Higher Order Spacing Distributions for a Class of Unitary Ensembles*

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We consider the nth-order spacing distribution, $P^n(s)$, in the statistical theory of energy levels of complex systems. Each P^n is written as a sum of multiple integrals over correlation functions. This procedure is used to establish the identity of the spacing distributions for all members of a class of Hamiltonian unitary ensembles. A power-series expansion of $P^n(s)$, valid for all n, is developed.

I. INTRODUCTION

A STATISTICAL theory has been developed[1-6] which has been applied to the problem of level spacing in heavy nuclei in a region of the excitation spectrum where the level density is approximately constant over, say, a hundred levels. A suitably chosen ensemble of N-dimensional Hamiltonian matrices is introduced, and one studies the distribution of the eigenvalues of ensemble members.

We are interested in developing approximation procedures for the calculation of energy level spacing distributions for a class of Hamiltonian matrix ensembles. To date, nearest-neighbor spacing distributions, $P^0(s)$ have been calculated, in the limit of large Hamiltonian matrix dimension N, for orthogonal, unitary, and symplectic ensembles[3-7]; the next-nearest-neighbor spacing distribution $P^1(s)$ has been calculated only for the orthogonal ensemble.[5,7] One can start the calculations by imposing restrictions on the matrix elements of members of the Hamiltonian ensemble. For matrix distribution functions $f(x_1,\cdots,x_N)$ which depend only on the eigenvalues x_1 to x_N, one obtains for the joint distribution function for the eigenvalues[1]

$$P_N(x_1,x_2,\cdots,x_N)=f(x_1,x_2,\cdots,x_N)\prod_{i<j}|x_i-x_j|^\beta, \quad (1)$$

where $\beta=1$, 2, 4 for the orthogonal, unitary, and symplectic ensembles, respectively.[5] The product factor arises from the Jacobian of the transformation from matrix to eigenvalue space and represents the volume of the former space associated with a given set of eigenvalues; it is responsible for the "repulsion effect."

Alternatively, one can immediately assume Eq. (1) as a form of the joint probability distribution of eigenvalues.[8] A particular $f(x_1,\cdots,x_N)$ does not uniquely determine the distribution of elements in the Hamiltonian matrix ensembles.

Members of the class of Hamiltonian ensembles in which $f(x_1,\cdots,x_n)$ is a product, $\prod_i [g(x_i)]^2$, have been extensively studied.[1-3,5,6,8] For example, the choices

$$\begin{aligned}
[g(x)]^2&=\exp(-x^2) && -\infty<x<\infty\,,\\
&=(1-x)^\mu(1+x)^\nu && \mu,\,\nu>-1;\quad |x|\leq 1\,,\\
&=x^\alpha e^{-x} && \alpha>-1;\quad 0\leq x<\infty\,,\\
&=1 && x=e^{i\theta},\quad 0\leq\theta\leq 2\pi\,,
\end{aligned}$$

lead to the so-called Gaussian, Jacobi, Laguerre, and circular ensembles, respectively.[9] The circular[5] and Gaussian[4-6] ensembles have been shown to have identical nearest-neighbor spacing distributions for $\beta=1$, 2, 4. Although the unitary ensembles, $\beta=2$ are of less physical interest than the orthogonal ensembles, they have been studied more extensively because the caluclations are easier. One hopes that certain results established for $\beta=2$ will lead to generalizations valid also for $\beta=1$.

In Sec. II, we discuss the nth-order spacing distribution, $P^n(s)$, which is the probability that between two levels separated by a distance s there are found exactly n levels. These distributions are, apart from their mathematical interest, of importance because of the availability of empirical data with which to investigate the range of validity of the theoretical models. It is shown that, in the flat region of the level density, and in the limit $N\to\infty$, the nth-order spacing distribution for all unitary ensembles associated with the classical orthogonal polynomials is identical with that of the circular ensemble.

In Sec. III, power series expansions of $P^n(s)$ are developed, valid for all $n\ll N$. Auxiliary mathematical results are derived in the Appendix.

II. EQUIVALENCE OF A CLASS OF UNITARY ENSEMBLES

The nth-order spacing distribution corresponding to the interval x to $x+s$ is given by

$$P^n(x,\,x+s)=\frac{N!}{(N-n-2)!n!}\left(\int_e\right)^{N-n-2}\left(\int_x^{x+s}\right)^n \times P_N(x,\,x+s,\,x_3,\,\cdots,\,x_N)d\tau_{3,N}. \quad (2)$$

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[2] E. P. Wigner, Ann. Math. **53**, 36 (1951); **55**, 7 (1952); **62**, 548 (1955); **65**, 203 (1957); **67**, 325 (1958).

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[7] P. B. Kahn, Nucl. Phys. **41**, 159 (1963).

[8] H. S. Leff, thesis, State University of Iowa SUI 63-23, 1963 (unpublished).

[9] The nomenclature in this field leaves something to be desired. The Gaussian ensemble is named for the weight function $[g(x)]^2$; "circular ensemble" describes the periodic property of the allowed range of variables; most of the remaining names (Jacobi, Laguerre, etc.) come from the orthogonal polynomials associated with the weight function and the allowed range.

Here $P_N(x_1,x_2,\ldots,x_N)$ is the joint distribution function; $d\tau_{q,r}$ is the partial volume element $(dx_q dx_{q+1}\cdots dx_r)$; $\int_e$ is an integral, with respect to any one of the x_k, over the entire range external to the interval x to $x+s$. The symmetry of the joint distribution function with respect to all permutations of the x_k allows us, here and below, to write the multiple integrations as symbolic powers, without specifying the variable associated with each integral. The symbol $\int$ will be used to designate integration over the entire range. (This range may differ from one ensemble to another.) The definition implies that

$$\int=\int_e+\int_x^{x+s}. \tag{3}$$

Wigner[10] established relations between the various spacing distributions and integrals over the correlation functions. We introduce

$$Q^k(x,x+s)=\frac{N!}{k!(N-k-2)!}\left(\int_x^{x+s}\right)^k\left(\int\right)^{N-k-2}\times P_N(x,x+s,x_3,\cdots,x_N)d\tau_{3,N}, \tag{4}$$

which differ from Wigner's functions,[10] I_k, only in normalization. Wigner developed the relationships

$$Q^k(x,x+s)=\sum_{n=k}^{N-2}\binom{n}{k}P^n(x,x+s). \tag{5}$$

That Eq. (5) follows from Eqs. (2) and (4) may be seen by observing that, where the integrand is symmetric, the binomial theorem may be applied to Eq. (3):

$$\left(\int\right)^{N-k-2}=\sum\binom{N-k-2}{n-k}\left(\int_e\right)^{N-n-2}\left(\int_x^{x+s}\right)^{n-k}. \tag{6}$$

One shows that

$$P^n(x,x+s)=\sum_{m=n}^{N-2}(-)^{m+n}\binom{m}{n}Q^m(x,x+s) \tag{7}$$

is a solution of the set of equations (5) by substituting (7) into (5).

The above results are applicable to any joint probability distribution which is symmetric in all the variables. In what follows, we restrict our attention to a particular class of unitary ensembles.

The factor $\prod_{i<j}(x_i-x_j)^2$ in Eq. (1) is equal to the square of the Vandemonde determinant.[3] Equation (1) may then be written as

$$P_N(x_1,\cdots,x_N)=\prod_i[g(x_i)]^2\begin{vmatrix}1 & 1 & \cdots & 1\\ x_1 & x_2 & \cdots & x_N\\ \vdots & & & \\ x_1{}^{N-1} & x_2{}^{N-1} & \cdots & x_N{}^{N-1}\end{vmatrix}^2. \tag{8}$$

Further calculations may be simplified if one follows a procedure developed by Mehta[3] for the Gaussian orthogonal ensemble. One may replace each row of the determinant in Eq. (8) by a linear combination of rows, obtaining the same set of linearly independent polynomials in every column. In particular, for a given $g(x_i)$ and an allowed range $a\leqslant x_i\leqslant b$, one generates in this manner a set of polynomials $\psi_n(x_i)$ orthogonal with weight $[g(x_i)]^2$ in this range.[5,8] One may now absorb the weight function $g(x_i)$ in every element of the ith column (every i); the n, i element is then $\varphi_n(x_i)\equiv g(x_i)\psi_n(x_i)$. The functions $\varphi_n(x_i)$ are now orthogonal in $a\leqslant x_i\leqslant b$ with constant weight. A constant outside the determinant allows the restriction that the φ_n are normalized in the range a to b.

Equation (8) may now be replaced by

$$P_N(x_1,\cdots,x_N)=C\,\mathrm{Det}[\varphi_n(x_i)]^2;\quad (i,n=1,2,\cdots,N). \tag{9}$$

The functions Q^{j-2} may be related to the j-level correlation function R_j:

$$R_j(x_1,\cdots,x_j)=N![(N-j)!]^{-1}\times\left(\int\right)^{N-j}P_N(x_1,\cdots,x_N)d\tau_{j+1,N}. \tag{10}$$

Equation (4) may be written as

$$Q^{j-2}(x,x+s)=[(j-2)!]^{-1}\left(\int_x^{x+s}\right)^{j-2}\times R_j(x,x+s,x_3,\cdots,x_j)d\tau_{3,j}. \tag{11}$$

The integrations in Eq. (10) have been carried out for the circular unitary ensemble by Dyson,[5] and for the Gaussian unitary case by Mehta and Dyson.[6] They obtain

$$R_j(x_1,\cdots,x_j)=\mathrm{Det}[K(x_l,x_m)];\quad (l,m=1,2,\cdots,j), \tag{12}$$

where $K(x_l,x_m)$ is the kernel function defined by

$$K(x_l,x_m)=\sum_{p=1}^{N}\varphi_p(x_l)\varphi_p(x_m). \tag{13}$$

From Eqs. (7), (11), and (12), it is readily seen that two ensembles with the same kernel will have the same set of spacing distributions.

Instead of starting either with a Hamiltonian ensemble or with Eq. (1), we may define a class of ensembles by Eq. (9), with the N functions $\varphi_n(x)$ chosen from any set which is orthonormal in some interval. [This class includes as special cases the ensembles defined by Eq. (8).] One can show that the derivation of Eqs. (12) and (13) from Eq. (9) holds for all ensembles of this class.

In what follows, we restrict ourselves to large N and

[10] E. P. Wigner, in *Statistical Properties of Spectra: Fluctuations*, edited by C. E. Porter (Academic Press Inc., New York, to be published), article 34.

to the flat region of the level density curve. The kernel has been derived for the circular unitary ensemble by Dyson[5] and for the Gaussian unitary ensemble by Wigner.[10] Their results are of the form

$$K(x_l,x_m)=\pi^{-1}[\sin B(x_l-x_m)]/(x_l-x_m). \tag{14}$$

Here B is a constant determined by the scale of the independent variables. Using the Christoffel-Darboux formula for the kernels and the appropriate asymptotic expansions of the resulting expressions,[11] we have obtained Eq. (14) for the ensembles generated by the Jacobi polynomials $[\psi_r(x)=P_{r-1}^{(\mu,\nu)}(x)$, any μ, $\nu]$ and by the Laguerre polynomials $[\psi_r(x)=L_{r-1}^{\alpha}(x)$, any $\alpha]$.

For the Jacobi ensembles, the result is valid if x_l and x_m are restricted to a neighborhood of the origin. In this neighborhood, the density has zero slope to order N_0/N (where N_0 is the number of levels in this region), and the error in the kernel is at most of this order, so that for a given accuracy N_0 is proportional to N.

The level density for the Laguerre ensembles has been derived by Bronk[12] and independently by Kahn, Porter, and Tang.[13] They found that the density has no point of zero slope in the finite region of x. For our purposes, we can define a "flat" region as the neighborhood of any $x=x_0$ (provided that x_0 is much larger than the mean spacing between the first two levels) if we restrict ourselves to a range of x over which the change in density may be neglected. Equation (14) is valid in this neighborhood if x_l and x_m are measured from a new origin at x_0. The constant B (and hence also the density at the new origin) is proportional to $x_0^{-1/2}$. For a fixed accuracy of the results, the size of the "flat" region is proportional to x_0, so that the number of levels increases as $x_0^{+1/2}$.

One concludes that for a unitary ensemble generated by any classical polynomial there exists a sufficiently large "flat" region in which Eq. (14) is valid. Hence, for these cases, each spacing distribution with $n\ll N$ is the same as that of the circular unitary ensemble.

III. POWER-SERIES EXPANSION

The development of a power-series expansion of P^n may be simplified by the introduction of a set of functions

$$G^j(x,x+s)=\binom{N}{j}\left(\int_x^{x+s}\right)^j\left(\int\right)^{N-j} \times P_N(x_1,x_2,\cdots,x_N)d\tau_{1,N}. \tag{15}$$

Dyson[5] introduced the notion of differentiating certain probability distributions to obtain the spacing distributions. (This technique has been used by other authors to obtain relations among various probability distribution functions.[8,14]) Following this method, the derivative of G^j with respect to s is obtained through the use of $(d/dx)(\int_a{}^b)^r=r(\int_a{}^b)^{r-1}(d/dx)\int_a{}^b$, which is valid for integrands that are symmetric in all variables and independent of x. One finds

$$\frac{dG^j(x,x+s)}{ds}=j\binom{N}{j}\left(\int_x^{x+s}\right)^{j-1}\left(\int\right)^{N-j} \times P_N(x+s,x_2,\cdots,x_N)d\tau_{2,N}. \tag{16}$$

Taking another derivative would involve differentiation under the integral sign. To avoid this difficulty, we restrict our attention to regions of x and s in which G^j (and hence also dG^j/ds) is independent of x. We may then shift x by $-s$, differentiate, and shift back again. The result is

$$d^2G^j(s)/ds^2=Q^{j-2}(s). \tag{17}$$

A series expansion of $P^m(s)$ may be obtained from expansions of the $G^j(s)$ through the use of Eqs. (7) and (17). Combining Eqs. (10), (12), and (15), we have

$$G^j(s)=(j!)^{-1}\left(\int_{-s/2}^{s/2}\right)^j \mathrm{Det}[K(x_l,x_m)]d\tau_{1,j}. \tag{18}$$

We assume the legitimacy of the series expansion

$$K(x_l,x_m)=\sum_{p,q} b(p,q)(x_l)^p(x_m)^q. \tag{19}$$

In the Appendix, the corresponding power-series expansion and subsequent multiple integration of G^j are carried out. The result may be expressed as

$$G^j(s)=2^j\sum_{r=0}^{\infty}(s/2)^{r+j}\sum_{\text{part}}\mathrm{Det}[b(p_l,q_m)] \times\mathrm{Det}[e(p_l,q_m)]. \tag{20}$$

Here the second summation is over partitions of the integer r into two sets of numbers, p_1 to p_j and q_1 to q_j. Each such partition of r uniquely determines two determinants: the l, m element of the first is the coefficient $b(p_l,q_m)$ in Eq. (19), while the corresponding element of the second is

$$e(p_l,q_m)=\begin{cases}0 & p_l+q_m \quad \text{odd}\\ (p_l+q_m+1)^{-1} & p_l+q_m \text{ even}.\end{cases} \tag{21}$$

From Eq. (21), it follows that $\mathrm{Det}[e(p_l,q_m)]$ will vanish if either two of the p_l or two of the q_m are equal. Then, for a given j, the leading term in Eq. (20) will, in general, be the one corresponding to the partition of r into two identical sets, $0,1,2,\cdots,j-1$. The minimum value of the exponent $r+j$ is therefore j^2. Hence, the leading term of the spacing distribution P^n is of degree $(n+2)^2-2$, in agreement with the result of Kahn and

[11] G. Szegö, *Orthogonal Polynomials* (American Mathematics Society Colloquium Publications, New York, 1959), Vol. 23, Eqs. (4.5.2), (8.21.10), (8.22.6), and (8.22.8).
[12] B. Bronk (to be published).
[13] P. B. Kahn, C. E. Porter, and Y. C. Tang (to be published).
[14] P. B. Kahn, Symposium on Statistical Properties of Complex Atomic and Nuclear Spectra, Stony Brook, 1963 (unpublished).

Porter.[15] The coefficient of this term is derived in the Appendix.

It follows from Eq. (18) that $G^j(s)$ has parity $(-)^j$.

The choice $B=\pi$ in the kernel given by Eq. (14) is such that the average nearest-neighbor spacing is one. With this kernel, the coefficients in Eq. (19) are

$$b(p,q)=(-)^{\frac{1}{2}(p-q)}\pi^{p+q}e(p,q)/p!q!. \tag{22}$$

Equation (21) then becomes

$$G^j(s)=(2/\pi)^j\sum_r(\pi s/2)^{r+j}\sum_{\text{part}}\{\mathrm{Det}[e(p,q)]\}^2 \times\{\prod_{i=1}^{j}(-)^{\frac{1}{2}(p_i-q_i)}p_i!q_i!\}^{-1}. \tag{23}$$

The series expansion of $G^2(s)$ is more easily obtained from the known result

$$Q^0(s)=R_2(x,\,x+s)=[1-(\sin\pi s)^2/s^2\pi^2]. \tag{24}$$

Instead of computing $G^3(s)$ from Eq. (23), one may find $Q^1(s)$ by an alternate method:

$$Q^1(s)=\pi^{-1}\sum_p(-)^p\frac{(2\pi s)^{2p-1}}{(2p+1)!}\left[\frac{(2p+1)^2}{2p-1}-4\sum_{j=1}^{2p}\frac{1}{j}\right]. \tag{25}$$

This expression is simpler than Eq. (23) for computation; however, the method used in the derivation is not tractable for higher $Q^k(s)$.

We expanded the infinite product expression[4,5,7] for $P^0(s)$ as a power series, to order s^{10}, and compared the coefficients with those of $P^0=Q^0-Q^1$. The validity of the latter expression to this order is guaranteed, since the leading term in $Q^2(s)$ is of order s^{14}. At the maximum of P^0 ($s\cong 0.9$), Q^1 contributes about 5% and Q^2 less than 0.02%. From Eq. (A6) one finds that Q^2 rapidly becomes important beyond $s=1.5$.

The power series converge slowly in the region where $P^n(s)$ is significantly different from zero; the method is unsuitable for a study of the behavior at very large s. However, the method has the advantage of adaptability for machine computation for many n.

APPENDIX

A determinant will be expressed here as an antisymmetrization of the product of diagonal elements. Thus Eq. (19) leads to

$$\mathrm{Det}[K(x_l,x_m)]=\prod_i\sum_{p_i,q_i}b(p_i,q_i)(x_i)^{q_i}A_P{}^{(j)}(x_i)^{p_i}. \tag{A1}$$

The implicit antisymmetrization on the left side of this equation involves permutations over the variables; on the right we use, instead, the operator $A_P{}^{(j)}$, which antisymmetrizes with respect to the j exponents p_1 to p_j. The latter operator can be commuted with the factor $(x_i)^{q_i}$ as well as with the multiple integration involved in the calculation of G^j. Substituting Eq. (A1) in Eq. (18) and carrying out the integrations, one finds

$$G^j(s)=\frac{2^j}{j!}\prod_i\sum_{p_i,q_i}b(p_i,q_i)A_P{}^{(j)}\left(\frac{s}{2}\right)^{p_i+q_i+1}e(p_i,q_i)$$
$$=\frac{2^j}{j!}\sum_{p_1\cdots p_j}\sum_{q_1\cdots q_j}\left(\frac{s}{2}\right)^{j+r}[\prod_i b(p_i,q_i)] \times[A_P{}^{(j)}\prod_i e(p_i,q_i)], \tag{A2}$$

where $e(p_l,q_m)$ is given in Eq. (21) and $r=\sum_{i=1}{}^j(p_i+q_i)$. We consider partitions of a given value of r into two sets of numbers p_1 to p_j, q_1 to q_j, without regard to order within each set. The sum over these indices in Eq. (A2) may be separated as follows: (1) For each partition of a given r, the sum over ordered values within the sets is expressed by the product of $S_Q{}^{(j)}$ and $S_P{}^{(j)}$, the symmetrization operators with respect to the members of the sets. (2) The sum is then taken over partitions of a given value of r. Finally, (3) the sum over r is taken, giving the sum over powers of s. Equation (A2) is now replaced by

$$G^j(s)=\frac{2^j}{j!}\sum_r\left(\frac{s}{2}\right)^{r+j}\sum_{\text{part}}S_Q{}^{(j)} \times\{S_P{}^{(j)}[\prod_i b(p_i,q_i)A_P{}^{(j)}\prod_i e(p_i,q_i)]\}. \tag{A3}$$

Using the known relationship $(A_P{}^{(j)}f_1)(A_P{}^{(j)}f_2)=S_P{}^{(j)}\times(f_1A_P{}^{(j)}f_2)$ the expression in braces in Eq. (A3) becomes

$$\{A_P{}^{(j)}\prod_i b(p_i,q_i)\}\{A_P{}^{(j)}\prod_i e(p_i,q_i)\} =\mathrm{Det}[b(p_i,q_i)]\,\mathrm{Det}[e(p_i,q_i)]. \tag{A4}$$

Since the product of determinants is completely symmetric with respect to the indices q_1 to q_j, the operation with $S_Q{}^{(j)}$ in Eq. (A3) results in a factor of $j!$. Combining these steps, we obtain Eq. (20).

From Eq. (21), one may show that $\mathrm{Det}[e(p_l,q_m)]$ has the following properties: (1) If two of the p_l (two of the q_m) are equal, the determinant will vanish. (2) If, in any partition, the sets p_i and q_i have unequal numbers of even integers, the determinant will vanish. (3) In a partition in which the number of even integers is the same in both sets, the determinant may be factored into a product of two determinants, one containing elements with even indices, the other those with odd indices.

These three points lead to considerable reduction in the labor involved in manual computation of the coef-

[15] P. B. Kahn and C. E. Porter, Nucl. Phys. 48, 385 (1963).

ficients in Eq. (23). In addition, they are used to find a general expression for the coefficient of the leading term of $G^j(s)$. We calculate the determinant $\mathrm{Det}[e(p_i,q_k)]$ for the case in which each set consists of the numbers 0, 1, 2, $\cdots$, $(j-1)$. If the rows and columns are arranged so that all the odd indices appear first, the determinant is in clearly factorable form, with zeros in all positions of the two off-diagonal (even-odd, odd-even) blocks. The dimensions of the factors will be equal or will differ by one, depending on whether j is even or odd. The p_iq_k element of either factor is $(p_i+q_k+1)^{-1}$. Either diagonal block is designated as $D(m)$, where m is the largest value of p_i or q_k. Evaluation of $\mathrm{D}(m)$ is straightforward and may be found in the treatise by Muir and Metzler[16]:

$$D(2u+\alpha)=\prod_{t=0}^{u}\left[\frac{(2t+\alpha)!}{(4t+2\alpha+1)!!}\right]^2 \times(4t+2\alpha+1)\,;\quad \alpha=0 \text{ or } 1\,. \tag{A5}$$

Whether j is odd or even, one may write $\mathrm{Det}[e(p_i,q_k)]$ as $D(j)D(j-1)$. Evaluating this product from Eq. (A5) and substituting the result in Eq. (23) yields, for the leading term of $G^j(s)$

$$\left(\frac{2}{\pi}\right)^j\left(\frac{\pi s}{2}\right)^{j^2}\left[\prod_{k=1}^{j}\frac{k!}{(2k+1)!!(2k-1)!!}\right]^2. \tag{A6}$$

[16] T. Muir and W. H. Metzler, *A Treatise on the Theory of Determinants* (Dover Publications, Inc., New York, 1960), p. 429.

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Gravitational Field: Equivalence of Feynman Quantization and Canonical Quantization

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The transition amplitude for the gravitational field as given by the Feynman sum over histories expression is analyzed in analogy to the electromagnetic transition amplitude. The analysis is based on an explicit representation of the Feynman sum by means of a lattice. The measure is found by consistency requirements and differs from those proposed by other workers. Particular attention is paid to the subsidiary conditions associated with the gauge group. It is shown, that the present approach is equivalent to the quantization by means of canonical variables as proposed by Dirac.

I. INTRODUCTION

THIS paper deals with the problem of assigning a well-defined meaning to

$$\sum_{\text{histories}} e^{iS}\,, \tag{I.1}$$

if S is the action for the free gravitational field. The present approach may actually be extended to the more general case of gravity interacting with matter. For simplicity we shall deal with the gravitational field only.

The prescription given by Feynman[1] to compute (I.1) is not completely straightforward, because the action for the gravitational field is degenerate. The presence of an invariance group generates various difficulties which are well known for the case of the electromagnetic field and its Abelian gauge group. The quantization of the electromagnetic field in the framework of the Feynman sum over histories is analyzed in some detail in Sec. II and constitutes the basis of the present approach to the quantization of the free gravitational field. In particular, we examine the subsidiary condition associated with the gauge group, which in the case of the electromagnetic transition amplitude states that this amplitude is invariant with respect to a gauge transformation of the potential at the initial and the final surface. Section III deals with the generalization of this discussion to the gravitational case in a purely formal and heuristic manner. A more precise framework for the evaluation of the gravitational amplitude is set up in Sec. IV and the derivation of the subsidiary conditions in this framework is given in Sec. V where we also proceed to convert them into differential form. Finally, it is shown in Sec. VI that the results obtained are equivalent to the results of the Hamiltonian quantization procedure as proposed by Dirac.[2] One could and should trace out in a similar way the connection between the sum over histories formulation and the canonical formalism given by Arnowitt, Deser, and Misner.[2] However, to treat this connection would lengthen the present account unduly.

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[2] P. A. M. Dirac, Proc. Roy. Soc. (London) **A246**, 333 (1958); Phys. Rev. **114**, 924 (1959); R. Arnowitt, S. Deser, and C. W. Misner, Phys. Rev. **113**, 745 (1959); **116**, 1322 (1959); **117**, 1595 (1960); **118**, 1100 (1960).