

Transport Theory for Electron-Phonon Interactions in Metals*

RICHARD E. PRANGE

Department of Physics and Astronomy, University of Maryland, College Park, Maryland

AND

LEO P. KADANOFF†

Department of Physics, University of Illinois, Urbana, Illinois

(Received 15 October 1963; revised manuscript received 13 January 1964)

By extending Migdal's approximation for electron-phonon interactions in metals to the nonequilibrium case, it is possible to derive a set of transport equations which are exact to order $(m/M)^{1/2}$. This coupled set of equations for the electron and phonon distribution functions is correct even in the situation in which the electronic excitation spectrum has considerable width and structure so that one might not expect *a priori* that there would be well-defined quasiparticles. Nonetheless, one of the forms of the electronic transport equation is identical to the transport equation suggested by Landau for the case in which the quasiparticle energy is well defined. The transport equations may be written in two different forms: In the first form, the electronic distribution function is labeled by a momentum vector; in the second, the labels are excitation energy and the position on the Fermi surface. Despite the width in the spectrum, the momentum-space form is identical with the Landau quasiparticle theory. The energy space form is slightly simpler because no wave function renormalization constants appear in the definition of the energies or in the scattering matrix elements. In fact, in the case in which there is space dependence but no time dependence this form of the transport equations looks identical to the weak-coupling Boltzmann equations. This identity is used to prove that to the accuracy of the adiabatic approximation the several transport coefficients are completely unchanged by the many-body effects of the electron-phonon interaction. These coefficients, which include the spin diffusivity and the viscosity as well as the ordinary conductivities and all the classical galvanomagnetic effects are thus correctly predicted by the standard weak-coupling theory. Many-body effects are also absent in $dn/d\mu$ and the spin susceptibility; however, they do appear in the specific heat and in the response to time-dependent disturbances.

I. INTRODUCTION

IT is remarkable that many experiments on many-body systems can be described by the independent-particle model. Especially for metals, it was unclear why many-body effects were not more pronounced, but a number of years ago, Landau,¹ in a famous series of papers, shed a great deal of light on the problem. Landau pointed out that the low-lying excited states of a Fermion system might very well have a natural one-to-one correspondence with the low-lying states of the noninteracting system. On the basis of this correspondence, he was able to obtain the equilibrium properties, and also the transport equations describing situations not far removed from equilibrium. The equations describe a set of weakly interacting quasiparticles. The main difference between this case and the limit of really weak interactions is that the quasiparticles have a modified energy-momentum relation and that there is a modification of the scattering matrix elements.

The Landau theory is phenomenological, since it introduces unknown functions as phenomenological parameters. Nevertheless, it is of enormous importance,

since it catalogs completely the phenomena of the systems that it describes, and since it provides a target for theorists who wish to start from a less phenomenological viewpoint. Several important advances have been made along this line. We mention the work of Luttinger and Nozières,² who show that as long as perturbation theory is formally correct, the Landau theory follows. That is, if the formal sum of perturbation series has certain crucial properties shared by individual terms in the series, then the Landau theory holds. Baym and Kadanoff³ made rather different arguments based upon the theory of Green's functions.

The theories of quasiparticles so far mentioned have relied upon the following idea: If one considers an electron of definite momentum near the Fermi surface, one finds that it has a component with well-defined energy, that is, its lifetime is very long. The reason is that the phase space for the possible decay products of the electron is very small, because of the Pauli principle.

In this paper, we investigate the question of whether a quasiparticle theory can be said to hold for the system of electrons and phonons. The previously mentioned work has been restricted to a system of Fermions interacting by means of instantaneous forces, as, for

* This research was supported in part by the U. S. Air Force Office of Scientific Research and the General Research Board of the University of Maryland.

† Alfred P. Sloan Foundation Fellow.

¹ L. Landau, *Zh. Eksperim. i Teor. Fiz.* **30**, 1058 (1956); **32**, 59 (1957); **35**, 97 (1958) [English transl.: *Soviet Phys.—JETP* **3**, 920 (1957); **5**, 101 (1957); **8**, 70 (1959)]; A. A. Abrikosov and I. M. Khalatnikov, *Rept. Progr. Phys.* **22**, 329 (1959); *Usp. Fiz. Nauk* **66**, 177 (1958) [English transl.: *Soviet Phys.—USP.* **1**, 68 (1958)].

² L. P. Pitaevskii, *Zh. Eksperim. i Teor. Fiz.* **37**, 1794 (1959) [English transl.: *Soviet Phys.—JETP* **10**, 1267 (1960)]; V. I. Karpman, *Zh. Eksperim. i Teor. Fiz.* **39**, 185 (1960) [English transl.: *Soviet Phys.—JETP* **12**, 133 (1961)]. J. M. Luttinger and P. Nozières, *Phys. Rev.* **127**, 1425, 1431 (1962); P. Nozières, *Theory of Interacting Fermion Systems* (W. A. Benjamin and Company, Inc., New York, 1963).

³ L. P. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin and Company, Inc., New York, 1963).

example, Coulomb repulsions. There have been suggestions recently that the quasiparticle picture fails for electron-phonon systems.^{4,5} Partially, this is a matter of semantics. We shall say that we have a quasiparticle theory if the system is described by a Boltzmann equation of the form suggested by Landau, even though various matrix elements, as well as the energy-momentum relations, are “renormalized.” We shall not regard it as a failure of the quasiparticle concept if it turns out that the matrix-element (wave function) renormalization is nontrivial.

A more profound objection to the use of the quasiparticle concept in the electron-phonon system is that an electron injected into the system with a definite momentum cannot be said to have a definite energy, if this energy is above the Fermi energy by as much as ω_0 , where ω_0 is a typical or maximum phonon energy (the Debye frequency). Such electrons so rapidly emit phonons that their energy cannot be defined. If such comparatively highly excited electronic states are not of importance, there is no difficulty in extending the usual arguments leading to the Landau theory to the electron-phonon case as well. This will be the case provided we consider only perturbations varying slowly enough in space and time, and provided we restrict the temperature to be well below the Debye temperature. On the other hand, if the highly excited states are important, the usual arguments justifying the quasiparticle concept fail.

There have been a number of previous attempts to justify the Boltzmann equations in the case in which the electron lifetime was not infinitely long.⁶ However, all of these attempts have been based on the approximation that the decay processes involve no change in the electronic energy. Thus, the electronic energies have to be at least well defined. In the electron-phonon system, this situation occurs at temperatures well above the Debye temperature. Our theory includes this as a special case.

There are many ways in which highly excited and short-lived electronic states can be practically obtained. One is to apply a microwave field whose frequency is comparable to the Debye frequency. Another is to apply a magnetic field so strong that the cyclotron frequency is comparable to the Debye frequency. A third is to raise the potential across a tunnel junction to a value of the order of the Debye frequency measured in electron volts. A fourth is to raise the temperature of the metal to the Debye temperature. We have been able to show that in the third and last cases, a quasiparticle theory (with a definite energy-momentum relation) holds and is essentially exact, in spite of the short lifetime of electrons of definite momentum. We

shall only treat the “high” temperature case in this paper, since the third case is a trivial extension of the temperature problem.

The results of this paper are based upon the approximations of Migdal^{7,8} which are believed to involve only errors allowed by the adiabatic (Born-Oppenheimer) approximation. This is not much of a restriction since the adiabatic approximation is always made in any theory of solids. This approximation regards as small the root of the ratio electron mass divided by ion mass. An equivalent small quantity is the ratio of the sound speed to Fermi velocity. Another equivalent parameter is the ratio of maximum phonon frequency (Debye frequency) to typical electron energy (Fermi energy).

Although the frequency scale of the phonon system is small in comparison with the electronic frequencies, the wavelength scale is quite comparable. Wavelengths of typical phonons or electrons are comparable with the lattice constants. We shall follow Migdal in exploiting these facts characteristic of the electron-phonon system.

In the next section we describe the starting point for our calculation, the Migdal approximation for the self-energy in the electron-phonon system and the exact transport theory of Kadanoff and Baym. In the third section the kinetic-energy variable is integrated out of the transport equation so that we may write an equation of motion for an electronic distribution function which depends upon the excitation energy and the direction of the momentum vector. Section IV describes how the energy space equation of motion may be transformed back into momentum space. The resulting transport equation is exactly of the Landau form. In Sec. V, Landau’s identifications of the densities and currents of conserved quantities is specialized to the electron-phonon case. The identifications are verified by a comparison with exact expressions for these densities and currents. In the final section we use these identifications and the transport equations in order to evaluate some thermodynamic derivatives and transport coefficients.

II. FORMULATION

The model electron-phonon system which we wish to study has the Hamiltonian

$$H = \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{q}} \omega_{\mathbf{q}}^0 (a_{\mathbf{q}}^\dagger a_{\mathbf{q}} + a_{-\mathbf{q}} a_{-\mathbf{q}}^\dagger) + \sum_{\mathbf{k}, \mathbf{q}} v^0(\mathbf{q}) c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}} (a_{\mathbf{q}} + a_{-\mathbf{q}}^\dagger). \quad (1)$$

We have made a number of assumptions in order not to encumber the notation. The electron spin and band

⁷ A. B. Migdal, Zh. Eksperim. i Teor. Fiz. **34**, 1438 (1958) [English transl.: Soviet Phys.—JETP **7**, 996 (1958)]; see also Ref. 5.

⁸ Migdal’s ideas are extended to finite temperature in A. A. Abrikosov, L. P. Gor’Kov, and I. E. Dzyaloshinski, *Methods of Quantum Field Theory in Statistical Physics*, translated by R. A. Silverman (Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1963), pp. 176–189.

⁴ L. P. Kadanoff, Phys. Rev. **132**, 2073 (1963); also, L. P. Kadanoff, in *Ravello 1963 Spring School Notes* (Academic Press Inc., to be published).

⁵ S. Engelberg and J. R. Schrieffer, Phys. Rev. **131**, 993 (1963).

⁶ J. S. Van Wieringen, Proc. Phys. Soc. (London) **A67**, 206 (1954); C. V. Chester and A. Thellung, *ibid.* **A73**, 745 (1959).

indices and the phonon polarization indices have been suppressed. The bare electron-phonon matrix element $v^0(\mathbf{q})$ has been taken to depend only on \mathbf{q} . It is supposed to have the usual properties and is not necessarily very small. Umklapp processes have been neglected so that "momentum" is conserved. Anharmonic effects (direct phonon-phonon interactions) have been ignored. None of these simplifications is essential to the subsequent argumentation.

In Eq. (1), $c_{\mathbf{k}}^\dagger$ is a creation operator for an electron of momentum \mathbf{k} , $\epsilon_{\mathbf{k}}$ is the electron kinetic energy (energy of the Bloch state), and $\omega_{\mathbf{q}}^0$ is the zero-order phonon frequency. We work in a box of unit volume and set $\hbar=1$. The $a_{\mathbf{q}}$ operator destroys a bare phonon of momentum \mathbf{q} .

A convenient way of proceeding is to introduce at once the physical phonon operators, defined by the transformation

$$\alpha_{\mathbf{q}} = u_{\mathbf{q}} a_{\mathbf{q}} + V_{\mathbf{q}} a_{-\mathbf{q}}^\dagger. \quad (2)$$

The transformation coefficients, which are to be determined self-consistently, are given by the relations

$$u_{\mathbf{q}}^2 - V_{\mathbf{q}}^2 = 1, \quad (2a)$$

$$u_{\mathbf{q}} = u_{-\mathbf{q}}, \quad V_{\mathbf{q}} = V_{-\mathbf{q}}, \quad (2b)$$

$$u_{\mathbf{q}}^2 + V_{\mathbf{q}}^2 = (\omega_{\mathbf{q}}^0/\omega_{\mathbf{q}}) + (\Pi_{\mathbf{q}}/\omega_{\mathbf{q}}^0), \quad (2c)$$

$$2u_{\mathbf{q}}V_{\mathbf{q}} = \Pi_{\mathbf{q}}/\omega_{\mathbf{q}}^0, \quad (2d)$$

and the relation between $\Pi_{\mathbf{q}}$ and $\omega_{\mathbf{q}}$ is

$$\omega_{\mathbf{q}}^2 = (\omega_{\mathbf{q}}^0)^2 + 2\Pi_{\mathbf{q}}\omega_{\mathbf{q}}. \quad (2e)$$

The frequency $\omega_{\mathbf{q}}$ will turn out to be the renormalized phonon frequency, while the quantity $\Pi_{\mathbf{q}}$ will have the significance of the polarization part. Under this transformation the Hamiltonian becomes

$$\begin{aligned} H = & \sum_{\mathbf{k}} \epsilon_{\mathbf{k}} c_{\mathbf{k}}^\dagger c_{\mathbf{k}} + \frac{1}{2} \sum_{\mathbf{q}} \omega_{\mathbf{q}} (\alpha_{\mathbf{q}}^\dagger \alpha_{\mathbf{q}} + \alpha_{-\mathbf{q}} \alpha_{-\mathbf{q}}^\dagger) \\ & + \sum_{\mathbf{k}, \mathbf{q}} v(\mathbf{q}) c_{\mathbf{k}+\mathbf{q}}^\dagger c_{\mathbf{k}} (\alpha_{\mathbf{q}} + \alpha_{-\mathbf{q}}^\dagger) \\ & - \frac{1}{2} \sum_{\mathbf{q}} \Pi_{\mathbf{q}} (\alpha_{\mathbf{q}}^\dagger + \alpha_{-\mathbf{q}}) (\alpha_{\mathbf{q}} + \alpha_{-\mathbf{q}}^\dagger). \quad (3) \end{aligned}$$

We have introduced the renormalized coupling constant

$$v(\mathbf{q})^2 = v^0(\mathbf{q})^2 (\omega_{\mathbf{q}}^0/\omega_{\mathbf{q}}).$$

In order to derive a transport theory, we define, following Ref. 3, the functions⁹

$$\begin{aligned} g_{\sigma}^{>}(\mathbf{k}, E; \mathbf{r}, t) = & \sum_{\mathbf{k}'} \int d t' e^{iE t' + i\mathbf{k}' \cdot \mathbf{r}} \\ & \times \langle c_{\mathbf{k}+\frac{1}{2}\mathbf{k}', \sigma}(t+\frac{1}{2}t') c_{\mathbf{k}-\frac{1}{2}\mathbf{k}', \sigma}^\dagger(t-\frac{1}{2}t') \rangle, \quad (4a) \end{aligned}$$

⁹ Here the averages indicate that the system has been perturbed from an equilibrium distribution in a way analogous to Eq. (8-18) of Ref. 3. However, we differ from this reference in looking only at times t after the disturbance has been shut off. This specialization is made solely to simplify the writing and can easily be removed.

$$\begin{aligned} g_{\sigma}^{<}(\mathbf{k}, E; \mathbf{r}, t) = & \sum_{\mathbf{k}'} \int d t' e^{iE t' + i\mathbf{k}' \cdot \mathbf{r}} \\ & \times \langle c_{\mathbf{k}-\frac{1}{2}\mathbf{k}', \sigma}^\dagger(t-\frac{1}{2}t') c_{\mathbf{k}+\frac{1}{2}\mathbf{k}', \sigma}(t+\frac{1}{2}t') \rangle, \quad (4b) \end{aligned}$$

which represent, respectively, the local density of electrons and holes with spin quantum number σ momentum \mathbf{k} and energy E in the neighborhood of the space-time point \mathbf{r}, t .

In the same way, we define

$$\begin{aligned} r^{<}(\mathbf{q}, \omega; \mathbf{r}, t) = & \sum_{\mathbf{q}'} \int d t' e^{i\omega t' + i\mathbf{q}' \cdot \mathbf{r}} \\ & \times \langle \alpha_{\mathbf{q}-\frac{1}{2}\mathbf{q}'}^\dagger(t-\frac{1}{2}t') \alpha_{\mathbf{q}+\frac{1}{2}\mathbf{q}'}(t+\frac{1}{2}t') \rangle, \quad (5) \end{aligned}$$

which gives the local number density of phonons of momentum \mathbf{q} and frequency ω . We shall also need the functions

$$\begin{aligned} g(\mathbf{k}, z) = & \int \frac{dE}{2\pi} \frac{g^{>}(\mathbf{k}, E) + g^{<}(\mathbf{k}, E)}{z - E} \\ = & \int \frac{dE}{2\pi} \frac{A(\mathbf{k}, E)}{z - E}, \quad (6) \end{aligned}$$

and

$$r(\mathbf{q}, z) = \int \frac{d\omega}{2\pi} \frac{r^{>}(\mathbf{q}, \omega) - r^{<}(\mathbf{q}, \omega)}{z - \omega}, \quad (7)$$

where z is a complex variable and we have not explicitly indicated the \mathbf{r}, t dependence or the spin dependence.

In the equilibrium case, $g(\mathbf{k}, z)$ and $r(\mathbf{q}, z)$ are independent of \mathbf{r}, t . There, it is more usual to define only the propagator D , given by

$$D(\mathbf{q}, t-t') = \frac{1}{2} t' T(\alpha_{\mathbf{q}}(t) + \alpha_{-\mathbf{q}}^\dagger(t)) (\alpha_{-\mathbf{q}}(t') + \alpha_{\mathbf{q}}^\dagger(t')).$$

Perturbation theory can be expressed entirely in terms of D in view of the form of the coupling terms in the Hamiltonian. Transport theory can be expressed entirely in terms of D also, but we have chosen to introduce r to make the analogy between the electron and phonon transport equations more transparent. We shall also have to define temporarily the functions $S^{>.<}(\mathbf{q}, \omega)$ and $S(\mathbf{q}, z)$ by

$$\begin{aligned} S^{<}(\mathbf{q}, \omega) = & \sum_{\mathbf{q}'} \int d t' \exp(i\mathbf{q}' \cdot \mathbf{r} + i\omega t') \\ & \times \langle \alpha_{\mathbf{q}-\frac{1}{2}\mathbf{q}'}^\dagger(t-\frac{1}{2}t') \alpha_{-\mathbf{q}-\frac{1}{2}\mathbf{q}'}^\dagger(t+\frac{1}{2}t') \rangle. \end{aligned}$$

The functions S will be shown to be negligible when the phonon renormalization is correctly carried out, as must be the case if the phonons are to be successfully interpreted as nearly independent elementary excitations.

In the equilibrium case, $g(\mathbf{k}, z)$ and $r(\mathbf{q}, z)$ can be determined with the aid of the Migdal approximation¹⁰

¹⁰ See Ref. 8, Eqs. (21.8) and (21.22). These approximations are here rewritten in the language of Ref. 3.

by

$$[z - \epsilon_{\mathbf{k}} - \Sigma_{\sigma}(\mathbf{k}, z)]g_{\sigma}(\mathbf{k}, z) = 1, \quad (8)$$

$$[z - \omega_{\mathbf{q}} - \Pi(\mathbf{q}, z)]r(\mathbf{q}, z) - \Pi(\mathbf{q}, z)S(\mathbf{q}, z) = 1, \quad (9a)$$

$$[-z - \omega_{\mathbf{q}} - \Pi(\mathbf{q}, z)]S(\mathbf{q}, z) - \Pi(\mathbf{q}, z)r(\mathbf{q}, z) = 0, \quad (9b)$$

where

$$\begin{aligned} \Sigma_{\sigma}(\mathbf{k}, z) &= \int \frac{dE}{2\pi} \frac{\Sigma_{\sigma}^{>}(\mathbf{k}, E) + \Sigma_{\sigma}^{<}(\mathbf{k}, E)}{z - E} \\ &= \int \frac{dE}{2\pi} \frac{\Gamma_{\sigma}(\mathbf{k}, E)}{z - E} \end{aligned} \quad (10)$$

and

$$\begin{aligned} \Pi(\mathbf{q}, z) &= \int \frac{d\omega}{2\pi} \frac{\Pi^{>}(\mathbf{q}, \omega) - \Pi^{<}(\mathbf{q}, \omega)}{z - \omega} - \Pi_{\mathbf{q}} \\ &= \int \frac{d\omega}{2\pi} \frac{\gamma(\mathbf{q}, \omega)}{z - \omega} - \Pi_{\mathbf{q}}. \end{aligned} \quad (11)$$

The widths $\Sigma^{>}$ and $\Pi^{>}$ are given by

$$\begin{aligned} \Sigma_{\sigma}^{>}(\mathbf{k}, E) &= \int \frac{d^3q d\omega}{(2\pi)^4} [v(\mathbf{q})]^2 [r^{>}(\mathbf{q}, \omega) \\ &\quad + r^{<}(-\mathbf{q}, -\omega)] g_{\sigma}^{>}(\mathbf{k} - \mathbf{q}, E - \omega) \end{aligned} \quad (12)$$

and

$$\begin{aligned} \Pi^{>}(\mathbf{q}, \omega) &= \sum_{\sigma} \int \frac{d^3k dE}{(2\pi)^4} [v(\mathbf{q})]^2 \\ &\quad \times g_{\sigma}^{>}(\mathbf{k} + \mathbf{q}, E + \omega) g_{\sigma}^{<}(\mathbf{k}, E). \end{aligned} \quad (13)$$

We have already neglected the contribution of $S^{>}, <$ in Eq. (12). The mathematical justification for this neglect, as well as for the neglect of $s(\mathbf{q}, z)$ in Eq. (9a) will be deferred until after we have had a chance to discuss the structure of $\Pi(\mathbf{q}, z)$. The widths $\Sigma^{<}$ and $\Pi^{<}$ are defined by equations identical to (12) except that $>$ and $<$ are everywhere interchanged.

The state of thermodynamic equilibrium at temperature β^{-1} and chemical potential μ is described by the solution of Eq. (8)–(13) with the subsidiary conditions

$$\begin{aligned} g^{<}(\mathbf{k}, E) &= \exp[-\beta(E - \mu)] g^{>}(\mathbf{k}, E), \\ r^{<}(\mathbf{q}, \omega) &= \exp[-\beta\omega] r^{>}(\mathbf{q}, \omega). \end{aligned} \quad (14)$$

If we follow the ideas of Ref. 3, we see that when $r^{>}, <$ and $g^{>}, <$ vary slowly in space and time (\mathbf{r} and t), Eqs. (8)–(13) remain equally true in the nonequilibrium case if we simply consider all the functions in these equations to depend upon \mathbf{r} and t . However, the detailed balancing conditions represented by Eq. (14) fails. These conditions must be replaced by the generalized Boltzmann equations:

$$\begin{aligned} [E - \epsilon_{\mathbf{k}} - \text{Re}\Sigma, g^{>}] - [\Sigma^{>}, \text{Reg}] \\ = -[E - \epsilon_{\mathbf{k}} - \text{Re}\Sigma, g^{<}] + [\Sigma^{<}, \text{Reg}] \quad (15) \\ = -\Sigma^{<}(\mathbf{k}, E)g^{>}(\mathbf{k}, E) + \Sigma^{>}(\mathbf{k}, E)g^{<}(\mathbf{k}, E) \end{aligned}$$

$$\begin{aligned} [\omega - \omega_{\mathbf{q}} - \text{Re}\Pi, r^{>}] - [\Pi^{>}, \text{Rer}] \\ = [\omega - \omega_{\mathbf{q}} - \text{Re}\Pi, r^{<}] - [\Pi^{<}, \text{Rer}] \quad (16) \\ = -\Pi^{>}(\mathbf{q}, \omega)r^{<}(\mathbf{q}, \omega) + \Pi^{<}(\mathbf{q}, \omega)r^{>}(\mathbf{q}, \omega). \end{aligned}$$

In writing Eqs. (15) and (16), we have used a generalized Poisson bracket notation in which, for Eq. (15),

$$[A, B] = \frac{\partial A}{\partial E} \frac{\partial B}{\partial t} - \frac{\partial A}{\partial t} \frac{\partial B}{\partial E} + \nabla_{\mathbf{r}} A \cdot \nabla_{\mathbf{k}} B - \nabla_{\mathbf{k}} B \cdot \nabla_{\mathbf{r}} A. \quad (17)$$

In Eq. (16) the brackets have the same meaning except that for $\partial/\partial E$ we read $\partial/\partial\omega$ and for $\nabla_{\mathbf{k}}$ we read $\nabla_{\mathbf{q}}$. Finally, in Eqs. (15) and (16) the notation Re means that the complex frequency variables in $g(\mathbf{k}, z)$, $r(\mathbf{q}, z)$, $\Sigma(\mathbf{k}, z)$, and $\Pi(\mathbf{q}, z)$ are to be moved onto the real axis and the integrals in Eqs. (6), (7), (10), and (11) are to be interpreted in the principal value sense. Thus, for example,

$$\text{Reg} = \text{Reg}(\mathbf{k}, E; \mathbf{r}, t) = \mathcal{P} \int \frac{dE' A(\mathbf{k}, E', \mathbf{r}, t)}{2\pi (E - E')}. \quad (18)$$

III. DERIVATION OF TRANSPORT EQUATIONS IN ENERGY SPACE

To make any progress with the equations of the previous section, we must employ a simplifying feature of this problem first noted by Migdal⁷ and subsequently used by many other authors.^{4,5} This simplifying feature results from the fact that all the phonon energies $\omega_{\mathbf{q}}^0$ are much smaller than μ . For this reason, the only electrons which participate in the electron-phonon interaction are those with energies very close to μ . In fact, we usually need only consider electrons with $|E - \mu|$ and $|\epsilon_{\mathbf{k}} - \mu|$ of the order of a typical thermal energy β^{-1} or a typical phonon energy ω_0 .

The smallness of the phonon energies is reflected in a great sensitivity of $\Sigma(\mathbf{k}, z; \mathbf{r}, t)$ to changes in z . In fact,

$$\frac{\partial}{\partial z} \Sigma(\mathbf{k}, z) = O \frac{\Sigma(\mathbf{k}, z)}{\omega_0}. \quad (19)$$

On the other hand, the momenta of the important phonons are quite comparable with electronic momenta. For this reason we cannot expect any corresponding sensitivity of Σ to changes in \mathbf{k} . We can see that

$$\nabla_{\mathbf{k}} \Sigma(\mathbf{k}, z) = O(\Sigma(\mathbf{k}, z)/k_0), \quad (20)$$

where k_0 is the typical electronic momentum, the Fermi momentum. From (20) it follows that

$$\frac{\partial}{\partial \epsilon_{\mathbf{k}}} \Sigma(\mathbf{k}, z) = O \frac{\Sigma(\mathbf{k}, z)}{\mu}, \quad (21)$$

where μ , the chemical potential, is far larger than ω_0 . [We remark that Eqs. (20) and (21) fail for superconductors because a new fundamental length, the coherence length, appears.]

Because of the insensitivity represented in Eq. (21), we shall consider $\Sigma(\mathbf{k}, z; \mathbf{r}, t)$ to be completely independ-

ent of ϵ_k . We replace \mathbf{k} in Σ by $\mathbf{k}_F(\hat{k})$ which indicates that we consider \mathbf{k} to be confined to the Fermi surface. Therefore it depends only on the unit vector $\hat{k}=\mathbf{k}/|\mathbf{k}|$. Henceforth we shall write $\Sigma(\mathbf{k},z; \mathbf{r},t)$ as $\Sigma(\hat{k},z; \mathbf{r},t)$ and consider Σ , $\Sigma^>$, and $\Sigma^<$ to be completely independent of ϵ_k .

This independence enabled Migdal to find $A(\mathbf{k},\omega)$ quite explicitly; it will permit us a considerable simplification of our transport equations. To begin this simplification we notice that Eqs. (6), (10), and (15) imply that

$$A(\mathbf{k},E; \mathbf{r},t) = \frac{\Gamma(\hat{k},E; \mathbf{r},t)}{[E-\epsilon_k - \text{Re}\Sigma(\hat{k},E,\mathbf{r},t)]^2 + \frac{1}{4}[\Gamma(\hat{k},E; \mathbf{r},t)]^2}. \quad (22)$$

The function A , in contrast to Σ and Γ , is strongly peaked in ϵ_k . It is just this peaking that limits our interest in the dependence of Σ on ϵ_k to values of ϵ_k in the neighborhood of E ($\sim\mu$). Any slowly varying function of \mathbf{k} multiplying $A(\mathbf{k},E)$ can be evaluated at $\mathbf{k}=\mathbf{k}_F$, with possible errors of order ω_0/μ being incurred. Since we can see explicitly all the dependence of A upon ϵ_k in (22), it is possible to compute the area under the peak as

$$(2\pi)^{-1} \int d\epsilon_k A(\mathbf{k},E; \mathbf{r},t) = 1. \quad (23)$$

These properties of $A(\mathbf{k},E,\mathbf{r},t)$ form the keystone of all our subsequent analyses just as their equilibrium analogs served as a keystone of Migdal's paper.

This method differs from the usual justifications of the Landau theory which are based on the smallness of the width Γ . It is well known that Γ is negligibly small if the conditions $|E-\mu| \ll \omega_0$ and $\beta^{-1} \ll \omega_0$ are satisfied. If these conditions are relaxed, as is done in this paper, $A(\mathbf{k},E,\mathbf{r},t)$ will have considerable width and structure when regarded as a function of E . The usual simplification of integrals over E will not then occur.

Equation (23) enables us to define a distribution function $f(\hat{k},E,\mathbf{r},t)$ by

$$(2\pi)^{-1} \int d\epsilon g_{\sigma}^<(\epsilon,\hat{k},E; \mathbf{r},t) = f_{\sigma}(\hat{k},E; \mathbf{r},t), \quad (24)$$

which is the density of electrons with total energy E with momentum in the direction \hat{k} . From (22),

$$(2\pi)^{-1} \int d\epsilon g_{\sigma}^>(\epsilon,\hat{k},E; \mathbf{r},t) = 1 - f_{\sigma}(\hat{k},E; \mathbf{r},t). \quad (25)$$

Since, in full thermodynamic equilibrium the condition (14) holds, it follows that in equilibrium, we have

$$f(\hat{k},E) = \{\exp[\beta(E-\mu)] + 1\}^{-1},$$

which is the usual Fermi distribution function.

Next we rewrite $\Pi^>^<$ and $\Sigma^>^<$ in terms of the distribution function f . To do this we use the fact that

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d\Omega}{4\pi} \int d\epsilon N(\epsilon,\hat{k}),$$

where $\int d\Omega$ is a solid angle integration. The ϵ integral contributes only very near $\epsilon=\mu$, so that we may replace

$$\int \frac{d^3k}{(2\pi)^3} = \int \frac{d\Omega}{4\pi} N_0(\hat{k}) \int d\epsilon, \quad (26)$$

where

$$N_0(\hat{k}) = N(\mu,\hat{k})$$

is the density of states in angle and energy evaluated at the Fermi surface, for electrons of one spin in the noninteracting system. If we make this replacement Eq. (11) implies that

$$\begin{aligned} \Sigma^>(\hat{k},E) &= (4\pi)^{-1} \int d\Omega' dE' N_0(\hat{k}) [v(\mathbf{k}_F - \mathbf{k}_{F'})]^2 \\ &\quad \times [1 - f(\hat{k}',E')] [r^>(\mathbf{k}_F - \mathbf{k}_{F'}, E - E') \\ &\quad + r^<(\mathbf{k}_{F'} - \mathbf{k}_F, E' - E)], \end{aligned} \quad (27a)$$

$$\begin{aligned} \Sigma^<(\hat{k},E) &= (4\pi)^{-1} \int d\Omega' dE' N_0(\hat{k}') [v(\mathbf{k}_F - \mathbf{k}_{F'})]^2 \\ &\quad \times f(\hat{k}',E') [r^<(\mathbf{k}_F - \mathbf{k}_{F'}, E - E') \\ &\quad + r^<(\mathbf{k}_{F'} - \mathbf{k}_F, E' - E)]. \end{aligned} \quad (27b)$$

Exactly the same manipulations applied to Eq. (12) give

$$\begin{aligned} \Pi^>(\mathbf{q},\omega) &= \Pi^<(-\mathbf{q}, -\omega) \\ &= \sum_{\sigma} (4\pi)^{-2} \int d\Omega d\Omega' N_0(\hat{k}) N_0(\hat{k}') \\ &\quad \times [v(q)]^2 (2\pi)^4 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} - \mathbf{q}) \\ &\quad \times [1 - f_{\sigma}(\hat{k}, E + \omega)] f_{\sigma}(\hat{k}', E). \end{aligned} \quad (28)$$

It is worth noting that $\Sigma_{\sigma}^>^<$ depends only on the density of electrons of one spin, whereas $\Pi^>^<$ depends on the distribution of both spins in equal measure.

As is explained in Ref. 8, Migdal's arguments for the approximation of $\Pi^>^<$ break down when \mathbf{q} is too small, i.e., only those phonons for which $q/k_0 > \omega_0/\mu$ can be treated accurately by the perturbation theory method. This is not a serious drawback because the long-wavelength phonons are few in number and do not contribute appreciably to the electron self-energies, or to transport processes, except in special circumstances. The long-wavelength sound waves can be treated *a posteriori* by means of the electron phonon transport equations themselves.

To find the electronic transport equation we integrate Eq. (15) over ϵ . Because of Eq. (20) we can neglect $\nabla_k \Sigma$ and $\nabla_k \Sigma^<$. Then Eq. (15) implies¹¹

$$\begin{aligned} [\omega - \epsilon_k - \Sigma(\hat{k},E), f(\hat{k},E)] - \left[\Sigma^<(\hat{k},E), \int \frac{d\epsilon}{2\pi} g(\epsilon,\hat{k},E) \right] \\ = -\Sigma^>(\hat{k},E) f(\hat{k},E) + \Sigma^<(\hat{k},E) [1 - f(\hat{k},E)]. \end{aligned} \quad (29)$$

¹¹ In this equation and below, we shall drop the notation Re from $\text{Re}\Sigma$ because we shall have no more use for the complex quantities themselves.

We can eliminate the term involving g in Eq. (29) if we notice that

$$\int \frac{d\epsilon}{2\pi} g(\epsilon, \hat{k}, E) = \mathcal{P} \int \frac{d\epsilon dE'}{(2\pi)^2} \frac{A(\epsilon, \hat{k}, E')}{E-E'} = \mathcal{P} \int \frac{dE'}{2\pi} \frac{1}{E-E'}$$

This is not, strictly speaking, well defined but all derivatives of this object certainly vanish. Therefore, Eq. (29) becomes

$$\left[1 - \frac{\partial}{\partial E} \Sigma_{\sigma}(\hat{k}, E) \right] \frac{\partial}{\partial t} f_{\sigma}(\hat{k}, E) + \frac{\partial \Sigma_{\sigma}}{\partial t}(\hat{k}, E) \frac{\partial f_{\sigma}}{\partial E}(\hat{k}, E) + (\nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F}) \cdot \nabla_{\mathbf{r}} f_{\sigma}(\hat{k}, E) = -\Sigma_{\sigma}^{>}(\hat{k}, E) f_{\sigma}(\hat{k}, E) + \Sigma_{\sigma}^{<}(\hat{k}, E) [1 - f_{\sigma}(\hat{k}, E)], \quad (30)$$

when the brackets are evaluated and terms like

$$\nabla_{\mathbf{k}} \Sigma \cdot \nabla_{\mathbf{r}} f = O[(\Sigma/k_0) |\nabla_{\mathbf{r}} f|]$$

are neglected. This neglect requires $\Sigma \ll \mu$ which is certainly very well satisfied in the metal.

The derivation of the phonon transport equation is slightly more complicated. We first study the quantity $\text{Re}\Pi(\mathbf{q}, \omega)$, given by

$$\text{Re}\Pi(q\omega) = \frac{\mathcal{P}}{2\pi} \int d\omega' \frac{\Pi^{>}(\mathbf{q}, \omega') - \Pi^{<}(\mathbf{q}, \omega')}{\omega - \omega'} - \Pi_{\mathbf{q}} \quad (31)$$

The values of ω of greatest interest to us are of order ω_0 . As Migdal has shown, the values of ω' contributing appreciably to the integral in (31) are much larger, of order μ . Consequently, the electron distributions on which (31) depends must be evaluated at energies remote from the Fermi energy. Far from the Fermi surface, the electron states are either full or empty, independent of the slight ripples on the Fermi sea which represent the deviation from equilibrium. Indeed, these electron states are uninfluenced by the electron-phonon interaction altogether. We may therefore compute $\text{Re}\Pi(q, \omega)$ with great accuracy by using the formula (12) for $\Pi^{>}, <(\mathbf{q}, \omega)$ and putting for the electron distribution functions in (12), the functions of the noninteracting system at zero temperature.

It follows from the above arguments that $\text{Re}\Pi(\mathbf{q}, \omega)$ is practically independent of frequency, for frequencies of order ω_0 . We choose it to vanish by picking $\Pi_{\mathbf{q}}$ to be

$$\Pi_{\mathbf{q}} = \frac{\mathcal{P}}{2\pi} \int d\omega' \frac{\Pi^{>}(\mathbf{q}, \omega') - \Pi^{<}(\mathbf{q}, \omega')}{-\omega'} \quad (32)$$

By this choice, we achieve the relation, valid for all ω ,

$$\text{Re}\Pi(\mathbf{q}, \omega)/\omega = O(\omega_0^2/\mu^2),$$

which allows us to neglect $\text{Re}\Pi$ completely.

Next we remark that the absorptive part of $\Pi(\mathbf{q}, z)$, which we have denoted by $\gamma(\mathbf{q}, z)$, is very small. It can be easily computed from Eq. (28) for the entire fre-

quency range. Migdal has computed this quantity in the equilibrium case, for which it is true that

$$\gamma_{\text{equil}}(\mathbf{q}, \omega)/\omega = O(\omega_0/\mu).$$

One finds an additional contribution to γ arising from the deviation of the system away from equilibrium. This contribution is easily estimated as

$$\gamma_{\text{nonequil}}(\mathbf{q}, \omega) = U\omega_0/\mu,$$

where U is a certain average displacement in energy of the electron states at the Fermi surface, away from their equilibrium values. This deviation from equilibrium we have assumed is small, so that we have $U \lesssim \omega_0$. It is important to note, and easy to verify, that the frequency dependence of γ is entirely contained in the equilibrium term which is simply proportional to ω . This is the case for the interesting range $|\omega| \ll \mu$.

Thus we conclude that both the dispersive and absorptive parts of $\Pi(\mathbf{q}, z)$ are very small. For nearly all purposes, we can completely neglect both. This result immediately justifies the neglect of the function $s(\mathbf{q}, z)$ which appears in Eq. (9). If z is allowed to approach a real positive frequency, the correction in Eq. (9a) can be estimated as being of order $[\Pi(q\omega)]^2/(\omega + \omega_0)$, which is completely negligible.

From Eq. (16) it follows that

$$r^{>}(\mathbf{q}, \omega) - r^{<}(\mathbf{q}, \omega) = \gamma(\mathbf{q}, \omega)/[(\omega - \omega_{\mathbf{q}})^2 + \frac{1}{4}\gamma^2], \quad (33)$$

which is strongly peaked as a function of frequency about $\omega_{\mathbf{q}}$. The peak has an area 2π . Thus, we are enabled to define the phonon distribution function as

$$N(\mathbf{q}; \mathbf{r}, t) = \int \frac{d\omega}{2\pi} r^{<}(\mathbf{q}, \omega; \mathbf{r}, t) \quad (34a)$$

and

$$N(\mathbf{q}; \mathbf{r}, t) + 1 = \int \frac{d\omega}{2\pi} r^{>}(\mathbf{q}, \omega; \mathbf{r}, t), \quad (34b)$$

where the integration is over the peaking of the integrand. Integrating Eq. (16) over the region of the peaking gives at once

$$\frac{\partial}{\partial t} N(\mathbf{q}; \mathbf{r}, t) + \nabla_{\mathbf{q}} \omega_{\mathbf{q}} \cdot \nabla_{\mathbf{r}} N(\mathbf{q}; \mathbf{r}, t) = \Pi^{<}(q\omega_{\mathbf{q}}) [N(\mathbf{q}) + 1] - \Pi^{<}(q\omega_{\mathbf{q}}) N(q). \quad (35)$$

We are able to drop the term $[\Pi^{<}, \text{Rer}]$, because $\Pi^{<}$ is smoothly varying at $\omega = \omega_{\mathbf{q}}$.

Equation (35) is the phonon Boltzmann equation. We shall require the form of $r^{>}, <(\mathbf{q}, \omega)$ for larger values of $\omega - \omega_{\mathbf{q}}$ later. It is evident from (33) that we expect these quantities to drop off as $(\omega - \omega_{\mathbf{q}})^{-2}$. Direct substitution shows that the relation

$$r^{>}, <(\mathbf{q}, \omega) = -\Pi^{>}, < \partial / \partial \omega (\text{Rer}) = \delta r^{>}, < \quad (36)$$

holds for large $\omega - \omega_{\mathbf{q}}$. The quantity Rer is

$$\text{Rer} = (\omega - \omega_{\mathbf{q}}) / [(\omega - \omega_{\mathbf{q}})^2 + \frac{1}{4}\gamma^2] \quad (37)$$

and can be approximated as $\mathcal{O}(1/(\omega-\omega_q))$ in view of the smallness of γ .

Thus, $r^>(\omega)$ for example, has approximately a Lorentzian shape, with tails dropping off as $(\omega-\omega_q)^{-2}$. There is some important structure in the tail, in the neighborhood of zero frequency, since $r^>$ must vanish exponentially as $e^{\beta\omega}$ for negative frequency. This structure depends on the deviation of the system from equilibrium. Only the response of the ions represented by the main peak of $r^><$ can truly be considered as a phonon response with a definite energy-momentum relation. The incoherent contribution of $\delta r^><$, which is rarely important at all, will turn out to give the ionic contribution to the electronic quasiparticles. We shall return to this point later, when we have seen the explicit role played by the line shape.

The fact that $A(\mathbf{k}, E)$ and $r^>-r^<$ are Lorentzian rather than true δ functions in the variables $\epsilon_{\mathbf{k}}$ and ω , respectively, is of no consequence in the calculation of those values of $\Sigma^><$ and $\Pi^><$ which enter the equations of motion, Eqs. (30) and (35). Thus we can immediately write,

$$\begin{aligned} \Sigma_{\sigma}^>(\hat{k}, E) = & 2\pi \int d^3q dE' \frac{d\Omega'}{4\pi} N_0(\hat{k}') [v(\mathbf{q})]^2 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} - \mathbf{q}) \\ & \times [1 - f_{\sigma}(\hat{k}', E')] \{ \delta(E - E' - \omega_q) [1 + N(\mathbf{q})] \\ & + \delta(E - E' + \omega_q) N(-\mathbf{q}) \} \quad (38) \end{aligned}$$

$$\begin{aligned} & \left[1 - \frac{\partial \Sigma_{\sigma}(\hat{k}, E)}{\partial E} \right] \frac{\partial f_{\sigma}(\hat{k}, E)}{\partial t} + \frac{\partial \Sigma_{\sigma}(\hat{k}, E)}{\partial t} \frac{\partial f_{\sigma}(\hat{k}, E)}{\partial E} + (\nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F}) \cdot \nabla_{\mathbf{r}} f_{\sigma}(\hat{k}, E) \\ & = -2\pi \int d^3q \frac{d\Omega'}{4\pi} dE' N_0 [v(\mathbf{q})]^2 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} - \mathbf{q}) \\ & \quad \times \{ \delta(E - E' - \omega_q) ([1 + N(\mathbf{q})] f_{\sigma}(\hat{k}, E) [1 - f_{\sigma}(\hat{k}', E')] - N(\mathbf{q}) [1 - f_{\sigma}(\hat{k}, E)] f_{\sigma}(\hat{k}', E')) \\ & \quad + \delta(E - E' + \omega_q) ([N(-\mathbf{q})] f_{\sigma}(\hat{k}, E) [1 - f_{\sigma}(\hat{k}', E')] - [1 + N(-\mathbf{q})] f_{\sigma}(\hat{k}', E') [1 - f_{\sigma}(\hat{k}, E)]) \}, \quad (41) \end{aligned}$$

$$\begin{aligned} & [(\partial/\partial t) + \nabla_{\mathbf{q}} \omega_{\mathbf{q}} \cdot \nabla_{\mathbf{r}}] N(\mathbf{q}) \\ & = -(2\pi)^4 \sum_{\sigma} \int \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} N_0(\hat{k}) N_0(\hat{k}') [v(\mathbf{q})]^2 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} - \mathbf{q}) \\ & \quad \times \{ N(\mathbf{q}) [1 - f_{\sigma}(\hat{k}, E)] f_{\sigma}(\mathbf{k}', E - \omega_q) - [1 + N(\mathbf{q})] f_{\sigma}(\hat{k}, E) [1 - f_{\sigma}(\hat{k}', E - \omega_q)] \}. \quad (42) \end{aligned}$$

Equations (41) and (42) represent one form of our basic transport equations. Notice that, except for the terms $\partial \Sigma/\partial t$ and $\partial \Sigma/\partial E$ these equations are identical in form with the standard weak-coupling transport equations.¹² The only difference is that where the kinetic-energy variable $\epsilon_{\mathbf{k}}$ appears in the weak-coupling equations the total energy variable E now appears. If the distribution function depends upon space but not upon time, the transport equations are identical to those of the weak-coupling theory.

¹² J. M. Ziman *Electrons and Phonons* (Oxford University Press, New York, 1962).

and

$$\begin{aligned} \Sigma_{\sigma}^<(\hat{k}, E) = & 2\pi \int d^3q dE' \frac{d\Omega'}{4\pi} N_0(\hat{k}') [v(\mathbf{q})]^2 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} + \mathbf{q}) \\ & \times f_{\sigma}(\hat{k}', E') \{ \delta(E - E' + \omega_q) [1 + N(\mathbf{q})] \\ & + \delta(E - E' - \omega_q) N(-\mathbf{q}) \}. \quad (39) \end{aligned}$$

The self-energy is

$$\begin{aligned} \Sigma_{\sigma}(\hat{k}, E) = & \mathcal{O} \int d^3q dE' \frac{d\Omega'}{4\pi} N_0 [v(\mathbf{q})]^2 \delta^3(\mathbf{k}_F - \mathbf{k}_{F'} - \mathbf{q}) \\ & \times \left\{ \frac{f_{\sigma}(\hat{k}', E' + \omega_q) - f_{\sigma}(\hat{k}', E' - \omega_q)}{E - E'} \right\}. \quad (40) \end{aligned}$$

Terms proportional to $N(\mathbf{q})$ do not appear in (40) because these terms are proportional to

$$\mathcal{O} \int dE' (E - E')^{-1},$$

which vanishes.

If we now finally collect all our results, the transport equations for the quasiparticle distribution function $N(\mathbf{q})$ and f become

IV. TRANSPORT EQUATIONS IN THE LANDAU FORM

We shall refer to Eqs. (41) and (42) as the energy-space or (E -space) form of the transport equations. This form does not look like the Landau equations for a Fermi liquid because Landau works with the momentum variable \mathbf{k} rather than E . In this section, we shall find equations which look similar to Landau's by transforming from E space into \mathbf{k} space.

In order to transform into \mathbf{k} space, we begin with a definition of the quasiparticle energy of the Landau theory

$$E_{\sigma}(\mathbf{k}; \mathbf{r}, t) = \epsilon_{\mathbf{k}} + \Sigma_{\sigma}(E(\mathbf{k}; \mathbf{r}, t), \hat{k}; \mathbf{r}, t). \quad (43)$$

We can *define* a new variable

$$\epsilon_\sigma(\hat{k}, E; \mathbf{r}, t) = E_\sigma - \Sigma_\sigma(E, \hat{k}; \mathbf{r}, t), \quad (44)$$

which can be used as a basic variable instead of E . This variable clearly has the significance of the local kinetic energy of a quasiparticle with total energy E .

For the remainder of this section, we assume that for each $\epsilon_{\mathbf{k}}$ there exists a unique $E(\mathbf{k}; \mathbf{r}, t)$ which solves Eq. (43). This uniqueness is known to fail in several cases of interest.¹³

A \mathbf{k} -space distribution function can be defined as

$$n_\sigma(\mathbf{k}; \mathbf{r}, t) = f_\sigma(E, \hat{k}; \mathbf{r}, t) \Big|_{E=E(\mathbf{k}, \mathbf{r}, t)}. \quad (45)$$

The transport equations can now be transformed into \mathbf{k} space. However, from (43)

$$dE_\sigma(\mathbf{k}; \mathbf{r}, t) = Z_\sigma(\mathbf{k}; \mathbf{r}, t) \times [d\epsilon_{\mathbf{k}} + d\Sigma_\sigma(E, \hat{k}; \mathbf{r}, t) \Big|_{E=E(\mathbf{k}, \mathbf{r}, t)}], \quad (46)$$

where $d\Sigma$ is to be computed at fixed E and

$$Z_\sigma(\mathbf{k}; \mathbf{r}, t) = \left[1 - \frac{\partial \Sigma_\sigma}{\partial E}(E, \hat{k}; \mathbf{r}, t) \Big|_{E=E(\mathbf{k}, \mathbf{r}, t)} \right]^{-1}. \quad (47)$$

This $Z(k)$ is called the wave function renormalization constant. From (46) it follows that

$$\nabla_{\mathbf{r}} E(\mathbf{k}) = Z(\mathbf{k}) [\nabla_{\mathbf{r}} \Sigma(\mathbf{k}, E)]_{E=E(\mathbf{k})}, \quad (48a)$$

$$\frac{\partial}{\partial t} E(\mathbf{k}) = Z(\mathbf{k}) [(\partial/\partial t) \Sigma(\hat{k}, E)]_{E=E(\mathbf{k})}, \quad (48b)$$

$$\nabla_{\mathbf{k}} E(\mathbf{k}) = Z(\mathbf{k}) \nabla_{\mathbf{k}} \epsilon_{\mathbf{k}}. \quad (48c)$$

The last relation follows because $\nabla_{\mathbf{k}} \Sigma(\hat{k}, E)$ is negligibly small. Equations (48) may be used to rewrite the transport equations, since

$$dn(\mathbf{k}) = [df(\hat{k}, E) + (\partial f/\partial E)(\hat{k}, E) dE(\hat{k})], \quad (49)$$

where $df(\hat{k}, E)$ is to be taken at fixed E . After some algebra we discover that the electronic transport equation, Eq. (41), may be expressed in \mathbf{k} space as

$$\begin{aligned} [(\partial/\partial t) + \nabla_{\mathbf{k}} E(\mathbf{k}) \cdot \nabla_{\mathbf{r}} - \nabla_{\mathbf{r}} E(\mathbf{k}) \cdot \nabla_{\mathbf{k}}] n(\mathbf{k}) \\ = -Z(\mathbf{k}) [\Sigma^>(\hat{k}, E(\mathbf{k})) n(\mathbf{k}) \\ - \Sigma^<(\hat{k}, E(\mathbf{k})) [1 - n(\mathbf{k})]], \end{aligned} \quad (50)$$

which is exactly the Landau form of the transport equation. Notice that we have not assumed that Γ is small compared to ω_0 or β^{-1} in the derivation of this equation. Therefore, we can conclude that the Landau transport equation is valid for all temperatures much lower than the degeneracy temperature. This is a much wider range of validity than we might have guessed *a priori*.

To complete the transformation into \mathbf{k} space, we rewrite $\Sigma^>$, $\Sigma^<$, $\Pi^>$, and $\Pi^<$ in terms of \mathbf{k} -space variables. From (48c)

$$\int \frac{d\Omega}{4\pi} dEN_0(\hat{k}) = \int d\epsilon_{\mathbf{k}} \frac{d\Omega}{4\pi} N_0 \frac{dE(\mathbf{k})}{d\epsilon_{\mathbf{k}}} = \int \frac{d^3 k}{(2\pi)^3} Z(\mathbf{k}). \quad (51)$$

Therefore, the collision terms in the Landau-Boltzmann equation may be rewritten with the aid of (41) and (51) as

$$\begin{aligned} Z(\mathbf{k}) \Sigma^>(\mathbf{k}, E(\mathbf{k})) &= (2\pi)^{-2} \int d^3 q d^3 k' [v(\mathbf{q})]^2 Z(\mathbf{k}) Z(\mathbf{k}') \delta^3(\mathbf{k} - \mathbf{k}' - \mathbf{q}) \\ &\quad \times [1 - n(\mathbf{k}')] \{ \delta(E(\mathbf{k}) - E(\mathbf{k}') - \omega_{\mathbf{q}}) [N(\mathbf{q}) + 1] + \delta(E(\mathbf{k}) - E(\mathbf{k}') + \omega_{\mathbf{q}}) N(-\mathbf{q}) \}, \end{aligned} \quad (52a)$$

$$\begin{aligned} Z(\mathbf{k}) \Sigma^<(\mathbf{k}, E(\mathbf{k})) &= (2\pi)^{-2} \int d^3 q d^3 k' [v(\mathbf{q})]^2 Z(\mathbf{k}) Z(\mathbf{k}') \delta^3(\mathbf{k} - \mathbf{k}' - \mathbf{q}) \\ &\quad \times n(\mathbf{k}') \{ \delta(E(\mathbf{k}) - E(\mathbf{k}') - \omega_{\mathbf{q}}) N(\mathbf{q}) + \delta(E(\mathbf{k}) - E(\mathbf{k}') + \omega_{\mathbf{q}}) [1 + N(-\mathbf{q})] \}, \end{aligned} \quad (52b)$$

These collision terms differ from those of the weak-coupling theory only in the appearance of the exact energies and of the extra factor $Z(\mathbf{k})Z(\mathbf{k}')$ in the scattering matrix element. In the weak-coupling theory this factor is replaced by unity.

In the transformation from E space to \mathbf{k} space, the only part of the phonon-transport equation, Eq. (42), that is modified is the collision terms $\Pi^>^<$. These

¹³ If the inverse to Eq. (44) is not unique, we must define several *branches* to the quasiparticle spectrum, such that integration over $\epsilon_{\mathbf{k}}$ and summation over the branch index is equivalent to integration over E . The interpretation of the several branches is similar in spirit to the interpretation of the several poles found on the second Riemann sheet of $g(\mathbf{k}, Z)$ by Engelsberg and Schrieffer (Ref. 15).

become

$$\begin{aligned} \Pi^>(\mathbf{q}, \omega) &= \sum_{\sigma} \int \frac{d^3 k d^3 k'}{(2\pi)^2} [v(\mathbf{q})]^2 Z(\mathbf{k}) Z(\mathbf{k}') \delta^3(\mathbf{q} - \mathbf{k} + \mathbf{k}') \\ &\quad \times \delta(\omega_{\mathbf{q}} - E(\mathbf{k}) + E(\mathbf{k}')) n(\mathbf{k}) [1 - n(\mathbf{k}')]; \end{aligned} \quad (53a)$$

$$\begin{aligned} \Pi^<(\mathbf{q}, \omega) &= \sum_{\sigma} \int \frac{d^3 k d^3 k'}{(2\pi)^2} [v(\mathbf{q})]^2 Z(\mathbf{k}) Z(\mathbf{k}') \delta^3(\mathbf{q} - \mathbf{k}' + \mathbf{k}) \\ &\quad \times \delta(\omega_{\mathbf{q}} - E(\mathbf{k}') + E(\mathbf{k})) n(\mathbf{k}') [1 - n(\mathbf{k})]. \end{aligned} \quad (53b)$$

Again the only change from the weak-coupling limit is in the modification of the energies and in the appearance of wave function renormalization constants.

All of the results of this section could have been derived by making the (unjustified) quasiparticle approximation¹⁴

$$\begin{aligned} g^>(\mathbf{k}, E) &= 2\pi\delta(\omega - E(\mathbf{k}))Z(\mathbf{k})[1 - n(\mathbf{k})], \\ g^<(\mathbf{k}, E) &= 2\pi\delta(\omega - E(\mathbf{k}))Z(\mathbf{k})n(\mathbf{k}) \end{aligned} \quad (54)$$

in the basic transport Eqs. (15) and (16).

Equations (54) are wrong because $A(k, E)$ has, when $\beta^{-1} \sim \omega_0$, considerable structure which is not reflected in the δ functions. Nonetheless, the quasiparticle theory exists despite the failure of Eq. (54).

V. CONSERVED QUANTITIES AND CONSERVATION LAWS

Since the Landau transport equation has been verified for the electron-phonon system in a metal, we might expect that Landau's identification of the densities and currents of conserved quantities might well be correct. According to Landau, the density of electrons with spin σ should be

$$n_\sigma(\mathbf{r}, t) = (2\pi)^{-3} \int d^3k n_\sigma(\mathbf{k}; \mathbf{r}, t), \quad (55a)$$

while the particle current for spin σ is

$$\mathbf{j}_\sigma(\mathbf{r}, t) = (2\pi)^{-3} \int d^3k [\nabla_{\mathbf{k}} E_\sigma(\mathbf{k}; \mathbf{r}, t)] n_\sigma(\mathbf{k}; \mathbf{r}, t). \quad (55b)$$

The time derivative of the energy density is

$$\begin{aligned} \frac{\partial}{\partial t} \epsilon(\mathbf{r}, t) &= \sum_\sigma \int \frac{d^3k}{(2\pi)^3} E_\sigma(\mathbf{k}; \mathbf{r}, t) \frac{\partial}{\partial t} n_\sigma(\mathbf{k}; \mathbf{r}, t) \\ &+ \frac{\partial}{\partial t} \int \frac{d^3q}{(2\pi)^3} \omega_q N(\mathbf{q}; \mathbf{r}, t) \end{aligned} \quad (55c)$$

with energy current

$$\begin{aligned} \mathbf{j}_\epsilon(\mathbf{r}, t) &= \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \left\{ \nabla_{\mathbf{k}} \frac{1}{2} [E_\sigma(\mathbf{k}; \mathbf{r}, t)]^2 \right\} n_\sigma(\mathbf{k}; \mathbf{r}, t) \\ &+ \int \frac{d^3q}{(2\pi)^3} \frac{1}{2} [\nabla_{\mathbf{q}} \omega_q^2] N(\mathbf{q}; \mathbf{r}, t). \end{aligned} \quad (55d)$$

The "momentum" density is

$$\begin{aligned} \mathbf{G}(\mathbf{r}, t) &= (2\pi)^{-3} \sum_\sigma \int d^3k \mathbf{k} n_\sigma(\mathbf{k}; \mathbf{r}, t) \\ &+ (2\pi)^{-3} \int d^3q \mathbf{q} N(\mathbf{q}; \mathbf{r}, t) \end{aligned} \quad (55e)$$

and the divergence of the stress tensor is

$$\begin{aligned} \nabla \cdot \mathfrak{T}(\mathbf{r}, t) &= \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \mathbf{k} [\nabla_{\mathbf{k}} E_\sigma(\mathbf{k}; \mathbf{r}, t) \cdot \nabla_{\mathbf{r}} n_\sigma(\mathbf{k}; \mathbf{r}, t) \\ &- \nabla_{\mathbf{r}} E_\sigma(\mathbf{k}; \mathbf{r}, t) \cdot \nabla_{\mathbf{k}} n_\sigma(\mathbf{k}; \mathbf{r}, t)] \\ &+ \int \frac{d^3q}{(2\pi)^3} \mathbf{q} \nabla_{\mathbf{q}} \omega_q \cdot \nabla_{\mathbf{r}} N(\mathbf{q}; \mathbf{r}, t). \end{aligned} \quad (55f)$$

We shall see that Eqs. (55) are, in fact, quite correct in the model which we have considered. In the more general model which is applicable to real metals, the concept of "momentum" density loses much of its meaning, since "momentum" is not conserved when umklapp processes play an important role. We shall specialize the Hamiltonian (1) still further, however, in order to bring out the relationship of the present theory of the electron-phonon system, with the ordinary Landau theory in which phonons are not involved. We shall assume spherical energy surfaces, i.e., we shall take $\epsilon_{\mathbf{k}} = k^2/2m$.

With this simplification, there is one result that we might want to take over from the Landau theory which is not, in fact, permissible. We might be tempted to identify the electronic mass flow current $m \sum_\sigma \mathbf{j}_\sigma(\mathbf{r}, t)$ with the electron quasiparticle part of the momentum current by writing

$$m \sum_\sigma \int \frac{d^3k}{(2\pi)^3} [\nabla_{\mathbf{k}} E_\sigma(\mathbf{k})] n_\sigma(\mathbf{k}) = \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \mathbf{k} n_\sigma(\mathbf{k}). \quad (56)$$

Equation (56) is false; we shall see this in detail below.

As a first step in verifying the identifications (55), we verify that the densities and currents defined in this manner satisfy the standard conservation laws.

These conservation laws follow from the integral invariances of the Boltzmann equation. By using the expansions (52) and (53) for $\Sigma^{>}, <$ and $\Pi^{>}, <$ we find the following integral invariants

$$\begin{aligned} \sum_\sigma \int \frac{d^3k}{(2\pi)^3} \left[\begin{array}{c} \delta_{\sigma, \sigma_0} \\ E_\sigma(\mathbf{k}) \\ \mathbf{k} \end{array} \right] Z_\sigma(\mathbf{k}) \{ \Sigma_\sigma^{>}(\mathbf{k}, E_\sigma) n_\sigma(\mathbf{k}) - \Sigma_\sigma^{<}(\mathbf{k}, E_\sigma) [1 - n_\sigma(\mathbf{k})] \} \\ + \int \frac{d^3q}{(2\pi)^3} \left[\begin{array}{c} 0 \\ \omega_q \\ \mathbf{q} \end{array} \right] \{ \Pi^{>}(\mathbf{q}, \omega_q) N(\mathbf{q}) - \Pi^{<}(\mathbf{q}, \omega_q) [1 + N(\mathbf{q})] \} = \left[\begin{array}{c} 0 \\ 0 \\ \mathbf{0} \end{array} \right], \end{aligned} \quad (57)$$

¹⁴ Reference 2 includes detailed discussions of the application of quasiparticle approximations like (67) for the situations in which these are valid.

which we have written in a vector notation. Equation (57) may be combined with the phonon Boltzmann equation (42) and the electronic k -space Boltzmann equation (50) to give the required conservation laws:

$$\partial n_\sigma(\mathbf{r},t)/\partial t + \nabla \cdot \mathbf{j}_\sigma(\mathbf{r},t) = 0, \quad (58a)$$

$$\partial \epsilon(\mathbf{r},t)/\partial t + \nabla \cdot \mathbf{j}_\epsilon(\mathbf{r},t) = 0, \quad (58b)$$

$$\partial \mathbf{G}(\mathbf{r},t)/\partial t + \nabla \cdot \mathfrak{Z}(\mathbf{r},t) = 0. \quad (58c)$$

Thus, we see that the identifications (55) are consistent in the sense that the quantities thus defined satisfy the right conservation laws. One cannot yet conclude that these identifications are correct—one can only say that if the densities are correctly identified, then the currents certainly are.

To get a more solid verification of (55) we must compare these expressions with more exact expressions for the densities and currents of conserved operators. To make this comparison, it is convenient to work in E space. The transformation of Eqs. (55) to E space is easily effected with the aid of Eqs. (48) and (49). The results of this transformation are

$$\frac{\partial n_\sigma}{\partial t} = \frac{\partial}{\partial t} \int \frac{d\Omega}{4\pi} \frac{dEN_0(\hat{k})}{4\pi} \left[1 - \frac{\partial}{\partial E} \Sigma_\sigma(\hat{k}, E) \right] f_\sigma(\hat{k}, E); \quad (59a)$$

$$\mathbf{j}_\sigma = \int \frac{d\Omega}{4\pi} \frac{dEN_0}{4\pi} \nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F} f_\sigma(\hat{k}, E); \quad (59b)$$

$$\frac{\partial \epsilon}{\partial t} = \sum_\sigma \int \frac{d\Omega}{4\pi} \frac{dEN_0}{4\pi} \left\{ \left[1 - \frac{\partial}{\partial E} \Sigma_\sigma(\hat{k}, E) \right] \left[\frac{\partial}{\partial t} f_\sigma(\hat{k}, E) \right] + \left[\frac{\partial}{\partial t} \Sigma_\sigma(\hat{k}, E) \right] \left[\frac{\partial}{\partial E} f_\sigma(\hat{k}, E) \right] \right\} E + \frac{\partial}{\partial t} \int \frac{d^3q}{(2\pi)^3} \omega_q N(q); \quad (59c)$$

$$\mathbf{j}_\epsilon = \sum_\sigma \int \frac{d\Omega}{4\pi} \frac{dEN_0 E}{4\pi} (\nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F}) f_\sigma(\hat{k}, E) + \int \frac{d^3q}{(2\pi)^3} \frac{1}{2} [\nabla_q \omega_q^2] N(q); \quad (59d)$$

$$\mathbf{G} = \sum_\sigma \int \frac{d\Omega}{4\pi} \frac{dEN_0 \mathbf{k}_F}{4\pi} \left[1 - \frac{\partial \Sigma_\sigma(\hat{k}, E)}{\partial E} \right] f_\sigma(\hat{k}, E) + \int \frac{d^3q}{(2\pi)^3} \mathbf{q} N(q); \quad (59e)$$

$$\mathfrak{Z}_{ij} = \sum_\sigma \int \frac{d\Omega}{4\pi} \frac{dE(\mathbf{k}_F)_i}{4\pi} (\nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F})_j f_\sigma(\hat{k}, E) + \int \frac{d^3q}{(2\pi)^3} q_i (\nabla_q \omega_q)_j N(q). \quad (59f)$$

Equations (59) are convenient alternative forms for the densities and currents. Notice that all three currents, \mathbf{j} , \mathbf{j}_ϵ , and \mathfrak{Z} take exactly the same form as in the weak-coupling theory except for the replacement of the dummy kinetic-energy variable ϵ of the weak-coupling theory by the dummy variable E in the exact theory. This form invariance will be important below where we compare the calculations of transport coefficients in the weak coupling and exact theories.

Equations (59) should be compared with the exact expressions for the densities and currents. Let us begin with the particle current because that is the simplest case. The exact expression is

$$\mathbf{j}_\sigma = \int \frac{d^3k}{(2\pi)^3} \frac{dE}{2\pi} \frac{\mathbf{k}}{m} - g_\sigma^<(\mathbf{k}, E), \quad (60)$$

which may be written in terms of $f(\hat{k}, E)$ with the aid of (24) and (26) as

$$\mathbf{j}_\sigma = \int \frac{d\Omega}{4\pi} \frac{\mathbf{k}_F}{4\pi} \frac{dE}{m} f_\sigma(\hat{k}, E).$$

In this way, the correctness of Eqs. (59b) and (55b) is verified. In particular, we may conclude that the number of electronic quasiparticles is the same as the number of electrons, and the charge of a quasiparticle is the electronic charge.

From the exact theory, the time derivative of the density is

$$\begin{aligned} \frac{\partial n_\sigma}{\partial t} &= \frac{\partial}{\partial t} \int \frac{d^3k}{(2\pi)^3} \frac{dE}{2\pi} g_\sigma^<(\mathbf{k}, E) \\ &= N_0 \int \frac{d\Omega}{4\pi} \frac{\partial}{\partial E} f_\sigma(\hat{k}, E). \end{aligned} \quad (61)$$

Equation (61) does not look the same as (59a). The difference between these is the term

$$\frac{\partial}{\partial t} \left\{ \int \frac{d\Omega}{4\pi} \frac{dEN_0}{4\pi} \frac{\partial \Sigma_\sigma(\hat{k}, E)}{\partial E} f_\sigma(\hat{k}, E) \right\}. \quad (62)$$

However, Eq. (40) for Σ may be used to verify the fact

that expression (62) vanishes. Therefore, Eqs. (55a), (59a), and (61a) all give equally valid expressions for $n(\mathbf{r}, t)$.

Next consider the "momentum" density which is exactly

$$\mathbf{G}(\mathbf{r}, t) = \sum_{\sigma} (2\pi)^{-4} \int d^3k dE \mathbf{k} g^{<}(\mathbf{k}, E) + (2\pi)^{-4} \int d^3q d\omega \mathbf{q} r^{<}(\mathbf{q}, \omega).$$

Equations (34), (36), and (60) enable us to simplify this as

$$\mathbf{G}(\mathbf{r}, t) = \sum_{\sigma} \int \frac{d\Omega}{4\pi} dEN_0 \mathbf{k}_F f(\hat{\mathbf{k}}, E; \mathbf{r}, t) + \int \frac{d^3q}{(2\pi)^3} \mathbf{q} N(\mathbf{q}; \mathbf{r}, t) + \int \frac{d^3q}{(2\pi)^3} \mathbf{q} \mathcal{P} \int \frac{d\omega}{2\pi} \frac{1}{\omega - \omega_q} \frac{\partial \Pi^{<}}{\partial \omega}. \quad (63)$$

The last term is the contribution of the structure of the line shape of $r^{>}(\mathbf{q}, \omega)$. The principal value has the effect of eliminating the contribution of this term in the immediate neighborhood of $\omega = \omega_q$, which contribution is taken into account by the second term.

We may substitute the identity

$$-\mathcal{P} \int \frac{d^3q d\omega}{(2\pi)^4} \frac{\mathbf{q}}{\omega - \omega_q} \frac{\partial \Pi^{<}}{\partial \omega}(\mathbf{q}, \omega) = \sum_{\sigma} \int \frac{d\Omega}{4\pi} dEN_0 f_{\sigma}(\hat{\mathbf{k}}, E) \mathbf{k}_F \frac{\partial \Sigma_{\sigma}}{\partial E}(\hat{\mathbf{k}}, E) \quad (64)$$

into (63) and immediately verify that the identification (59e) is correct. Equation (64) follows from Eqs. (28) and (40) after a few manipulations.

We can now understand why Eq. (56) is not true. This equation would set the electronic mass flow current equal to the part of the momentum density which comes from electronic quasiparticles. However, the quasiparticle is part of the time a combination of electron and ion motion, so that the quasiparticle momentum is partially electronic, and partially ionic. In the usual Landau theory, the quasiparticle momentum belongs partially to one electron and partially to its surrounding cloud of correlated electrons, but all of the momentum is electronic.

The following interpretation can be given to the terms of Eq. (63). The first term is the electronic mass flow. The second term is the part of the ionic momentum that we associate with the true phonons which have the dispersion relation ω_q . The final term is the part of the ionic momentum which cannot be associated with the phonons, but which we have been able to incorporate as part of the quasiparticle momentum.

The great virtue of the Landau theory is that it manages to associate all the necessary physical quanti-

ties with the quasiparticles, in this case, with the electrons and phonons. Because of the definite energy-momentum relations, calculations are simplified and made more intuitive. There are no terms in the Landau theory, besides the collision terms, which are not associated definitely with one or the other type of quasiparticles. In the straightforward Green's function theory, on the other hand, contributions like the last term of (63) are commonplace. These off-energy-shell, away-from-Fermi-surface contributions are difficult to deal with.

One may ask why it is that it is possible to lump such quantities into the quasiparticle terms. Why are we able to find such miraculous identities as Eqs. (62) and (64)? The answer seems to lie in the existence of the conservation laws. On the one hand the Green's function theory satisfies the conservation laws whether or not we take into account the peaking of the distribution functions $g^{<}$ and $r^{<}$. On the other hand, we have seen that the areas under these peaks, namely, $f(\hat{\mathbf{k}}, E)$ and $N(\mathbf{q})$, must also satisfy the conservation equations. Since it is virtually impossible that two distinct sets of "additive" conserved quantities exist,¹⁵ there must be identities connecting the two forms of the expressions for the conserved quantities.

Since we have verified the Landau-theory expression for the momentum density, it follows from the conservation law that Eqs. (55f) and (59f) for the stress tensor must be correct.

To verify the Landau-like expressions for the energy density and energy current, we must employ arguments rather similar to those above, which unfortunately involve much more algebra. We shall not present these arguments here but reserve them for an Appendix.

VI. EVALUATION OF PHYSICAL QUANTITIES

We now have two forms of the electronic transport theory: a momentum space form which looks essentially identical to that of the Landau theory, and an energy space form which is somewhat simpler and easier to use. Despite the greater simplicity of the energy space results, it is instructive to study in detail the result of applying the standard Landau approach to our system.

As we have just noted, there is one major point of difference between the Fermi liquid case and the electron-phonon system: the failure of Eq. (56) which results from the contribution of $\delta r^{<}$ to the momentum density. This failure in turn invalidates two important results of the Landau Fermi liquid theory in our case:

$$\frac{m}{m^*} = 1 - \sum_{\sigma, \sigma'} N_0 \int \frac{d\Omega'}{4\pi} f_{\sigma, \sigma'}(\mathbf{k}_F \cdot \mathbf{k}_{F'}) \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}' / 2 \quad (65)$$

and also the relationship of the chemical potential to

¹⁵ L. D. Landau and E. M. Lifshitz, *Statistical Physics* (Addison-Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), p. 11.

electronic density,

$$\frac{\partial n}{\partial \mu} = \frac{\pi^2}{mk_0} + \sum_{\sigma, \sigma'} (4\pi)^{-1} \int d\Omega' f_{\sigma, \sigma'}(\mathbf{k}_F; \mathbf{k}_{F'}) \times (1 - \hat{\mathbf{k}} \cdot \hat{\mathbf{k}}')/4, \quad (66)$$

where $f_{\sigma, \sigma'}(\hat{\mathbf{k}}_F; \hat{\mathbf{k}}_{F'})$ is defined by¹⁶

$$f_{\sigma, \sigma'}(\hat{\mathbf{k}}; \hat{\mathbf{k}}') = \delta E_{\sigma}(\mathbf{k}) / \delta n_{\sigma'}(\mathbf{k}'). \quad (67)$$

To see how these expressions must be modified in the electron-phonon case we calculate $\partial n / \partial \mu$ by using Eqs. (55a) and (61) to find the response to a slow variation of the chemical potential with time. From (63),

$$\frac{\partial n}{\partial t} = \frac{\partial n}{\partial \mu} \frac{\partial \mu}{\partial t} = 2 \int \frac{d\Omega}{4\pi} dE N_0 \frac{\partial f(\hat{\mathbf{k}}, E)}{\partial t}.$$

However, in equilibrium,

$$f(\hat{\mathbf{k}}, E) = f_0(E) = [\exp^{\beta(E - \mu)} + 1]^{-1},$$

so that

$$\frac{\partial}{\partial t} f(\hat{\mathbf{k}}, E) = \frac{\partial f_0(E)}{\partial \mu} \frac{\partial \mu}{\partial t} = - \frac{\partial f_0(E)}{\partial E} \frac{\partial \mu}{\partial t}.$$

Thus, Eq. (61) implies that

$$\frac{\partial n}{\partial \mu} = 2 \int \frac{d\Omega}{4\pi} N_0 = \frac{mk_0}{\pi^2}, \quad (68)$$

which is exactly the same result as in the weak-coupling theory.¹⁷ For electron-phonon interactions, Eq. (68) replaces Eq. (66) of the Fermi liquid theory.

Equation (55a) may be used to find an alternative evaluation of $\partial n / \partial \mu$. From (55a), we have

$$\begin{aligned} \frac{\partial n}{\partial \mu} &= 2 \int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial \mu} f_0(E(\mathbf{k})) \\ &= -2 \int \frac{d^3k}{(2\pi)^3} \frac{\partial}{\partial E} f_0(E) \left[1 - \frac{\partial}{\partial \mu} E(\mathbf{k}) \right] \\ &= 2 \int d\epsilon k N_0 \frac{\partial f_0(E)}{\partial E} \left[-1 + \int \frac{d^3k'}{(2\pi)^3} f(\mathbf{k}; \mathbf{k}') \frac{\partial n(\mathbf{k}')}{\partial \mu} \right]. \end{aligned} \quad (69)$$

Since, in our case, $f_{\sigma, \sigma'}$ is diagonal, $f_{\sigma, \sigma'}(\mathbf{k}, \mathbf{k}') = \delta_{\sigma, \sigma'} \times f(\mathbf{k}, \mathbf{k}')$. At temperatures such that β^{-1} is much smaller than typical phonon energies we can follow Landau in making the replacement

$$\int \frac{d^3k'}{(2\pi)^3} \frac{\partial n(\mathbf{k}')}{\partial \mu} f(\mathbf{k}; \mathbf{k}') = \frac{1}{2} \frac{\partial n}{\partial \mu} \int \frac{\partial \Omega'}{\partial \mu} f(\mathbf{k}_F; \mathbf{k}_{F'})$$

¹⁶ One should notice that $f_{\sigma, \sigma'}(\mathbf{k}; \mathbf{k}')$ is diagonal in spin space in our case.

¹⁷ An identical argument leads to D. Simkin's result that the spin susceptibility is unaffected by electron-phonon interactions [thesis, University of Illinois, 1963 (unpublished)].

and

$$\int d\epsilon = \int dE \left(\frac{d\epsilon}{dE} \right) = \frac{m}{m^*} \int dE$$

to find

$$\frac{\partial n}{\partial \mu} = \frac{m^* k_0}{\pi^2} \left[1 - \frac{1}{2} \int \frac{d\Omega'}{4\pi} f(\mathbf{k}_F; \mathbf{k}_{F'}) \frac{\partial n}{\partial \mu} \right]. \quad (70)$$

A comparison of Eqs. (68) and (70) indicates that

$$\frac{m}{m^*} = 1 - \frac{1}{2} N_0 \int \frac{d\Omega'}{4\pi} f(\mathbf{k}_F; \mathbf{k}_{F'}). \quad (71)$$

For the case of the electron-phonon interaction, Eq. (71) may be used at low temperatures in place of Eq. (65) of the Fermi liquid theory.

The specific heat may be evaluated with the aid of Eqs. (55c) and (59c) if we interpret the time derivative in this equation to refer to a gradual change in the temperature T . It is natural to split the specific heat into two terms. The phonon contribution is

$$C^P = \frac{\partial}{\partial T} \int \frac{d^3q}{(2\pi)^3} \omega_q \frac{1}{e^{\beta\omega_q} - 1}. \quad (72)$$

Since ω_q is independent of temperature, this is exactly the usual weak-coupling phonon specific heat.

The electronic quasiparticle specific heat is given by

$$\begin{aligned} C^e &= 2 \int \frac{d\Omega}{4\pi} dE N_0 E \left\{ \left[1 + \frac{\partial \Sigma(E)}{\partial E} \right] \frac{\partial f_0(E)}{\partial T} \right. \\ &\quad \left. + \frac{\partial \Sigma(\hat{\mathbf{k}}, E)}{\partial T} \frac{\partial f_0(E)}{\partial E} \right\} \\ &= 2 \int \frac{d^3k}{(2\pi)^3} E(\mathbf{k}) \frac{\partial}{\partial T} f_0(E(\mathbf{k})). \end{aligned} \quad (73)$$

The second equality gives us

$$C^e = \frac{2}{3} \pi^2 (m^*/m) T N_0 \quad (74)$$

at temperatures far below the typical phonon temperatures. The first equality implies an expression for the electronic specific heat which is valid at all temperatures, namely,

$$\begin{aligned} C^e &= \frac{2}{3} \pi^2 T N_0 + 2 \frac{\mathcal{P}}{T} \int dE dE' \frac{d\Omega}{4\pi} \frac{d\Omega'}{4\pi} N_0(\hat{\mathbf{k}}) N_0(\hat{\mathbf{k}}') \\ &\quad \times \frac{\partial f_0(E)}{\partial E} \frac{\partial f_0(E')}{\partial E'} \left\{ [v(q)]^2 \frac{(E - E')^2}{E - E' + \omega_q} \right\}. \end{aligned} \quad (75)$$

It ought to be remarked that the electronic contribution to the specific heat can be distinguished experimentally from the ionic contribution only for temperatures much less or rather greater than the Debye temperature, since both the magnitude and temperature variation of the lattice specific heat overwhelms the

electronic contribution in the intermediate temperature range. In the low-temperature case, the result can be shown to agree with that found by Eliashberg.^{8,18} In the range above the Debye temperature, the result is given by (74), but with m^* set equal to m .

Finally, we consider some of the transport coefficients: the viscosity, the thermal conductivity and the spin-diffusion coefficient. These are usually defined by considering the local equilibrium situation in which the distribution functions are such that the collision terms in the Boltzmann equations vanish, and the distribution functions depend upon space but not upon time.¹⁹ In our case, the local equilibrium solutions are²⁰

$$\tilde{f}_\sigma(\mathbf{k}, E) = \{\exp(\beta(\mathbf{r})[E + \mathbf{k}_F \cdot \mathbf{v}(\mathbf{r}) - \mu_\sigma(\mathbf{r})]) + 1\}^{-1}, \quad (76a)$$

$$\tilde{N}(\mathbf{q}) = \{\exp(\beta(\mathbf{r})[\omega_q + \mathbf{q} \cdot \mathbf{v}(\mathbf{r})]) - 1\}^{-1}. \quad (76b)$$

Because of the spatial dependence of these terms they produce forcing terms on the Boltzmann equations (41) and (42) in that the left-hand sides of these equations become, respectively, $\nabla_{\mathbf{k}_F} \epsilon_{\mathbf{k}_F} \cdot \nabla \mathbf{r} \tilde{f}$ and $\nabla_q \omega_q \cdot \nabla \mathbf{r} \tilde{N}(\mathbf{q})$. The transport coefficients are then defined in terms of the currents which appear in response to these forcing terms. Thus, for example, the thermal conductivity κ is defined as the coefficient which connects the energy current and the gradient of the temperature

$$\mathbf{j}_\epsilon(\mathbf{r}) = -\kappa \nabla T(\mathbf{r}).$$

But, in this situation in which there is no time dependence, the Boltzmann equations (54) and (55), the expressions (73b), (73d) and (72f) for the currents are precisely identical in form to those which appear in the weak-coupling limit. The only change is the renaming of the variable of integration, $\epsilon \rightarrow E$. Thus the exact transport coefficients which emerge from the solutions to these exact equations must be the same as those which are found by solving the weak-coupling Boltzmann equations. We may conclude that the spin diffusion coefficient, the viscosity (which is related to the damping of low-frequency phonons), and the thermal conductivity are all predicted quite correctly by the standard weak-coupling theory. Other transport coefficients which are unaffected by the interaction are listed in the final section.

VII. CONCLUSIONS AND DISCUSSION

We have seen that for the model considered, the electron-phonon system can be described by the traditional Boltzmann equation, with corrections of the Landau type, and with matrix elements renormalized. Thus the concept of quasiparticles can be considerably extended, even into a region where an energy-momentum relation does not obviously exist. That a precise

energy-momentum dispersion relation can be defined at all rests upon the circumstances that the momentum dependence of many important functions is weak. In a certain sense, therefore, the quasiparticle states of this theory are states of definite excitation energy and definite position on the Fermi surface, but are smeared out in an unimportant way in momentum space about the Fermi surface.

The importance of this result is that it leads one to believe that there is nothing to worry about in the use of the Boltzmann equation throughout the entire temperature range in metals. With the Landau corrections, ordinary transport properties should be capable of description to a very high accuracy. These Landau corrections have, to our knowledge, been written down for the electron-phonon system for the first time, although the equations are exactly what an educated guesser might come up with.

However, there exists a simpler statement of the transport theory for this case than the Landau \mathbf{k} -space description. In E space, the transport equations have an extra simplicity resulting from the disappearance of matrix-element renormalizations. Many-body effects only appear in the time derivative terms in the transport equation.

This extra simplicity has enabled us here to show that the spin diffusivity, the thermal conductivity, and the viscosity are all correctly predicted by the standard weak-coupling theory in the sense that there are no many-body electron-phonon interaction corrections to these quantities. For just this same reason, the tunneling rate,⁴ the spin-lattice relaxation time,⁴ the dc electrical conductivity,²¹ the anomalous skin effect,²¹ and the de Haas-van Alphen effect,²² $\partial n/\partial \mu$ and the spin susceptibility¹⁷ are all unaffected by these many-body effects. In addition, it easily follows from our equations (generalized to include the effects of an external static electromagnetic field) that the thermopower and all the galvano-magneto-thermal coefficients are correctly calculated by weak-coupling theory.

On the other hand, it is known that the low-field cyclotron resonance frequency²³ and the specific heat^{8,18} are indeed modified by electron-phonon interaction effects. Thus, considerable progress has already been made in sorting out the effects of electron-phonon interactions in normal metals.

The question arises how well our simplified model reflects the properties of real metals. There are at least three important points. First, Coulomb effects have

²¹ S. Nakajima and M. Watabe, *Progr. Theoret. Phys. (Kyoto)* **29**, 341 (1963); see also, K. Baumann and J. Ranninger, *Ann. Phys. (N.Y.)* **20**, 157 (1962).

²² S. Nakajima and M. Watabe, *Progr. Theoret. Phys. (Kyoto)* **30**, 271 (1963).

²³ A calculation of the cyclotron resonance frequency is mentioned in Ref. 22. From our point of view this calculation can be understood by considering $\mathbf{F} = m\mathbf{a}$; i.e., $d\mathbf{G}/dt = -\mathbf{j} \times \mathbf{B}/c$. Since G has in it an electron-phonon interaction correction [Eq. (72e)] while \mathbf{j} does not [Eq. (72b)], the resonance frequency must be modified.

¹⁸ G. M. Eliashberg, *Zh. Eksperim. i Teor. Fiz.* **43**, 1005 (1962) [English transl.: *Soviet Phys.—JETP* **16**, 780 (1963)].

¹⁹ Chapman and Cowling, *Mathematical Theory of Non-Uniform Gases* (Cambridge University Press, New York, 1960), Chap. 7.

²⁰ Reference 3, Eq. (10.13).

to be included. It is known that with Coulomb effects alone one can obtain the Landau equations.² These strong effects lead to self-energy and vertex corrections, etc., which are weakly varying in *both* momentum and frequency. It seems highly plausible, therefore, that one can build the electron-phonon Landau theory on an underlying Coulomb quasiparticle theory. This combined theory, as well as the relationship of the present theory to graphical analysis, is presently being studied.

The same sort of remarks apply to the additional complications introduced by the presence of impurities, which are present in all real metals.²⁴

Thirdly, lattice effects have been glossed over. The most important of these is perhaps the existence of umklapp processes. There is no difficulty in extending the theory to take these into account although, if umklapp processes are present, the concept of crystal momentum density loses much of its meaning. The result is as expected; all that is needed is to include the umklapp processes in the collision and self-energy expressions. Some interesting problems can be anticipated if the structure right at the zone boundaries are important, or if there are usually small band splittings, caused by spin-orbit effects, say. In this case, the distribution function cannot be diagonalized in advance, but has to be regarded as a density matrix with a small number of dimensions.

Other extensions of the theory will be needed to take into account disturbances of high frequency (of order ω_0). Disturbances arising from weak magnetic fields, and slowly varying electric fields are easily included in the natural way. The response of the system to a high-frequency microwave field may not have a simple transport description, since even the energy levels become poorly defined. However, the problem might well be soluble, since Migdal's arguments still apply. On the other hand, there are arguments which indicate that the classical effects of a strong magnetic field (cyclotron frequency comparable with the Debye frequency) can be incorporated in the natural way into the Landau-Boltzmann theory. Thus, for example, the high-field magnetoresistance can be calculated on the weak-coupling model. These arguments will be presented in a future communication.

Finally, it is tempting to speculate that the normal fluid in superconductors might, after all, be susceptible to a nearly exact description of the Landau type.

ACKNOWLEDGMENTS

One of the authors (R. E. P.) wishes to express his appreciation to the General Research Board of the University of Maryland for a grant under which this work was initiated. A travel grant from the Fulbright Foundation is also gratefully acknowledged. He also wishes to thank Professor J. Leite Lopes for his generous hospitality at the Centro Brasileiro de Pesquisas

Fisicas. The criticism of Professor D. Falk in reading the manuscript was very helpful.

APPENDIX: VERIFICATION OF LANDAU-LIKE EXPRESSION FOR THE ENERGY DENSITY

In the main body of this paper we claimed that the equivalent expressions (55c) and (59c) defined the time derivative of the energy density. In this appendix we investigate this claim and establish its plausibility.

We begin by rewriting the Landau expression for the energy density by integrating Eq. (59c) by parts to obtain

$$\partial \epsilon^L / \partial t = \partial (\epsilon_1 + \epsilon_2) / \partial t + I + \partial \epsilon_p / \partial t, \quad (A1)$$

where

$$\epsilon_1 = \sum_{\sigma} (4\pi)^{-1} \int d\Omega dE N_0 E f_{\sigma}(\hat{k}, E), \quad (A2)$$

$$\epsilon_2 = - \sum_{\sigma} (4\pi)^{-1} \int d\Omega dE N_0 E \left(\frac{\partial \Sigma_{\sigma}}{\partial E} \right) f_{\sigma}(\hat{k}, E), \quad (A3)$$

$$\epsilon_p = (2\pi)^{-3} \int d^3q \omega_q N(\mathbf{q}), \quad (A4)$$

and

$$I = - \sum_{\sigma} (4\pi)^{-1} \int d\Omega dE f_{\sigma}(\hat{k}, E) \partial \Sigma_{\sigma} / \partial t. \quad (A5)$$

Notice that I does not even appear to be a total time derivative. If our identification of the time derivative of the energy density is to make any sense at all we must show that I is a total time derivative. If we use Eq. (53) for Σ we can show after a little manipulation that

$$I = \partial \epsilon_3 / \partial t, \quad (A6)$$

where

$$\epsilon_3 = - \frac{1}{2} (4\pi)^{-1} \sum_{\sigma} \int d\Omega dE N_0 \Sigma_{\sigma}(\hat{k}, E) f_{\sigma}(\hat{k}, E). \quad (A7)$$

Therefore, we can integrate Eq. (A1) to construct a quantity,

$$\epsilon^L = \epsilon_1 + \epsilon_2 + \epsilon_3 + \epsilon_p + C. \quad (A8)$$

Here, C is a constant of integration which is independent of time and which therefore makes no contributions to changes in ϵ^L .

The quantity ϵ^L is dimensionally an energy. It satisfies a local conservation law

$$\partial \epsilon^L / \partial t + \nabla \cdot \mathbf{j}_{\epsilon} = 0.$$

Since there exists only a limited number of conserved quantities, it is hardly conceivable that ϵ^L can be anything else but the energy density.

To make this point more firmly, we consider the expression for the energy density

$$\begin{aligned} \epsilon = \sum_{\sigma} (2\pi)^{-4} \int d^3k dE \frac{1}{2} (E + \epsilon_{\mathbf{k}}) g_{\sigma} <(\mathbf{k}, \epsilon) \\ + (2\pi)^{-4} \int d^3q d\omega \omega r <(\mathbf{q}, \omega), \quad (A9) \end{aligned}$$

which is exact when $S <$ is negligibly small.

²⁴ J. M. Langer, Phys. Rev. **120**, 714 (1960).

Unfortunately, we cannot easily evaluate expression (A9), in general, because we do not know $\int d\epsilon \epsilon g^<$. Consequently, we specialize to the case of local thermodynamic equilibrium²⁵ in which

$$g_{\sigma}^<(\mathbf{k}, E) = A_{\sigma}(\mathbf{k}, E) \tilde{f}_{\sigma}(\mathbf{k}, E), \quad (\text{A10})$$

where \tilde{f} is the local equilibrium distribution function of the form (88a) where we allow β , v , and μ_{σ} to be functions of r and t . The consideration of this case is sufficient for establishing the correctness of our formula (87) for the electronic specific heat. Since Eq. (22) implies that

$$\int d\epsilon (E - \epsilon - \Sigma_{\sigma}) A_{\sigma} = 0, \quad (\text{A11})$$

we may rewrite expression (A9) as

$$\begin{aligned} \epsilon &= \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \frac{dE}{2\pi} E g_{\sigma}^<(\hat{k}, E) \\ &\quad - \frac{1}{2} \sum_{\sigma} \int \frac{d^3k}{(2\pi)^3} \frac{dE}{2\pi} \Sigma_{\sigma}(\hat{k}, E) g_{\sigma}^<(\hat{k}, E) \\ &\quad + \int \frac{d^3q}{(2\pi)^3} \frac{d\omega}{2\pi} \omega r^<(q, \omega), \\ \epsilon &= \sum_{\sigma} (2\pi)^{-4} \int d^3k dE E g^<(\mathbf{k}, E) \\ &\quad - \frac{1}{2} \sum_{\sigma} (2\pi)^{-4} \int d^3k dE \Sigma(\hat{k}, E) g^<(\mathbf{k}, E) \\ &\quad + (2\pi)^{-4} \int d^3q d\omega \omega r^<(q, \omega). \quad (\text{A12}) \end{aligned}$$

²⁵ *Note added in proof.* One of us (R. E. P.) has succeeded in generalizing the proof of this Appendix to the general nonequilibrium case. This proof then permits us to conclude that Eqs. (55c) and (59c) are always correct expressions for $\partial\epsilon/\partial t$.

With the aid of the replacement

$$(2\pi)^{-3} \int d^3k \rightarrow (4\pi)^{-1} \int d\Omega d\epsilon N_0$$

and the use of the sum rule

$$(2\pi)^{-1} \int d\epsilon A(\mathbf{k}, E) = 1,$$

we can see that the first two terms in (A12) are, respectively, identical to ϵ_1 and ϵ_3 except for a time-independent constant of integration. Therefore, we discover that

$$\epsilon - \epsilon^L = (2\pi)^{-4} \int d^3q d\omega \omega r^<(q, \omega) - \epsilon_2 - \epsilon_p \quad (\text{A13})$$

when we neglect the irrelevant constant of integration.

Equation (36) enables us to express $r^<$ in terms of $N(\mathbf{q})$ and $\delta r^< = -[\partial(\text{Re}r)/\partial\omega]\Pi^<$. Therefore,

$$\epsilon - \epsilon^L = \int \frac{d^3q d\omega}{(2\pi)^4} \frac{1}{\omega - \omega_q} \frac{\partial}{\partial\omega} [\omega \Pi^<(\mathbf{q}, \omega)] - \epsilon_2. \quad (\text{A14})$$

A substitution of expression (28) for $\pi^<$ leads, after considerable manipulation, to the result that the right-hand side of (A14) vanishes.

Therefore, in local thermodynamic equilibrium, our heuristic identification of $\partial\epsilon/\partial t$ is identical with the exact expression for this quantity. If our identification of the energy density is wrong, then there must exist a new conserved quantity²⁰ whose density is $\epsilon - \epsilon^L$. This new quantity must vanish whenever the system is in local thermodynamic equilibrium. It seems impossible to us that such an object can exist, and consequently we take the Landau-like identification of the energy density to be correct.