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Aharonov-Bohm Paradox*

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It is remarked that even in the classical limit the behavior of a charged particle interacting with a stationary current distribution exhibits paradoxical behavior. This comes from the inequality of action and reaction forces. It is shown that the vector potential term appearing in the expression for the canonical momenta represents the electromagnetic field momentum.

I. INTRODUCTION

IT has been shown by Aharonov and Bohm[1] that according to the general principles of quantum mechanics the behavior of a charged particle will be affected by the presence of a static vector potential even though the particle should be constrained to move in a region of space where the magnetic field, $\mathbf{B}=\nabla\times\mathbf{A}$, is zero. This paradoxical situation has since been discussed by several authors,[2] and Furry and Ramsey[2] have discussed a similar effect involving the scalar potential.

The effect follows from the canonical quantization procedure wherein the momentum canonically conjugate to the position, $\mathbf{p}=m\mathbf{v}+q/c\mathbf{A}$, is represented by $(\hbar/i)\nabla$ operating on the wave function. $\mathbf{p}$ then has a fundamental significance in that it determines the wave number of the Schrödinger waves.

In the two-slit experiment considered by Aharonov and Bohm one has a solenoid between the two slits such that a magnetic flux is enclosed between paths which go from the source to the detector through the different slits, but the magnetic field is zero on all such accessible paths. Then the classical motion of charged particles going through either of the slits is unaffected by the solenoidal currents; but there is a phase difference, equal $\hbar^{-1}(q/c)\oint\mathbf{A}\cdot d\mathbf{s}$, in the waves going through one or the other slits which causes a shift of the diffraction pattern dependent on the value of the solenoidal flux. The sense of paradox arises from the dependence of the quantum behavior on a physical parameter, the enclosed flux, which the corresponding classical particle behavior is seemingly independent of. It may be said that this shows a peculiar quantum significance of the vector potential but this is not quite true. The total momentum, angular momentum, and energy of a system have fundamental quantum-mechanical significance as the generators of space translations, rotations, and time translations, and the potentials appearing in nonrelativistic problems represent the contributions of the electromagnetic field to these physical quantities.

Classically, the physical significance of the $q/c\mathbf{A}(r)$ term in the canonical momentum is that it is the "interaction" momentum of the electromagnetic field, just as $q\phi(\mathbf{r})$ is the interaction energy of the electromagnetic field, $q/c\mathbf{r}\times\mathbf{A}(\mathbf{r})$ is the interaction field angular momentum. For an isolated charged particle we know that a portion of its momentum, mass, etc., reside in the surrounding field and the above terms represent the change in these quantities when the particles are brought into interaction. (It is of course only in the nonrelativistic regime that the field properties may be "integrated out" and expressed as potentials depending on the positions and velocities of the particles.)

If, then, a charged particle is incident on a system of currents $\mathbf{P}=m\mathbf{v}+M\mathbf{V}+q/c\mathbf{A}(\mathbf{r}-\mathbf{R})$, is the total momentum, where m, $\mathbf{v}$, and $\mathbf{r}$ are the mass, velocity, and position of the particle, and M, $\mathbf{V}$, and $\mathbf{R}$ represent the corresponding quantities for the body carrying the currents. $\mathbf{P}$ is of course conserved, whereas $m\mathbf{v}+M\mathbf{V}$ is usually not (action forces usually not equal to reaction forces in a system of charged particles and currents).

The invariance of physical laws under space translations leads to the identification of the operator for the total momentum with the generator of space translations,

$$\mathbf{P}=\sum_i m_i\mathbf{v}_i+(q_i/c)\mathbf{A}(\mathbf{r}_i-\mathbf{R})+M\mathbf{V}\rightarrow(\hbar/i)(\sum_i\nabla_i+\nabla_R),$$

where we have generalized slightly to allow for the presence of several charged particles. With the assumption that if M is indefinitely large then the state vector may be represented as a function of the $\mathbf{r}_i$ with $\mathbf{R}$ entering only as a parameter then it becomes reasonable to identify $(\hbar/i)\nabla_i$ with $\mathbf{p}_i=m_i\mathbf{v}_i+q_i/c\mathbf{A}(\mathbf{r}_i-\mathbf{R})$. In the classical limit the mechanical momentum $m\mathbf{v}$ and the interaction field momentum are separately measurable but of course in the quantum limit the separation is limited by the uncertainty relation. Naturally we cannot "go behind" the identification of $\mathbf{p}$ with $(h/i)\nabla$, our point, which is modest, is only that the two parts of $\mathbf{p}$ are classically "physical" momenta.

II. THE CLASSICAL LIMIT

We now wish to derive the classical results alluded to in Sec. I, and incidentally to show that in the classical

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[1] Y. Aharonov and D. Bohm, Phys. Rev. **115**, 485 (1959).
[2] W. H. Furry and N. F. Ramsey, Phys. Rev. **118**, 623 (1960); M. Peshkin, I. Talmi, and L. Tassie, Ann. Phys. (N. Y.) **12**, 426 (1961); L. Tassie and M. Peshkin, Ann. Phys. (N. Y.) **16**, 177 (1961).

limit the systems considered exhibit a behavior fully as paradoxical as that of the quantum systems.

We consider a system of stationary currents and a particle of mass m, charge q, and velocity $\mathbf{v}$ incident on the currents. To avoid irrelevant complications we suppose that electrostatic imaging effects and the change in the currents due to induced emf's are negligible. If the currents should be "wound" such that the magnetic flux is entirely contained within a certain region of space, e.g., the interior of a tightly wound infinitely long solenoid, or a toroid, then if the particle does not penetrate this region there is no force exerted on it by the currents and it maintains constant velocity. However, the particle does exert a force on the currents and will cause the current carrying body to move as it passes by.

The force on the currents to first order in v/c is

$$\begin{aligned}\mathbf{F}_j&=(q/c^2)\int \mathbf{j}(\mathbf{r}')\times(\mathbf{v}\times\nabla_r)|\mathbf{r}-\mathbf{r}'|^{-1}d^3r',\\ &=-q/c(\mathbf{v}\times\nabla)\times\mathbf{A}(\mathbf{r}),\\ &=-q/c[(\mathbf{v}\cdot\nabla)\mathbf{A}(\mathbf{r})+\mathbf{v}\times(\nabla\times\mathbf{A})-\mathbf{v}(\nabla\cdot\mathbf{A})],\end{aligned}\tag{1}$$

where

$$\mathbf{A}(\mathbf{r})=c^{-1}\int \mathbf{j}(r')|\mathbf{r}-\mathbf{r}'|^{-1}d^3r'\tag{2}$$

is the vector potential at the particle's position $\mathbf{r}$ due to the currents. We have supposed the currents stationary, then from $\nabla\cdot\mathbf{j}=0$, $\nabla\cdot\mathbf{A}=0$ follows.

The force on the particle is

$$\mathbf{F}_p=(q/c)\mathbf{v}\times(\nabla\times\mathbf{A}),\tag{3}$$

and so from Eqs. (1) and (3)

$$\mathbf{F}_p+\mathbf{F}_j=-q/c(\mathbf{v}\cdot\nabla)\mathbf{A}=-(d/dt)(q/c)\mathbf{A}.\tag{4}$$

We have tacitly assumed that the current-carrying body was so massive it did not move, however, if it has finite mass M, velocity $\mathbf{V}$, and c.m. position $\mathbf{R}$ then Eqs. (1), (3), and (4) remain correct to first order in v/c and V/c if the substitutions $\mathbf{v}\rightarrow\mathbf{v}-\mathbf{V}$, $\mathbf{r}\rightarrow\mathbf{r}-\mathbf{R}$ are made in Eqs. (1)–(4).[3]

We see from Eq. (4) that forces between a charged particle and a system of currents are usually not equal and opposite (e.g., neutron-electron interaction in the classical limit). The most striking examples are those for which there is no force on the particle then from Newton's equation and Eq. (4)

$$M\mathbf{V}=-(q/c)\mathbf{A}(\mathbf{r}-\mathbf{R}).\tag{5}$$

For an infinite straight solenoid carrying flux Φ, $\mathbf{A}(\rho,\varphi)=\Phi(2\pi\rho)^{-1}\hat{\varphi}$, thus a charged particle placed at a distance ρ will cause the solenoid to circle around it with angular velocity $\omega=(q/c)\Phi(2\pi\rho^2M)^{-1}$. Similarly, a charged particle a distance z along the axis of a ring solenoid will give the ring a z momentum $=(q/c)\Phi a^2(z^2+a^2)^{-3/2}$, where a is the radius of the ring.

From the general conservation laws of electrodynamics the sum of the forces on the particle and currents is equal and opposite to the rate of change of the momentum of the electromagnetic field, and so from Eq. (4)

$$\frac{d}{dt}(4\pi c)^{-1}\int \mathbf{E}\times\mathbf{B}d^3r=\frac{d}{dt}(q/c)\mathbf{A}(\mathbf{r}).\tag{6}$$

We now have

$$\mathbf{p}+M\mathbf{V}=\text{const},\tag{7}$$

where,

$$\mathbf{p}=m\mathbf{v}+(q/c)\mathbf{A}(\mathbf{r}),\tag{8}$$

and the $(q/c)\mathbf{A}(\mathbf{r})$ term with $\mathbf{A}$ given by (2) represents the field momentum (strictly speaking it is the "interaction momentum" obtained by crossing the $\mathbf{E}$ from the particle into the $\mathbf{B}$ from the currents and vice versa).

We now give the direct elementary derivations for the field momentum $\mathbf{p}_f$ and angular momentum $\mathbf{L}_f$. We suppose the currents stationary and choose the gauge $\nabla\cdot\mathbf{A}=0$. Neglecting terms of second order in v/c, $\nabla\times\mathbf{E}=0$. It is then elementary to show that $\mathbf{E}\times\mathbf{B}=4\pi\rho\mathbf{A}+\nabla\mathbf{E}\cdot\mathbf{A}-\nabla\cdot\mathbf{E}\mathbf{A}-\nabla\cdot\mathbf{A}\mathbf{E}$. In integrations over all space the terms involving ∇ equal zero, and we obtain

$$\mathbf{p}_f=\frac{1}{4\pi c}\int E\times Bd\tau=(q/c)\mathbf{A}(\mathbf{r}),\tag{9}$$

$$\mathbf{L}_f=\frac{1}{4\pi c}\int(\mathbf{r}-\mathbf{r}_1)\times(\mathbf{E}\times\mathbf{B})d\tau=(q/c)(\mathbf{r}-\mathbf{r}_1)\times\mathbf{A}(\mathbf{r}).\tag{10}$$

In Eq. (10) we have computed the angular momentum about a fixed point $\mathbf{r}_1$.

Note added in proof. The discussion given above of the significance of the vector potential term in the canonical momenta is similar to one given by Murray Peshkin,[4] which I was not aware of at the time this paper was written.

[3] This follows most easily from the fact that forces on a stationary body are invariant under Lorentz transformations except for terms of the order V^2/c^2, see R. C. Tolman, *Relativity, Thermodynamics, and Cosmology* (Oxford University Press, New York, 1934), p. 45.

[4] Murray Peshkin, Proceedings of the Midwest Conference on Theoretical Physics held at the Argonne National Laboratory, June 1962.