Low-Energy Neutron-Neutron Scattering Parameters*

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A precise evaluation of the spectrum in the reaction $\pi^- + D \to 2n + \gamma$, including final-state interactions is presented with a view of determining the neutron-neutron scattering length. Approximations of previous calculations are examined and avoided where their effect is found to be significant. Sufficiently accurate experiments should be capable of distinguishing this parameter to an error of 1 F.

I. INTRODUCTION

THE determination of the low-energy neutron-neutron scattering parameters is of interest for two reasons. First, it has been suggested that the difference between these parameters for $n-n$, $n-p$, and $p-p$ scattering may be due to a breakdown of charge independence of nuclear forces caused by the $\pi^\pm - \pi^0$ mass difference,[1] and a future theory of the two-nucleon potential should account for these differences. Second, in any process where two neutrons appear in the final state these parameters are needed to evaluate final- or initial-state interactions.

In this article we shall elaborate on a method for determining these parameters, specifically the $n-n$ scattering length and effective range, discussed by McVoy.[2] The method involves the study of the energy and angle distribution in the process:

$$\pi^- + D \to n + n + \gamma, \qquad (1)$$

where the π^- is captured from an S state of the deuteron. The shape of the energy-angle spectrum of the two final neutrons shows a marked sensitivity to the assumed scattering length.

The reason for re-examining the calculation of Ref. 2 is to check the effect of certain approximations made therein. This has relevance as current experimental techniques permit the determination of the scattering length to an accuracy of ~ 1 F.[3]

The difference between this work and that of Ref. 2 may be summarized by the following points:

(1) The validity of the impulse approximation for process (1) is checked by evaluating the next-order corrections as given by Chew and Goldberger.[4]

(2) Instead of using a Hulthén S-state and an asymptotic D-state wave function for the deuteron, a wave function obtained by a numerical solution of the deuteron problem was used.[5]

(3) The final-state interaction between the two neutrons was put in via the Jacob-Omnès-Mahoux method,[6] where no recourse had to be made to a zero-range approximation. Likewise no model was needed for the scattering-state wave function of the two neutrons.

(4) A check on the magnitude of the enhancement due to P-wave neutron-neutron scattering was made.

The other approximations used in Ref. 2 were shown to give negligible corrections and were not further examined.

Sections II, III, and the Appendix are devoted to the mathematical formulation of the problem, and the results are presented and discussed in Sec. IV.

A word should be added about a more general question of the validity of the impulse approximation for this problem. In our approximation we have considered multiple scatterings of the pion from either proton or neutron, as well as proton-neutron scattering inside the deuteron. However, we have not taken into account interactions of the pion or photon with virtual pion currents inside the deuteron. The only motivation we give for the neglect of these terms is our ignorance of reliable methods of calculating them. As shall be shown, the effect of the first-order corrections to the impulse approximation turn out to have negligible effect on the shape of the spectra studied; it may be hoped that likewise these incalculable terms will not contribute sizeably. It should be kept in mind that although some of these corrections, both the ones discussed above and those not calculated, may make small differences in the total rates, as we are interested only in spectral shapes, and normalize all our curves to have a definite value at certain points, many of these corrections become totally insignificant. This point shall be discussed again in Sec. IV.

II. THE IMPULSE APPROXIMATION AND FIRST-ORDER CORRECTIONS

For convenience we shall write down the formulas of Ref. 4 for the impulse approximation. If we consider a scatterer composed of N particles, interacting through a potential U, and an incident particle which interacts with the kth particle of the scatterer by a potential V_k,

* Work supported by the U. S. Atomic Energy Commission.
[1] D. Y. Wong and H. P. Noyes, Phys. Rev. 126, 1866 (1962).
[2] K. W. McVoy, Phys. Rev. 121, 1401 (1961).
[3] R. P. Haddock to H. P. Noyes (private communication).
[4] G. F. Chew and M. L. Goldberger, Phys. Rev. 87, 778 (1952).
[5] N. K. Glendenning and G. Kramer, Phys. Rev. 126, 2159 (1962). The wave function used in this work is the solution of potential No. 8 of this reference. Thanks are due Dr. Glendenning for having kindly provided us with the numerical values of the wave function.

[6] M. Jacob, G. Mahoux, and R. Omnès, Nuovo Cimento 23, 838 (1962).

the total Hamiltonian has the form:

$$H = H_0 + V, \qquad (2a)$$

with

$$H_0 = K + U, \qquad (2b)$$

where k is the total kinetic energy and

$$V = \sum_{k=1}^{N} V_k. \qquad (3)$$

Let $\omega_k^{(\pm)}$ be a two-particle wave operator involving the incident and kth particle of the scatterer, $\omega^{\pm}$ satisfies the equation:

$$\omega_i^{(+)} = 1 + (E - K - V_k + i\eta)^{-1} V_k,$$
$$\omega_k^{(-)} = 1 + V_k (E - K - V_k - i\eta)^{-1}. \qquad (4)$$

We may define the two-particle scattering operator

$$t_k^{(+)} = V_k \omega_k^{(+)},$$
$$t_k^{(-)} = \omega_k^{(-)} V_k. \qquad (5)$$

With the aid of the t_k's and ω_k's we may express the total scattering operator as

$$T^{(+)} = \sum_k t_k^{(+)} + \sum_{k,k'} (\omega_{k'}^{(-)} - 1)[U, \omega_k^{(+)}]$$
$$+ \sum_{k \neq k'} t_k^{(-)} (\omega_k^{(+)} - 1). \qquad (6)$$

The first term of (6) represents the usual impulse approximation; the second term gives the correction due to the diminution of the incident wave as it crosses the scatterer and of the influence of the binding potential U on the individual two-body scatterings. The last term represents the multiple scattering corrections.

Before proceeding with the examination of each of the terms of (6) for process (1) let us establish the notation used. Units are such that $\hbar = c = 1$. The deuteron wave function in momentum space is written as:

$$\varphi_{ij}^{(m)}(\mathbf{q}) = \left\{ \left[u(q) + \frac{w(q)}{\sqrt{8}} \left(\frac{3\boldsymbol{\sigma}_1 \cdot \mathbf{q}\, \boldsymbol{\sigma}_2 \cdot \mathbf{q}}{q^2} - \boldsymbol{\sigma}^1 \cdot \boldsymbol{\sigma}^2 \right) \right] \chi_1^m \right\}_{ij}, \qquad (7)$$

where χ_1^m is the product of two Pauli spinors in a total spin-one state, with magnetic quantum number m. $U(q)$ and $w(q)$ are the Bessel transforms of the S and D
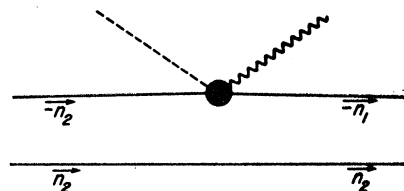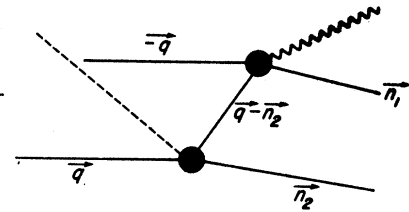

FIG. 1. First-order impulse approximation.

FIG. 2. Multiple scattering correction.

position space wave functions.

$$u(q) = \frac{1}{\sqrt{2}\pi} \int u(q) j_0(qr) r\, dr,$$
$$w(q) = \frac{1}{\sqrt{2}\pi} \int w(r) j_2(qr) r\, dr, \qquad (8)$$

with the normalization

$$\int (u^2(r) + w^2(r))\, dr = 1. \qquad (9)$$

The above wave function is used in the first-order evaluation with $u(r)$ and $w(r)$ taken from Ref. 5. As the corrections are expected to be small compared to the first-order terms, a simpler wave function was used to evaluate them, namely, the Hulthén[7] function for the S-state, and the D-state contribution was neglected. The Hulthén function in position space is

$$u(r) = (2\alpha\beta(\alpha+\beta))^{1/2}(e^{-\alpha r} - e^{-\beta r})(\beta - \alpha)^{-1}, \qquad (10)$$

with

$$\alpha = 0.3274 m_\pi \quad \text{and} \quad \beta = 1.54 m_\pi.$$

The two-body scattering amplitudes used were for the processes:

$$\pi^- + p \rightarrow n + \gamma, \qquad (11a)$$
$$\pi^+ + N \rightarrow \pi^+ + N. \qquad (11b)$$

As in Ref. 2 the amplitude for (11a) for a relative proton-pion momentum $\mathbf{q}$ going to a neutron of momentum $\mathbf{q}_f$ is taken as the low-energy limit of the amplitude due to Chew, Goldberger, Low, and Nambu,[8]

$$\epsilon_\mu T_{ij}{}^\mu(\mathbf{q}_i, \mathbf{q}_f) \sim i(\boldsymbol{\sigma} \cdot \boldsymbol{\epsilon})_{ij}. \qquad (12)$$

(In the corrections we have neglected processes going via $\pi^0 + n \rightarrow n + \gamma$ which go to zero for zero-incident momentum and are negligible at the energies due to the motion of the particles inside the deuteron.)

For (11b) we take the effective range approximation for $\pi - N$ scattering.[9] The scattering matrix for a rela-

[7] L. Hulthén and M. Sugawara, in *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1957), Vol. 39, p. 5.
[8] G. F. Chew, M. L. Goldberger, F. E. Low, and Y. Nambu, Phys. Rev. **106**, 1345 (1957).
[9] See, for example, J. Hamilton, *The Theory of Elementary Particles* (Oxford University Press, New York, 1959).
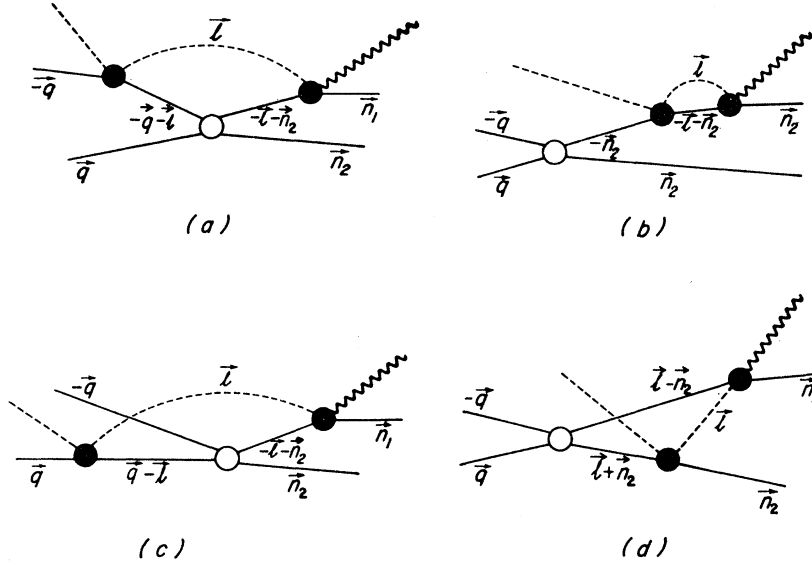
Fig. 3. Impulse assumption corrections.

tive nucleon-pion relative momentum $\mathbf{q}$ and final-nucleon momentum $\mathbf{q}_f$ is

$$T_{ij}(\mathbf{q}_i,\mathbf{q}_f) = \frac{m_N + m_\pi}{m_N m_\pi} \frac{a_\alpha}{4\pi^2} \delta_{ij},$$

(13)

$$a_3 = -0.11/m_\pi, \quad a_1 = 0.17/m_\pi.$$

The kinematics of the problem are such that we have two independent variables. We shall use either $\mathbf{Q}$, relative $n-n$ momentum, and θ_Q, the angle between $\mathbf{Q}$ and the photon direction, or $\mathbf{p}$, the momentum of one of the neutrons and ψ, its angle with the photon direction. We shall also use $E_Q = Q^2/2m_N$, $E = p^2/2m_N$, and ω, the photon energy.

In the subsequent discussion it will be useful to make recourse to a diagramatic presentation of the terms of Eq. (6). In these diagrams a solid line indicates either an external nucleon on nucleon-energy denominator; a dashed line indicates an external pion or pion-energy denominator; a wiggly line indicates the final photon. A solid circle denotes a relevant two-body scattering operator, and an open circle the $n-p$ binding potential U. We shall evaluate the transition element for process (1) for a zero relative $\pi - D$ momentum, with an initial deuteron with magnetic quantum number m, to two neutrons of momenta $\mathbf{n}_1$, $\mathbf{n}_2$, with spin indices α, β,

respectively, and photon of momentum γ and polarization index μ.

A. First-Order Impulse Approximation

This term is represented in Fig. 1, and may be written as

$$\varphi_{i\beta}{}^{(m)}(-\mathbf{n}_2)T_{i\alpha}{}^\mu(-\mathbf{n}_2;\mathbf{n}_1).$$

(14)

(The Pauli principle for the two neutrons shall be taken into account later by projecting out even angular momentum waves for singlet states, and odd waves for the triplet case.)

B. Multiple Scattering Term

Neglecting $\pi^0 + n \rightarrow \pi^0 + \gamma$ the only contribution comes from Fig. 2 and is

$$\int d\mathbf{q}\, \varphi_{ij}{}^{(m)}(-\mathbf{q}) \frac{T_{i\alpha}{}^{\mu*}(-\mathbf{q};\mathbf{n}_1)T_{j\beta}(\mathbf{q};\mathbf{n}_2)}{(n_2{}^2 - q^2)/2m_N - (\mathbf{q} - \mathbf{n}_2)^2/2m_\pi + i\epsilon}.$$

(15)

C. Impulse Assumption Errors

These contributions come from the second term in (6), and due to the commutators each contribution gives rise to two diagrams. We take the difference of Figs. 3(a) and 3(b) and of 3(c) and 3(d). From Fig. 3(a) we have

$$\int d\mathbf{q}d\mathbf{l} \left\{ \varphi_{ij}{}^{(m)}(-\mathbf{q}) \left[\frac{T_{m\alpha}{}^{\mu*}(-\mathbf{l}-\mathbf{n}_2;\mathbf{n}_1)\langle -\mathbf{l}-\mathbf{n}_2, m;\mathbf{n}_2,\beta|U|-\mathbf{q}-\mathbf{l}, n;\mathbf{q}, j\rangle}{n_1{}^2/2m_N + \omega + (\mathbf{l}+\mathbf{n}_2)^2/2m_N - l^2/2m_\pi - m_\pi - i\epsilon} \right] \right.$$

$$\left. \times \left[\frac{T_{in}(-\mathbf{q};-\mathbf{q}-\mathbf{l})}{(\mathbf{q}+\mathbf{l})^2/2m_N + l^2/2m_\pi - q^2/2m_N + i\eta} \right] \right\}.$$

(16)

From 3(b) we have

$$\int d\mathbf{q}\,dl\,\varphi_{ij}{}^{(m)}(-\mathbf{q})\langle-\mathbf{n}_2,\,n;\,\mathbf{n}_2,\,\beta\,|\,U\,|\,\mathbf{q},\,i;\,-\mathbf{q},\,j\rangle$$

$$\times\left\{\frac{T_{m\alpha}{}^{\mu*}(-\mathbf{l}-\mathbf{n}_2;\,\mathbf{n}_1)T_{nm}(-\mathbf{n}_2,\,-\mathbf{l}-\mathbf{n}_2)}{[(1+\mathbf{n}_2)^2/2m_N+l^2/2m_\pi-n_2/2m_N-i\epsilon][n_1{}^2/2m_N+\omega-(1+\mathbf{n}_2)^2/2m_N-l^2/2m_\pi+i\eta-m_\pi]}\right\}. \quad (17)$$

Figure 3(c) contributes

$$\int d\mathbf{q}\,dl\,\varphi_{ij}{}^{(m)}(-\mathbf{q})\left\{\frac{T_{m\alpha}{}^{\mu*}(-\mathbf{l}-\mathbf{n}_2;\,\mathbf{n}_1)}{n_1{}^2/2m_N+\omega-l^2/2m_\pi-(1+\mathbf{n}_2)^2/2m_N-m_\pi-i\epsilon}\right.$$

$$\left.\times\langle-\mathbf{l}-\mathbf{n}_2,\,m;\,\mathbf{n}_2,\,\beta\,|\,U\,|-\mathbf{q},\,i;\,\mathbf{q}-\mathbf{l},\,n\rangle\frac{T_{jn}(\mathbf{q},\,\mathbf{q}-\mathbf{l})}{(\mathbf{q}-\mathbf{l})^2/2m_N+l^2/2m_\pi-q^2/2m_N+i\eta}\right\}, \quad (18)$$

and from 3(d)

$$\int d\mathbf{q}\,dl\,\varphi_{ij}{}^{(m)}(-\mathbf{q})\left\{\frac{T_{m\alpha}{}^{\mu*}(-\mathbf{l}-\mathbf{n}_2;\,\mathbf{n}_1)T_{n\beta}(\mathbf{l}+\mathbf{n}_2;\,\mathbf{n}_2)}{[n_1{}^2/2m_N+\omega-(1+\mathbf{n}_2)^2/2m_N+l^2/2m_\pi-m_\pi-i\epsilon][n_2{}^2/2m_N-(1+\mathbf{n}_2)^2/2m_N+l^2/2m_\pi+i\epsilon]}\right.$$

$$\left.\times\langle-\mathbf{l}-\mathbf{n}_2,\,m;\,\mathbf{l}+\mathbf{n}_2,\,n\,|\,U\,|-\mathbf{q}_1 i;\,q_1 j\rangle\right\}, \quad (19)$$

where $\langle\mathbf{q}_1,\alpha_1;\,\mathbf{q}_2,\alpha_2\,|\,U\,|\,\mathbf{q}_3,\alpha_3;\,\mathbf{q}_n,\alpha_n\rangle$ denotes the matrix element of the potential between nucleons on momenta $\mathbf{q}_1,\,\mathbf{q}_2\cdots$ and spin indices $\alpha_1,\,\alpha_2,\,\cdots$, respectively.

We shall make one further approximation. In the evaluation of the correction terms, which are small compared to the first-order amplitude (as shall be verified by the subsequent calculation), we shall neglect terms of order m_π/m_N. This simplification permits an evaluation of these corrections without a recourse to an explicit model for the potential. Then, as shall be shown in the Appendix, Eqs. (16) and (17) cancel and (18) minus (19) are of order m_π/m_N and may consistently be dropped. The only correction left is (15).

Using the above expressions and some of the evaluations presented in the Appendix the transition amplitude may be written as

$$\langle\alpha_1\beta\,|\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}\left(u(n_2)-\frac{w(n_2)}{\sqrt{8}}\frac{3\boldsymbol{\sigma}_1\cdot\mathbf{n}_2\boldsymbol{\sigma}_2\cdot\mathbf{n}_2}{n_2{}^2}-\boldsymbol{\sigma}_1\cdot\boldsymbol{\sigma}_2\right)|\,\chi_1{}^m\rangle$$

$$-a_3\langle\alpha_1\beta\,|\,\boldsymbol{\sigma}_1\cdot\boldsymbol{\varepsilon}\,|\,\chi_1{}^m\rangle\frac{(2\alpha\beta(\alpha+\beta))^{1/2}}{(\beta-\alpha)}\frac{1}{n_2}$$

$$\times\left[\arctan\frac{n_2}{\alpha}-\arctan\frac{n_2}{\beta}\right]. \quad (20)$$

Expression (20) is further decomposed into its partial waves (even for triplet $n-n$, odd for singlet) and into the two-photon polarization directions. Suppressing spinor and polarization indices

$$T(Q_1\cos\theta_Q)=\sum_l(2l+1)T_l(Q;\omega)P_l(\cos\theta_Q), \quad (21)$$

where we have put in the redundant variable ω in the expression for the partial-wave amplitudes for future convenience. The summation ranges over odd or even l.

III. FINAL-STATE INTERACTIONS

The enhancement of process (1) due to the strong low-energy scattering of the two neutrons in the final state will be inserted into the matrix element by a method discussed in Ref. 6. If $\delta_l(Q)$ is the scattering phase shift for two neutrons of relative momentum Q, the enhanced amplitude becomes

$$T_{l,\text{enh}}(Q;\omega)=T_l(Q;\omega)+\frac{\exp[g_l(Q)+i\delta_l(Q)]}{\pi}$$

$$\times\int_{-\infty}^{\infty}dQ^1\frac{T_l(Q^1;\omega)\exp[-g_l(Q^1)\sin\delta_l(Q^1)]}{Q^1-Q}, \quad (22)$$

where

$$g_l(Q)=\frac{1}{\pi}P\int_{-\infty}^{\infty}dQ^1\frac{\delta_l(Q^1)}{Q^1-Q}. \quad (23)$$

In (22) the $T_l(Q^l;\omega)$ are evaluated keeping ω fixed. As in the process considered the pion is captured at rest, thus ω and Q are related; keeping ω and varying Q corresponds to evaluating the matrix element for incident pions of finite momenta, or in the $\pi-D$ center-of-mass system this corresponds to letting the initial energy vary.

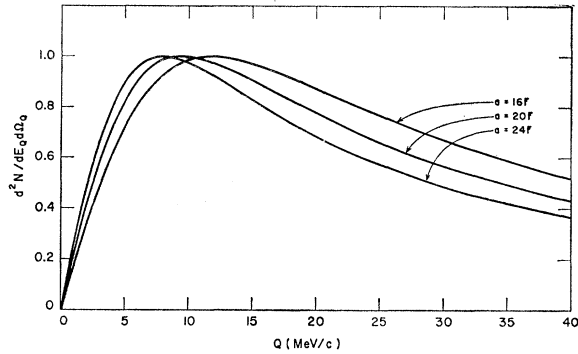For $l=0, 1$ it turns out that $T_l(Q;\omega)$ has a slow variation as a function of Q and we may take it out of

FIG. 4. Neutron-neutron relative energy spectrum.

the integrand and approximate (22) by[10]

$$T_l(Q)_{\text{enh}} = T_l(Q)e^{g_l(Q)+i\delta_l(Q)}, \qquad (24)$$

where we have dropped the photon-energy variable. The error caused by this assumption is less than 0.5% in the range of Q values considered.

We now discuss the forms of $\delta_l(Q)$ investigated. For the S-state scattering we are interested in the scattering length and effective range parameters. The effective range expansion is

$$Q \cot\delta_0(Q) = 1/a + \tfrac{1}{2}Q^2 r. \qquad (25)$$

A generalized form due to Noyes,[11] derived from a dispersion calculation taking into account the one-pion exchange cut, was also examined.

$$Q \cot\delta_0(Q) = 1/a + Q^2 r/2 + cQ^4/(1+dQ^2). \qquad (26)$$

As we shall not be interested in absolute rates but only in spectrum shapes normalized to a definite value at some point, the calculation using the phase shifts (25) and (26) differed absolutely insignificantly. Thus, for simplicity we employed (25) in most of the numerical work. From (25) we find

$$\exp(2g_0(Q)) = \left(Q^2 + \frac{((1+2r/a)^{1/2}+1)^2}{r^2}\right)$$

$$\times \left(Q^2 + \frac{((1+2r/a)^{1/2}-1)^2}{r^2}\right)^{-1}. \qquad (27)$$

Although we expect the S wave to dominate the scattering for small energies, in connection with point 4 of the Introduction, the P-wave triplet enhancement was computed. The P-wave phase shifts were taken from a semiphenomenological fit to the $p-p$ date, with the electromagnetic correction left out.[12]

The ratio of P- to S-wave enhancement is $\sim 0.2\%$ for

[10] J. Jackson, in *Dispersion Relations*, edited by G. R. Screaton (Oliver and Boyd, Ltd., Edinburgh, 1961).
[11] H. P. Noyes, Phys. Rev. **130**, 2025 (1963).
[12] H. P. Noyes (private communication).

small Q and grows to $1\%-2\%$ for $Q=50$ MeV/c. Thus in the subsequent calculations P-wave enhancements were not considered.

The expression for the spectrum becomes

$$dN = (|T_0^{(1)}(Q)|^2 + |T_0^{(2)}(Q)|^2)e^{2g_0(Q)}\Omega, \qquad (28)$$

where the superscripts indicate the two-photon polarizations. Ω is the phase space which is proportional to either

$$\frac{\omega Q}{(1+\omega/2m_N)}dE_Q d(\cos\theta_Q) \qquad (29)$$

or

$$\frac{\omega P}{(1+P/m_N+P\cos\psi/m_N)}dEd(\cos\psi).$$

IV. DISCUSSION OF RESULTS

Restricting ourselves to photon energies near the maximum we have seen that this process goes almost entirely to an S state of the two final neutrons. Thus $dN/dE_Q d\Omega_Q$ is independent of θ_Q. The results for several values of the parameters are presented in Figs. 4 and 5. From Fig. 4 we see that the determination of the energy of the maximum value of the spectrum may serve to indicate the scattering length. However, from Fig. 5 we see that this determination is not unambiguous as increasing the effective range may bring the spectrum to coincide with one for a higher scattering length and smaller effective range. If we restrict ourselves to effective ranges less than 3.5 F this method would yield the scattering length to an accuracy of 1 F. To determine both parameters one would have to go to larger Q values where effects would make this calculation unreliable.

For comparison with Ref. 2, we have plotted the rates as a function of E and ψ (Figs. 6 and 7). Using the results of Ref. 2 would give a scattering length 1–2 F smaller than those obtained using this calculation. Although the multiple scattering corrections discussed
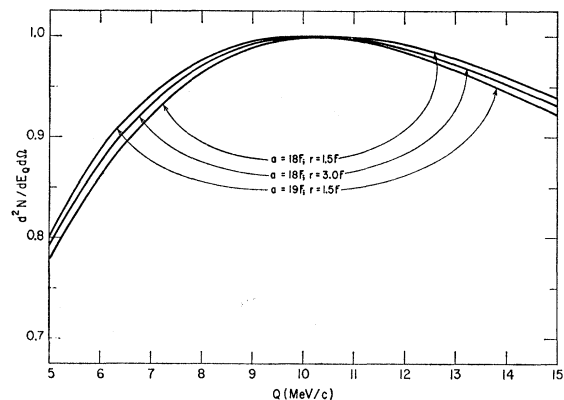

FIG. 5. Neutron-neutron relative energy spectrum near maximum.

before give rise to about a 6% larger total rate, when the spectrum is normalized it is indistinguishable to the one obtained from only the first-order terms. The difference between these two calculations comes mainly from our treatment of the final-state corrections, where we had to make recourse to neither the zero-range approximation nor assume any model for the scattering wave function.

Summarizing, we see that a sufficiently accurate determination of the spectrum $dN/dE_Q d\Omega_Q$ ($\sim 3\%$ in the energy variable and $\sim 1\%$ in the rate) would determine the $n-n$ scattering length to ~ 1 F. A determination of the effective range would involve a calculation of this process valid for larger Q.

ACKNOWLEDGMENT

It is a pleasure to thank Professor H. Pierre Noyes for the suggestion of this problem and for many informative discussions during its preparation.

APPENDIX

In this section we shall sketch the evaluation of some of the integrals presented in Sec. II, and specifically show some of the cancellations mentioned there.

Putting (12) into (15) we obtain an integral of the form

$$\int d\mathbf{q} \frac{u(q)}{(n_2{}^2-q^2)/2m_N-(\mathbf{q}-\mathbf{n}_2)^2/2m_\pi-i\epsilon}, \quad (A1)$$
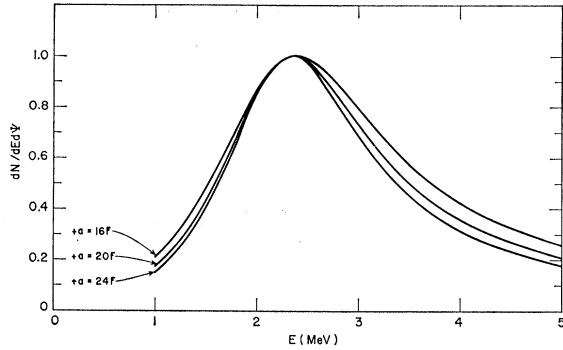
where $u(q)$ is the Bessel transform of the Hulthén

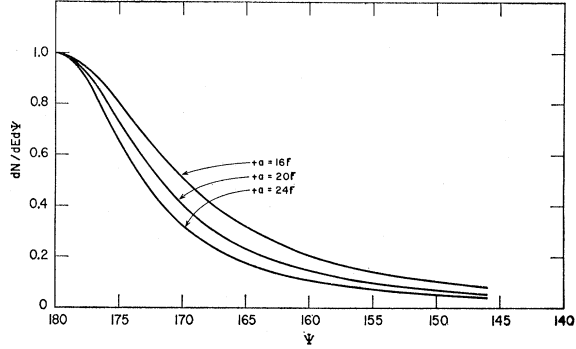Fig. 6. Single neutron energy spectrum for neutrons emitted at 172° to the photon direction.

Fig. 7. Neutron angle spectrum for neutron of an energy of 2.4 MeV.

function. Neglecting m_π/m_N we obtain for (A1)

$$(A1)=\frac{(2\pi)^{3/2}}{n_2}\left[\arctan\left(\frac{n_2}{\alpha}\right)-\arctan\left(\frac{n_2}{\beta}\right)\right]. \quad (A2)$$

Equation (16) contributes a term proportional to

$$\int \frac{d\mathbf{q} dl d\mathbf{r}_1 d\mathbf{r}_2 e^{i(\mathbf{l}+\mathbf{n}_2+\mathbf{q})\cdot\mathbf{r}_1}e^{-i\mathbf{q}\cdot\mathbf{r}_2}V(\mathbf{r}_1)\psi(\mathbf{r}_2)}{[\omega-m_\pi-l^2/2m_\pi](l^2/2m_\pi)}, \quad (A3)$$

where $V(r_1)$ is the $n-p$ potential and $\psi(r_2)$ is the 3-dimensional Fourier transform of $u(q)$. The contribution of (17) is exactly the same and the two terms cancel.

The difference of (18) and (19) is proportional to

$$\int \frac{d\mathbf{q} dl d\mathbf{r}_1 d\mathbf{r}_2 e^{-i\mathbf{q}\cdot\mathbf{r}_2}e^{i(\mathbf{n}_2-\mathbf{q})\cdot\mathbf{r}_1}V(\mathbf{r}_1)\psi(\mathbf{r}_2)}{[\omega-m_\pi-l^2/2m_\pi](l^2/2m_\pi)}(1-e^{i\mathbf{l}\cdot\mathbf{r}}). \quad (A4)$$

Carrying out some of the integrations we obtain

$$\int \frac{dl d\mathbf{r} e^{i\mathbf{n}_2\cdot\mathbf{r}}V(\mathbf{r})\psi(\mathbf{r})}{[\omega-m_\pi-l^2/2m_\pi](l^2/2m_\pi)}(1-e^{i\mathbf{l}\cdot\mathbf{r}}). \quad (A5)$$

Noticing that $V(\mathbf{r})$ operates next to $\psi(\mathbf{r})$ we may use the Schrödinger equation and show that this term is proportional to

$$\frac{m_\pi}{m_N}\frac{1}{n_2}\frac{(\beta^2-\alpha^2)}{n_2{}^2+\beta^2}\arctan\frac{n_2}{2\alpha+\beta+2m_\pi(m_\pi-\omega)}, \quad (A6)$$

which is m_π/m_N smaller than (A2).