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Scattering Formalism for Singular Potential Theory

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When using power-series expansions in the coupling constant(s) for the computation of scattering amplitudes in singular potential theory, it is necessary to introduce some cutoff in order to avoid divergences of the individual terms in this series. In earlier work this was done by cutting off the potential. It is here shown that the general scattering formalism actually contains an intrinsic limiting operation which makes it unnecessary to introduce a cutoff as an extraneous computational device. For regular potentials one tacitly and justifiedly exchanges this operation with another step in the derivations. It is not clear whether such a "built-in" cutoff procedure also exists in unrenormalizable field theory.

1. INTRODUCTION

RECENTLY there has been a revival of interest in the problem of scattering by a static singular potential.[1–3] These explorations were motivated by a desire to have some understanding of unrenormalizable field theories. In order to make contact with field theory, attempts have been made to expand in powers of the potential strength. For this purpose, cutoffs are usually introduced by modifying the singular potential, or equivalently by considering the singular potential as the limit of a sequence of nonsingular potentials. It is the purpose of this paper to show that an intrinsic limiting operation is already contained in the scattering formalism, so that the introduction of an extraneous cutoff as a mathematical device is actually unnecessary.

We confine ourselves to potentials for which

$$\int_c^\infty r^2|V(r)|dr<\infty \qquad (1.1)$$

for any $c>0$, and for which $V(r)$ is bounded for all $r\geqslant c>0$. To begin with we consider s-wave scattering. Let $\psi(k,r)$ be the required radial wave function for wave number k which satisfies a suitable boundary condition near $r=0$, and which behaves for large r asymptotically as

$$\psi(k,r)\sim\tfrac{1}{2}ik^{-1}[e^{-ikr}-S(k)e^{ikr}]. \qquad (1.2)$$

For more detailed definitions see Sec. 2. In terms of the Jost function[4] $f(\lambda;k,r)$, where λ denotes the strength of the potential, Eq. (1.2) may be rewritten as

$$\psi(k,r)=\tfrac{1}{2}ik^{-1}[f(\lambda;k,r)-S(k)f(\lambda;-k,r)]. \qquad (1.3)$$

The scattering amplitude $A(k)$ is

$$A(k)=(2ik)^{-1}[S(k)-1], \qquad (1.4)$$

so that

$$\psi(k,r)=g(\lambda;k,r)+A(k)f(\lambda;-k,r), \qquad (1.5)$$

where

$$g(\lambda;k,r)=(-2ik)^{-1}[f(\lambda;k,r)-f(\lambda;-k,r)]. \qquad (1.6)$$

For the conventionally so-called regular potentials, i.e., potentials that satisfy

$$\int_0^\infty r|V(r)|dr<\infty, \qquad (1.7)$$

in addition to (1.1), the following relation between f, g and A is well known:

$$A(k)=-\lim_{r\to0}g(\lambda;k,r)/\lim_{r\to0}f(\lambda;-k,r). \qquad (1.8)$$

It must be emphasized that the validity of Eqs. (1.2)–(1.6) is independent of the behavior of $V(r)$ in the neighborhood of $r=0$, and in particular does not require the existence of the integral (1.7). In other words, Eqs. (1.2)–(1.6) also hold for singular potentials, i.e., potentials that are not regular. On the other hand, Eq. (1.8) does not have such general validity. For example, for a repulsive r^{-4} potential, neither of the limits on the right-hand side of (1.8) exist.

It will be shown in Sec. 2, however, that the following modification of Eq. (1.8) holds for regular as well as for singular potentials:

$$A(k)=-\lim_{r\to0}[g(\lambda;k,r)/f(\lambda;-k,r)]. \qquad (1.9)$$

Equation (1.9) is derived in Sec. 2 by a judicious choice of the Green's function in the scattering integral equation, whereby $A(k)$ appears explicitly in the integral equation. If the right-hand side of Eq. (1.8) exists, as in the case of regular potentials, then Eq. (1.9) reduces to Eq. (1.8). This reduction is discussed in some detail in Sec. 3. If the right-hand side of Eq. (1.8) fails to make any sense, then one must use the limit of the quotient Eq. (1.9) instead of the quotient of the

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[1] N. N. Khuri and A. Pais, Rev. Mod. Phys. **36**, 590 (1964).
[2] G. Tiktopoulos and S. Treiman (to be published).
[3] A. Pais and T. T. Wu, J. Math. Phys. **5**, 799 (1964).
[4] R. Jost, Helv. Phys. Acta **20**, 256 (1947).

limits. In Sec. 4 examples will be found of different ways in which Eq. (1.8) fails.

The substitution of Eq. (1.9) into Eq. (1.4) immediately yields

$$S(k)=\lim_{r\to 0}[f(\lambda;k,r)/f(\lambda;-k,r)]. \qquad (1.10)$$

However, Eq. (1.9) is more general than Eq. (1.10) since Eq. (1.10) has no content for $k=0$.

Equation (1.9) also suggests a new classification of potentials into the regular and the singular varieties. We propose that a potential is called regular if and only if Eq. (1.8) holds in addition to Eq. (1.9). Potentials that satisfy Eq. (1.7) [in addition to Eq. (1.1)] are always regular in this new sense, but Eq. (1.7) is by no means a necessary condition, as shown in example C of Sec. 4. It seems that this new classification is more natural physically, because there does not seem to be a simple physical interpretation of the absolute value of a potential.

It should be emphasized that Eq. (1.9) is not an *ad hoc* prescription for the order of limits. This equation is rather a straightforward consequence of the theory. It also leads to some new insight into the nature of the limiting processes for singular potentials.

The point is this. In singular potential theory, power-series expansions in the coupling strength, such as the Born series, have no meaning as every term in the series is divergent. In earlier work, one first introduced a cutoff in the potential, then legitimately expanded in a Born series, then resummed this series to get a finite answer as the cutoff tends to zero. Now we shall see in Sec. 2 that it is possible to find $A(k)$ by using power series expansions in the coupling constant *without ever to have to introduce a cutoff*. The reason for this is that $g(\lambda,k,r)$ and $f(\lambda,-k,r)$ in Eq. (1.6) are, for given $r\neq 0$ and given k, entire functions in the coupling constant. This property has played an important role[5] in the study of convergence of expansions in scattering theory for regular potentials. But it is also true for singular potentials that we may legitimately use Born series for the functions g and f, for any $r>0$. Then what operation takes over the role of the limit operation on a cutoff? This is precisely the limit process which appears in Eq. (1.9) *prescribed by the theory as an intrinsic rather than an added prescription.* In order to make effective use of this intrinsic property, it appears essential to use integral equations with boundary values which are independent of the coupling constants.

It would be a major step to have a formulation of unrenormalizable field theory with a "built-in" limiting process, as we have found here for potential theory. We do not know how to achieve this in general. However, we hold it likely that procedures similar to the present ones can be developed for Bethe-Salpeter equations. We hope to come back to this elsewhere.

[5] R. Jost and A. Pais, Phys. Rev. **82**, 840 (1951).

2. SCATTERING FORMALISM

We consider, for clarity, the case of s-wave scattering first. Let $\psi(k,r)$ be the s-wave part of the wave function multiplied by r. Then ψ satisfies, in the range $(0,\infty)$,

$$[(d^2/dr^2)+k^2-\lambda V(r)]\psi(k,r)=0. \qquad (2.1)$$

Here $k\geqslant 0$. We shall put no restriction on the behavior of $V(r)$ near $r=0$. However, to avoid unnecessary complications, we retain the restrictions stated at the beginning of the introduction, in particular Eq. (1.1). As $r\to\infty$,

$$\psi(k,r)\sim k^{-1}\sin kr+A(k)e^{ikr}, \qquad (2.2)$$

where $A(k)$ remains to be determined. If $k=0$, the first term on the right-hand side of (2.2) is to be interpreted as r.

The condition near $r=0$ has been discussed by Kramers.[6] We shall take the following formulation. Consider the set of all possible nontrivial solutions to (2.1) without any boundary condition. Since (2.1) is a linear differential equation of second order, this set may be parametrized by two complex numbers. We distinguish two possible cases.

Case I. One *cannot* divide the set in two parts, such that, as $r\to 0$, the solutions ψ_1 in one part are small compared to the solutions ψ_2 in the other part, so that

$$\lim_{r\to 0}\frac{\psi_1(r)}{\psi_2(r)}=0 \qquad (2.3)$$

is *not* true. This case has been discussed in detail by Case[7] and we shall not consider it any further.

Case II. One *can* divide the set as indicated, and Eq. (2.3) is true. Let $\psi_s(k,r)$ be any solution of the "small" kind.[8] Without loss of generality we choose $\psi_s(k,r)$ to be real.

We use as a condition on $\psi(k,r)$ that

$$\psi(k,r)=\text{const}\,\psi_s(k,r). \qquad (2.4)$$

Equations (2.1), (2.2), and (2.4) determine $\psi(k,r)$ uniquely.

Write Eq. (2.1) in the form

$$[(d^2/dr^2)+k^2]\psi(k,r)=\lambda V(r)\psi(k,r), \qquad (2.5)$$

and consider the right-hand side as the source term. Let

$$\psi_0(k,r)=\psi(k,r)-\lambda\int_r^\infty dr'k^{-1}\sin k(r'-r)V(r')\psi(k,r'). \qquad (2.6)$$

Then $\psi_0(k,r)$ satisfies the homogeneous equation

$$[(d^2/dr^2)+k^2]\psi_0(k,r)=0. \qquad (2.7)$$

[6] H. A. Kramers, *Quantum Mechanics* (Interscience Publishers, Inc., New York, 1957), pp. 183, 184.

[7] K. M. Case, Phys. Rev. **80**, 797 (1950).

[8] Of course, any other solution of the "small" kind is a constant multiple of $\psi_s(k,r)$.

By (2.6), (2.7), and (2.2) we find that

$$\psi_0(k,r)=k^{-1}\sin kr+A(k)e^{ikr}. \quad (2.8)$$

Equation (2.6), with (2.8), is an integral equation for $\psi(k,r)$. Note that in this integral equation $A(k)$ appears explicitly, and is yet to be determined by Eq. (2.4).

Define $f(\lambda;k,r)$ and $g(\lambda;k,r)$ by

$$f(\lambda;k,r)-\lambda\int_r^\infty dr'k^{-1}\sin k(r'-r)\times V(r')f(\lambda;k,r')=e^{-ikr}, \quad (2.9)$$

and

$$g(\lambda;k,r)-\lambda\int_r^\infty dr'k^{-1}\sin k(r'-r)\times V(r')g(\lambda;k,r')=k^{-1}\sin kr. \quad (2.10)$$

Then Eq. (1.5) follows from Eq. (2.6). Here $f(\lambda;k,r)$ is the Jost function.[4] Since Eqs. (2.9) and (2.10) are Volterra integral equations, for any fixed $r>0$, $f(\lambda;k,r)$, and $g(\lambda,k,r)$ are entire functions of λ by Eq. (1.1). From this it does not follow that $\psi(k,r)$ has the same property [see Eq. (1.5)] because the dependence of $A(k)$ on λ changes in general the analyticity properties of ψ as compared to f and g.

Let $\psi_s'(k,r)$ be a solution of (2.1) linearly independent of $\psi_s(k,r)$. Since $f(\lambda;-k,r)$ and $g(\lambda;k,r)$ both satisfy (2.1), they are linear combinations of ψ_s and ψ_s':

$$f(\lambda;-k,r)=\alpha\psi_s(k,r)+\alpha'\psi_s'(k,r), \quad (2.11)$$

and

$$g(\lambda;k,r)=\beta\psi_s(k,r)+\beta'\psi_s'(k,r). \quad (2.12)$$

A comparison of Eq. (2.4) with Eqs. (1.5), (2.11), and (2.12) gives

$$\beta'+A\alpha'=0. \quad (2.13)$$

It is not possible for both α' and β' to vanish, because, in that case, f and g would be linearly dependent, which contradicts Eqs. (2.9) and (2.10). We thus consider two cases: (a) $\alpha'=0$ and (b) $\alpha'\neq 0$.

Case (a): $\alpha'=0$. In this case, $\psi(k,r)$ does not exist, as Eq. (2.13) cannot be satisfied. Moreover, $f(\lambda;-k,r)$ is a constant multiple of a real function. Hence the right-hand side of Eq. (2.9) is a constant multiple of a real function. This requires $k=0$. Thus case (a) corresponds to the situation of a zero-energy bound state, $\psi_s(0,r)$ being the bound-state wave function. In this case, we say that A is infinite.

Case (b): $\alpha'\neq 0$. In this case, there is no bound state at this energy, and

$$A(k)=-\beta'/\alpha'. \quad (2.14)$$

By Eqs. (2.2), (2.11), and (2.12), Eq. (2.14) can be written in the form Eq. (1.9). Moreover, this limit is guaranteed to exist in this case by (2.14).

Thus we have now shown what was contended in Sec. 1. The simple trick by which we have reached our aim is to use in Eq. (2.6) a Green's function which "avoids the origin." This in turn is possible due to the choice Eq. (2.8) of the boundary value.

Once again, as in earlier work on singular potentials[1] (and on unrenormalizable field theory[9]) it appears expedient to split the sought for scattering wave function in two parts. However, the present splitting is not identical with the one used earlier. To see this consider the zero-energy equations

$$f(\lambda;0,r)-\lambda\int_r^\infty dr'(r'-r)V(r')f(\lambda;0,r')=1, \quad (2.15)$$

$$g(\lambda;0,r)-\lambda\int_r^\infty dr'(r'-r)V(r')g(\lambda;0,r')=r, \quad (2.16)$$

which follow from Eqs. (2.9) and (2.10). From these equations it is readily checked[10] that the split used earlier is as follows. One part is a linear combination of f and g, the other exactly soluble part is essentially A/r. With this last method the introduction of a cutoff could not be avoided.

Finally, for $l\geqslant 0$, Eq. (2.1) is replaced by

$$\left[\frac{d^2}{dr^2}+k^2-\frac{l(l+1)}{r^2}-\lambda V(r)\right]\psi_l(k,r)=0. \quad (2.17)$$

As $r\rightarrow\infty$,

$$\psi_l(k,r)\sim k^{-1}\sin(kr-\tfrac{1}{2}l\pi)+A_l(k)e^{ikr}. \quad (2.18)$$

The discussion of the condition near $r=0$ is the same as before. The integral equations (2.9) and (2.10) are generalized to

$$f_l(\lambda;k,r)+\lambda\int_r^\infty dr'(rr')^{1/2}[J_{l+\frac{1}{2}}(kr)Y_{l+\frac{1}{2}}(kr')-J_{l+\frac{1}{2}}(kr')Y_{l+\frac{1}{2}}(kr)]V(r')f_l(\lambda;k,r')=(\tfrac{1}{2}\pi kr)^{1/2}(-i)^{l+1}H_{l+\frac{1}{2}}^{(2)}(kr), \quad (2.19)$$

and

$$g_l(\lambda;k,r)+\lambda\int_r^\infty dr'(rr')^{1/2}[J_{l+\frac{1}{2}}(kr)Y_{l+\frac{1}{2}}(kr')-J_{l+\frac{1}{2}}(kr')Y_{l+\frac{1}{2}}(kr)]V(r')g_l(\lambda;k,r')=(\tfrac{1}{2}\pi k^{-1}r)^{1/2}J_{l+\frac{1}{2}}(kr), \quad (2.20)$$

while Eq. (1.9) is essentially not changed:

$$A_l(k)=-\lim_{r\to 0}\frac{g_l(\lambda;k,r)}{f_l(\lambda;-k,r)}. \quad (2.21)$$

[9] G. Feinberg and A. Pais, Phys. Rev. **133**, B477 (1964).

[10] The detailed connection is as follows. Using the quantities Ψ_1 and Ψ_2 defined in Ref. 1, Eq. (2.6) and $\Psi_2^{(1)}$, $\Psi_2^{(2)}$ *ibid.* Eq. (2.12) one has $r\Psi_2^{(1)}=g$, $r\Psi_2^{(2)}+1=f$, $r\Psi_1=A$, $r\Psi_2=g+A(f-1)$.

3. REGULAR POTENTIALS

We consider in this section potentials which in addition satisfy Eq. (1.7). In this case, considering again only the s wave, we define ψ_s and ψ_s' as the solutions of the integral equations

$$\psi_s(k,r)+\lambda\int_0^r dr'k^{-1}\sin k(r-r')\times V(r')\psi_s(k,r')=k^{-1}\sin kr, \quad (3.1)$$

and

$$\psi_s'(k,r)+\lambda\int_0^r dr'k^{-1}\sin k(r-r')\times V(r')\psi_s'(k,r')=\cos kr. \quad (3.2)$$

By the theory of Volterra integral equations,[11] the Born series converges uniformly in r for both (3.1) and (3.2). Moreover

$$\lim_{r\to0}\psi_s(k,r)=0, \quad (3.3)$$

and

$$\lim_{r\to0}\psi_s'(k,r)=1. \quad (3.4)$$

Thus they satisfy all the conditions previously prescribed for ψ_s and ψ_s'. With this choice of ψ_s and ψ_s', Eqs. (2.11) and (2.12) yield

$$\lim_{r\to0}f(\lambda;-k,r)=\alpha' \quad (3.5)$$

and

$$\lim_{r\to0}g(\lambda;k,r)=\beta'. \quad (3.6)$$

Therefore, by Eq. (2.14) we obtain Eq. (1.8) for case II(b). Thus, for regular potentials, Eq. (1.8) holds in addition to Eq. (1.9).

4. EXAMPLES

We give here three examples to illustrate the various possibilities.

Example A

As a first example let $k=0$, and

$$V(r)=r^{-m}, \quad (4.1)$$

with $m>3$. In this case[1]

$$f(\lambda;0,r)=(\nu\lambda^{1/2})^{-\nu}\Gamma(1+\nu)r^{1/2}I_\nu(z), \quad (4.2)$$

and

$$g(\lambda;0,r)=(\nu\lambda^{1/2})^{\nu}\Gamma(1-\nu)r^{1/2}I_{-\nu}(z), \quad (4.3)$$

where

$$\nu=(m-2)^{-1}, \quad (4.4)$$

and

$$z=2\nu\lambda^{1/2}r^{-1/(2\nu)}. \quad (4.5)$$

Thus this belongs to case I if $\lambda<0$. If $\lambda>0$, this belongs to case II(b). In the latter case, it is easily verified that both $f(\lambda,0,r)$ and $g(\lambda;0,r)$ are unbounded as $r\to0$, while

$$A(0)=-\lim_{r\to0}\frac{g(\lambda;0,r)}{f(\lambda;0,r)}=-(\nu\lambda^{1/2})^{2\nu}\frac{\Gamma(1-\nu)}{\Gamma(1+\nu)}, \quad (4.6)$$

as obtained before.[1]

[11] We are indebted to Professor H. McKean for a helpful discussion on this point.

Example B

As a second example, let

$$V(r)=\begin{cases} r^{-2}, & r<1, \\ 0. & r>1. \end{cases} \quad (4.7)$$

It is easily verified in this case that, for $r<1$,

$$f(\lambda;-k,r)=\tfrac{1}{2}\pi r^{1/2}e^{ik}\{k[Y_\nu'(k)J_\nu(kr)-Y_\nu(kr)J_\nu'(k)] -(\tfrac{1}{2}+ik)[Y_\nu(k)J_\nu(kr)-Y_\nu(kr)J_\nu(k)]\}, \quad (4.8)$$

and

$$g(\lambda;k,r)=\tfrac{1}{2}\pi r^{1/2}\{\sin k[Y_\nu'(k)J_\nu(kr)-Y_\nu(kr)J_\nu'(k)] -(\tfrac{1}{2}k^{-1}\sin k+\cos k)[Y_\nu(k)J_\nu(kr)-Y_\nu(kr)J_\nu(k)]\}, \quad (4.9)$$

where

$$\nu=(\tfrac{1}{4}+\lambda)^{1/2}. \quad (4.10)$$

This belongs to case I if $\lambda<-\frac{1}{4}$. If $\lambda\geqslant-\frac{1}{4}$, it belongs to case II(b). Again in the latter case, as $r\to0$,

$$f(\lambda;-k,r)\sim\tfrac{1}{2}\pi r^{1/2}e^{ik}\times[(\tfrac{1}{2}+ik)J_\nu(k)-kJ_\nu'(k)]Y_\nu(kr), \quad (4.11)$$

and

$$g(\lambda;k,r)\sim\tfrac{1}{2}\pi r^{1/2}\times[(\tfrac{1}{2}k^{-1}\sin k+\cos k)J_\nu(k)-\sin kJ_\nu'(k)]Y_\nu(kr). \quad (4.12)$$

If $0>\lambda\geqslant-\frac{1}{4}$, then

$$\lim_{r\to0}f(\lambda;-k,r)=\lim_{r\to0}g(\lambda;k,r)=0. \quad (4.13)$$

Thus Eq. (1.8) is meaningless, but Eq. (1.9) gives

$$A(k)=-e^{-ik}\frac{(\frac{1}{2}k^{-1}\sin k+\cos k)J_\nu(k)-\sin kJ_\nu'(k)}{(\frac{1}{2}+ik)J_\nu(k)-kJ_\nu'(k)}. \quad (4.14)$$

Thus we have now seen two possible ways in which Eq. (1.8) can fail. It does so in example A because we get ∞/∞ and in example B with $0>\lambda\geqslant-\frac{1}{4}$ because we get 0/0. In either case Eq. (1.9) works well.

If $\lambda>0$, then, near $r=0$, $f(\lambda;-k,r)$ is unbounded, and $g(\lambda;k,r)$ is unbounded unless

$$(\tfrac{1}{2}k^{-1}\sin k+\cos k)J_\nu(k)-\sin kJ_\nu'(k)=0. \quad (4.15)$$

These are just the points where the phase shift is a multiple of π. Again Eq. (1.8) is meaningless but Eq. (1.9) gives Eq. (4.14).

Example C

Consider, as a last example, the rather pathological potential

$$V(r)=\begin{cases} r^{-2}\sin r^{-1}, & r<1, \\ 0 & r>1. \end{cases} \tag{4.16}$$

In this case, we can define $\psi_s(k,r)$ and $\psi_s'(k,r)$ by Eqs. (3.1) and (3.2). It can be verified that the two Born series converge, and Eqs. (3.3)–(3.6) and (1.8) hold. Thus, for this potential, all the properties of a regular potential are obtained even though it does not satisfy Eq. (1.2).

Note added in proof. We want to thank Dr. M. Bég for drawing our attention to a paper by N. Limic [Nuovo Cimento **26**, 581 (1962)] which contains the statement that for singular potentials the S-matrix element for given l is the limit of the quotient of two Jost functions.

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S-Matrix Poles Close to Threshold*

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Two-channel processes are studied to determine whether sizeable peaks can be produced in elastic scattering for one of the channels by threshold effects arising from the second channel (taken to be in an S-wave state). The problem is first examined by means of a simple model whose analytic properties can easily be deduced. It is found that, when all the particles are stable, large cusps occur if there is a pole of the S matrix on an unphysical sheet in the vicinity of the inelastic threshold. The cusps become "woolly" when one of the particles in the second channel is allowed to be unstable. Similar results are obtained in a calculation using an ND^{-1} formulation. These S-matrix poles correspond to virtual states of the particles in the inelastic channel, their positions on the unphysical sheets depending on the force of interaction between the particles. It is further suggested that some of the peaks observed in experiment may be of this type, having their origins in inelastic thresholds rather than direct particle resonances. In particular, the Y_0^* at 1815 MeV and the K_1K_1 peak near threshold may be manifestations of this.

I. INTRODUCTION

MANY authors have discussed threshold effects, or cusps, in elementary particle reactions, including the case of a threshold for the production of an unstable particle.[1–4] Questions naturally arise as to whether these threshold effects can be responsible for sizeable peaks in cross sections; and if so, whether such peaks should be classified as elementary particles or as phenomena of an essentially different character. The purpose of this paper is to call attention to a situation in which threshold effects do indeed produce sizeable peaks; namely, when there exists a pole in the S matrix close to an S-wave threshold on the unphysical sheet reached by passing through the branch cut associated with the threshold. Moreover, we conjecture that this situation is very likely to be responsible whenever a threshold effect manifests itself as a peak comparable to those associated with particles. From the point of view of S-matrix theory, a threshold effect of this nature can quite properly be called a particle since it arises from a pole in the S matrix.

In Sec. II, we shall discuss these points in more detail by considering some examples. The simplest example, given in Sec. IIA, of the type of threshold effect we are discussing is the "virtual state" occurring in the ^1S state of the neutron-proton system. In Sec. IIB, the case of two channels involving only stable particles is discussed, and in Sec. IIC, two channels where one of the particles in the second channel is unstable. The latter case is an extension of the work of Nauenberg and Pais.[3] In Sec. III, we consider threshold effects within the framework of a dynamical model, using the matrix ND^{-1} formalism. Some clarification is thereby obtained of the work by Ball and Frazer on peaks in cross sections near the threshold for production of an unstable particle.[2] Lastly, in Sec. IV we discuss some possible experimental manifestations of threshold effects.

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