convenient to introduce the invariant variables,

$$\begin{aligned} s&=(p_1+p_2)^2=4(q_s^2+\mu^2)\,,\\ t&=(p_1+p_3)^2=-2q_s^2(1-\cos\theta_s)\,,\\ u&=(p_1+p_4)^2=-2q_s^2(1+\cos\theta_s)\,,\end{aligned}\tag{I.1}$$

where q_s is the barycentric momentum of one of the pions and θ_s is the barycentric scattering angle (see Fig. 1).

With the aid of energy-momentum conservation, s, t, and u are not all independent but satisfy

$$s+t+u=4\mu^2.\tag{I.2}$$

For the reaction $(1+2\to 3+4)$ to be a physical process, we require

$$q_s^2>0 \quad\text{and}\quad |\cos\theta_s|\leq 1\,,$$

which means $s>4\mu^2$, $t<0$, and $u<0$. Similarly, we have two other sets of physical regions corresponding to two other reactions: For the reaction $(1+3\to 2+4)$, $t>4\mu^2$, $s<0$, and $u<0$. For the reaction $(1+4\to 2+3)$, $u>4\mu^2$, $s<0$, and $t<0$.

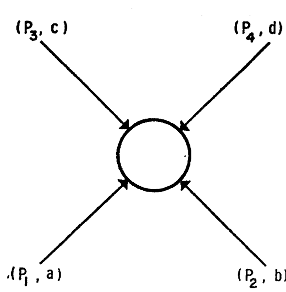


FIG. 1. Pion-pion elastic scattering, $\pi+\pi\to\pi+\pi$.

* Supported in part by the National Science Foundation.
† Present address: Department of Physics, The University of Michigan, Ann Arbor, Michigan.
[1] G. F. Chew, *S-Matrix Theory of Strong Interactions* (W. A. Benjamin and Company, Inc., New York, 1961).

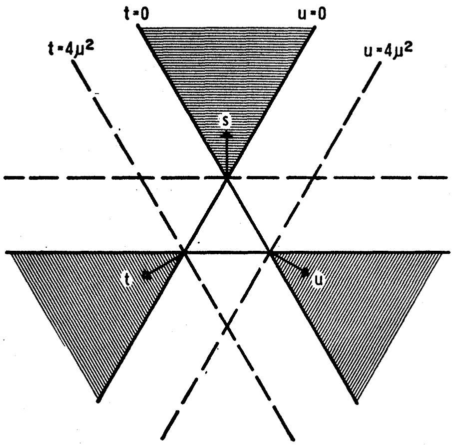


FIG. 2. The Mandelstam diagram for $\pi\pi \to \pi\pi$. The shaded regions are the three physical regions.

As was stated before, the discontinuities in the scattering amplitude come from the requirement of unitarity.

The Heisenberg S matrix is related to the invariant amplitude $F(s,t,u)$ by the relation

$$S_{fi}=\delta_{fi}+4\pi i(2\pi)^4\delta^4(p_f-p_i)\frac{F(s,t,u)}{(\omega_1\omega_2\omega_3\omega_4)^{1/2}}, \quad \text{(I.3)}$$

and F is now related to the T matrix by

$$F(s,t,u)=(\omega_1\omega_2\omega_3\omega_4)^{1/2}T_{fi}, \quad \text{(I.4)}$$

where ω_1, ω_2 and ω_3, ω_4 are the energies of the initial and final pions, respectively. Unitarity expressed in terms of the T matrix reads

$$i\langle f|T^\dagger-T|i\rangle=4\pi(2\pi)^4\sum_n\langle f|T^\dagger|n\rangle \times\langle n|T|i\rangle\delta^4(p_i-p_n). \quad \text{(I.5)}$$

This can be rewritten as

$$\mathrm{Im}\langle f|T|i\rangle=(2\pi)^5\sum_n\langle f|T^\dagger|n\rangle\langle n|T|i\rangle\delta^4(p_i-p_n), \quad \text{(I.6)}$$

and

$$\mathrm{Im}\langle i|T|i\rangle=(2\pi)^5\sum_n|\langle n|T|i\rangle|^2\delta^4(p_i-p_n) =(1/2\pi)(q_s/W_s)\sigma_{\mathrm{tot}}, \quad \text{(I.7)}$$

where W_s is barycentric total energy, i.e., $\sqrt{s}$ in the process $1+2\to 3+4$. The relation (I.7) is the well-known optical theorem for the forward scattering. From (I.4) the total cross section is given by

$$\sigma_{\mathrm{tot}}(s)=(8\pi/q_s\sqrt{s})\,\mathrm{Im}F(s,t-u=0). \quad \text{(I.8)}$$

If the energy is so low that only the elastic channel is open, then the elastic unitarity relation for the invariant amplitude can be written

$$\mathrm{Im}F(q_s^2,\cos\theta_s)=\frac{1}{4\pi}\frac{2q_s}{\sqrt{s}}\int d\Omega' F^*(q_s^2,\cos\theta') \times F(q_s^2,\cos(\theta_s,\theta')), \quad \text{(I.9)}$$

for

$$q_s^2>0, \quad |\cos\theta_s|\le 1,$$

where

$$\cos\theta_s=1+(t/2q_s^2),$$
$$\cos(\theta_s,\theta')=\cos\theta_s\cos\theta'+\sin\theta_s\sin\theta'\cos\phi',$$

and

$$d\Omega'=\sin\theta' d\theta' d\phi'.$$

The three physical regions corresponding to the three channels are well illustrated by the Mandelstam diagram in Fig. 2.

Let us show that unitarity requires the presence of a branch cut on the real axis, say, in s for $s\ge 4\mu^2$. Consider $F(s,t-u)$ at a fixed real $t-u$. From the basic postulate of the S-matrix approach that the amplitude is analytic except for requirement of unitarity, if the unitarity relation (I.6) holds in a neighborhood $\mathrm{Re}s\le 4\mu^2$, then we deduce that $F(s,t-u)$ is real for $\mathrm{Re}s\le 4\mu^2$ and is analytic in this neighborhood. Thus, the amplitude satisfies the general theorem of the reflection property,

$$F(s^*,t-u)=F^*(s,t-u). \quad \text{(I.10)}$$

Now because of unitarity, $F(s,t-u)$ has a nonvanishing imaginary part for $s>4\mu^2$. These result that $F(s,t-u)$ must have a branch point at $s=4\mu^2$, and it is clearly convenient to take the cut on the real axis for $s>4\mu^2$. The discontinuity across the cut is

$$\mathrm{Im}F(s,t-u) =[F(s+i\epsilon,t-u)-F(s-i\epsilon,t-u)]/2i. \quad \text{(I.11)}$$

The right-hand cut (physical cut) extends in s from $4\mu^2$ to ∞. It is an accident of the equal mass case that those branch cuts associated with t and u channels are coincident. From G-parity conservation in strong interactions,[2] only the production of pairs of pions is allowed; the threshold for inelastic scattering with two additional pions occurs at $s=16\mu^2(q_s^2=3\mu^2)$ on the right-hand cut and at $s=-12\mu^2(q_s^2=-4\mu^2)$ on the left-hand cut (see Fig. 3).

Unitarity takes a simple form if we consider the partial-wave amplitude

$$f_l(q_s^2)=\frac{1}{2}\int_{-1}^{1}d(\cos\theta_s)p_l(\cos\theta_s)F(q_s^2,\cos\theta_s). \quad \text{(I.12)}$$

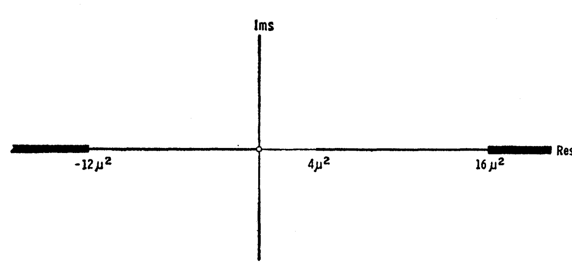


FIG. 3. Singularities in the complex s plane.

[2] T. D. Lee and C. N. Yang, Nuovo Cimento 3, 749 (1956).

From (I.9), we obtain

$$\mathrm{Im} f_l(q_s^2)=(2q_s/\sqrt{s})\,|f_l(q_s^2)|^2\,, \qquad \text{(I.13)}$$

for $q_s^2>0$. Elastic unitarity is satisfied if we write the partial-wave amplitude as

$$f_l(q_s^2)=(\sqrt{s}/2q_s)e^{i\delta_l(s)}\sin\delta_l(s)\,, \qquad \text{(I.14)}$$

where the phase shifts are real.

At higher energies, $s>16\mu^2$ or $q_s^2>3\mu^2$, the phase shifts become complex but we can generally specify unitarity by

$$\mathrm{Im} f_l(q_s^2)=(2q_s/\sqrt{s})\,|f_l(q_s^2)|^2 R_l(q_s^2)\,, \qquad \text{(I.15)}$$

or

$$\mathrm{Im}(1/f_l(q_s^2))=-(2q_s/\sqrt{s})R_l(q_s^2)\,, \qquad \text{(I.16)}$$

where

$$R_l(q_s^2)=\sigma_l(\text{total})/\sigma_l(\text{elastic})\,,$$

which is 1 for $q_s^2<3\mu^2$. The discontinuity of the partial-wave amplitude across the right-hand cut is thus given by unitarity (I.15) or (I.16). The discontinuity across the left-hand cut where $q_s^2<-\mu^2$ is not obtained directly from unitarity because the unitarity relation cannot be used for negative values of q_s^2. However, the fact that the unphysical cut is associated with the physical processes of the second and the third channels is a guiding idea to obtain the discontinuity across the left-hand cut. Since the amplitude can be continued from one physical region to another, it is meaningful to discuss the behavior of the amplitude under a transformation crossing, which takes us from one region to another.

By assuming charge independence, F may be written as the sum of three terms,

$$F(s,t,u)=A(s,t,u)\delta_{ab}\delta_{cd}+B(s,t,u)\delta_{ac}\delta_{bd}+C(s,t,u)\delta_{ad}\delta_{bc}\,, \qquad \text{(I.17)}$$

where a, b and c, d are the isospin indices running from 1 to 3 to label the pion-charge degree of freedom of the initial and final pions, respectively.

It is well known that when there are two or more identical particles among the four involved in the scattering, the exchange of two identical particles at most changes the sign of the amplitude. Such an interchange means switching two of the s, t, u variables, leaving the third alone. In π-π scattering, the amplitude is symmetric under the exchange of two particles. The physical significance of crossing symmetry is even clearer if we introduce invariant amplitudes corresponding to the well-defined isotopic quantum number I. Using the three projection operators

$$\begin{aligned} p^{(0)}&=\tfrac{1}{3}(\mathbf{I}_1\cdot\mathbf{I}_2+1)(\mathbf{I}_1\cdot\mathbf{I}_2-1)\,,\\ p^{(1)}&=-\tfrac{1}{2}(\mathbf{I}_1\cdot\mathbf{I}_2+2)(\mathbf{I}_1\cdot\mathbf{I}_2-1)\,,\\ p^{(2)}&=\tfrac{1}{6}(\mathbf{I}_1\cdot\mathbf{I}_2+1)(\mathbf{I}_1\cdot\mathbf{I}_2+2)\,,\end{aligned} \qquad \text{(I.18)}$$

corresponding to the three isotopic spin states $I=0$, 1, 2, the three invariant amplitudes in (I.17) can be identified (see Appendix I).

$$\begin{aligned} A(s,t,u)&=\tfrac{1}{3}(A^0(s,t,u)-A^2(s,t,u))\,,\\ B(s,t,u)&=\tfrac{1}{2}(A^1(s,t,u)+A^2(s,t,u))\,,\\ C(s,t,u)&=-\tfrac{1}{2}(A^1(s,t,u)-A^2(s,t,u))\,,\end{aligned} \qquad \text{(I.19)}$$

where A^0, A^1, and A^2 correspond to the total isotopic spin quantum numbers $I=0$, 1, and 2, respectively.

The permutations of s, t, and u give three sets of simple symmetry properties.

$$\begin{aligned} A(s,t,u)&=A(s,u,t)\,, \\ B(s,t,u)&=C(s,u,t)\,,\end{aligned} \quad \text{under} \quad \begin{aligned} s&\leftrightarrow s\,,\\ t&\leftrightarrow t\,;\end{aligned} \qquad \text{(I.20a)}$$

$$\begin{aligned} A(s,t,u)&=C(u,t,s)\,, \\ B(s,t,u)&=B(u,t,s)\,,\end{aligned} \quad \text{under} \quad \begin{aligned} s&\leftrightarrow u\,,\\ t&\leftrightarrow t\,;\end{aligned} \qquad \text{(I.20b)}$$

$$\begin{aligned} A(s,t,u)&=B(t,s,u)\,, \\ C(s,t,u)&=C(t,s,u)\,,\end{aligned} \quad \text{under} \quad \begin{aligned} s&\leftrightarrow t\,,\\ u&\leftrightarrow u\,.\end{aligned} \qquad \text{(I.20c)}$$

From (I.19), we see that (I.20a) is an expression of the generalized Pauli principle for the s channel.

$$A^I(s,t,u)=(-1)^I A^I(s,u,t)\,, \qquad \text{(I.21)}$$

which means that A^0 and A^2 are even functions of $t-u$ or $\cos\theta_s$ while A^1 is an odd function which is in accordance with Bose statistics.

Equations (I.20b) and (I.20c) are similar expressions of the principle for t and u channels, respectively. To see this, let us introduce the following invariant amplitudes:

$$\begin{aligned} X_0&=A+B+C\,,\\ X_1&=A-C\,,\\ X_2&=A+C-2B\,.\end{aligned} \qquad \text{(I.22)}$$

(I.20b) gives

$$X_i(s,t,u)=(-1)^i X_i(u,t,s)\,. \qquad \text{(I.23)}$$

It is also seen that the third set of symmetry (I.20c) is automatically satisfied if (I.21) and (I.23) are satisfied. Moreover, under (I.20c), the transformation of $X_i(s,t,u)$ gives

$$A^I(t,s,u)=\sum_{I'=0}^{2}\chi_{II'}A^{I'}(s,t,u)\,, \qquad \text{(I.24)}$$

where

$$(\chi_{II'})=\begin{bmatrix} \frac{1}{3} & 1 & \frac{5}{3}\\ \frac{1}{3} & \frac{1}{2} & -\frac{5}{6}\\ \frac{1}{3} & -\frac{1}{2} & \frac{1}{6}\end{bmatrix}. \qquad \text{(I.25)}$$

The crossing relation (I.24) is only useful in practice if the right-hand side or left-hand side is evaluated in or very near a physical region. There are two cases of interest: (i) the right hand is in, say, the s channel. (ii) We are very near all three channels, i.e., near the maximum symmetry point $s=t=u=\frac{4}{3}\mu^2$. Let us consider case (i) first. The partial-wave expansion of the right-hand side of (I.24) does not converge for the region $\mathrm{Re}\,s<0$, since as soon as s reaches zero, there are singularities from physical t and u channels. Analytic continuation in s below $\mathrm{Re}s=0$ still gives, however, the function $A_l^I(s)$ from (I.12). Consider fixed $s<0$ where we have two cuts $t>4\mu^2$ and $u>4\mu^2$ (see Fig. 2). There will thus occur a discontinuity of $A_l^I(s)$ for $s<0$. Some parts of the two cuts overlap each other below $s<-4\mu^2$. The discontinuity can be easily obtained from the crossing relation (I.24). From (I.24), we obtain

$$A^I(s,t,u)=\sum_{I'}\chi_{II'}A^{I'}(t,s,u)\,, \qquad \text{(I.26)}$$

$$A^I(s,t,u)=(-1)^I\sum_{I'}\chi_{II'}A^{I'}(u,s,t)\,, \qquad \text{(I.27)}$$

where (I.27) is obvious from (I.21). It will be convenient to use (I.26) for $s<0$ and $t>4\mu^2$ and (I.27) for $s<0$ and $u>4\mu^2$. We also notice that approaching the cut from above in s plane, at fixed negative real part, corresponds to approaching the physical cut in t plane from below [see (I.37)]. For the range of s from 0 to $-4\mu^2$, the angular integration ranges over three regions at a fixed s, from $u=0$ to $t=4\mu^2$ where the t channel contributes, from $t=4\mu^2$ to $u=4\mu^2$ where no singularities occur and from $u=4\mu^2$ to $t=0$ where the u channel contributes. Therefore, at a fixed $s<0$,

$$\mathrm{Im}A_l^I(s)=\tfrac{1}{2}\int_{-1}^{A(t=4\mu^2)}d(\cos\theta_s)P_l(\cos\theta_s)\,\mathrm{Im}A^I(s,\cos\theta_s)+\tfrac{1}{2}\int_{A(t=4\mu^2)}^{B(u=4\mu^2)}d(\cos\theta_s)P_l(\cos\theta_s)\,\mathrm{Im}A^I(s,\cos\theta_s)$$
$$+\tfrac{1}{2}\int_{B(u=4\mu^2)}^{1}d(\cos\theta_s)P_l(\cos\theta_s)\,\mathrm{Im}A^I(s,\cos\theta_s)\,, \qquad \text{(I.28)}$$

where $A(t=4\mu^2)$, $B(u=4\mu^2)$ are two points on $t=4\mu^2$, $u=4\mu^2$ for the fixed s, $-4\mu^2<s<0$, respectively. The second integral of (I.28) vanishes for $-4\mu^2<s<0$. Thus, we have

$$\mathrm{Im}A_l^I(s)=-\tfrac{1}{2}\int_{-1}^{A(t=4\mu^2)}d(\cos\theta_s)P_l(\cos\theta_s)\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(t,\cos\theta_t)$$
$$-\tfrac{1}{2}(-1)^I\int_{B(u=4\mu^2)}^{+1}d(\cos\theta_s)P_l(\cos\theta_s)\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(u,\cos\theta_u)\,, \qquad \text{(I.29)}$$

where

$$\cos\theta_t=1+s/2q_t^2\,,\quad \cos\theta_u=1+s/2q_u^2\,.$$

Remembering

$$\cos\theta_s=1+t/2q_s^2=-1-u/2q_s^2\,,$$

(I.29) can be rewritten

$$\mathrm{Im}A_l^I(s)=-\frac{1}{4q_s^2}\int_{-4q_s^2}^{4\mu^2}dt\,P_l\!\left(1+\frac{t}{2q_s^2}\right)\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(t,\cos\theta_t)$$
$$+(-1)^I\frac{1}{4q_s^2}\int_{4\mu^2}^{-4q_s^2}du\,P_l\!\left(-1-\frac{u}{2q_s^2}\right)\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(u,\cos\theta_u)\,. \qquad \text{(I.30)}$$

Since odd I contains only odd angular momentum states from (I.21), two terms of (I.30) can be combined.

$$\mathrm{Im}A_l^I(s)=\frac{1}{2q_s^2}\int_{4\mu^2}^{-4q_s^2}dt\,P_l\!\left(1+\frac{t}{2q_s^2}\right)\sum_{I'}\chi_{II'}\,\mathrm{Im}A^{I'}(t,\cos\theta_t)\,, \qquad \text{(I.31)}$$

or expanding $\mathrm{Im}A^I(t,\cos\theta_t)$ in terms of partial-wave amplitudes,

$$\mathrm{Im}A_l^I(s)=\frac{1}{2q_s^2}\int_{4\mu^2}^{-4q_s^2}dt\,P_l\!\left(1+\frac{t}{2q_s^2}\right)\sum_{I'}\chi_{II'}\sum_{l'}(2l'+1)P_{l'}\!\left(1+\frac{s}{2q_t^2}\right)\mathrm{Im}A_{l'}^{I'}(t)\,. \qquad \text{(I.32)}$$

The series (I.32) can still be used below $s=-4\mu^2$, if we allow for some cancellations between overlapping parts of the cuts, which will be possible according to the double spectral representation for $s>-32\mu^2$.[3,4] If we introduce, in

[3] G. F. Chew and S. Mandelstam, Phys. Rev. **119**, 467 (1960). These authors derived (I.25) and (I.34) from the double dispersion relations.

[4] See, for example, Ref. 3, or L. A. P. Balazs, Phys. Rev. **128**, 1939 (1962).

the physical t channel, the variables

$$t=4(q_t^2+\mu^2),\quad s=-2q_t^2(1-\cos\theta_t),\quad u=-2q_t^2(1+\cos\theta_t),\tag{I.33}$$

then (I.32) can be rewritten

$$\mathrm{Im}A_l^I(q_s^2)=\frac{2}{q_s^2}\int_0^{-q_s^2-\mu^2}d(q_t^2)P_l\left(1+2\frac{q_t^2+\mu^2}{q_s^2}\right)\sum_{I'}\chi_{II'}\sum_{l'}(2l'+1)P_{l'}\left(1+2\frac{q_s^2+\mu^2}{q_t^2}\right)\mathrm{Im}A_{l'}^{I'}(q_t^2)\tag{I.34}$$

for $q_s^2<0$. If q_s^2 is sufficiently close to $-\mu^2$ so that $\mathrm{Im}A_l^I(q_t^2)$ in (I.34) may be argued to behave as $q_t^{2l'}$, then the function $\mathrm{Im}A_l^I(q_s^2)$ for $q_s^2<\mu^2$ is well approximated by the lowest one or two partial waves in the sum over l'. When q_s^2 is large, the contribution from many partial waves with relatively small phase shifts may not be negligible in the sum over l'. Thus, certain low partial waves which are dominant to the physical scattering amplitude may not dominate in (I.34) except for $-q_s^2$ very close to μ^2.

Now let us consider the second application of the crossing relation (I.24). The partial wave expansion converges in the neighborhood of the symmetry point

$$s=t=u=\tfrac{4}{3}\mu^2\tag{I.35}$$

as will the corresponding expansions for channels t and u. The common region of convergence for both sides of (I.24) can be denoted by the small triangle in Fig. 2, bounded by $s=0$, $t=0$, $u=0$, in which

$$A^I(s,z_s)=\sum_{l=0}(2l+1)P_l\left(\frac{z_s}{s-4\mu^2}\right)A_l^I(s),\tag{I.36}$$

where $z_s=4q_s^2\cos\theta_s=t-u$. From (I.21), even l's appear in the sum of (I.36) for even isotopic spin states and odd l's for odd isotopic spin state. Under the crossing $s\leftrightarrow t$, $u\leftrightarrow u$, the new variables t, z_t become

$$t=-\tfrac{1}{2}(s-4\mu^2)+\tfrac{1}{2}z_s,\quad z_t=\tfrac{1}{2}(3s-4\mu^2)+\tfrac{1}{2}z_s.\tag{I.37}$$

From the oddness of X_1 of (I.22), we obtain at the symmetry point

$$A^0(s=\tfrac{4}{3}\mu^2,\,z_s=0)-\tfrac{5}{2}A^2(s=\tfrac{4}{3}\mu^2,\,z_s=0)=0,\quad A^1(s=\tfrac{4}{3}\mu^2,\,z_s=0)=0.\tag{I.38}$$

Differentiating X_i with respect to s and z_s and evaluating at the symmetry point, we get

$$\frac{\partial A^1}{\partial z_s}=\frac{1}{2}\frac{\partial A^0}{\partial s}=-\frac{\partial A^2}{\partial s}.\tag{I.39}$$

Proceeding further differentiations, at the symmetry point,

$$\frac{\partial^2A^0}{\partial s^2}-\frac{5}{2}\frac{\partial^2A^2}{\partial s^2}=-\frac{9}{2}\frac{\partial^2A^1}{\partial s\partial z_s},\quad \frac{\partial^2A^0}{\partial z_s^2}-\frac{5}{2}\frac{\partial^2A^2}{\partial z_s^2}=\frac{3}{2}\frac{\partial^2A^1}{\partial s\partial z_s},\quad \frac{\partial^2A^0}{\partial z_s^2}-7\frac{\partial^2A^2}{\partial z_s^2}=-\frac{\partial^2A^0}{\partial s^2}+\frac{\partial^2A^2}{\partial s^2},\tag{I.40}$$

is obtained. Such a process can go indefinitely to give an infinity of symmetry point conditions, keeping in mind that any even times of differentiations of A^1 and any odd times of differentiations of A^0, A^2 with respect to z_s vanish at the symmetry point. We hope that well inside the convergence region, very high partial waves can be neglected so that the higher derivative conditions are not of practical interest, since we can limit ourselves, at low energy, to a finite number of partial waves which satisfy elastic unitarity. Thus, we have obtained from the crossing symmetry relation (I.24), the relation (I.34) which holds for $\mathrm{Re}s<0$ and (I.38), (I.39), and (I.40) which hold in the common region of convergence,[5] bounded by $s=0$, $t=0$, $u=0$ in Fig. 2.

In Sec. II, the derivation of the amplitude on the basis of analyticity, elastic unitarity, and crossing symmetry is given. The further assumptions that are made to get numerical answers are discussed.

In Sec. III, our results are compared with those of others. In doing so, the difference in methods and assumptions used by others are briefly reviewed.

Finally, Sec. IV includes discussions of future applications of our calculations, and possibility of our approximation scheme converging toward a unique answer.

II. CONSTRUCTION OF THE PARTIAL-WAVE AMPLITUDES

The lth partial wave of the total amplitude for a given isotopic spin state I is given, from (I.11), by

$$A_l^I(\nu)=\frac{1}{2}\int_{-1}^{1}d(\cos\theta)P_l(\cos\theta)A^I(\nu,\cos\theta),\tag{II.1}$$

[5] Conditions (I.39) and (I.40) are exactly the same as those of G. F. Chew and S. Mandelstam, Nuovo Cimento **19**, 752 (1961).

where

$$\nu=q_s^2=\tfrac{1}{4}s-1\,,$$
$$\cos\theta=\cos\theta_s=(t/2q_s^2)+1=-(u/2q_s^2)-1\,,$$

and the mass of the pion μ is taken to be unity. We write, in the elastic region,

$$A_l^I(\nu)=\frac{\nu^l}{M_l^I(\nu)-i[\nu^{2l+1}/(\nu+1)]^{1/2}\theta(\nu)}\,, \qquad \text{(II.2)}$$

where

$$M_l^I(\nu)=[\nu^{2l+1}/(\nu+1)]^{1/2}\cot\delta_l^I(\nu) \qquad \text{(II.3)}$$

for $\nu>0$, and $\theta(\nu)$ is the usual step function. Equation (II.2) implies the convenience of using the inverse amplitude. The discontinuity of the inverse partial wave amplitude across the right-hand cut is given by (I.16)

$$\mathrm{Im}(\nu^l/A_l^I(\nu))=-(\nu^{2l+1}/\nu+1)^{1/2}R_l^I(\nu)\,, \qquad \text{(II.4)}$$

for $0<\nu<3$, $R_l^I(\nu)=1$ and a simple calculation gives

$$R_l^I(\nu)=\frac{2(1-\eta(\nu)\cos 2\,\mathrm{Re}\delta_l^I)}{\eta^2(\nu)+1-2\eta(\nu)\cos 2\,\mathrm{Re}\delta_l^I}\,, \qquad \text{(II.5)}$$

where

$$\eta(\nu)=e^{-2\,\mathrm{Im}\delta_l^I(\nu)}\,,$$
$$0\leq\eta(\nu)\leq 1\,.$$

The discontinuity of the inverse amplitude across the left-hand cut is given by

$$\mathrm{Im}\left(\frac{\nu^l}{A_l^I}\right)=-\frac{\nu^l\,\mathrm{Im}A_l^I(\nu)}{|A_l^I(\nu)|^2}\,, \qquad \text{(II.6)}$$

where the function $\mathrm{Im}A_l^I(\nu)$ may be calculated from the crossing relation (I.34) for $\nu<-1$. The proper threshold behavior of the partial-wave amplitude has been accounted for in (II.2). Branch cuts extend from 0 to $+\infty$ and from -1 to $-\infty$ in ν. Let us denote the inverse amplitude by

$$F_l^I(\nu)=\nu^l/A_l^I(\nu)\,. \qquad \text{(II.7)}$$

The partial-wave $A_l^I(\nu)$ is bounded at infinity in virtue of (I.14). $F_l^I(\nu)$ goes to a nonvanishing constant at $\nu=0$ and shares the same branch points and cuts as $A_l^I(\nu)$. We assume in this article that A_l^I has no zero.

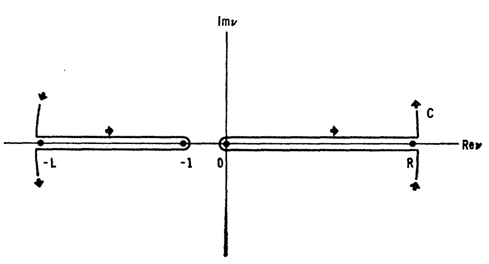


FIG. 4. Singularities of $F_l^I(\nu)$ in the ν plane. The contour C is the one to which Cauchey's theorem is applied.

We feel it may be useful not to consider high-energy contributions explicitly, but to cut off at some finite values of the energy. It is also hoped that sensitive dependence on the cutoff values of the dispersion integrals is eased by introducing regulated discontinuity functions of $F_l^I(\nu)$ across the cuts.

Let us write down integral equations for $F_l^I(\nu)$ applied to the contour C in Fig. 4.

$$F_l^I(\nu)=\sum_{p=0}^{q_l}a_p\nu^p+\frac{\nu^{N+1}}{\pi}\int_0^R\frac{\rho_l^I(\nu')}{\nu'^{N+1}(\nu'-\nu)}d\nu' +\frac{\nu^{M+1}}{\pi}\int_{-L}^{-1}\frac{\rho_l'^I(\nu')}{\nu'^{M+1}(\nu'-\nu)}d\nu' +\frac{1}{2\pi i}\int_{\text{circle}}\frac{F_l^I(\nu')}{\nu'-\nu}d\nu'\,, \qquad \text{(II.8)}$$

where

$$\rho_l^I(\nu)=-\theta(\nu)[\nu^{2l+1}/(\nu+1)]^{1/2}R_l^I(\nu)\,, \qquad \text{(II.9)}$$

$$\rho_l'^I(\nu)=-(\nu^l\,\mathrm{Im}A_l^I(\nu)/|A_l^I(\nu)|^2)\theta(-\nu-1)\,, \qquad \text{(II.10)}$$

$$q_l=\max(N,M)\geq l$$

and we have used

$$\frac{1}{\pi}\int\frac{\mathrm{Im}F(x')}{x'-x}dx' =\frac{x^N}{\pi}\int\frac{\mathrm{Im}F(x')}{x'^N(x'-x)}dx'+\sum_{p=0}^{N-1}a_p'x^p\,. \qquad \text{(II.11)}$$

The last term in (II.8) is the integral over the large "circle." The actual threshold for inelastic process occurs at $\nu=3$ but generally it is believed that the inelastic effect is small up to $\nu\approx 10$. The upper limit of the first integral in (II.8) is taken at some value around $\nu=10$ and $R_l^I(\nu)$ is set equal to 1. $\rho_l^I(\nu)$ is regulated by ν^{N+1} where N is an integer such that the integral is insensitive to the cutoff value R. It is interesting to notice from (II.9) that the minimum value of such an integer N is l. The lower limit $-L$ of the second integral in (II.8) may be taken from the range of energy values for which the crossing relation (I.34) converges, as we have no good, simple way to calculate the left-hand cut beyond that point. The regulation factor ν^{M+1} on $\rho_l'^I(\nu)$ is introduced for the same purposes as in the first integral.

The polynomial has q_l+1 terms where q_l is the larger integer between N and M. The contributions from the large circle of the contour C of Fig. 4 can always be approximately absorbed into this polynomial as long as ν is not near to the circle.

In Appendix B, several cases of inelastic effects are sketched. Among other things, we consider two types of inelastic effect: (i) $R_l^I(\nu)$ tends to a constant as $\nu\to\infty$. (ii) $R_l^I(\nu)$ blows up, but less rapidly than ν^2. In these cases the minimum order of regulation for $\rho_l^I(\nu)$ would be $N=l$ or $l+1$, respectively, to eliminate sensitive dependence on R for large R. As for $\rho_l'^I(\nu)$, its high-energy behavior is also discussed in Appendix B, using the results from Regge theory. However, *let us assume that with the use of lth-order polynomial and regulation $N=M=l$, the amplitude will be sufficiently accurate for $|\nu|\ll 10$.* (In particular, we need an accurate representation for $|\nu|\lesssim 1$ and roughly correct expressions for $|\nu|\lesssim 5$.) Our severely approximated forms for left-hand discontinuity and elastic unitarity on the right happen to give convergent integrals with $M=l$, and for calculational convenience, we take $R=+\infty$ and $-L=-\infty$ although this is not necessary.

Let us also assume that s and p waves are dominant in both direct and indirect channels. Then (II.10) can be approximated to a reasonable accuracy only for ν very close to -1. Since we are assuming that the major part of the two-pion forces comes from the relative s and p states of the exchanged pair, the remaining part of two-pion forces as well as multipion forces may be represented by the *strength parameters* approximating (II.10) near $\nu=-1$ such that the forms of the discontinuity are exact there, together with those *subtraction parameters* introduced in (II.8). In approximating (II.10) near -1, the p-wave in the right-hand side of (I.34) is not contributing much compared to the s waves. The left-hand cut form is approximated by (I.34) in which s waves are taken in the right-hand side in such a way that it has a correct form near $\nu=-1$ with a parameter γ_I mentioned above. Namely

$$\rho_0'^{I=0,2}(\nu)=\gamma_{I=0,2}\frac{3\nu+1}{\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}, \qquad \text{(II.12)}$$

$$\rho_1'^{I=1}(\nu)=\gamma_{I=1}\frac{5\nu+1}{\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}. \qquad \text{(II.13)}$$

In Appendix C, the effect of the p-wave resonance is included to approximate the left-hand cut forms and the values of the parameters are calculated by repeating essentially the same calculation as the one we are about to explain.

Under the assumptions we made, we get for the two s waves,

$$F_0^I(\nu)=\alpha_I-\frac{\nu}{\pi}P\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{d\nu'}{\nu'(\nu'-\nu)}+\gamma_I\frac{\nu}{\pi}P\int_{-\infty}^{-1}\left(\frac{-\nu'-1}{-\nu'}\right)^{3/2}\frac{(3\nu'+1)d\nu'}{\nu'^2(\nu'-\nu)}$$
$$+i\left[-\left(\frac{\nu}{\nu+1}\right)^{1/2}\theta(\nu)+\gamma_I\frac{3\nu+1}{\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}\theta(-\nu-1)\right], \quad (I=0,\,2) \qquad \text{(II.14)}$$

and for the p wave,

$$F_1^{I=1}(\nu)=\alpha_1+\beta\nu-\frac{\nu^2}{\pi}P\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{d\nu'}{\nu'(\nu'-\nu)}+\gamma_1\frac{\nu^2}{\pi}P\int_{-\infty}^{-1}\left(\frac{-\nu'-1}{-\nu'}\right)^{3/2}\frac{(5\nu'+1)d\nu'}{\nu'^3(\nu'-\nu)}$$
$$+i\left[-\left(\frac{\nu^3}{\nu+1}\right)^{1/2}\theta(\nu)+\gamma_1\frac{5\nu+1}{\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}\theta(-\nu-1)\right]. \qquad \text{(II.15)}$$

The integrals in (II.14) and (II.15) are easily carried out;

$$-\frac{\nu}{\pi}P\int_0^\infty\left(\frac{\nu'}{\nu'+1}\right)^{1/2}\frac{d\nu'}{\nu'(\nu'-\nu)}\equiv f(\nu)=\frac{2}{\pi}\left(\frac{\nu}{\nu+1}\right)^{1/2}\ln[(\nu+1)^{1/2}+(\nu)^{1/2}] \quad \text{for} \quad \nu>0 \quad \text{or} \quad \nu<-1,$$
$$=\frac{2}{\pi}\left(\frac{-\nu}{1+\nu}\right)^{1/2}\tan^{-1}\left(\frac{1+\nu}{-\nu}\right)^{1/2} \quad \text{for} \quad -1<\nu<0, \qquad \text{(II.16)}$$

$$\frac{\nu}{\pi}P\int_{-\infty}^{-1}\left(\frac{-\nu'-1}{-\nu'}\right)^{3/2}\frac{(3\nu'+1)d\nu'}{\nu'^2(\nu'-\nu)}=(3\nu+1)f_L(\nu)-\frac{2}{5\pi}, \qquad \text{(II.17)}$$

and

$$\frac{\nu^2}{\pi}P\int_{-\infty}^{-1}\left(\frac{-\nu'-1}{-\nu'}\right)^{3/2}\frac{(5\nu'+1)d\nu'}{\nu'^3(\nu'-\nu)}=(5\nu+1)\left[f_L(\nu)-\frac{2}{5\pi}\right]+\frac{4}{35\pi}\nu, \qquad \text{(II.18)}$$

where

$$f_L(\nu)=-\frac{2}{\pi}\frac{1}{\nu}\left\{\frac{1}{3}+\frac{\nu+1}{\nu}+\left(\frac{\nu+1}{\nu}\right)^{3/2}\ln[(-\nu-1)^{1/2}-(-\nu)^{1/2}]\right\} \quad \text{for} \quad \nu>0 \quad \text{or} \quad \nu<-1,$$
$$=-\frac{2}{\pi}\frac{1}{\nu}\left\{\frac{1}{3}+\frac{\nu+1}{\nu}+\left(\frac{1+\nu}{-\nu}\right)^{3/2}\tan^{-1}\left(\frac{-\nu}{1+\nu}\right)^{1/2}\right\} \quad \text{for} \quad -1<\nu<0. \qquad \text{(II.19)}$$

The s-wave phase shifts $\delta_0^{I=0,2}$ in the physical region are given by

$$M_0^{I=0,2}(\nu)=\left(\frac{\nu}{\nu+1}\right)^{1/2}\cot\delta_0^{I=0,2}=\alpha_{I=0,2}+f(\nu)+\gamma_{I=0,2}\left\{(3\nu+1)f_L(\nu)-\frac{2}{5\pi}\right\}, \tag{II.20}$$

while p-wave phase shift δ_1^1 is given by

$$M_1^{I=1}(\nu)=\left(\frac{\nu^3}{\nu+1}\right)^{1/2}\cot\delta_1^{I=1}=\alpha_1+\beta\nu+\nu f(\nu)+\gamma_1\left\{(5\nu+1)\left[f_L(\nu)-\frac{2}{5\pi}\right]+\frac{4\nu}{35\pi}\right\}. \tag{II.21}$$

The p-wave state can develop a resonance behavior for suitable values of α_1 and β. It appears that a positive α_1 and a negative β will produce a resonance in the physical region. It also seems that the magnitude of β is likely to be the order of 1.[6] The p-wave amplitude obtained from (II.15) can be written in the physical region

$$\frac{A_1^1(\nu)}{\nu}=\frac{\Gamma}{\nu_R-\nu\left\{1-\Gamma f(\nu)-\Gamma\gamma_1\left[\frac{5\nu+1}{\nu}\left(f_L(\nu)-\frac{2}{5\pi}\right)+\frac{4}{35\pi}\right]\right\}-i\Gamma\left(\frac{\nu^3}{\nu+1}\right)^{1/2}\theta(\nu)}, \tag{II.22}$$

where

$$\nu_R=-\alpha_1/\beta=\alpha_1\Gamma. \tag{II.23}$$

Our amplitudes (II.14) and (II.15) contain seven parameters of which four are subtraction constants and three are the strength parameters related to the contributions from the left-hand singularities.

From crossing symmetry, we obtained (I.34) which relates the physical region of the crossed channels to the unphysical region of the direct channel, and relations (I.38), (I.39), and (I.40) between amplitudes at the maximum symmetry point inside the common region of convergence of the partial wave expansion. If we consider all higher partial waves than $l=1$ to be small, the crossing symmetry relations (I.38, I.39, I.40) give almost exact relations between s and p amplitudes at the symmetry point;

$$A_0^0(\nu=-\tfrac{2}{3})=\tfrac{5}{2}A_0^2(\nu=-\tfrac{2}{3}), \tag{II.24}$$

$$\partial A_0^0/\partial\nu=-2(\partial A_0^2/\partial\nu)=-9A_1^1, \tag{II.25}$$

and

$$\frac{\partial^2 A_0^0}{\partial\nu^2}-\frac{5}{2}\frac{\partial^2 A_0^2}{\partial\nu^2}=27A_1^1+18\frac{\partial A_1^1}{\partial\nu}. \tag{II.26}$$

In obtaining the second derivative relation (II.26), a small effect from d waves has already been included.[7]

Giving the value of the s-wave amplitude at the symmetry point as[8]

$$A^0(-\tfrac{2}{3})=\tfrac{5}{2}A^2(-\tfrac{2}{3})=-5\lambda, \tag{II.27}$$

we get

$$-\lambda\simeq\tfrac{1}{5}A_0^0(\nu=-\tfrac{2}{3})=\tfrac{1}{2}A_0^2(\nu=-\tfrac{2}{3}). \tag{II.28}$$

Thus, for a given λ, the symmetry point conditions (II.25), (II.26), and (II.28) consist of five independent conditions. On the other hand, from our construction of amplitudes, the discontinuity across the left-hand cut is given by

$$\mathrm{Im}A_l^I(\nu)=-\mathrm{Im}(A_l^I)^{-1}/|A_l^{I-1}|^2 \tag{II.29}$$

in which the parameters are contained. The crossing relation (I.34) evaluated by inserting our amplitude (II.14) and (II.15) in the right-hand side, is compared with (II.29) at some point $\nu=\nu_F$ near -1, for each of the three partial waves. Thus, we have enough conditions to determine all seven parameters not counting λ, with one condition left over.

In order to demonstrate how our simple model gives rather satisfactory results without too much involved work, an example of the determination of parameters is briefly sketched in the following. For a given value of λ, (II.28) gives

$$-1/5\lambda=\alpha_0+f(\nu=-\tfrac{2}{3})-\gamma_0[f_L(\nu=-\tfrac{2}{3})+(2/5\pi)], \tag{II.30}$$

$$-1/2\lambda=\alpha_2+f(\nu=-\tfrac{2}{3})-\gamma_2[f_L(\nu=-\tfrac{2}{3})+(2/5\pi)]. \tag{II.31}$$

[6] If we apply Cauchey's theorem to

$$G_1^1(\nu)=\frac{\nu}{A_1^1(\nu)(\nu-\nu_0)(\nu-\nu_1)}=\frac{F_1^1(\nu)}{(\nu-\nu_0)(\nu-\nu_1)}$$

for the contour C of Fig. 4, $F_1^1(\nu)$ turns out, neglecting left-cut contributions

$$F_1^1(\nu)=h_1(\nu_0\nu_1)+\nu h_2(\nu_0\nu_1)+\nu f(\nu)+i\rho_1^1(\nu)\theta(\nu),$$

where $f(\nu)$ is given by (II.16) and

$$h_1(\nu_0\nu_1)=[\nu_0\nu_1/(\nu_0-\nu_1)][f(\nu_0)-f(\nu_1)],$$
$$h_2(\nu_0\nu_1)=[1/(\nu_0-\nu_1)][\nu_1(f\nu_1)-\nu_0 f(\nu_0)].$$

Comparing with (II.15), $h_2(\nu_0\nu_1)$ is identified as β. Taking $\nu_0=-\frac{2}{3}$, and varying ν_1 from -1 to -10, $h_2(\nu_0\nu_1)$ turns out to be always order of 1. In the first conjecture of the p-wave π-π resonance on the basis of nucleon electromagnetic structure, W. R. Frazer and J. R. Fulco, Phys. Rev. Letters 2, 364 (1956), also took β to be order of unity. The numerical calculation by B. H. Bransden and J. W. Moffat, of Ref. 13 confirms this. We are indebted to Dr. Moffat for informing us of this.

[7] The d waves in the low-energy region are taken into account as

$$A_2^{I=0,2}=C^{I=0,2}(s-4)^2.$$

From (I.40), it follows that

$$\frac{\partial^2 A_0^0}{\partial s^2}-\frac{5}{2}\frac{\partial^2 A_0^2}{\partial s^2}=-4\frac{\partial^2 A_1^1}{\partial s\partial z_s},$$

from which (II.26) follows.

[8] Some authors call it the renormalized coupling constant. See Refs. 3, 10, and 18.

From the first condition of (II.25),

$$(-5\lambda)^2\left\{\frac{df}{d\nu}+\gamma_0\frac{d}{d\nu}\left[(3\nu+1)f_L-\frac{2}{5\pi}\right]\right\}_{\nu=-2/3}=-2(2\lambda)^2\left\{\frac{df}{d\nu}+\gamma_2\frac{d}{d\nu}\left[(3\nu+1)f_L-\frac{2}{5\pi}\right]\right\}_{\nu=-2/3}, \quad \text{(II.32)}$$

which gives a linear relation between γ_0 and γ_2 or α_0 and α_2. Conditions of (II.29) give, in the vicinity of $\nu=-1$.

$$\mathrm{Im}A_0{}^{I=0,2}(\nu)=-\frac{\gamma_{I=0,2}[(3\nu+1)/\nu][(-\nu-1)/-\nu]^{3/2}}{\left\{\alpha_{I=0,2}+f(\nu)+\gamma_{I=0,2}\left[(3\nu+1)f_L(\nu)-\frac{2}{5\pi}\right]\right\}^2+\left\{\gamma_{I=0,2}\frac{3\nu+1}{\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}\right\}^2}, \quad \text{(II.33)}$$

from which a relation for γ_0/γ_2 may be deduced when (I.34) is used to evaluate $\mathrm{Im}A_0{}^{I=0,2}(\nu)$ near $\nu=-1$, in the left-hand side of (II.33). When this ratio is compared with that out of (II.30) and (II.31), keeping in mind the relation between α_0 and α_2 from (II.32), we have one equation to be solved containing only one parameter, say α_0, for a given value of λ. Thus, the s-wave parameters are completely determined which, in turn, can be used for the determination of the parameters of the p wave, using the second condition (II.25), (II.26), and (II.29).

The range of λ values are taken from -0.5 to $+0.5$. We find a p-wave resonance is generated for negative values of λ with $|\lambda|<0.39$. Our solution for negative λ also show strong s-wave attraction for $I=0$, $l=0$ which may accord with the well-known ABC experiment.[9] It is very amusing to notice that our solutions exhibit the development of $I=0$ s-wave bound states for negative values of λ with $|\lambda|\gtrsim 0.44$ and of $I=1$ p-wave bound states for negative values of λ, with $|\lambda|\gtrsim 0.39$ but neither bound states nor resonance for the $I=2$ s wave. Table I shows the list of the parameters for values of from -0.5 to $+0.5$ and some of the results are plotted on the Figs. 5 and 6. The consistency of our solution is shown in Figs. 7 and 8. Scattering lengths are plotted in Fig. 9. We have enough conditions, in principle, to determine all parameters. The final condition, however, as illustrated by Fig. 7, does not provide a unique determination of the final parameter λ. Instead one sees that $0.1\geq\lambda\geq -0.3$.

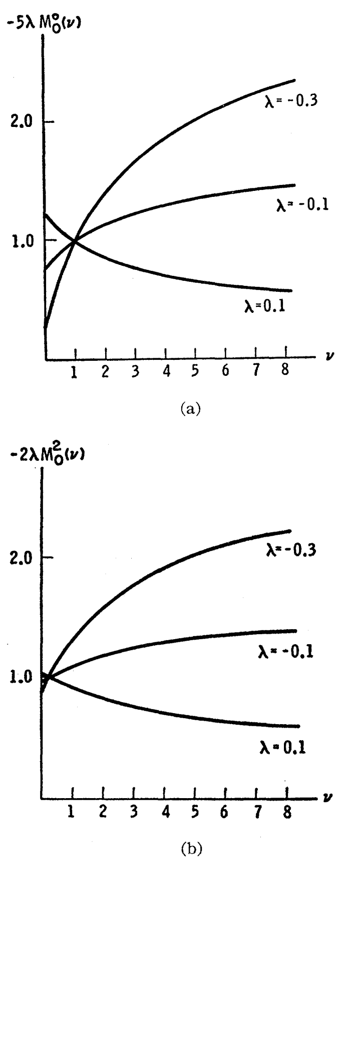


FIG. 5. The two s-wave phase shifts. Our results are approximately the same as those of Chew, Mandelstam, and Noyes for $|\lambda|\leq 0.3$.

III. COMPARISON WITH PREVIOUS CALCULATIONS

Okubo[10] has shown that the s-dominant case of π-π scattering corresponds to a conventional Feynman method with interaction Hamiltonian of the form of

TABLE I. Subtraction constants. Notice that scattering length $a_l{}^I=1/\alpha_I$ where $l=0$ for $I=0$, 2 and $l=1$ for $I=1$.

λ	α_0	α_2	α_1	β
0.08	−2.9639	−6.4787	390.0	39.2039
0.10	−2.4635	−5.2300	255.7898	33.6828
0.15	−1.7959	−3.5663	122.1975	28.7884
0.20	−1.4617	−2.7356	73.1599	23.1536
0.25	−1.2609	−2.2382	53.0	27.6281
0.30	−1.1269	−1.9069	35.8799	15.4024
0.40	−0.9593	−1.4932	21.5400	10.7157
0.50	−0.8588	−1.2447	14.3600	7.7368
−0.08	2.0323	6.0330	266.8872	−174.4035
−0.09	1.7543	5.3393	204.4493	−149.5733
−0.10	1.5318	4.7845	161.00	−128.9902
−0.15	0.8641	3.1211	59.4699	−80.7574
−0.20	0.5298	2.2908	26.4800	−59.3598
−0.25	0.3291	1.7930	12.3700	−47.7179
−0.30	0.1956	1.4602	5.4400	−40.4750
−0.35	0.1006	1.2214	2.00	−34.6974
−0.40	0.0302	1.0396	−0.5700	−32.3264
−0.45	−0.0080	0.8465	−2.0000	−43.9904
−0.50	−0.0658	0.7771	−3.5	−50.0

[9] A. Abashian, N. E. Booth, and K. M. Crowe, Phys. Rev. Letters **5**, 258 (1960).
[10] S. Okubo, Phys. Rev. **118**, 357 (1960).

$4\pi\lambda(\phi_\mu\phi_\mu)^2$, including only the chain diagrams. Bincer and Sakita[11] deduced an inequality relation for λ on the basis of a dispersion relation for the scattering phase shift

$$-0.29 \lesssim \lambda \lesssim 0.15 .$$

Chew and Mandelstam[3] estimated an inequality relation for λ from s-dominant solutions, by requiring the absence of zeros of the denominator function of $I=0$ s wave, on the negative real axis $|\nu|<10$, neglecting,

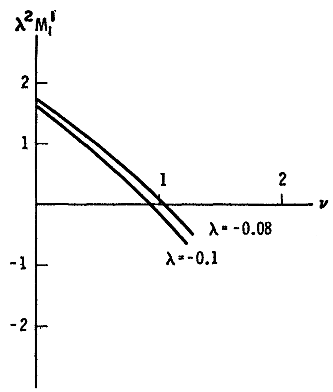


FIG. 6. Development of the p-wave resonance for physically acceptable λ values.

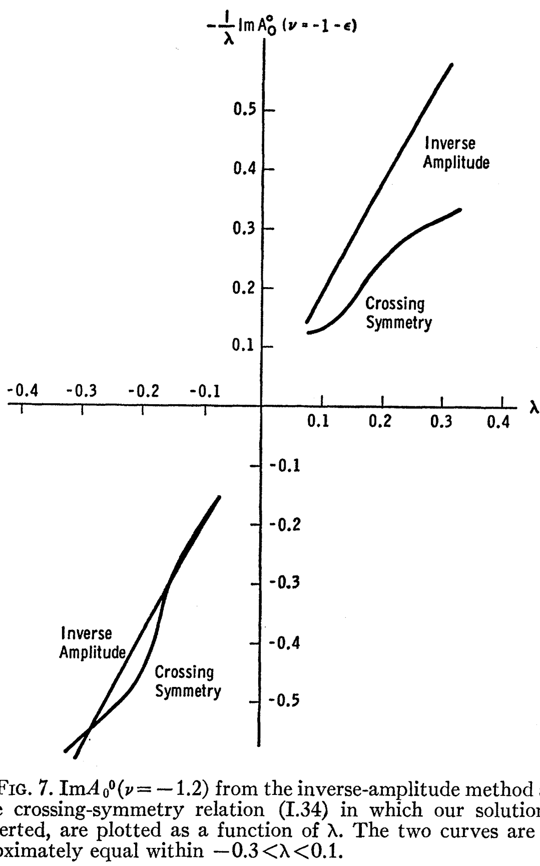
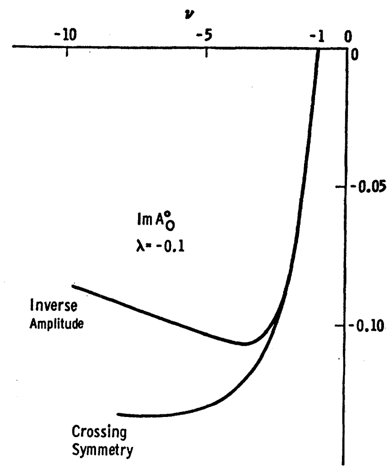


FIG. 7. $\mathrm{Im}A_0{}^0(\nu=-1.2)$ from the inverse-amplitude method and the crossing-symmetry relation (I.34) in which our solution is inserted, are plotted as a function of λ. The two curves are approximately equal within $-0.3<\lambda<0.1$.

[11] A. M. Bincer and B. Sakita, Phys. Rev. **129**, 1905 (1963).

FIG. 8. $\mathrm{Im}A_0{}^0$ from two methods (see caption of Fig. 8) for $\lambda=-0.1$. The two curves are in good agreement up to $\nu\approx-3$.



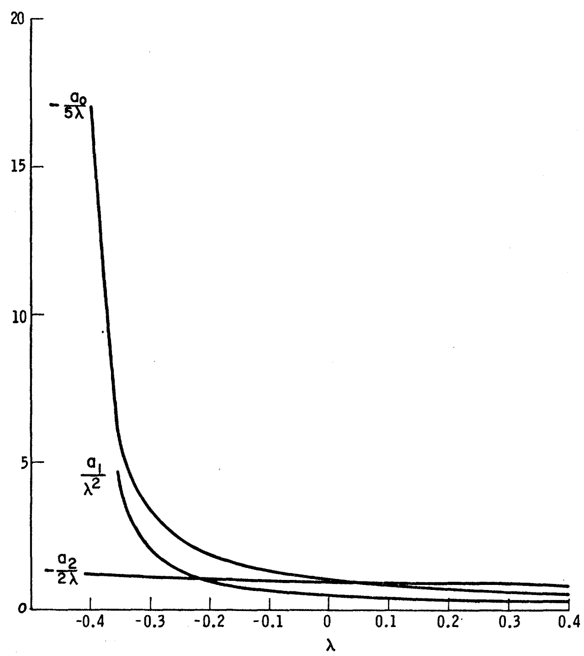


FIG. 9. The scattering lengths as a function of λ.

however, contribution from the left-cut singularities,

$$-1.5 \lesssim a_0 \lesssim 1.8 ,$$

or

$$-0.36 \lesssim \lambda \lesssim 0.3 .$$

Bransden and Moffat[12,13] obtained solutions using the inverse amplitude derived by Moffat[14] for negative λ with $|\lambda|\leq 0.5$ by numerical calculations, but with positive λ, the existence of solutions was not sure for $\lambda<0.25$ and for $\lambda>0.25$, solutions did not exist.

We have varied λ from -0.5 to $+0.5$. For all positive

[12] B. H. Bransden and J. W. Moffat, Nuovo Cimento **21**, 505 (1961); Phys. Rev. Letters **6**, 708 (1961).
[13] B. H. Bransden and J. W. Moffat, Phys. Letters **8**, 145 (1962).
[14] J. W. Moffat, Phys. Rev. **121**, 926 (1961).

λ, the solutions fail to give p-wave resonances. For negative λ with $|\lambda|<0.39$ the solutions show p-wave resonance behavior. It appears also that as the value of $|\lambda|$ increases, the position of the p-wave resonance energy moves towards zero energy, and the p wave becomes bound for $|\lambda|\gtrsim 0.39$. For negative λ with $|\lambda|\geq 0.44$, there occurs a bound state in $I=0$ s wave which confirms Desai's argument[15] of the large enhancement by strong final-state s-wave interaction corresponding to the well-known ABC anomaly.[9] Our solutions are in good agreement with Regge pole analysis of π-π scattering[16] that a strong attractive s-wave potential might well produce a bound p state.

Although some authors[17] have made an analysis of π-π s-wave phase shifts from π-N scattering and production data to get the values of s-wave scattering lengths, because of the question of validity of such an analysis, we feel it is better to say, solely from the existence of the ρ resonances, that the solutions for negative λ with $|\lambda|\leq 0.3$ are all physically acceptable.

The inverse amplitude technique was first applied to π-π problem by Moffat[14] and practical calculations were carried out by Bransden and Moffat.[13] The left-hand-cut contributions were obtained by a numerical iteration scheme keeping s and p waves only in the crossed channel.

As we have seen in Sec. I, the crossing-symmetry relation (I.34) may be safely used up to $\nu=-2$, however, below $\nu<-2$, the cancellation of the overlapping singularities is not clear, at the moment. Furthermore, the crossing symmetry relation (I.34) may be well approximated by certain lower partial waves only when ν is sufficiently close to -1. For larger $-\nu$, many partial waves with relatively small phase shifts may not be negligible due to the fact that the argument of $P_{l'}$ ranges from -1 to $-\infty$ as explained in Sec. I. Thus, we feel that using the form of the unphysical cut near $\nu=-1$ with appropriate parameters will not lose the general character of the problem when we keep just the lowest partial wave in the crossed channel. The polynomial terms in Moffat's inverse amplitude arise from consideration of threshold behavior of the amplitudes near the zero-energy region, while, in our case, the high-energy contribution is suppressed by the cutoff of the integrals as well as by the polynomials in which the contribution from the large circle, the higher partial waves in the crossed channel and the multipion exchange effect are hoped to be absorbed. We emphasize that the number of the terms in the polynomial is not necessarily $l+1$ for the lth partial wave as in Moffat's case, but may vary according to the behavior of the unphysical cut at higher energies and the behavior of the inelastic factor $R_l^I(\nu)$ as we have discussed in Sec. II and Appendix II.

Uretsky and Smith[18] have recently obtained solutions for pion-pion scattering with the partial-wave dispersion relation in the Mandelstam representation taking up to second-order terms of the perturbation series based on $L_{\text{int}}=\lambda\phi^4$ as the left-hand singularities. They obtained solutions very similar to those of Chew, Mendelstam, and Noyes[19] who used the crossing symmetry relation (I.34) to estimate the left-hand discontinuities keeping only s waves in the crossed channel. Both solutions are obtained from the N/D technique and are characterized by small p-wave scattering for physically acceptable values of λ, unlike our case. They point out the analogy of the partial wave dispersions to the Schrödinger equation of nonrelativistic quantum theory and the analogy of the left-hand-cut singularity to an interaction Hamiltonian, in other words, to a form of the generalized potential.[1] But the analogy is imperfect; in the N/D method, information as to the existence and uniqueness of solutions, as we have in the Schrödinger problem, is almost lacking. The solution may fail to exist if the N/D solution involves a spurious "ghost" pole not present in the dispersion relation. A brief examination by us indicates, however, that our new results are to be associated with the handling of the left-hand cut and crossing relations rather than use of T^{-1} instead of N/D methods. A calculation shows that if we change our procedure and use a two-pole approximation for the left-hand cut of the p-wave amplitude, then we obtain a solution similar to the dominant s-wave solution obtained by others using the N/D approach. The solutions obtained by Uretsky and Smith[18] are based on unitarity and analyticity conditions but they hope that crossing symmetry is satisfied, to the extent that the notion can be meaningful in a perturbation treatment. Their main interest is to investigate the dynamics of the $\lambda\phi^4$ type and perhaps to show the similarity in results of this model to those of the S-matrix approach. However, there still remain questions on this,[20] and also the convergence of the perturbation series is not at all clear. The third-order calculation by Uretsky and Saperstein[21] leads a fundamental difference between their results and those of the S-matrix approach such as an attractive p state from repulsive s state.

Since they could not get the p-wave resonance for physically acceptable values of λ, some have thought to start with a resonant p wave in the crossed channel.

[15] B. Desai, Phys. Rev. Letters **6**, 497 (1961).

[16] G. F. Chew, S. C. Frautschi, and S. Mandelstam, Phys. Rev. **126**, 1202 (1962).

[17] H. J. Schnitzer, Phys. Rev. **125**, 1059 (1962); J. Kirz, J. Schwartz, and R. D. Tripp, *ibid.* **126**, 763 (1962). J. Hamilton, P. Menotti, G. C. Oades, and L. L. J. Vick, *ibid.* **128**, 1881 (1962).

[18] K. Smith and J. L. Uretsky, Phys. Rev. **131**, 861 (1963).

[19] G. F. Chew, S. Mandelstam, and H. P. Noyes, Phys. Rev. **119**, 478 (1960).

[20] A derivative coupling scheme is also shown to be possible in π-π problem by Y. Miyamoto, Progr. Theoret. Phys. (Kyoto) **24**, 840 (1960) and A. V. Efremov, Chou Hung-Yuan, and D. V. Shirkov, Zh. Eksperim. i Teor. Fiz. **41**, 603 (1961) [English transl.: Soviet Phys.—JETP **14**, 432 (1962)]. Our form for the p-wave amplitude was shown to be associated with $L_{\text{int}}=2\pi a(\phi_\alpha\partial_\mu\phi_\beta-\phi_\beta\partial_\mu\phi_\alpha)^2$ by Miyamoto, while s waves are from $L_{\text{int}}=4\pi\lambda(\phi_\mu\phi_\mu)^2$.

[21] A. Saperstein and J. L. Uretsky, Phys. Rev. **133**, B1340 (1964).

Ball and Wong[22] obtained a four-parameter resonance form of p wave using the N/D technique not only by inserting the ρ resonance in the crossed channel in p-wave state, but also by replacing the left-cut contribution by a pole term, in each s-wave state, together with numerically calculated nearby left-hand cut contributions up to a cutoff point keeping only s and p waves. However, their solutions cannot be represented by the usual two parameter resonance form, and correspond to a nearby left-hand discontinuity that is an order of magnitude larger than that estimated by Chew and Mandelstam,[23] perhaps due to the fact that they do not require the second-derivative condition at the symmetry point. There are two points which have to be mentioned. First of all, the p wave $[\nu^3/(\nu+1)]^{1/2}\cot\delta_1{}^1$ is increasing near the threshold as energy increases. This is due to the fact that the left-hand discontinuity is too large as mentioned above. Secondly, their solution shows a zero in the $I=2$ s-wave state amplitude. A zero in the amplitude is not possible in the present formulation of our inverse amplitude. However, we have developed the possibility of describing such zeros in the inverse amplitude formulation using inelastic corrections. This will be reported in a subsequent paper. We are unable to discuss this point at present.

Balázs[24] made a further simplification of Ball and Wong's technique: He kept only the p-wave resonant state both in direct and indirect channels, neglecting all other states.[25] The left-hand-cut contribution is replaced by two-pole terms, according to an argument which involves interpolating the denominator, hoping the imaginary part of amplitude at high energy to be varying very slowly. The p-wave amplitude and its derivatives obtained in this way with the N/D technique are compared with those from the fixed momentum transfer dispersion relation at $\nu=-2$ to determine parameters. We have noticed that for physically acceptable values of λ, s waves are very significant, thus the idea of the bootstrap method should be viewed with caution.[26] Moreover, the crossing symmetry relation near $\nu=-1$ is completely neglected in his calculation. The interpolation of the denominator of the integrand without considering the inelastic behavior carefully at high energies should not be overlooked. However, since his approach was aimed mainly at predicting the ρ resonance, it is possible that it is not necessary to satisfy conditions in detail in the low-energy region.

In our calculation we have taken the view that the position and the width and even the existence of the p-wave resonance need not be put into the formulation, but that we can determine them through our conditions. Our work is essentially a one-parameter formulation, and if we need more subtractions due to the behavior of left-hand singularities in the higher energy region or due to the behavior of the inelastic coefficient $R_l{}^I(\nu)$, we can develop more conditions to determine all except perhaps one. Eventually, we feel that even this last parameter might be approximately determined unambiguously. The possible development of more conditions will be discussed in the next section.

The strength parameters γ_I will approach 1 as the forms of left-hand cuts become more and more accurate, namely, by iterating our solutions.

Figure 5 shows that our s-wave results are approximately the same as those of Chew, Mandelstam, and Noyes[19] for $|\lambda|\leq 0.3$. Figure 6 shows the development of the p-wave resonance for physically acceptable λ values. As $-\lambda$ increases, the resonance position moves toward the zero energy. Figure 9 shows the scattering lengths as a function of λ. A weakly attractive p wave appears when s wave is strongly attractive.

It is very difficult to compare our results with experiment at the moment. But it is still interesting to compare with some of the analysis based on pion-nucleon scattering data, to give at least a qualitative idea on the scattering lengths. Schnitzer[17] obtained scattering lengths

$$a_0=0.5\,,\quad a_2=0.16\,,\quad a_1=0.07\,,$$

using the Chew-Low extrapolation technique. This s-wave analysis can be fitted by making the choice $\lambda=-0.08$ in our solutions, for which we obtain

$$a_0=0.5\,,\quad a_2=0.168\,,\quad a_1=0.0035\,.$$

Uretsky and Smith's solutions show for $\lambda=-0.09$

$$a_0=0.5\,,\quad a_2=0.17\,,\quad a_1=0.003\,,$$

while ours, for this choice of λ, are

$$a_0=0.54\,,\quad a_2=0.189\,,\quad a_1=0.006\,.$$

It becomes most interesting when we compare our results with those of Bransden and Moffat's. The p-wave subtraction constants α_1, and β correspond to $\xi_1\nu_0$ and $(1/a_1)-f(\nu_0)-\xi_1$, respectively, in their notation. At $\lambda=-0.1$,

$$\text{B-M:}\quad a_0=0.67\,,\quad a_2=0.2\,,\quad a_1=0.0075\,,$$
$$\text{ours:}\quad a_0=0.653\,,\quad a_2=0.209\,,\quad a_1=0.0062\,.$$

This tells us that our simple approximation give the same values of scattering length as the numerical calculations of Bransden and Moffat without losing the general character of the problem. Moreover, our simple

---

[22] J. S. Ball and D. Y. Wong, Phys. Rev. Letters **7**, 390 (1961).
[23] See footnote of Ref. 15.
[24] L. A. P. Balázs, Phys. Rev. **128**, 1939 (1962).
[25] The philosophy behind the bootstrap method in strong interaction is well discussed by M. Udgaonkar, Proceedings of the 1963 Midwest Conference on Theoretical Physics, May 31–June 1, 1963, p. 65 (unpublished).
[26] F. Zachariasen and C. Zemach, Phys. Rev. **128**, 849 (1962) showed that in some model, large s waves have little effect on the p wave. However, if one stresses a detailed satisfaction of the conditions in a limited region as in our case, the s waves are playing an important role. In this connection, it is very interesting to notice that the s state itself in the crossed channel cannot be neglected in a self-consistent calculation of the s-wave scattering by L. A. P. Balázs, Phys. Rev. **129**, 872 (1963).

model exhibits the resonance behavior of the p wave for physically acceptable values of λ, (say at $\lambda=-0.08$, $\bar{\nu}_R=400$ MeV) which one could not get from the perturbation theory[18,21] or s-dominant solutions of N/D technique.[19]

We also mention that our solutions satisfy Shirkov's[27] crossing symmetry condition

$$A_1{}^1(\nu=-1)=(1/18)(-2(1/\alpha_0)+5(1/\alpha_2)),$$

which was obtained for the forward scattering direction under the second crossing symmetry set (I.20b).

We conclude this section by saying that we feel our simple model gives a physically reasonable answer for low-energy π-π scattering, but that correct values of the position and the width of the p-wave resonance will be predicted only when the left-hand cut and the inelastic effect are more adequately handled.

IV. CONCLUSIONS AND FURTHER REMARKS

We have constructed the scattering amplitude on the basis of two essential factors in addition to unitarity, analyticity and crossing symmetry. First of all, the partial-wave expansion of (I.36) is terminated at l_m. In particular the d waves are neglected in carrying out the conditions (I.38) and (I.39) while for the relations of (I.40), the d waves are included in the scattering length approximation. The estimated d-wave amplitudes are not so significant as to affect the conditions (I.38) and (I.39) at the symmetry point.[28] Secondly, the explicit consideration of higher energy region (in both positive and negative directions) is suppressed by using q_lth order of polynomial and regulating the integrals as discussed in the text following (II.11).

We need $l_m\rightarrow\infty$ to satisfy crossing symmetry completely in the small triangle of Fig. 2, because of divergence of (I.36) as $\nu\rightarrow-1$. According to others, there may be a principle allowing us to adopt a small q_l, but (i) the left-hand-cut prescription is uncertain in all but the nearby region, (ii) the right-hand cut depends on unknown inelastic processes. We choose to incorporate this uncertainty in the polynomial. The better we handle (i) and (ii), the smaller q_l will be needed. Unfortunately, we know of no simple prescription as to the q_l and l_m. We propose to experiment with increased q_l and increased l_m. We have not investigated the uniqueness or convergence problem of our formulation which is in some sense opposite of Balachandran's problem.[29] Increasing q_l and l_m will give solutions satisfying crossing more and more accurately. Thus, we hope these solutions will converge toward a definite answer, e.g., the lower partial waves near threshold will converge to definite values.

It seems obvious however that this procedure is practically limited. If we wish to predict higher partial waves or distant resonances, we will be dealing with tiny effects in the symmetry point region, e.g., we need to extend R (explicitly treat inelastic processes) if we hope to describe higher resonances. Furthermore, although we have unlimited crossing conditions in the small triangle, since the higher derivative conditions are more and more sensitive to higher partial waves, they may not give any new conditions to lower partial waves.

We remark that instead of fully using all conditions in our present formulation, we preferred to keep one parameters λ free. The last condition may be used to select out sensible values of λ. Figure 7 shows us that the $I=0$ s-wave absorptive parts from our solution and from crossing symmetry (I.34) in which our solution is inserted, are in good agreement for $-0.3\leq\lambda\leq0.1$.

If we keep the same l_m but increase q_l, then we can use for example the crossing symmetry (I.20c) in a particular direction, say, backward. From (I.24), we obtain

$$\sum_l(2l+1)(-1)^lA_l{}^I(s)=\sum_{I'}\sum_{l'}\chi_{II'}(2l'+1)(-1)^{l'}A_{l'}{}^{I'}(t),$$

with $s+t=4\mu^2$. This relation shows the symmetry of singularities around $s=2\mu^2$. One can easily see that Shirkov's crossing relation is a special case of this relation.[30]

Future applications of our solutions may be classified as follows: (i) We increase number of partial waves. (ii) We increase q_l or improve construction of right-hand and left cuts. In this category: (a) We are working on an approximate representation of inelastic effects; (b) we are working on the nearby left-hand cut through the relation (I.34) by machine calculation, so that we may not need to increase q_l very much in order to obtain much better amplitude.

We will find our procedure convincing if upon making an improved evaluation we find similar results.

ACKNOWLEDGMENTS

The author wishes to thank Professor Marc H. Ross for suggesting this problem and for patient direction and encouragement. He is also indebted to Dr. J. W. Moffat of RIAS for some stimulating discussions.

[27] A. V. Efremov, V. A. Meshcheryakov, D. V. Shirkov, and H. Chou, *Proceedings of the 1960 Annual International Conference on High-Energy Physics at Rochester* (Interscience Publishers, Inc., New York, 1961), p. 279.

[28] See footnote 7. From the relations (I.40), the two d-wave constants $C^{I=0,2}$ of footnote 7 turn out to be

$$C^0=\frac{1}{20}\frac{\partial^2A_0{}^2}{\partial s^2},$$

and

$$C^2=-\frac{1}{100}\left(\frac{1}{2}\frac{\partial^2A_0{}^2}{\partial s^2}-\frac{\partial^2A_0{}^0}{\partial s^2}\right),$$

respectively. At the symmetry point $C^0\sim10^{-4}$ and $C^2\sim10^{-5}$.

[29] A. P. Balachandran, University of Chicago, EFINS-63-40 (unpublished).

[30] J. R. Higgins (private communication).

APPENDIX A

Using the projection operators of (I.18) for the isotopic spin states $I=0$, 1, and 2, respectively, one can calculate the matrix element

$$F(s,t,u)=\langle cd|\sum_{I=0}^{2}p^{(I)}A^I(s,t,u)|ab\rangle, \tag{A1}$$

with

$$\langle a|I_i|b\rangle=-i\epsilon_{iab}, \tag{A2}$$

where the antisymmetric tensor ϵ_{iab} satisfies

$$\epsilon_{ijk}\epsilon_{ipq}=\delta_{jp}\delta_{kq}-\delta_{pk}\delta_{qj}. \tag{A3}$$

From (A2) and (A3), we obtain

$$(\mathbf{I}_1\cdot\mathbf{I}_2)_{(cd)(ab)}=\delta_{cb}\delta_{ad}-\delta_{ab}\delta_{cd}, \tag{A4}$$

$$(\mathbf{I}_1\cdot\mathbf{I}_2)^2{}_{(cd)(ab)}=\delta_{cd}\delta_{ab}+\delta_{ca}\delta_{db}, \tag{A5}$$

and thus

$$\langle cd|p^{(0)}|ab\rangle=\tfrac{1}{3}\delta_{cd}\delta_{ab}, \tag{A6}$$

$$\langle cd|p^{(1)}|ab\rangle=\tfrac{1}{2}(\delta_{ac}\delta_{bd}-\delta_{ad}\delta_{bc}), \tag{A7}$$

$$\langle cd|p^{(2)}|ab\rangle=\tfrac{1}{2}(\delta_{ac}\delta_{bd}+\delta_{ad}\delta_{bc}-\tfrac{2}{3}\delta_{ab}\delta_{cd}), \tag{A8}$$

and finally

$$F(s,t,u)=\tfrac{1}{3}(A^0-A^1)\delta_{ab}\delta_{cd}+\tfrac{1}{2}(A^1+A^2)\delta_{ac}\delta_{bd}-\tfrac{1}{2}(A^1-A^2)\delta_{ad}\delta_{bc}, \tag{A9}$$

which identifies the three invariant amplitudes of (I.19)

$$A=\tfrac{1}{3}(A^0-A^2),\quad B=\tfrac{1}{2}(A^1+A^2),\quad C=-\tfrac{1}{2}(A^1-A^2), \tag{A10}$$

or

$$A^0=3A+B+C,\quad A^1=B-C,\quad A^2=B+C. \tag{A11}$$

APPENDIX B

It is perhaps interesting to sketch the inelastic effects in $F_l{}^I(\nu)$ of (II.8). If we were to extend the contour to infinity on the positive real axis we would have to deal with the integral

$$\frac{\nu^{N+1}}{\pi}\int_R^\infty\frac{\rho_l{}^I(\nu')}{\nu'^{N+1}(\nu'-\nu)}d\nu' =-\frac{\nu^{N+1}}{\pi}\int_R^\infty\frac{R_l{}^I(\nu')[\nu'/(\nu'+1)]^{1/2}}{\nu'^{N-l+1}(\nu'-\nu)}d\nu', \tag{B1}$$

where $R_l{}^I(\nu)$ is given by (II.5). We consider some typical cases: If $\eta=0$ at high energy, then $R_l{}^I(\nu)=2$ so that $N=l$ will make the integral (B.1) converge.

Also from the analogy of potential scattering[31] at infinity, in certain cases, δ_R becomes $n\pi$. Then $R_l{}^I$ remains finite unless $\eta\to1$ and $(1-\cos2\delta_R)/(1-\eta)\to0$.

We obtain from (II.5) when $\eta\to1$ and $(1-\cos2\delta_R)/(1-\eta)\to0$

$$R_l{}^I(\nu)\approx\frac{\delta_I}{(\delta_R-n\pi)^2+\delta_I{}^2}\to\frac{1}{\delta_I}, \tag{B2}$$

where $\delta_R=\mathrm{Re}\,\delta_l{}^I(\nu)$ and $\delta_I=\mathrm{Im}\,\delta_l{}^I(\nu)$. The asymptotic behavior δ_I may now be determined in this case from the dispersion relation

$$A_l{}^I(\nu)=\frac{1}{\pi}\int_0^\infty\frac{[(\nu'+1)/\nu']^{1/2}\eta(\nu')\sin^2\delta_R(\nu')}{\nu'-\nu}d\nu' +\frac{1}{\pi}\int_3^\infty\frac{[(\nu'+1)/\nu']^{1/2}(1-\eta(\nu'))}{2(\nu'-\nu)}d\nu' +\text{left-hand cut} \tag{B3}$$

apart from subtraction terms. We find that δ_I goes to zero less rapidly than $1/\nu^2$ so that if $N=l+1$ the integral (B1) will converge.

Omnes[32] has shown that the discontinuity of $A^I(\nu)/\nu^l$ at infinity is determined by the position of the leading Regge pole at zero total energy $\alpha^I(0)$ and not by the spin of the physical bound states or resonances, for a partial wave $A_l{}^I(\nu)$ of the amplitude $A(s,t)$ which satisfies the Mandelstam representation and has the asymptotic behavior of the Regge type (i.e., one Regge pole behavior). Both discontinuities in $\mathrm{Im}F_l{}^I(\nu)^{-1}$ at $\nu\to\pm\infty$ behaves as $\nu^{\alpha^I(0)-l-1}$ up to logarithmic factors and in fact $A_l{}^I(\nu)/\nu^l\to\nu^{\alpha^I(0)-l-1}$ when $|\nu|\to\infty$ for both real and imaginary parts. Thus,

$$\rho_l{}^I(\nu)\underset{\nu\to\infty}{\sim}\nu^lR_l{}^I(\nu)\underset{\nu\to\infty}{\sim}\nu^{-\alpha^I(0)+l+1}, \tag{B4}$$

and

$$\rho_l'{}^I(\nu)=\mathrm{Im}F_l{}^I(\nu)\underset{\nu\to-\infty}{\longrightarrow}\nu^{-\alpha^I(0)+l+1}. \tag{B5}$$

From the Regge pole analysis of Chew, Frautschi, and Mandelstam,[16]

$$\alpha^0(0)=1,\quad \alpha^1(0)\approx\tfrac{1}{2},\quad \alpha^2(0)\approx-\tfrac{1}{2}.$$

Thus, we see that the minimum orders of regulation for (B1) and for the corresponding expression for the left-hand side is $N=M=l$, i.e., for $I=0$ and $I=1$. For $I=2N=M=l+1$ is needed.

APPENDIX C

It is also interesting to consider the effect of the p-wave resonance on the values of the parameters. [The p wave was neglected on the right-hand side of (I.34).] From the resonance formula (II.22), we can see that the actual resonance position is slightly shifted from ν_R and

[31] N. Levinson, Phys. Rev. **75**, 1445 (1949); G. Frye and R. L. Warnock, *ibid.* **130**, 478 (1963).

[32] R. Omnes, Phys. Rev. **133**, B1543 (1964).

given by

$$\bar{\nu}_R=\frac{\nu_R}{1-\Gamma f(\nu_R)-\Gamma[((5\nu_R+1)/\nu_R)(f_L(\nu_R)-(2/5\pi))-(4/35\pi)]\gamma_1}. \tag{C1}$$

Since the effective width from our first calculation is small, we may approximate the absorptive part of the p wave by the zero width resonance formula,

$$\mathrm{Im}A_1{}^1(\nu)=\frac{\Gamma^2\nu[\nu^3/(\nu+1)]^{1/2}}{\left\{\nu_R-\nu\left[1-\Gamma f(\nu)-\Gamma\left[\frac{5\nu+1}{\nu}\left(f_L(\nu)-\frac{2}{5\pi}\right)-\frac{4}{35\pi}\right]\gamma_1\right]^2+\Gamma^2\frac{\nu^3}{\nu+1}\right\}}. \tag{C2}$$

$$\approx\pi\Gamma\bar{\nu}_R\delta(\bar{\nu}_R-\nu).$$

Inserting (C2) into (I.34), we notice that there occurs a new cut starting at $-\bar{\nu}_R-1$ due to the p-wave exchange in the crossed channel,

$$\mathrm{Im}A_l{}^I(\nu<-1)=\frac{2}{\nu}\int_0^{-\nu-1}d\nu' P_l\left(1+2\frac{\nu'+1}{\nu}\right)(\chi_{I0}\,\mathrm{Im}A^0(\nu')+\chi_{I2}\,\mathrm{Im}A^2(\nu'))$$
$$+\frac{6\pi\Gamma}{\nu}\chi_{I1}\left(1+2\frac{\nu+1}{\bar{\nu}_R}\right)P_l\left(1+2\frac{\bar{\nu}_R+1}{\nu}\right)\theta(-\nu-1-\bar{\nu}_R)\theta(\bar{\nu}_R). \tag{C3}$$

This behavior of symmetry of singularities about $\nu=-\frac{1}{2}$ (or $s=2$) has been shown independently by several other investigators.[27,30] Again, by making severe approximations for the first term of (C3) near $\nu=-1-\epsilon$ for each partial wave, we repeat the same procedures discussed in Sec. II to determine the parameters. For example, in case of $\lambda=-0.1$, we approximate $\rho_l{}'^I(\nu)$ by

$$\rho_0{}'^{I=0,2}(\nu)=\begin{pmatrix}0.4296\gamma_0\\0.2986\gamma_2\end{pmatrix}\left[\frac{3\nu+1}{3\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}+\frac{1}{5\nu}\left(\frac{-\nu-1}{-\nu}\right)^{5/2}+\frac{1}{7\nu}\left(\frac{-\nu-1}{-\nu}\right)^{7/2}\right]$$
$$-\begin{pmatrix}\gamma_0\\\gamma_2\end{pmatrix}\frac{6\pi\Gamma\bar{\nu}_R}{\nu}\left(1+2\frac{\nu+1}{\bar{\nu}_R}\right)\theta(-\nu-1-\bar{\nu}_R)\theta(\bar{\nu}_R), \tag{C4}$$

$$\rho_1{}'^{I=1}(\nu)=0.1056\gamma_1\left[\frac{5\nu+1}{3\nu}\left(\frac{-\nu-1}{-\nu}\right)^{3/2}+\frac{2\nu+1}{5\nu}\left(\frac{-\nu-1}{-\nu}\right)^{5/2}\right]$$
$$-\gamma_1\frac{3\pi\Gamma\bar{\nu}_R}{\nu}\left(1+2\frac{\bar{\nu}_R+1}{\nu}\right)\left(1+2\frac{\nu+1}{\bar{\nu}_R}\right)\theta(-\nu-1-\bar{\nu}_R)\theta(\bar{\nu}_R). \tag{C5}$$

Inserting $\nu_R=0.83$ (≈ 380 MeV) and $\Gamma=0.008$ (see Table I) into (C4) and (C5), after repeating the same procedure as in the text with newly obtained $F_l{}^I(\nu)$, we get for $\lambda=-0.1$,

$$1/\alpha_0=0.661,\quad 1/\alpha_2=0.206,\quad 1/\alpha_1=0.0059,\quad \beta=-55.3697,$$

and

$$\bar{\nu}_R=1.3,\quad \Gamma=0.018.$$

We notice that as we improve the left-hand singularities, the magnitude of β becomes smaller[6] but without much change of the value of scattering length in both s and p states.