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Generalization of the Boguliubov Method Applied to Mixtures of Bose-Einstein Particles*

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The method developed by Boguliubov for an imperfect Bose-Einstein gas is generalized in order to permit examination of a mixture of an arbitrary number of different species of bosons. The generalized method involves diagonalization of a matrix which has a Hermitian-like property by a unitary-like transformation in a space with an indefinite metric. When a neutral mixture of two types of charge bosons is examined, it is found that two types of elementary excitations exist; one having the energy-momentum dispersion relation associated with plasma oscillations at low momenta and another having the energy-momentum relationship characteristic of a free particle with a modified mass at very low and high momenta. Further investigation shows that the plasma-type excitation consists of an oscillation in charge density while the free-particle-like excitation consists of an oscillation in mass density.

INTRODUCTION

IN 1947, Boguliubov[1] developed a method which, when applied to an imperfect Bose-Einstein gas, yielded the first semiquantitative explanation of the phenomenon of superfluidity. The essence of the method was the bilinearization of the Hamiltonian by a reasonable approximation and the introduction of quasi-particles through a canonical transformation which diagonalized the Hamiltonian. These essential elements of the method have since been applied with some success to a wide range of problems.[2–12]

The Boguliubov method was developed in order to investigate the properties of a system of a single species of bosons interacting through a two-particle potential which is a function only of the interparticle distance. In order to permit the investigation of perhaps an even wider ranger of problems we consider here a generalization of the method to deal with systems consisting of several different species of bosons interacting through more general two-body potentials.

In the Boguliubov method the system is considered in second quantization with the one-particle states labeled by a quantum number k (such as the individual particle momenta) which is additively conserved in a two-body interaction. Basic to the approximation is the assumption that there exists one such one-particle state, say $k=0$, which is populated on the average by a large number of particles approximately equal to the total number of particles in the system. The average occupation number N^0 of this state is then treated as a c number, and the creation and destruction operators

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for particles in this state are also treated as c numbers, equal to $(N^0)^{1/2}$. The second assumption made in the treatment is the neglect of all interaction terms which are not of at least second order in the creation and destruction operators for particles in the $k=0$ state. The Hamiltonian is thereby reduced to a sum of expressions each of which is a quadratic form in the creation and destruction operators only in the states k and $-k$ for fixed k (with no interaction between different k values). Boguliubov then shows that it is possible to diagonalize the reduced Hamiltonian by a simple canonical transformation.

GENERALIZATION OF THE BOGULIUBOV METHOD

Let us use the symbol k to represent collectively those quantum numbers characterizing one-particle states which are additively conserved in a collision and the symbol α to represent collectively the remaining quantum numbers required to characterize a one-particle state. We require further that the spectrum of k be symmetric about $k=0$, that $k=0$ be in the spectrum, and that $k=0$ be for given α, that one-particle state which, in the absence of interaction, has the lowest energy. Thus, k may represent the linear momentum of a particle or the z component of angular momentum of a particle in an axially symmetric potential, while α may label a particle species or a component of spin momentum of a particle. Our subsequent nomenclature will be chosen to correspond to the case where k represents linear momentum and α represents particle species, but the extension to more general situations is usually obvious.

We shall be using a second quantized representation. We let $a_\alpha(k)$, $a_\alpha^+(k)$ represent, respectively, the destruction and creation operators for a particle in the state (k,α), and $t_\alpha(k)$ the energy of a single particle in this state. The fact that our particles are bosons is then expressed in the commutation relations

$$[a_\alpha(k),a_\beta(k')]=[a_\alpha^+(k),a_\beta^+(k')]=0,$$
$$[a_\alpha(k),a_\beta^+(k')]=\delta_{\alpha\beta}\delta_{kk'}. \qquad (1)$$

The interaction between two particles leads to collisions in which a pair of particles with quantum numbers $(k,\alpha;k',\beta)$ undergoes a transition to a state with quantum numbers $(k+K,\gamma;k'-K,\delta)$. We represent the matrix element of the interaction between such pairs of states by $V^{-1}\,g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)$. Here V represents the volume in the case of a gas of interacting particles so that g is independent of V (provided surface effects are ignored). We adopt the convention that the subscripts label incoming particles, the superscripts the outgoing particles, the first argument corresponds to the momentum of the particle labeled by the first subscript, the second argument the momentum of the particle labeled by the second subscript, and the last argument corresponds to the momentum transfer, e.g., $g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)$ is the matrix element for the interaction shown in Fig. 1. Symmetry considerations lead to the following relations between the matrix elements:

$$\begin{aligned}g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)&=g_{\beta\alpha}{}^{\delta\gamma}(k',k,-K)\\&=g_{\alpha\beta}{}^{\delta\gamma}(k,k',k'-k-K)\\&=g_{\beta\alpha}{}^{\gamma\delta}(k',k,k+K-k'). \qquad (2)\end{aligned}$$

Furthermore, Hermiticity requires

$$g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)=g_{\gamma\delta}{}^{\alpha\beta*}(k+K,k'-K,-K). \qquad (3)$$

The Hamiltonian in second quantization takes the form

$$\begin{aligned}H=&\sum_\alpha\sum_k t_\alpha(k)a_\alpha^+(k)a_\alpha(k)\\&+\frac{1}{2V}\sum_{\alpha,\beta,\gamma,\delta}\sum_{k,k',K}g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)a_\delta^+(k'-K)\\&\times a_\gamma^+(k+K)a_\beta(k')a_\alpha(k). \qquad (4)\end{aligned}$$

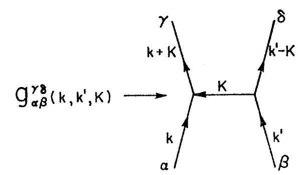
The generalization of the Boguliubov approximation required for a particular problem depends in part on the constraints imposed on the system (either internally through selection rules, or externally) with respect to the total number of particles present belonging to each species α. We shall assume here that such constraints take the form of fixing the total number of particles of each species:

$$\sum_k a_\alpha^+(k)a_\alpha(k)=N_\alpha=n_\alpha V, \qquad (5)$$

where N_α is large. In this case the Boguliubov approximation consists first in assuming that the state $k=0$ for each α is macroscopically occupied so that $a_\alpha(0)$, $a_\alpha^+(0)$ can be approximated by the numbers $(N_\alpha{}^0)^{1/2}$. $N_\alpha{}^0$ is the average number of α particles in the $k=0$ state. Secondly we drop those terms in the interaction part of the Hamiltonian which do not contain at least two creation or destruction operators for the state $k=0$. We are left with a single sum and for convenience we write

$$H=H_0+\sum_{k\neq0}H_k,$$
$$H_k=H_{-k}=\tfrac{1}{2}(h_k+h_{-k}),$$
$$\begin{aligned}h_k=&\sum_\alpha t_\alpha(k)a_\alpha^+(k)a_\alpha(k)\\&+\tfrac{1}{2}\sum_{\alpha,\beta}[(n_\alpha n_\beta)^{1/2}\{2g_{\alpha\beta}(k)a_\beta^+(k)a_\alpha(k)\\&+f_{\alpha\beta}(k)a_\beta^+(-k)a_\alpha^+(k)\\&+f_{\alpha\beta}^*(k)a_\beta(-k)a_\alpha(k)\}],\end{aligned}$$
$$H_0=\sum_\alpha t_\alpha(0)N_\alpha+\sum_{\alpha,\beta}(N_\beta N_\alpha n_\beta n_\alpha)^{1/2}\times[g_{\alpha\beta}(0)+f_{\alpha\beta}(0)]. \qquad (6)$$

FIG. 1. A pictorial representation of the matrix element $g_{\alpha\beta}{}^{\gamma\delta}(k,k',K)$.



In the above

$$g_{\alpha\beta}(k)=\sum_{\gamma,\delta}\left(\frac{n_\gamma n_\delta}{n_\alpha n_\beta}\right)^{1/2} g_{\gamma\alpha}{}^{\beta\delta}(0,k,k)\,, \tag{7}$$

$$f_{\alpha\beta}(k)=\sum_{\gamma,\delta}\left(\frac{n_\gamma n_\delta}{n_\alpha n_\beta}\right)^{1/2} g_{\gamma\delta}{}^{\alpha\beta}(0,0,k)\,, \tag{8}$$

where use has been made of the symmetry and Hermiticity relations (2) and (3).

The next task to be undertaken is the diagonalization by a canonical transformation of the operators H_k. To achieve this end it is convenient to introduce a new notation by introducing operators $X_i(k)$, $X_i^+(k)$, where the index i runs through twice as many values as does the index α. Thus, let the values taken by α be designated 1, 2, 3, $\cdots\nu$. We now define

$$X_i(k)=a_i(k)\,,\quad X_i^+(k)=a_i^+(k) \quad \text{for} \quad i=1,2,3\cdots\nu$$

and (9)

$$X_i(k)=a_{i-\nu}^+(-k)\,,\quad X_i^+(k)=a_{i-\nu}(-k) \quad \text{for} \quad i=\nu+1,\ \nu+2,\ \cdots 2\nu\,,$$

where the index i is defined modulo 2ν. Note that according to these definitions

$$X_{i+\nu}^+(-k)\equiv X_i(k)\,,\qquad X_{i+\nu}(-k)\equiv X_i^+(k)\,. \tag{10}$$

The commutation relations (1) then assume the form

$$[X_i(k),X_j(k)]=[X_i^+(k),X_j^+(k)]=0\,,\qquad [X_i(k),X_j^+(k)]=\beta_{ij}\,. \tag{11}$$

Here β_{ij} is a $2\nu\times 2\nu$ diagonal matrix whose elements are

$$\begin{aligned}\beta_{ij}&=0 &&\text{for}\quad i\neq j\\ &=1 &&\text{for}\quad i=j=1,2,3,\cdots\nu\\ &=-1 &&\text{for}\quad i=j=\nu+1,\ \nu+2,\ \cdots 2\nu.\end{aligned} \tag{12}$$

We now need not indicate the argument k of the X_i, X_i^+ operators for what immediately follows. The partial Hamiltonian H_k takes the following form in terms of the new operators.

$$H_k=\sum_{i,j,l} X_i^+\beta_{ij}\eta_{jl}X_l+E_k{}^0\,, \tag{13}$$

where

$$E_k{}^0=-\tfrac{1}{2}\sum_\alpha \{t_\alpha(-k)+n_\alpha g_{\alpha\alpha}(-k)\} \tag{14}$$

arises from the commutation of operators required to bring all X^+ operators standing to the left of X operators. The η_{jl} are defined by

$$\begin{aligned}\eta_{jl}&=\tfrac{1}{2}[t_j(k)\delta_{jl}+(n_jn_l)^{1/2}g_{jl}(k)] && j,\,l=1,2,\cdots\nu\\ &=-\tfrac{1}{2}[t_j(-k)\delta_{jl}+(n_jn_l)^{1/2}g_{jl}(-k)] && j,\,l=\nu+1,\ \nu+2,\ \cdots 2\nu\\ &=\tfrac{1}{2}(n_jn_l)^{1/2}f_{lj}(k) && j=1,2,\cdots\nu;\quad l=\nu+1,\ \nu+2,\ \cdots 2\nu\\ &=-\tfrac{1}{2}(n_jn_l)^{1/2}f_{jl}{}^*(k) && j=\nu+1,\ \nu+2,\ \cdots 2\nu;\quad l=1,2,\cdots\nu,\end{aligned} \tag{15}$$

with $t_{\alpha+\nu}$, $g_{\alpha+\nu,\ \beta+\nu}$, $n_{\alpha+\nu}$, etc., given by t_α, $g_{\alpha\beta}$, n_α, etc. It is important to note that if j and l are considered matrix indices, then the matrix η and the matrix β satisfy the relation

$$(\beta\eta)^+=\eta^+\beta=\beta\eta\,, \tag{16}$$

or

$$\eta^+=\beta\eta\beta^{-1}=\beta\eta\beta \quad (\text{since } \beta^{-1}=\beta)\,.$$

If $X_1,\cdots X_{2\nu}$ are considered as elements of a column matrix X and $X_1^+,\cdots X_{2\nu}^+$ as elements of a row matrix X^+, then H_k becomes in matrix notation

$$H_k=X^+\beta\eta X+E_k{}^0\,. \tag{17}$$

The required generalization of the Boguliubov canonical transformation consists now in introducing new operators Y_i related to the X_i by a linear transformation.

$$X=UY\quad X^+=Y^+U^+\,. \tag{18}$$

This transformation is to bring H_k to the form

$$H_k=Y^+\beta\lambda Y+E_k{}^0 \tag{19}$$

with λ a diagonal matrix. However, in order for this transformation to be canonical, it must leave invariant the commutation relations (11). In particular, this requires

$$[Y_i,Y_j^+]=\beta_{ij}\,, \tag{20}$$

and

$$[X_k,X_l^+]=\sum_{i,j}[U_{ki}Y_i,Y_j^+U_{jl}^+]=\sum_{i,j}U_{ki}\beta_{ij}U_{jl}^+=\beta_{kl} \tag{21}$$

or, in matrix form,

$$U\beta U^+=\beta\,. \tag{22}$$

Alternatively,

$$\beta U^+=U^{-1}\beta\,.$$

Taking the inverse of both sides, noting that $\beta^{-1}=\beta$, one obtains on a slight rearrangement

$$\beta=U^+\beta U\,. \tag{23}$$

Thus, it immediately follows that

$$X^+\beta X=Y^+U^+\beta UY=Y^+\beta Y\,.$$

This result can be given a geometric interpretation: We may regard $X^+\beta X$ as the scalar product of the vector X with itself; or more generally, define the scalar product of X with X' by

$$(X,X')=X^+\beta X'.$$

The associated linear-vector space then does not possess a positive-definite metric but, instead, the metric β. A linear transformation on the operators X_i which is canonical corresponds to a "pseudo-unitary" transformation U which, by (23), leaves the metric invariant. A matrix η satisfying (16) may be regarded as a "pseudo-Hermitian" matrix in this space in the sense that

$$\begin{aligned}(X,\eta X')=X^+\beta\eta X'=X^+\eta^+\beta X'\\ =(\eta X)^+\beta X'=(\eta X,X').\end{aligned} \quad (24)$$

Thus, the problem of bringing H_k to the form (19) is reduced to finding a pseudo-unitary matrix U which diagonalizes the pseudo-Hermitian matrix η.

$$U^+\beta\eta U=\beta\lambda \quad (\lambda \text{ diagonal}). \quad (25)$$

The symmetry properties of η, i.e.,

$$\eta_{ij}(k)=-n_{i+\nu,j+\nu}(-k),$$

and

$$n_{i,j+\nu}(k)=-n_{i+\nu,j}(-k) \quad i,\ j=i,\ \cdots\nu,$$

allow one to order the λ's in such a way that the Y's will be interrelated in the same way as the X's, i.e.,

$$\begin{aligned}Y_{i+\nu}^+(-k)\equiv Y_i(k),\\ Y_{i+\nu}(-k)\equiv Y_i^+(k).\end{aligned} \quad (26)$$

The U matrix will then have the property

$$\begin{aligned}U_{ij}&=U_{i+\nu,j+\nu}^*,\\ U_{i+\nu,j}&=U_{i,j+\nu}^+ \qquad \text{for} \quad i,\ j=1\cdots\nu.\end{aligned}$$

This choice is consistent with Eq. (23). Equation (25) can be rewritten in the forms

$$\begin{aligned}\beta U^+\beta\eta U&=\lambda,\\ U^{-1}\eta U&=\lambda,\\ \eta U&=U\lambda.\end{aligned} \quad (27)$$

Thus,

$$\sum_j \eta_{ij}U_{jk}=\sum_j \delta_{ij}U_{jk}\lambda_k=U_{ik}\lambda_k. \quad (28)$$

Thus, the columns of the matrix U (if it exists) are linearly independent and orthogonal eigenvectors of the matrix η, and the diagonal elements of λ are the associated eigenvalues. Thus, a solution to the problem exists and can be found if there exist 2ν linearly independent orthogonal eigenvectors of the matrix η. At present we have not been able to determine the conditions for the existence of such a set of eigenvectors.

If a solution exists one may then, using Eq. (26) or defining

$$A_\alpha(k)\equiv Y_\alpha(k)\equiv Y_{\alpha+\nu}^+(-k), \quad (29)$$

write the partial Hamiltonian (19) as

$$H_k=\sum_\alpha \{E_k^0-\lambda_{\alpha+\nu}+\lambda_\alpha A_\alpha^+(k)A_\alpha(k)\}.$$

A NEUTRAL GAS OF TWO SPECIES OF CHARGED BOSONS

The theory will now be applied to a neutral mixture of two types of charged bosons. It will be shown that two types of elementary excitations exist; one having at low momenta the energy-momentum dispersion relation associated with plasma oscillations, the other a dispersion relation characteristic of a free particle with a modified mass at low and high momenta. The special case where the particles of the two species have equal masses and equal, but opposite, charges will be examined in detail. It is shown here that one of the excitations corresponds to oscillations in mass density and the other to oscillations in charge density. Finally it will be shown that the criterion for the validity of the Boguliubov approximations is essentially the same as that found by Foldy for the one species case. It is not asserted here that there is not a collapsed state of the system with energy lower than the ground state found by this method. The following calculation in any event will serve to illustrate the preceding method.

We consider a volume V containing N_1 bosons with a positive charge q_1 and N_2 bosons with a negative charge q_2. The case of a mixture of two different types of bosons with charges of the same sign, neutralized by a uniform background charge is essentially identical with this. Assuming the particles to be spinless and assuming only two-body Coulomb interactions, the Hamiltonian in second quantized representation is

$$\begin{aligned}H=&\sum_k [t_1(k)a_1^+(k)a_1(k)+t_2(k)a_2^+(k)a_2(k)]\\ &+(1/2V)\sum_{k'',k',k}\{g_{11}(k)a_1^+(k''-k)a_1^+(k'+k)\\ &\qquad\times a_1(k'')a_1(k')\\ &+g_{22}(k)a_2^+(k''-k)a_2^+(k'+k)a_2(k'')a_2(k')\\ &+2g_{12}(k)a_2^+(k''-k)a_1^+(k'+k)a_2(k'')a_1(k')\},\end{aligned} \quad (30)$$

where

$$\begin{aligned}t_i(k)&=\hbar^2k^2/2m_i,\\ g_{ij}(k)&=4\pi q_iq_j/k^2,\end{aligned} \quad (31)$$

and

$$m_{\alpha+\nu}=m_\alpha, \quad q_{\alpha+\nu}=q_\alpha.$$

Using the commutation properties of the boson operators and the neutrality condition $(q_1N_1+q_2N_2=0)$, it is easily seen that the $k=0$ term vanishes.

Introducing the X notation and making the Boguliubov approximations for both species yields the following partial Hamiltonian:

$$H_k=\sum_{i,j,l} X_i^+\beta_{ij}\eta_{jl}X_l+E_k^0 \quad (32)$$

with

$$E_k^0=-\tfrac{1}{2}\{t_1(k)+n_1^0g_{11}(k)+t_2(k)+n_2^0g_{22}(k)\}\,, \quad (33)$$

and the η matrix is given by (15) with

$$f_{ij}(\pm k)=f_{ji}(\pm k)=f_{ij}^*(\pm k)=g_{ij}(k). \quad (34)$$

The secular equation (28) for this case is

$$\lambda^4-\tfrac{1}{4}\lambda^2[t_1^2+t_2^2+2t_1n_1^0g_{11}+2t_2n_2^0g_{22}] +\tfrac{1}{16}[t_1^2t_2^2+2t_1^2t_2n_2^0g_{22}+2t_1t_2^2n_1^0g_{11}]=0. \quad (35)$$

This biquadratic equation is easily solved for the four λ_k with the result

$$\lambda=\pm\tfrac{1}{2}\{\tfrac{1}{2}[t_1^2+t_2^2+2(t_1n_1^0g_{11}+t_2n_2^0g_{22})] \pm\tfrac{1}{2}([t_1^2-t_2^2+2(t_1n_1^0g_{11}-t_2n_2^0g_{22})]^2 +16t_1t_2n_1^0n_2^0g_{12}^2)^{1/2}\}^{1/2} =\pm\tfrac{1}{2}\epsilon^{\pm}(k). \quad (36)$$

The transformed partial Hamiltonian is now

$$H_k=E_k^0+Y^+\beta\lambda Y =E_k^0+\lambda_1Y_1^+Y_1+\lambda_2Y_2^+Y_2-\lambda_3Y_3^+Y_3 -\lambda_4Y_4^+Y_4. \quad (37)$$

Again taking cognizance of the commutation properties of the Y's and also ordering the λ's such that the Y's satisfy Eq. (26) we find

$$H_k=E_k^0+\tfrac{1}{2}\epsilon^++\tfrac{1}{2}\epsilon^-+\tfrac{1}{2}\epsilon^+(Y_1^+Y_1+Y_3Y_3^+) +\tfrac{1}{2}\epsilon^-(Y_2^+Y_2+Y_4Y_4^+)$$

and

$$H=\sum_k\,(E_k^0+{}^-\epsilon^++\tfrac{1}{2}\epsilon\tfrac{1}{2}+\epsilon^+A_1^+A_1+\epsilon^-A_2^+A_2)\,. \quad (38)$$

The transformation matrix could be found in accordance with the procedure described in the preceding section. This however proves quite cumbersome algebraically and will not be done here.

It is interesting to note the forms of the pseudo-particle excitation energies in the limit of very small and very large momentum.

$$\lim_{k\to 0}\epsilon^+(k)\to(4\pi\hbar^2)^{1/2}\left(\frac{n_1^0q_1^2}{m_1}+\frac{n_2^0q_2^2}{m_2}\right)^{1/2} +\frac{\hbar^3}{16(\pi)^{1/2}(m_1m_2)^{3/2}}\left[\frac{n_1^0q_1^2m_2^3+n_2^0q_2^2m_1^3}{(n_1^0q_1^2m_2+n_2^0q_2^2m_1)^{3/2}}\right]k^4, \quad (39)$$

$$\lim_{k\to\infty}\epsilon^+(k)\to\hbar^2k^2/2m_1. \quad (40)$$

This is the plasma type excitation, exhibiting an energy gap and going over, asymptotically for large momentum, into the normal energy-momentum relation.

For the second type of excitation we find

$$\lim_{k\to 0}\epsilon^-(k)\to\frac{\hbar^2k^2}{2(m_1m_2)^{1/2}}\left[\frac{n_1q_1^2m_1+n_2q_2^2m_2}{n_1q_1^2m_2+n_2q_2^2m_1}\right]^{1/2} \quad (41)$$

and

$$\lim_{k\to\infty}\epsilon^-(k)\to\hbar^2k^2/2m_2.$$

Here there is no energy gap and the energy-momentum relationship is that characteristic of free particles. Algebraic complexity prohibits further analysis of these excitations. More insight into their character can be gained in the following more specific case.

THE SYMMETRICAL TWO SPECIES NEUTRAL CHARGED GAS

We consider here the same system as in the preceding section where it is now specified that the two species of bosons have equal masses and equal, but opposite charges. Since the number of particles is not conserved in the Boguliubov approximation the neutrality condition takes the form

$$q_1\langle N_1\rangle+q_2\langle N_2\rangle=0\,,$$

or $\quad (42)$

$$q_1\langle N_1^0\rangle+q_1\langle\sum_k a_1^+(k)a_1(k)\rangle+q_2\langle N_2^0\rangle +q_2\langle\sum_k a_2^+(k)a_2(k)\rangle=0.$$

The fact that initially a fixed total number of particles were considered to be in the box now takes the form

$$\langle N_1\rangle+\langle N_2\rangle=\text{constant}\,,$$

or $\quad (43)$

$$\langle N_1^0\rangle+\langle\sum_k a_1^+(k)a_1(k)\rangle+\langle N_2^0\rangle +\langle\sum_k a_2^+(k)a_2(k)\rangle=\text{constant}.$$

By letting $N_2^0=N_1^0+\Delta$ in the Hamiltonian and calculating the quantities above it can easily be seen that the only way of satisfying these conditions is to have

$$\langle N_1^0\rangle=\langle N_2^0\rangle. \quad (44)$$

The preceding calculation will therefore be considered with the additional conditions

$m_1=m_2$ [and therefore $t_1(k)=t_2(k)=t$],

$q_1=-q_2$ [and therefore $g_{11}(k)=-g_{12}(k)=g_{22}(k)=g$],

and $n_1^0=n_2^0=n^0$.

The eigenvalues now take the much simpler form

$$\lambda=\pm\tfrac{1}{2}\{t^2+2tn^0g\pm 2tn^0g\}^{1/2}=\pm\tfrac{1}{2}\epsilon^{\pm} \quad (45)$$

and the ground-state energy is

$$U_0=\tfrac{1}{2}\sum_{k\neq 0}[(t^2+4tn^0g)^{1/2}-(t+2n^0g)]. \quad (46)$$

In this case the algebra involved in carrying out the procedure for explicitly obtaining the transformation matrix is greatly simplified, with the result

$$U=\begin{bmatrix} A & \sqrt{2}/2 & -B & 0 \\ -A & \sqrt{2}/2 & B & 0 \\ -B & 0 & A & \sqrt{2}/2 \\ B & 0 & -A & \sqrt{2}/2 \end{bmatrix}, \tag{47}$$

$$A=\left\{\frac{t+2n^0g+\epsilon^+}{4\epsilon^+}\right\}^{1/2}, \quad B=\left\{\frac{t+2n^0g-\epsilon^+}{4\epsilon^+}\right\}^{1/2}. \tag{48}$$

From the commutation relations it is easily shown that

$$U^{-1}=\beta U^+\beta, \tag{49}$$

and thus the Y operators can now be expressed in terms of the original operators. The result is

$$\begin{aligned} Y_1&=A(X_1-X_2)+B(X_3-X_4), \\ Y_2&=\sqrt{2}/2(X_1+X_2), \\ Y_3&=B(X_1-X_2)+A(X_3-X_4), \\ Y_4&=\sqrt{2}/2(X_3+X_4). \end{aligned} \tag{50}$$

Obviously, as was arranged,

$$Y_4^+(-k)=Y_2(k),$$

and

$$Y_3^+(-k)=Y_1(k). \tag{51}$$

The partial Hamiltonian is

$$H_k=E_k^0+\tfrac{1}{2}\epsilon^++\tfrac{1}{2}\epsilon^-+\tfrac{1}{2}\epsilon^+(Y_1^+Y_1+Y_3Y_3^+) +\tfrac{1}{2}\epsilon^-(Y_2^+Y_2+Y_4Y_4^+) \tag{52}$$

and, using (51) or (29),

$$H=\sum_k\left[(E_k^0+\tfrac{1}{2}\epsilon^++\tfrac{1}{2}\epsilon^-)+\epsilon^+Y_1^+Y_1+\epsilon^-Y_2^+Y_2\right]$$

or

$$H=\sum_k\left[(E_k^0+\tfrac{1}{2}\epsilon^++\tfrac{1}{2}\epsilon^-)+\epsilon^+A_1^+A_1+\epsilon^-A_2^+A_2\right]. \tag{53}$$

The asymptotic forms of the energy-momentum relationships for the pseudoparticles are now quite simple:

$$\lim_{k\to 0}\epsilon^+(k)\to \hbar q\left(\frac{8\pi n^0}{m}\right)^{1/2}+\left(\frac{\hbar^3}{16(2\pi n^0q^2m^3)^{1/2}}\right)k^4,$$

$$\lim_{k\to\infty}\epsilon^+(k)\to\frac{\hbar^2k^2}{2m} \tag{54}$$

(the plasma-type excitation relations),

and

$$\epsilon^-(k)=\frac{\hbar^2k^2}{2m} \quad \text{for all } k \quad \text{(free particle)}. \tag{55}$$

In order to gain some insight into the nature of these excitations consider the expectation values of the mass density and of the charge density in a wave packet composed of superpositions of states of various numbers of the two types of excitations with a single momentum. Such wave packets will be given by

$$|\phi_1\rangle=\sum_{n_1(k)}C_{n_1(k)}|n_1(k)\rangle$$

and

$$|\phi_2\rangle=\sum_{n_2(k)}C_{n_2(k)}|n_2(k)\rangle, \tag{56}$$

where $\hbar k$ is the momentum of the quasiparticle and the $C_{n(k)}$ are the weighting factors for the various states in the distribution.

The number-density operator is given in terms of the field operators by

$$\rho=\psi^+(r)\psi(r), \tag{57}$$

or for a composite system

$$\rho=\sum_\alpha\psi_\alpha^+(r)\psi_\alpha(r), \tag{58}$$

where $\psi(r)$ destroys a particle at the point r.

For a two-component system the mass and charge densities are given by

$$\rho_{\text{mass}}=\sum_\alpha m_\alpha\psi_\alpha^+(r)\psi_\alpha(r), \tag{59}$$

and

$$\rho_{\text{charge}}=\sum_\alpha q_\alpha\psi_\alpha^+(r)\psi_\alpha(r). \tag{60}$$

To obtain the second quantized representation of these operators we expand the field operators as follows:

$$\psi_\alpha(r)=\sum_k a_\alpha(k)U_{k,\alpha}(r), \tag{61}$$

where the $U_{k\alpha}(r)$'s form a complete set of single-particle wave functions. The $a_\alpha(k)$'s are the usual destruction operators. In particular we may choose the U's to be plane waves in which case we find

$$\psi^+(r)\psi(r)=\sum_{k,k'}a^+(k')a(k)e^{-i(k'-k)r}. \tag{62}$$

If $\hbar\omega_k$ is the energy of a single quasiparticle of momentum k, then the time evolution of an eigenstate corresponding to the excitation of n such quasiparticles is given by

$$\begin{aligned} |n(k,t)\rangle&=|n(k,0)\rangle e^{-in\omega_kt} \\ &=|n(k)\rangle e^{-in\omega_kt}. \end{aligned} \tag{63}$$

It is now a simple matter to calculate the mass and charge density expectation values in states composed of superpositions of such wave functions. The results, for $n^0\gg n$, are

$$\begin{aligned} &\langle\phi_1|\rho_{\text{mass}}|\phi_1\rangle=2n^0, \\ &\langle\phi_1|\rho_{\text{charge}}|\phi_1\rangle\sim\cos(kr-\omega_kt+\phi), \\ &\langle\phi_2|\rho_{\text{mass}}|\phi_2\rangle-2n^0\sim\cos(kr-\omega_kt+\phi), \\ &\langle\phi_2|\rho_{\text{charge}}|\phi_2\rangle=0. \end{aligned} \tag{64}$$

(The constants of proportionality and the phase angle ϕ depend on the chosen C_n's.) Thus, the plasma-type excitations consist of oscillations in charge density, while the free-particle excitations consist of mass-density oscillations.

A similar interpretation arises from calculations of the transition probabilities between the ground state and states with one of the two types of pseudoparticles. Again it is found that

$$\begin{aligned} \langle 0|\rho_{\text{mass}}|k_1\rangle &= 0, \\ \langle 0|\rho_{\text{charge}}|k_1\rangle &\sim e^{i(kr-\omega_k t)}, \\ \langle 0|\rho_{\text{mass}}|k_2\rangle &\sim e^{i(kr-\omega_k t)}, \\ \langle 0|\rho_{\text{charge}}|k_2\rangle &= 0, \end{aligned} \tag{65}$$

where $|k_i\rangle$ denotes a state with one pseudoparticle of type i with momentum k and $\langle 0|$, of course, corresponds to the state with no pseudoparticles present. Thus, excitations of the first type could be induced by an electromagnetic field and excitations of the second type by a gravitational field.

In order that the Boguliubov approximations be applicable the number of particles in excited states must be a small fraction of the total number of particles or

$$\langle N-N_0/N_0\rangle = \langle n-n_0/n_0\rangle \ll 1 \tag{66}$$

for both species of bosons.

In both cases this reduces to

$$\frac{1}{\pi^2 n_0}\int_0^\infty B^2k^2dk \ll 1, \tag{67}$$

where B is given by Eq. (48).

This integral corresponds to that of Foldy[5] if his n_0 is replaced by $2n_0$. Thus,

$$\langle n-n_0/n_0\rangle = 2^{5/3}Qr_{s0}, \tag{68}$$

where

$$\begin{aligned} Q &= 1.905, \\ r_{s0} &= (3/4\pi)^{1/3}[mq^2/\hbar^2(2n_0)^{1/3}], \end{aligned} \tag{69}$$

and again the approximations are valid at high densities.

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Verdet Constant of the "Active Medium" in a Laser Cavity

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It is shown that the frequency of the intensity modulation observed when the output of a gas laser in a homogeneous axial magnetic field is veiwed through a polarizer is simply related to the Verdet constant of the "active medium." This method is used to determine the Verdet constant of "active" neon at 0.633 μ. A value of 5.9×10^{-7} rad/cm-Oe is obtained. Theoretical expressions for the Verdet constant of a dilute monatomic gas at a frequency close to the center of a Doppler-broadened line are derived for the three allowed transitions, $\Delta J=0$ and $\Delta J=\pm1$. The results of the experiment and the theory are used to estimate the threshold values of the absorption coefficient and the population inversion density in the present case.

I. INTRODUCTION

MORE than a century ago Michael Faraday discovered the effect which today bears his name. He observed that when plane-polarized light is passed through matter which has been placed in a homogeneous longitudinal magnetic field, the plane of polarization of the emergent light is rotated through some angle θ with respect to the incident beam. The amount of rotation per unit field strength per unit optical path length is commonly referred to as the Verdet constant V.

Any optical activity exhibited by a medium, be it this magnetically induced type or its natural counterpart, is indicative of a nonzero value of the quantity $n_r(\nu)-n_l(\nu)$, where $n_r(\nu)$ and $n_l(\nu)$ are the indices of refraction of the medium for right and left circularly polarized light of frequency ν, respectively. In terms of θ, the optical path length d, and the vacuum speed of light c, we may write for $n_r(\nu)-n_l(\nu)$

$$n_r(\nu)-n_l(\nu) = (c/\pi\nu)(\theta/d), \tag{1}$$

and in terms of V and the magnitude of the longitudinal