faces ("fast" and "slow" waves) so formed is shown in Fig. 5.

In these diagrams the functions $\beta_\pm$ are given by

$$2\beta_\pm = 1 \pm \left[1 - \left(\frac{2v}{1+\kappa v^2}\right)^2\right]^{1/2}.$$

In both extremes of vanishing and very large wave number the fast wave becomes isotropic. For small wavelengths the fast wave becomes the vacuum electrodynamic mode $\omega^2 = c^2k^2$, while for large wavelengths the fast wave collapses to the nonpropagating mode $\omega^2 = \kappa\Omega_0^2$. Similarly the slow wave, in the limit of small wavelengths, becomes a nonpropagating anisotropic wave $\omega^2 = \Omega_0^2(1+\chi\sin^2\theta)$, while in the limit of large wavelengths it becomes a propagating anisotropic wave $(\omega^2/k^2) = (c^2/\kappa)(1+\chi\sin^2\theta)$. These surfaces are sketched in Fig. 6.

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Quantum Theory of Domain-Wall Motion*

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Several workers have examined the enhancement of nuclear magnetic resonance within a Bloch wall, and have demonstrated the existence of both bound and free "spin-wave" excitations on the Bloch wall structure. The free states correspond to precessional excitations akin to ordinary spin-wave excitations, while the bound states form a convenient basis for the representation of domain-wall motion. We derive the spectrum of both types of excitations, including exchange, anisotropy, and dipole field contributions for an infinite uniaxial ferromagnet. In contrast to earlier treatments, we treat the dipole field exactly (in the magnetostatic approximation), and show that this leads to a translational spectrum in which many states are degenerate with the "uniform translation," which is the translational mode excited by a uniform external magnetic field. The existence of such degeneracy is required for damping by imperfections to occur. The precessional spectrum is greatly different from the usual spin-wave spectrum, and, in particular, is not a symmetric function of **k**. The dipole fields lead to strong interactions, not conserving momentum, between the precessional modes; such interactions may explain the increase in ferromagnetic-resonance linewidth which is observed experimentally in the presence of a domain wall (in low dc magnetic fields). The motion of the domain wall, when it is bound to a certain position in the crystal by linear restoring forces, is studied by a Green's function technique. The domain-wall effective mass so obtained is identical to the expression given by Döring, and the domain-wall damping parameter proves to be simply related to the energy dispersion of the uniform translational mode. We calculate this energy dispersion due to scattering by the dipole fields, and due to "fluctuations," as used by Clogston *et al.* to explain the linewidth in disordered systems, such as the ferrites. The damping due to intrinsic scattering processes is proportional to T^2, while the damping due to "fluctuations" is essentially temperature-independent. In disordered systems, such as ferrite, the resonance linewidth and domain-wall damping due to "fluctuations" should agree to within a factor of order unity. The motion is *not* describable by the Landau-Lifshitz equation. This communication is intended to demonstrate that a formulation for the quantum-mechanical study of domain-wall motion exists, and has the properties necessary to explain the losses which occur during such motion; it is not intended to lead to any quantitative results which can be directly compared with experiment. We also consider the specific heat contribution due to the domain wall, and we find that this is proportional to T above about 10^{-2} °K. It should be possible to observe such a specific heat contribution in YIG below 1°K.

I. INTRODUCTION

SEVERAL workers[1–4] have considered the "spin-wave" excitations on the Bloch wall structure, both in ferro- and antiferromagnetic systems. It appears to be generally true that there exist two types of these excitations: Those bound to the wall, corresponding to translation of the wall (these all tend to zero well into the domains); and those which tend to plane waves well into the domains, corresponding to precessional modes in the domain-wall (DW) configuration. Previous work with these excitations has been aimed at evaluating the contribution to the nuclear magnetic resonance linewidth due to the presence of the Bloch wall; the

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[1] F. Boutron, Compt. Rend. **252**, 3955 (1961).

[2] J. M. Winter, Phys. Rev. **124**, 452 (1961).

[3] D. I. Paul, Phys. Rev. **126**, 78 (1962).

[4] D. I. Paul, Phys. Rev. **131**, 178 (1963).

present work is based on the realization that the bound excitations, the "translational modes," form a convenient set of basis functions for the quantum-mechanical analysis of DW motion. The amplitude of a particular translational mode, the "uniform translation," is directly proportional to the displacement of the wall, and our final object is to calculate quantum-mechanically the amplitude of the mode in response to an external magnetic field. Such a treatment is convenient for calculation of losses and the associated DW damping.

We choose to discuss a uniaxial ferromagnet of infinite extent. Provided that the limit of infinite sample volume is properly obtained, a stable, planar domain-wall configuration exists[5]; we choose this configuration as a ground state, and consider the excitations on this ground state. The spectra of these excitations are derived from a Hamiltonian including exchange, anisotropy, and dipole field contributions. An exact treatment of the dipole field, within the magnetostatic approximation, generalizes the calculation beyond those given previously.[1,2] In order to facilitate this treatment of the dipole field, the entire calculation is carried out in the continuum approximation, where we work with an angular momentum density rather than with a lattice of spins. This treatment shows that the uniform translational mode is degenerate with a number of other translational modes when the DW is bound to some position in the lattice by linear restoring forces. Such degeneracy plays a major role in theory of DW damping due to imperfections in a fashion similar to the theory of the ferromagnetic resonance linewidth.[6]

In Sec. II, the general formulation of the problem is discussed, and the operators for small deviations from static structure are introduced through the Holstein-Primakoff transformation.[7] In Sec. III, the Hamiltonian is diagonalized to obtain the energies of both translational and precessional excitations, plus terms describing the interactions among these excitations. In Sec. IV, we discuss the equilibrium properties of the system, and we find that the temperature dependence of the saturation magnetization $M_s(T)$ depends on position in the sample, though this effect is probably not measurable. In Sec. V, the equation of motion of the domain wall is derived using a Green's function technique, and finally, in Sec. VI, we consider some processes which can contribute to the DW damping.

II. GENERAL FORMULATION

We envision an infinite plate of a uniaxial ferromagnet, with the easy axis, chosen to be the x axis, lying in the plane of the plate, and the z axis normal to the plane (Fig. 1). We take the plate thickness to be $2L$, and let $L\rightarrow\infty$. The magnetization at the plane $z=-L$ is constrained to lie in the $+x$ direction, while it lies in the $-x$ direction at $z=+L$. If α is the exchange constant, β the anisotropy constant, as defined in Eq. (4) below, and φ_s is the angle between the magnetization and the x axis, it is well known[1,2] that, in the limit $L\rightarrow\infty$, the free energy is extremal if

$$\sin\varphi_s(z)=\operatorname{sech}\frac{z-z_0}{d},\quad d=\left[\frac{\alpha}{\beta}\right]^{1/2},\tag{1}$$

where z_0 is the value of z for which $\varphi_s=\pi/2$ (coordinate of the DW center). Brown[5] has shown that this solution is stable, or minimizes the free energy, *provided that* pinned-spin boundary conditions are maintained on the planes $z=\pm L$, where $L\rightarrow\infty$. Furthermore, there are no surface poles, and the internal magnetic field, which we call the dipole field, satisfies

$$\nabla\cdot\mathbf{h}_{\text{dip}}=-\nabla\cdot\mathbf{M};\quad \nabla\times\mathbf{h}_{\text{dip}}=0.\tag{2}$$

It is necessary to approach infinite volume in the manner outlined above in order to guarantee the stability of the DW structure (the pinned-spin boundary conditions prevent the ferromagnet from relaxing to the state of uniform magnetization), and to eliminate internal fields which depend on the sample geometry.

The problem may be quantized, in the continuum approximation, by treating the components M_1, M_2, M_3 of the magnetization as components of a vector angular-momentum density operator, with the commutation relations[8]

$$[M_i(\mathbf{r},t),M_j(\mathbf{r}',t)]=-i\gamma\hbar\epsilon_{ijk}M_k(\mathbf{r},t)\delta(\mathbf{r}-\mathbf{r}'),\tag{3}$$

in which ϵ_{ijk} is the unit antisymmetric tensor, and γ is the magnitude of the gyromagnetic ratio $\gamma=g|e|/2m$. We also treat the components of the dipole field, as determined from Eq. (2), as quantum-mechanical operators. Use of the continuum approximation facilitates the solution of Eq. (2).

The Hamiltonian of the ferromagnet, assuming isotropic exchange, is taken to be

$$\mathcal{H}=\int\{\tfrac{1}{2}\alpha[(\nabla M_1)^2+(\nabla M_2)^2+(\nabla M_3)^2]-\tfrac{1}{2}\beta M_1{}^2+\tfrac{1}{2}\mu_0 h_{\text{dip}}{}^2\}dV,\tag{4}$$

when the x axis corresponds to the easy axis, where α is the exchange constant, and β the anisotropy constant. In addition, the Hamiltonian will contain a term $-\mu_0\int\mathbf{H}_0\cdot\mathbf{M}dV$ due to the external field $\mathbf{H}_0(t)$; we neglect this term for the time being, and consider its effects in Sec. V below.

It is very convenient for our purposes to formulate the problem in terms of deviations from the static DW

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[5] W. F. Brown, Jr., *Magnetostatic Principles in Ferromagnetism* (North-Holland Publishing Company, Amsterdam, 1962), Chap. 7, Secs. 5 and 6.

[6] A. M. Clogston, H. Suhl, L. R. Walker, and P. Anderson, J. Phys. Chem. Solids **1**, 129 (1956).

[7] T. Holstein and H. Primakoff, Phys. Rev. **58**, 1098 (1940).

[8] I. A. Akhiezer, V. G. Bar'Yakhtar, and M. I. Kaganov, Usp. Fiz. Nauk **71**, 533 (1960) [English transl.: Soviet Phys.—Usp. **3**, 567, 661 (1961)]; Usp. Fiz. Nauk **72**, 3 (1960) [English transl.: Soviet Phys.—Usp. **3**, 661 (1961).

structure given by Eq. (1). We accomplish this by going into the wall (primed) coordinates $(\mathbf{a}_1,\mathbf{a}_2,\mathbf{a}_3)$ in Fig. 1, where the $3'$ direction lies along $\mathbf{M}$ as given by Eq. (1). The primed coordinates are helical coordinates obtained by a space-dependent rotation, as follows: let R be an operator giving infinitesimal rotations about the z axis, so that

$$[R,M_1(\mathbf{r},t)]=i\hbar M_2(\mathbf{r},t); \qquad (5)$$
$$[R,M_2(\mathbf{r},t)]=-i\hbar M_1(\mathbf{r},t); \quad [R,M_3(\mathbf{r},t)]=0.$$

Then the operators giving the deviations from static structure, in the primed coordinates, are

$$\begin{aligned} M_1'(\mathbf{r},t)&=e^{-i\varphi_s R/\hbar}M_2(\mathbf{r},t)e^{i\varphi_s R/\hbar},\\ M_2'(\mathbf{r},t)&=e^{-i\varphi_s R/\hbar}M_3(\mathbf{r},t)e^{i\varphi_s R/\hbar}, \qquad (6)\\ M_3'(\mathbf{r},t)&=e^{-i\varphi_s R/\hbar}M_1(\mathbf{r},t)e^{i\varphi_s R/\hbar}. \end{aligned}$$

It can be shown that the M_i' satisfy the commutation relations [Eq. (3)], or in other words that these commutation relations are invariants under space-dependent rotations of coordinates. The advantage in using the M_i' lies in the fact that for small deviations from static structure, M_1' and M_2' are expected to be small, while $M_3'\simeq M_0$, where M_0 is the magnitude of the magnetization vector $[M_0(M_0+1)\approx M_0^2]$. Because of the relative sizes of the operators M_i', and because of the invariance of the commutation relations, we may introduce the Holstein-Primakoff[7] transformation to the operators of a Bose field:

$$M_1'(\mathbf{r},t)+iM_2'(\mathbf{r},t)=(2\gamma\hbar M_0)^{1/2}a^\dagger\left(1-\frac{\gamma\hbar a^\dagger a}{2M_0}\right)^{1/2};$$
$$M_1'(\mathbf{r},t)-iM_2'(\mathbf{r},t)=(2\gamma\hbar M_0)^{1/2}\left(1-\frac{\gamma\hbar a^\dagger a}{2M_0}\right)^{1/2}a; \qquad (7)$$
$$M_3'(\mathbf{r},t)=M_0-\gamma\hbar a^\dagger a,$$

where

$$[a(\mathbf{r},t)a^\dagger(\mathbf{r}',t)]=\delta(\mathbf{r}-\mathbf{r}'), \qquad (8)$$

and we obtain the second-quantized Hamiltonian by keeping only the first few terms in the expansion of the square roots in Eq. (7).

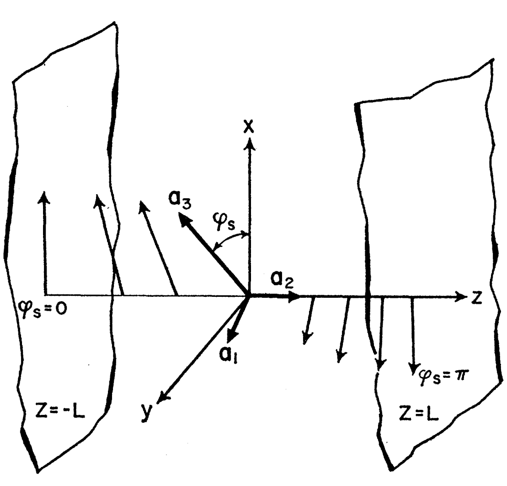

FIG. 1. Coordinate systems.

In order to consider small displacements of the DW about its equilibrium position, we introduce into the Hamiltonian [Eq. (4)] the term[2]

$$\int K(M_1')^2 dV. \qquad (9)$$

It will be verified below that this term leads to linear restoring forces acting on the wall.

Suppose that the wall is bound to the point $z_0=0$, and consider small excursions about the point $z_0=0$. Then, if $\langle\ \rangle$ denotes the average value of an operator over a canonical ensemble, using Eq. (1),

$$\begin{aligned}\langle M_1(\mathbf{r},t)\rangle&=M_0\cos\varphi_s[z-z_0(t)]\simeq M_0\cos\varphi_s(z)\\ &\quad+M_0z_0(t)\sin\varphi_s(z)\frac{d\varphi_s(z)}{dz}\\ &=M_0\cos\varphi_s(z)+\frac{M_0z_0(t)}{d}\sin^2\varphi_s(z). \qquad (10)\end{aligned}$$

On the other hand, Eqs. (6) may be inverted to give

$$M_1(\mathbf{r},t)=M_3'(\mathbf{r},t)\cos\varphi_s(z)-M_1'(\mathbf{r},t)\sin\varphi_s(z). \qquad (11)$$

Taking the canonical average of Eq. (11), and comparing to Eq. (10), we find[2]

$$\begin{aligned}\langle M_1'(\mathbf{r},t)\rangle&=-\frac{M_0z_0(t)}{d}\sin\varphi_s(z)\\ &=-\frac{M_0z_0(t)}{d}\operatorname{sech}\left(\frac{z}{d}\right). \qquad (12)\end{aligned}$$

Equation (12) connects the wall displacement $z_0(t)$ to a calculable quantum-mechanical average. By finding how $\langle M_1'(\mathbf{r},t)\rangle$ depends on an external magnetic field, we obtain the DW displacement from Eq. (12).

It is not necessary to restrict ourselves to small excursions of the DW from an equilibrium position in order to apply the formalism of Eqs. (6) and (7). When the DW may assume any position in the crystal (z_0 arbitrary), we can get small deviations and hence expand the square roots in Eq. (7) by letting the primed coordinate system move with the domain wall and treating small deviations from static structure in a coordinate system in which the DW is stationary. The Hamiltonian is the same as that given in Eq. (4) for small DW velocities since the lattice of spins has been replaced by continuous fields in the continuum approximation and an observer at the center of the wall cannot say whether he is moving with respect to these continuous fields. The equations governing the motion of the wall are obtained by setting $\langle M_1'\rangle=0$ in the moving coordinate system. However, this situation is physically uninteresting, since a DW is always, in reality, bound to an equilibrium position (a freely translating wall is not equivalent to a wall which has broken free of restraining influences—the coercive force is zero in the former case), and we shall not mention it further. We merely wish to point out that the formalism developed above is also applicable to a freely translating wall, and presumably, to a wall which has broken free of constraints, though we do not discuss either case here.

The Hamiltonian [Eq. (4)] is transformed to a Fourier representation by expansion in the functions[1,2]

$$\psi_k=(1/\sqrt{2})e^{i\mathbf{k}_t\cdot\mathbf{r}}\operatorname{sech}(z/d);\qquad\phi_k=\left[\frac{ik_zd-\tanh(z/d)}{ik_zd+1}\right]e^{i\mathbf{k}\cdot\mathbf{r}}.\tag{13}$$

These are the approximate eigenfunctions of the problem; ψ_k is a translational eigenfunction, ϕ_k a precessional eigenfunction. Quantity $\mathbf{k}$ is the wave vector, and $\mathbf{k}_t$ is a transverse wave vector ($k_z=0$). For an infinite crystal, the orthogonality properties satisfied by ψ_k and ϕ_k are

$$\begin{aligned}&\int\psi_k(\mathbf{r})\psi_{k'}(\mathbf{r})dV=(2\pi)^2d\delta(\mathbf{k}_t-\mathbf{k}_t');\\&\int\phi_k(\mathbf{r})\phi_{k'}(\mathbf{r})dV=(2\pi)^3\delta(\mathbf{k}-\mathbf{k}');\\&\int\psi_k(\mathbf{r})\phi_{k'}(\mathbf{r})dV=0.\end{aligned}\tag{14}$$

The transformation to the Fourier representation is effected by writing

$$\begin{aligned}a(\mathbf{r},t)&=\frac{1}{(2\pi)^2d}\int\zeta_k(t)\psi_k(\mathbf{r})d\mathbf{k}_t+\frac{1}{(2\pi)^3}\int a_k(t)\phi_k(\mathbf{r})d\mathbf{k};\\a^\dagger(\mathbf{r},t)&=\frac{1}{(2\pi)^2d}\int\zeta_k^\dagger(t)\psi_k^*(\mathbf{r})d\mathbf{k}_t+\frac{1}{(2\pi)^3}\int a_k^\dagger(t)\phi_k^*(\mathbf{r})d\mathbf{k}.\end{aligned}\tag{15}$$

The operators ζ_k, $\zeta_k^\dagger$, and a_k, $a_k^\dagger$ satisfy the equal time commutation relations

$$\begin{aligned}&[\zeta_k,\zeta_{k'}^\dagger]=(2\pi)^2d\delta(\mathbf{k}_t-\mathbf{k}_t'),\\&[a_k,a_{k'}^\dagger]=(2\pi)^3d\delta(\mathbf{k}-\mathbf{k}'),\\&[\zeta_k,a_{k'}^\dagger]=[\zeta_k^\dagger,a_{k'}^\dagger]=0.\end{aligned}\tag{16}$$

The operators ζ_k, $\zeta_k^\dagger$ are Bose operators for translation (quantum analogues of the translational mode amplitudes), while a_k, $a_k^\dagger$ are Bose operators for precession. Choice of Eq. (13) as basis functions for a Fourier representation greatly simplifies the diagonalization of the Hamiltonian. Because of the presence of the wall, the problem is spatially inhomogeneous, but the use of Eq. (13), rather than, say plane waves, eliminates the difficulties associated with this inhomogeneity, at least as far as the translational modes are concerned.

III. DIAGONALIZATION OF THE HAMILTONIAN

When we express the Hamiltonian (4) in terms of the field operators by use of Eq. (15), we obtain terms involving products of two or more operators. The Hamiltonian is approximately diagonalized when all the two-body terms have been re-expressed in terms of number operators $\zeta_k^\dagger\zeta_k$ or $a_k^\dagger a_k$. Products of three or more operators, provided that they are small, are to be treated as interactions among the excitations defined by the two-body Hamiltonian.

Use of the basis functions [Eq. (13)] automatically diagonalizes all the two-body terms arising from the exchange and anisotropy contributions to Eq. (4), but does not diagonalize the dipole-field contribution. In order to find the dipole field, and to facilitate the treatment of the magnetostatic condition $\nabla\times\mathbf{h}_{\text{dip}}=0$, we expand $\mathbf{h}_{\text{dip}}$ in plane waves, writing

$$\mathbf{h}_{\text{dip}}(\mathbf{r},t)=\frac{1}{(2\pi)^3}\int\mathbf{h}_k(t)e^{i\mathbf{k}\cdot\mathbf{r}}d\mathbf{k}.\tag{17}$$

Then $\mathbf{k}\times\mathbf{h}_k=0$, and

$$\mathbf{h}_k(t)=\frac{i\mathbf{k}}{k^2}\int(\nabla\cdot\mathbf{M})e^{-i\mathbf{k}\cdot\mathbf{r}}dV.\tag{18}$$

The contribution to the Hamiltonian is

$$\mathcal{H}_{\text{dip}}=\tfrac{1}{2}\mu_0\int h_{\text{dip}}^2dV=\frac{\mu_0}{2(2\pi)^3}\int\mathbf{h}_k\cdot\mathbf{h}_{-k}d\mathbf{k}.\tag{19}$$

We write $\mathbf{h}_k$ in terms of the operators ζ_k, a_k by writing the components of $\mathbf{M}$ in terms of these operators, from Eqs. (7) and (15), and putting the results into Eq. (18). Using the Fourier transform pairs

$$\begin{aligned}&\int e^{ikz}\operatorname{sech}(z/d)dz=\pi d\operatorname{sech}(\pi kd/2);\\&\int e^{ikz}\tanh(z/d)dz=i\pi d\operatorname{csch}(\pi kd/2);\\&\int e^{ikz}\operatorname{sech}^2(z/d)dz=\pi kd^2\operatorname{csch}(\pi kd/2),\end{aligned}\tag{20}$$

which are pairs Nos. 625, 612, and 607.8, respectively, of Ref. 9, and working to fourth order in the translation

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[9] G. A. Campbell and R. M. Foster, *Fourier Integrals for Practical Application*, (D. Van Nostrand, Inc., Princeton, New Jersey, 1948).

operators, second order in the precession operators, we obtain for Eq. (19)

$$\begin{aligned}\mathcal{H}_{\mathrm{dip}}=\mathrm{const}&+\frac{\gamma\hbar M_0}{(2\pi)^3}\int\Big[R(\mathbf{k})a_k{}^\dagger a_k-\frac{\tilde{R}(\mathbf{k})}{2}(a_k{}^\dagger a_{-k}{}^\dagger+a_{-k}a_k)\Big]d\mathbf{k}+\frac{1}{(2\pi)^6}\int d\mathbf{k}d\mathbf{k}'\tilde{X}(\mathbf{k};\mathbf{k}')a_k{}^\dagger a_{k'}\\
&+\frac{1}{(2\pi)^6}\int d\mathbf{k}d\mathbf{k}'\tilde{X}_1(\mathbf{k}\mathbf{k}')a_k{}^\dagger a_{k'}{}^\dagger+\mathrm{conj}+\frac{1}{(2\pi)^3}\int\tilde{Y}(\mathbf{k})a_k{}^\dagger\zeta_k d\mathbf{k}+\mathrm{conj}\\
&+\frac{1}{(2\pi)^3}\int\tilde{Y}_1(\mathbf{k})a_k{}^\dagger\zeta_{-k}{}^\dagger d\mathbf{k}+\mathrm{conj}+\frac{\gamma\hbar M_0}{(2\pi)^2d}\int d\mathbf{k}_t\Big[P(\mathbf{k}_t)\zeta_k{}^\dagger\zeta_k-\frac{Q(\mathbf{k}_t)}{2}(\zeta_k{}^\dagger\zeta_{-k}{}^\dagger+\zeta_{-k}\zeta_k)\Big]\\
&+\frac{1}{(2\pi)^6d^3}\int\tilde{\Phi}(\mathbf{k}_{t1}\mathbf{k}_{t2};\mathbf{k}_{t3})\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}-\mathbf{k}_{t3})\zeta_1{}^\dagger\zeta_2{}^\dagger\zeta_3d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}+\mathrm{conj}\\
&+\frac{1}{(2\pi)^8d^4}\int\tilde{\Psi}(\mathbf{k}_{t1}\mathbf{k}_{t2};\mathbf{k}_{t3}\mathbf{k}_{t4})\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}-\mathbf{k}_{t3}-\mathbf{k}_{t4})\zeta_1{}^\dagger\zeta_2{}^\dagger\zeta_3\zeta_4d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}d\mathbf{k}_{t4}+\mathrm{conj}\\
&\qquad+\frac{1}{(2\pi)^8d^4}\int\tilde{\Psi}_1(\mathbf{k}_{t1}\mathbf{k}_{t2}\mathbf{k}_{t3};\mathbf{k}_{t4})\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}+\mathbf{k}_{t3}-\mathbf{k}_{t4})\zeta_1{}^\dagger\zeta_2{}^\dagger\zeta_3{}^\dagger\zeta_4d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}d\mathbf{k}_{t4}+\mathrm{conj}\,,\end{aligned}\tag{21}$$

where "conj" denotes the Hermitian conjugate of the term immediately preceding, and where

$$\begin{aligned}P(\mathbf{k}_t)&=\frac{\mu_0\pi d}{4}\int_0^\infty\frac{k^2dk}{k_t{}^2+k^2}\Big[(k_xd)^2\operatorname{csch}^2\Big(\frac{\pi kd}{2}\Big)+(1+k_yd)^2\operatorname{sech}^2\Big(\frac{\pi kd}{2}\Big)\Big];\\
Q(\mathbf{k}_t)&=-\frac{\mu_0\pi d}{4}\int_0^\infty\frac{k^2dk}{k_t{}^2+k^2}\Big\{(k_xd)^2\operatorname{csch}^2\Big(\frac{\pi kd}{2}\Big)-[1-(k_yd)^2]\operatorname{sech}^2\Big(\frac{\pi kd}{2}\Big)\Big\};\\
R(\mathbf{k})&=\frac{\mu_0}{2k^2d^2[1+(k_zd)^2]}[(k_z{}^2d^2-k_yd)^2+k^2d^2(1+k_yd)^2];\\
\tilde{R}(\mathbf{k})&=\frac{\mu_0}{2k^2d^2[1+(k_zd)^2]}[(k_z{}^4d^4-k_y{}^2d^2)+k^2d^2(1-k_y{}^2d^2)];\end{aligned}\tag{22}$$

$$\tilde{\Psi}(12;34)\simeq-(\gamma\hbar)^2\frac{\mu_0\pi d}{3};\quad\tilde{\Psi}_1(123;4)\simeq-(\gamma\hbar)^2\frac{\mu_0\pi d}{3}.$$

The approximation used in obtaining the expressions for $\tilde{\Psi}$ and $\tilde{\Psi}_1$ is the "Winter approximation" $\mathbf{h}_{\mathrm{dip}}=-M_2'(\mathbf{r},t)\mathbf{1}_z$, which is discussed in more detail below. We shall not need numerical expressions for $\tilde{Y}$, $\tilde{Y}_1$, $\tilde{X}$, $\tilde{X}_1$, and $\tilde{\Phi}$ in what follows, and we therefore do not give expressions for these quantities.

The Hamiltonian is now diagonalized by the method of Bogoliubov.[10] We introduce the unitary transformations

$$\begin{aligned}\zeta_k&=u_kt_k+v_k{}^*t_{-k}{}^\dagger;\\ a_k&=w_kc_k+x_k{}^*c_{-k}{}^\dagger,\end{aligned}\tag{23}$$

and choose the c numbers u_k, v_k, w_k, and x_k so that t_k and c_k satisfy commutation relations like Eq. (16), and also so that the two-body Hamiltonian, excepting the terms in $\tilde{X}$, $\tilde{X}_1$, $\tilde{Y}$, and $\tilde{Y}_1$, is diagonal in the number operators

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[10] N. Bogoliubov, Zh. Eksperim. i Teor. Fiz. 19, 256 (1948).

$t_k^\dagger t_k$ and $c_k^\dagger c_k$. We find

$$\mathcal{H}_0=\text{const}+\mathcal{H}_{\text{trans}}+\mathcal{H}_{\text{prec}}+\mathcal{H}_{\text{int}}$$

$$\begin{aligned}
\mathcal{H}_{\text{prec}}={}&\frac{1}{(2\pi)^3}\int \epsilon_k c_k^\dagger c_k d\mathbf{k}+\frac{1}{(2\pi)^6}\int X(1;2)c_1^\dagger c_2 d\mathbf{k}_1 d\mathbf{k}_2+\text{conj}+\frac{1}{(2\pi)^6}\int X_1(12)c_1^\dagger c_2^\dagger d\mathbf{k}_1 d\mathbf{k}_2+\text{conj};\\
\mathcal{H}_{\text{trans}}={}&\frac{1}{(2\pi)^2 d}\int e_k t_k^\dagger t_k d\mathbf{k}_t+\frac{1}{(2\pi)^6 d^3}\int \Phi(12;3)\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}-\mathbf{k}_{t3})t_1^\dagger t_2^\dagger t_3 d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}+\text{conj}\\
&+\frac{1}{(2\pi)^6 d^3}\int \Phi_1(123)\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}+\mathbf{k}_{t3})t_1^\dagger t_2^\dagger t_3^\dagger d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}+\text{conj}\\
&+\frac{1}{(2\pi)^8 d^4}\int \Psi(12;34)\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}-\mathbf{k}_{t3}-\mathbf{k}_{t4})t_1^\dagger t_2^\dagger t_3 t_4 d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}d\mathbf{k}_{t4}+\text{conj}\\
&+\frac{1}{(2\pi)^8 d^4}\int \Psi_1(123;4)\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}+\mathbf{k}_{t3}-\mathbf{k}_{t4})t_1^\dagger t_2^\dagger t_3^\dagger t_4 d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}d\mathbf{k}_{t4}+\text{conj}\\
&+\frac{1}{(2\pi)^8 d^4}\int \Psi_2(1234)\delta(\mathbf{k}_{t1}+\mathbf{k}_{t2}+\mathbf{k}_{t3}+\mathbf{k}_{t4})t_1^\dagger t_2^\dagger t_3^\dagger t_4^\dagger d\mathbf{k}_{t1}d\mathbf{k}_{t2}d\mathbf{k}_{t3}d\mathbf{k}_{t4}+\text{conj};\\
\mathcal{H}_{\text{int}}={}&\frac{1}{(2\pi)^3}\int Y(\mathbf{k})c_k^\dagger t_k d\mathbf{k}+\text{conj}+\frac{1}{(2\pi)^3}\int Y_1(\mathbf{k})c_k^\dagger t_{-k}^\dagger d\mathbf{k}+\text{conj}.
\end{aligned}\tag{24}$$

The terms X, X_1, Y, Y_1, Φ, Φ_1 are obtained from the corresponding terms $\tilde{X}$, $\tilde{X}_1$, etc., of Eq. (21) through the transformation (23), and Ψ, Ψ_1, and Ψ_2 are obtained similarly from Eq. (22), provided that $\beta\ll\mu_0$, $2K\ll\mu_0$ (the four-body terms arising from exchange and anisotropy can then be neglected); this condition is almost always satisfied in real crystals. The other parameters in Eqs. (23) and (24) are given by

$$\begin{aligned}
\frac{e_k}{\gamma\hbar M_0}&=\frac{P(\mathbf{k}_t)-P(-\mathbf{k}_t)}{2}+\left\{\left[\frac{P(\mathbf{k}_t)+P(-\mathbf{k}_t)+2K}{2}\right]^2-[Q(\mathbf{k}_t)-K]^2+[P(\mathbf{k}_t)+P(-\mathbf{k}_t)+2K]\alpha k_t^2+(\alpha k_t^2)^2\right\}^{1/2};\\
\frac{\epsilon_k}{\gamma\hbar M_0}&=\frac{R(\mathbf{k})-R(-\mathbf{k})}{2}+\left\{\left[\frac{R(\mathbf{k})+R(-\mathbf{k})+2K}{2}\right]^2-[\tilde{R}(\mathbf{k})-K]^2+[R(\mathbf{k})+R(-\mathbf{k})+2K](\beta+\alpha k^2)+(\beta+\alpha k^2)^2\right\}^{1/2};\\
u_k=u_{-k}&=\frac{Q(\mathbf{k}_t)-K}{\{[Q(\mathbf{k}_t)-K]^2-[P(\mathbf{k}_t)+K+\alpha k_t^2-e_k/\gamma\hbar M_0]^2\}^{1/2}};\\
v_k=v_{-k}&=\frac{P(\mathbf{k}_t)+K+\alpha k_t^2-e_k/\gamma\hbar M_0}{\{[Q(\mathbf{k}_t)-K]^2-[P(\mathbf{k}_t)+K+\alpha k_t^2-e_k/\gamma\hbar M_0]^2\}^{1/2}};\\
w_k=w_{-k}&=\frac{\tilde{R}(\mathbf{k})-K}{\{[\tilde{R}(\mathbf{k})-K]^2-[R(\mathbf{k})+K+\beta+\alpha k^2-\epsilon_k/\gamma\hbar M_0]^2\}^{1/2}};\\
x_k=x_{-k}&=\frac{R(\mathbf{k})+K+\beta+\alpha k^2-\epsilon_k/\gamma\hbar M_0}{\{[\tilde{R}(\mathbf{k})-K]^2-[R(\mathbf{k})+K+\beta+\alpha k^2-\epsilon_k/\gamma\hbar M_0]^2\}^{1/2}}.
\end{aligned}\tag{25}$$

[The constant K is the restoring-force constant introduced in Eq. (9).]

The operators $c_k^\dagger$ and c_k are creation and destruction operators for precessional excitations, while $t_k^\dagger$ and t_k are creation and destruction operators for translational excitations. The quantity e_k is the energy of the translational state with wave vector $\mathbf{k}_t$, and ϵ_k is the energy of the precessional state with wave vector $\mathbf{k}$. The remaining terms in the Hamiltonian (24) describe the interactions or scattering among these excitations. The terms in X and X_1 in the precessional Hamiltonian describe interactions between the precessional modes, in which momentum (components of $\mathbf{k}$) are not conserved. When these terms are small, as they are for $\mathbf{k}$ nearly perpendicular to the plane of the DW, they may be treated as scattering terms; these terms probably offer an explanation of the enhancement of the resonance linewidth due to the presence of the wall. If these terms become sufficiently large, however, the precessional Hamiltonian cannot be regarded as diagonalized (the excitations at ϵ_k are too short-lived). We are not directly concerned with the precessional states here, except as they may act as a reservoir for the scattering of the translational modes through the terms of $\mathbf{H}_{\text{int}}$, and we do not consider this problem further.

We show in Sec. V that a uniform external magnetic field excites only the translational mode with $\mathbf{k}=0$, the "uniform translation," and does not excite any precessional modes in the first order. The scattering due to the terms in $\mathbf{H}_{\text{int}}$ conserves the transverse wave vector $\mathbf{k}_t$, and also conserves energy, since $\mathbf{H}_{\text{int}}$ is Hermitian. Because $\epsilon_k > e_k$ for all $\mathbf{k}_t$, as we show below, such scattering does not occur, and the precessional modes are completely decoupled and unexcited in the first order. Hence we may neglect $\mathbf{H}_{\text{prec}}$ and $\mathbf{H}_{\text{int}}$ altogether, and concentrate on the translational states, in order to obtain the first-order response of the system to an external field.

Finally, the terms Φ, Φ_1, Ψ, Ψ_1, and Ψ_2 in the translational Hamiltonian describe interactions involving three or more translational modes, in which the momentum is conserved. We show in Sec. VI that only the terms in Ψ contribute to the DW damping.

In order to obtain the translational spectrum, we must find the integrals

$$I_1(a)=\int_{-\infty}^{\infty}\frac{x^2}{x^2+a^2}\operatorname{csch}^2x\,dx;\quad I_2(a)=\int_{-\infty}^{\infty}\frac{x^2}{x^2+a^2}\operatorname{sech}^2x\,dx, \tag{26}$$

in terms of which P, Q, and e_k are

$$\begin{aligned}
P(\mathbf{k}_t)&=(\mu_0/4)(k_td\cos\varphi_k)^2I_1(\pi k_td/2)+(\mu_0/4)(1+k_td\sin\varphi_k)^2I_2(\pi k_td/2);\\
Q(\mathbf{k}_t)&=-(\mu_0/4)(k_td\cos\varphi_k)^2I_1(\pi k_td/2)+(\mu_0/4)(1-k_t^2d^2\sin^2\varphi_k)I_2(\pi k_td/2);\\
e_k/\gamma\hbar M_0&=(\mu_0/2)k_tdI_2(\pi k_td/2)\sin\varphi_k+\{[\alpha k_t^2+(\mu_0/2)I_2(\pi k_td/2)][2K+\alpha k_t^2+(\mu_0/2)k_t^2d^2\\
&\qquad\times(I_1(\pi k_t/2)\cos^2\varphi_k+I_2(\pi k_td/2)\sin^2\varphi_k)]\}^{1/2}.
\end{aligned}\tag{27}$$

With the help of the relations

$$\frac{1}{x^2+a^2}=\frac{1}{2a}\int_{-\infty}^{\infty}e^{-a|y|}e^{ixy}dy;$$

$$\operatorname{sech}^2x=\frac{1}{2}\int_{-\infty}^{\infty}\frac{y}{\sinh(\pi y/2)}e^{-ixy}dy;$$

$$I_2(a)=I_1(a)-2I_1(2a),$$

we find

$$I_1(a)=\begin{cases}\dfrac{\pi}{a}-2+2\displaystyle\sum_{m=1}^{\infty}m(-1)^{m+1}\zeta(m+1)\left(\frac{a}{\pi}\right)^m, & a<\dfrac{\pi}{2};\\[2ex] \dfrac{\pi^2}{3a^2}, & a\rightarrow\infty;\end{cases}$$

$$I_2(a)=\begin{cases}2+2\displaystyle\sum_{m=1}^{\infty}(-1)^m(2^{m+1}-1)m\zeta(m+1)\left(\frac{a}{\pi}\right)^m, & a<\dfrac{\pi}{2};\\[2ex] \dfrac{\pi^2}{6a^2}, & a\rightarrow\infty,\end{cases}\tag{28}$$

where $\zeta(m)$ is the Riemann zeta function.[11] The integrals $I_1(a)$ and $I_2(a)$ are plotted in Fig. 2, and the spectrum $e_k/\gamma\hbar M_0$ is plotted against k_td in Fig. 3. The spectrum is not symmetrical ($e_k\neq e_{-k}$), which we emphasize by plotting e_{-k} in the left-hand quadrant in Fig. 3(a). The minimum energy occurs for $\mathbf{k}\neq 0$, $\varphi_k=3\pi/2$, and the form of the minimum is shown in Fig. 3(b) for several values of β/μ_0 and $2K/\mu_0$. All e_k curves pass through the same value, $e_0=\gamma\hbar M_0(2\mu_0K)^{1/2}$ for $\mathbf{k}_t=0$; however, since $e_k<e_0$ for some $\mathbf{k}_t$, there exist states in the spectrum which are degenerate with the uniform translation $\mathbf{k}_t=0$. The existence of this de-

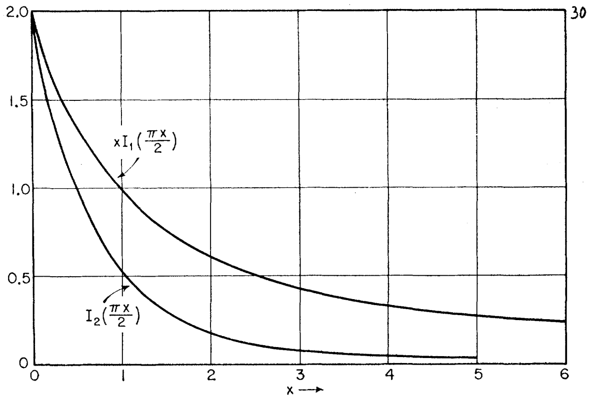

FIG. 2. The quantities $I_2(\pi x/2)$ and $xI_1(\pi x/2)$.

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[11] E. Jahnke and F. Emde, *Tables of Functions* (Dover Publications, New York, 1945).

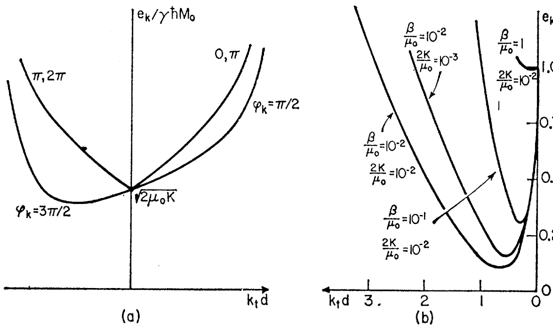

FIG. 3. Translational eigenvalue spectrum: (a) complete spectrum; (b) detail of spectrum for $\varphi_k=3\pi/2$.

generacy essentially solves the problem of the origin of DW damping.

Winter[2] has obtained a translational spectrum somewhat similar to that shown in Fig. 3 by writing the dipole field in the form $\mathbf{h}_{\text{dip}}=-M_2'(\mathbf{r},t)\mathbf{1}_z$. This is a long-wavelength approximation, which we can obtain from our results by setting I_1 and I_2 equal to their values at $\mathbf{k}_t=0$. This follows since

$$\nabla\cdot\mathbf{M}=\frac{\partial M_3'}{\partial x}\cos\varphi_s-\frac{\partial M_1'}{\partial x}\sin\varphi_s+\frac{\partial M_3'}{\partial y}\sin\varphi_s+\frac{\partial M_1'}{\partial y}\cos\varphi_s+\frac{\partial M_2'}{\partial z}, \quad (29)$$

which we get from (6). For $\mathbf{k}_t=0$, $\partial/\partial x=\partial/\partial y=0$, and Winter's results follow immediately. In this approximation, which we call the "Winter approximation," the degeneracy in the translational spectrum is removed, and we use the Winter approximation for the dipole field wherever this degeneracy is not essential in the present work. We have already used it in obtaining approximate forms for the quantities Ψ and Ψ_1 in Eq. (22), since these terms are merely scattering cross sections.

The precessional spectrum is obtained from Eq. (25), if θ_k is the angle between $\mathbf{k}$ and the z axis, φ_k the angle between $\mathbf{k}_t$ and the x axis, as

$$\frac{\epsilon_k}{\gamma\hbar M_0}=\frac{\mu_0 kd\sin^3\theta_k\sin\varphi_k}{(1+k^2d^2\cos^2\theta_k)}+\left\{\mu_0^2\frac{(1+k^2d^2)(1+k^2d^2\cos^4\theta_k)}{(1+k^2d^2\cos^2\theta_k)^2}\sin^2\theta_k\sin^2\varphi_k\right.$$
$$\left.+2K(\mu_0+\beta+\alpha k^2)+(\beta+\alpha k^2)\left[\mu_0\frac{(1+\sin^2\theta_k\sin^2\varphi_k+k^2d^2\cos^4\theta_k+k^2d^2\sin^2\theta_k\sin^2\varphi_k)}{(1+k^2d^2\cos^2\theta_k)}+\beta+\alpha k^2\right]\right\}^{1/2}, \quad (30)$$

and is shown in Fig. 4. The precessional spectrum is also asymmetrical; the smallest ϵ_k occur at $\mathbf{k}=0$, $\theta_k=0$, and is $\epsilon_0/\gamma\hbar M_0=[(2K+\beta)(\mu_0+\beta)]^{1/2}$. Since $\beta>0$, it follows that $\epsilon_0>e_0$; since both ϵ_k and e_k increase no faster than αk^2, it follows that $\epsilon_k>e_k$. There is thus no value of $\mathbf{k}_t$ for which $\epsilon_k=e_k$, and no interactions occur between the translational and precessional modes.

The precessional spectrum ϵ_k does not reduce to the ordinary spin-wave spectrum in the limit $d\to\infty$, as it should. This occurs because, in this limit, both X and X_1 of Eq. (24) approach δ functions. When this is taken into account, and the precessional Hamiltonian is properly diagonalized, we recover the ordinary spin-wave spectrum.

Neither the precessional nor translational spectrum is symmetric under the operation $k_y\to -k_y$. Because the chosen DW structure is degenerate with another, different structure (obtained by putting $\varphi_s\to-\varphi_s$, or $M_y\to -M_y$), this lack of symmetry does not violate any general spatial or time-reversal symmetry considerations. We can understand how the lack of symmetry is induced by the dipole field by considering a long-wavelength ($kd\ll1$) precessional excitation with k in the y direction:

$$M_x=M_0\cos\varphi_s(z)-A_ke^{iky}\sin\varphi_s(z)\cos\varphi_s(z)$$
$$M_y=M_0\sin\varphi_s(z)+A_ke^{iky}\cos^2\varphi_s(z)$$
$$M_z=B_ke^{iky}\cos\varphi_s(z).$$

We satisfy the long-wavelength condition by setting $d\to 0$, in which case $\sin\varphi_s(z)\simeq 0$, $\cos^2\varphi_s(z)\simeq 1$, and

$$\cos\varphi_s(z)\simeq\begin{cases}+1, & z<0\\ -1, & z>0\end{cases}$$

(the deviation M_z always points toward the wall); the dipole field is then the solution of

$$\nabla\times\mathbf{h}=0;\quad \nabla\cdot\mathbf{h}=-\nabla\cdot\mathbf{M}=2B_k\delta(z)-ikA_k,$$

or

$$h_z=B_k\cos\varphi_s(z)-ikzA_k.$$

For $kz\ll1$, this dipole field also points toward the wall. The energy of the spins in the dipole field is proportional to $-\mathbf{M}\cdot\mathbf{h}$, and this increases for positive k, decreases for

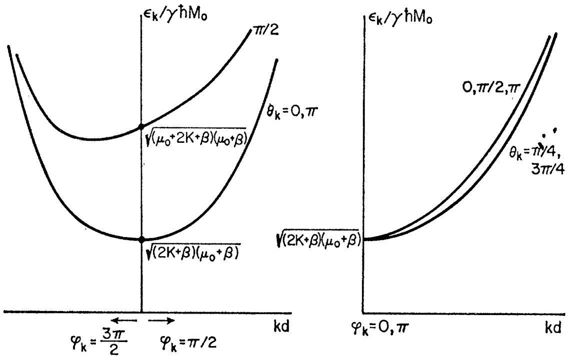

FIG. 4. Precessional eigenvalue spectrum.

negative k, so that the spectrum is as shown in Fig. 4. The same sort of thing happens for the translational excitations; the lack of symmetry occurs because the z-directed dipole field depends on k_y, which in turn occurs because of the peculiar form of the excitations. The z dependence of these excitations is precisely what is required to produce a dipole field h_z such that the product $-M_z h_z$ increases for positive k_y and decreases for negative k_y. Furthermore, every excitation consists of a propagating wavelike disturbance plus a translation of the wall so that while the disturbance is propagating along the $+y$ direction, say, the *domain wall is moving out from underneath it.* The field seen by the disturbance in this situation is so nonuniform that all bets are off regarding the symmetry of the spectrum.

IV. EQUILIBRIUM PROPERTIES

We now consider the evaluation of the saturation magnetization $M_s(T)$ of the sample. In thermal equilibrium we may write

$$\begin{aligned}\langle t_k{}^\dagger t_{k'}\rangle &= (2\pi)^2 d\delta(\mathbf{k}_t-\mathbf{k}_t{}')n_k = \frac{(2\pi)^2 d\delta(\mathbf{k}_t-\mathbf{k}_t{}')}{\exp(e_k/k_BT)-1}\,;\\ \langle c_k{}^\dagger c_{k'}\rangle &= \frac{(2\pi)^3\delta(\mathbf{k}-\mathbf{k}')}{\exp(\epsilon_k/k_BT)-1}\,,\end{aligned} \tag{31}$$

where, as before, $\langle\ \rangle$ denotes the average over a canonical ensemble. The only nonvanishing $\langle M_i{}'\rangle$ is $\langle M_3{}'\rangle$, for which

$$\begin{aligned}\langle M_3{}'\rangle = M_0 &- \frac{\gamma\hbar\,\mathrm{sech}^2(z/d)}{2(2\pi)^2 d}\int |v_k|^2 d\mathbf{k}_t - \frac{\gamma\hbar}{(2\pi)^3}\int |x_k|^2\left(\frac{k_z{}^2d^2+\tanh^2(z/d)}{k_z{}^2d^2+1}\right)d\mathbf{k}\\ &- \frac{\gamma\hbar\,\mathrm{sech}^2(z/d)}{2(2\pi)^2 d}\int \frac{|u_k|^2+|v_k|^2}{\exp(e_k/k_BT)-1}d\mathbf{k}_t - \frac{\gamma\hbar}{(2\pi)^3}\int \frac{|w_k|^2+|x_k|^2}{\exp(\epsilon_k/k_BT)-1}\left(\frac{k_z{}^2d^2+\tanh^2(z/d)}{k_z{}^2d^2+1}\right)d\mathbf{k}\,.\end{aligned} \tag{32}$$

The last two integrals in Eq. (32) have different temperature dependence. Well into the domains, only the last contributes to $M_3{}'$, and we recover the Bloch $T^{3/2}$ law.[12] Within the wall, however, the next-to-last integral also contributes, and adds the term

$$+\frac{\gamma\hbar\mu_0}{16\pi\alpha d}\left(\frac{k_BT}{e_0}\right)\ln[1-e^{e_0/k_BT}]\,\mathrm{sech}^2\left(\frac{z}{d}\right) \tag{33}$$

in the Winter approximation. It is doubtful that the existence of such a spatially inhomogeneous temperature dependence of the magnetization could be verified experimentally, since one cannot measure the magnetization within a domain wall. However, the specific heat contribution of the domain wall for $k_BT \gg e_0$ is $C_v = Nk_B{}^2T\zeta(2)/2\pi\gamma\hbar M_0\alpha$ (in joules/°K-m^3), where N is the number of walls per unit length in the z direction, and it is possible that such a linear term in the specific heat could be detected at sufficiently low temperatures in insulating ferromagnets. Putting in the numbers for YIG, for example, and assuming $e_0/h \simeq 100$ Mc/sec $N=1$ wall/micron, we get $C_v = 0.5T$ erg/cm^3-°K for $T \gg 10^{-2}$°K. The magnitude and temperature dependence of C_v depend on the details of the binding mechanism, and measurements on good single crystals are indicated.

V. GREEN'S FUNCTION THEORY OF RESPONSE TO APPLIED FIELDS

A small, uniform magnetic field applied along the x direction leads to the perturbing Hamiltonian

$$\mathcal{H}_1 = -\mu_0 H_0(t)\int [M_3{}'\cos\varphi_s(z) - M_1{}'\sin\varphi_s(z)]dV\,. \tag{34}$$

If we put in the Fourier expansions of $M_1{}'$ and $M_3{}'$ according to Eqs. (7) and (15), we find that the $M_1{}'$ term excites only the translational modes, and, to first order in $t_k{}^\dagger$ and t_k, excites only the uniform translation, $\mathbf{k}_t=0$. The $M_3{}'$ term excites only precessional modes, and does so only in the second order in $c_k{}^\dagger$ and c_k. We neglect the excitation of precessional modes, and consider only the first-order terms in $t_k{}^\dagger$ and t_k arising from Eq. (34). By going to the interaction representation where

$$\mathcal{H}_1(t) = \exp(i\mathcal{H}_0t/\hbar)\mathcal{H}_1\exp(-i\mathcal{H}_0t/\hbar) \tag{35}$$

we find that $\langle M_1{}'\rangle$ and $\langle M_2{}'\rangle$ are given by[13]

$$\langle M_i{}'\rangle = \frac{i}{\hbar}\int_{-\infty}^{t}\langle[\mathcal{H}_1(t'),M_i{}'(t)]\rangle dt'\,,\quad i=1,2\,. \tag{36}$$

Let $M_{k1}{}'(t)$ and $M_{k2}{}'(t)$ be the operator coefficients in an expression of $M_1{}'(t)$ and $M_2{}'(t)$ in the translational eigenfunctions. To first order in $t_k{}^\dagger$ and t_k,

$$\begin{aligned}M_{k1}{}' &= \frac{(2\gamma\hbar M_0)^{1/2}}{2}(u_k+v_k)(t_{-k}{}^\dagger+t_k)\,;\\ M_{k2}{}' &= \frac{(2\gamma\hbar M_0)^{1/2}}{2i}(u_k-v_k)(t_{-k}{}^\dagger-t_k)\,.\end{aligned} \tag{37}$$

Defining the Green's functions

$$(2\pi)^2 dP_{i1}(\mathbf{k}\mathbf{k}';t) = (i/\hbar)\theta(t)\langle[M_{ki}{}'(t),M_{k'1}{}'(0)]\rangle\,, \tag{38}$$

where

$$\theta(t) = \begin{cases}1, & t>0\\ 0, & t<0,\end{cases} \tag{39}$$

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[12] F. Bloch, Z. Physik **61**, 206 (1930).

[13] D. N. Zubarev Usp. Fiz. Nauk **71**, 71 (1960) [English transl.: Soviet Phys.—Usp. **3**, 320 (1960)].

we write Eq. (36) in the form

$$\langle M_i'(t)\rangle=-\mu_0\,\mathrm{sech}(z/d)\int d\mathbf{k}_t e^{i\mathbf{k}_t\cdot\mathbf{r}}\times\int_{-\infty}^{\infty}dt' H_0(t')P_{i1}(\mathbf{k}0;t-t'),\quad i=1,2. \tag{40}$$

Introducing the Fourier transforms

$$\begin{aligned}H_0(t)&=(1/2\pi)\int_{-\infty}^{\infty}H_0(\omega)e^{-i\omega t}d\omega;\\ P_{i1}(\mathbf{kk}';t)&=(1/2\pi)\int_{-\infty}^{\infty}P_{i1}(\mathbf{kk}';\omega)e^{-i(\omega-i0^+)t}d\omega,\end{aligned} \tag{41}$$

with the slightly negative imaginary part of ω included to guarantee convergence of the integrals for $t>0$, we finally obtain from Eq. (40)

$$\langle M_i'\rangle=-\frac{\mu_0}{2\pi}\,\mathrm{sech}(z/d)\int d\mathbf{k}_t e^{i\mathbf{k}_t\cdot\mathbf{r}}\times\int_{-\infty}^{\infty}d\omega H_0(\omega)P_{i1}(\mathbf{k}0;\omega),\quad i=1,2. \tag{42}$$

Comparing Eq. (42) with Eq. (12), we obtain the result

$$z_0(t)=\frac{\mu_0 d}{2\pi M_0}\int d\mathbf{k}_t e^{i\mathbf{k}_t\cdot\mathbf{r}}\int_{-\infty}^{\infty}d\omega H_0(\omega)P_{11}(\mathbf{k}0;\omega). \tag{43}$$

Results are complete when we find P_{11} and P_{21}.

It is easy to verify that the terms $\langle[t_k^\dagger(t),t_{k'}^\dagger(0)]\rangle$ and $\langle[t_k(t),t_k(0)]\rangle$ are higher order terms in the scattering, and vanish when the Hamiltonian includes only the term in $t_k^\dagger t_k$. Neglecting these two Green's functions, Eqs. (37) and (38) reduce to

$$\begin{aligned}P_{11}(\mathbf{kk}';\omega)&=\frac{\gamma\hbar M_0}{2}(u_k+v_k)(u_{k'}+v_{k'})\times[G_1(\mathbf{kk}';\omega)+G_2(\mathbf{kk}';\omega)];\\ P_{21}(\mathbf{kk}';\omega)&=\frac{\gamma\hbar M_0}{2i}(u_k-v_k)(u_{k'}+v_{k'})\times[G_1(\mathbf{kk}';\omega)-G_2(\mathbf{kk}';\omega)],\end{aligned} \tag{44}$$

in which

$$\begin{aligned}(2\pi)^2 d\, G_1(\mathbf{kk}';t)&=(i/\hbar)\theta(t)\langle[t_{-k}^\dagger(t),t_{k'}(0)]\rangle;\\ (2\pi)^2 d\, G_2(\mathbf{kk}';t)&=(i/\hbar)\theta(t)\langle[t_k(t),t_{-k'}^\dagger(0)]\rangle.\end{aligned} \tag{45}$$

In an approximation equal in accuracy to the kinetic equation approach, it can be shown[13,14] that

$$\begin{aligned}G_1(\mathbf{kk}';\omega)&\simeq\frac{\delta(\mathbf{k}+\mathbf{k}')}{e_{k'}+\hbar\omega+i\Gamma_{k'}};\\ G_2(\mathbf{kk}';\omega)&\simeq\frac{\delta(\mathbf{k}+\mathbf{k}')}{e_k-\hbar\omega-i\Gamma_k},\end{aligned} \tag{46}$$

neglecting corrections to e_k due to the scattering. The quantity Γ_k, which we consider in more detail in Sec. VI, is the energy dispersion of the state $\mathbf{k}_t$, and is closely related to the probability per unit time of a transition out of this state [Eqs. (46) are valid only for collisions which conserve momentum]. Putting Eq. (46) into Eq. (43),

$$z_0(t)=\gamma^2\mu_0^2M_0 d\int_{-\infty}^{\infty}\frac{d\omega}{2\pi}H_0(\omega)e^{-i\omega t}\times\left[\left(\frac{e_0^2+\Gamma_0^2}{\hbar^2}-\omega^2\right)-i\frac{2\Gamma_0}{\hbar}\omega\right]^{-1}, \tag{47}$$

corresponding to the equation of motion

$$\mu\ddot{z}_0+\eta\dot{z}_0+\kappa z_0=2\mu_0M_0H_0(t), \tag{48}$$

where

$$\begin{aligned}\mu&=2/\gamma^2\mu_0 d;\\ \eta&=4\Gamma_0/\gamma^2\mu_0\hbar d;\\ \kappa&=2(e_0^2+\Gamma_0^2)/\gamma^2\hbar^2\mu_0 d.\end{aligned} \tag{49}$$

We identify μ as the DW effective mass, identical to the mass, identical to the expression given by Döring,[15] η as the DW damping parameter, proportional to the dispersion of the uniform translation, and κ as the restoring-force constant. If $e_0\gg\Gamma_0$, as is necessary for Eq. (46) to be true, we find

$$\kappa=4KM_0^2/d, \tag{50}$$

verifying that the expression in Eq. (9) describes a linear restoring force. Equations (49) may be regarded as a derivation of the DW effective mass and damping parameter from the first principles.

The equations of motion for $\langle M_1'\rangle$ and $\langle M_2'\rangle$, as obtained from Eq. (42), are

$$\begin{aligned}\frac{\partial\langle M_1'\rangle}{\partial t}+\frac{\Gamma_0}{\hbar}\langle M_1'\rangle&=-\gamma M_0\mu_0\langle M_2'\rangle\\ \frac{\partial\langle M_2'\rangle}{\partial t}+\frac{\Gamma_0}{\hbar}\langle M_2'\rangle&=\gamma\mu_0M_0H_0\sin\varphi_s(z)\\ &\quad+2\gamma M_0\mu_0^2K\langle M_1'\rangle.\end{aligned} \tag{51}$$

The damping is of Bloch-Bloembergen form

$$[\mathbf{M}\cdot(\partial\mathbf{M}/\partial t)\neq0],$$

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[14] L. Kadanoff and G. Baym, *Quantum Statistical Mechanics* (W. A. Benjamin, Inc., New York, 1962), Chap. IV.

[15] W. Döring, Z. Naturforsch. **3a**, 373 (1948).

and we are forced to conclude that DW motion cannot properly be described by the Landau-Lifshitz[16] equation unless $\Gamma_0=0$. This is rather curious, since Eq. (48) *can* be derived from the Landau-Lifshitz equations, but only by assuming that $\langle M_1'\rangle \propto \dot{z}_0$.

VI. SOME CONTRIBUTIONS TO Γ_0

The contributions to Γ_0 arising from the terms in Φ, Φ_1, Ψ, Ψ_1, and Ψ_2 in the translational Hamiltonian (24) are, in the second order of perturbation theory,

$$\begin{aligned}\Gamma_0=&\frac{1}{(2\pi)^5d^3}\int|\Phi(10;-1)|^2(n_1+n_{-1}+1)\\&\times\delta(e_1+e_{-1}-e_0)d\mathbf{k}_{t1}+\frac{4}{(2\pi)^2d^4}\int|\Psi(0,1;2,1-2)|^2\\&\times[n_1(n_2+1)(n_{1-2}+1)-(n_1+1)n_2n_{1-2}]\\&\times\delta(e_2+e_{1-2}-e_1-e_0)d\mathbf{k}_{t1}d\mathbf{k}_{t2},\end{aligned}\tag{52}$$

where

$$n_k=[\exp(e_k/k_BT)-1]^{-1}.\tag{53}$$

(These results may be compared to the scattering arising from similar terms in the theory of magnetic resonance.[8]) Since $e_k+e_{-k}>e_0$, the first term in Eq. (52) is zero, and the entire contribution to Γ_0 arises from the second term. We can rewrite this term in the form

$$\begin{aligned}\Gamma_0=&\frac{4}{(2\pi)^7d^4}[\exp(e_0/k_BT)-1]\\&\times\int|\Psi(0,1;2,1-2)|^2(n_1+1)n_2n_{1-2}\\&\times\delta(e_2+e_{1-2}-e_1-e_0)d\mathbf{k}_{t1}d\mathbf{k}_{t2}.\end{aligned}\tag{54}$$

We get an order-of-magnitude estimate of Γ_0 by writing $e_k\simeq e_0+\gamma\hbar M_0\alpha k^2{}_t$. Then

$$\frac{\Gamma_0}{e_0}\simeq\frac{1}{256(2\pi)^3}\left(\frac{\mu_0}{2K}\right)^2\left(\frac{\gamma\hbar\mu_0}{M_0\alpha d}\right)^2\times\begin{cases}\dfrac{k_BT}{e_0}e^{-e_0/k_BT}, & \dfrac{k_BT}{e_0}\ll1;\\[2ex]\left(\dfrac{k_BT}{e_0}\right)^2, & \dfrac{k_BT}{e_0}\gg1.\end{cases}\tag{55}$$

Assuming $2K/\mu_0=10^{-2}$, $\alpha=10^{-7}$ erg/cm, $d=10^{-5}$ cm, and $\mu_0M_0=1000$ G, so that $e_0/h\simeq200$ Mc/sec, this becomes

$$\begin{aligned}\frac{\Gamma_0}{e_0}&\simeq10^{-9}Te^{-10^{-2}/T}, && T\ll10^{-2}\ {}^\circ\mathrm{K}\\&\simeq10^{-7}T^2, && T\gg10^{-2}\ {}^\circ\mathrm{K}.\end{aligned}\tag{56}$$

The damping corresponding to Eq. (56) becomes quite appreciable above about 100°K. However, measured DW mobilities are usually found to increase with temperature, while Eq. (56) leads to a mobility which decreases with increasing temperature. It is important to note that Γ_0 measures the linewidth of the DW resonance, while mobility is usually obtained experimentally for a wall which has broken free of constraints. The discrepancy in temperature dependence presumably arises because the assumed binding is a poor approximation to physical reality, but it would be interesting to see how the linewidth of the DW resonance in the initial permeability spectrum depends upon temperature.

We also suppose that DW damping can arise from extrinsic sources (impurities, imperfections, internal fields dependent on sample shape, etc.), and consider as an example the "fluctuations in internal fields" proposed by Clogston *et al.*[6] as a possible source of scattering in disordered systems. We add to the Hamiltonian the scattering term

$$\begin{aligned}\mathcal{H}_{\mathrm{scat}}=&\int\delta D(\mathbf{r},\mathbf{r}')\Big\{\mathbf{M}(\mathbf{r},t)\cdot\mathbf{M}(\mathbf{r}',t)\\&-3\frac{[M(\mathbf{r},t)\cdot(\mathbf{r}-\mathbf{r}')][\mathbf{M}(\mathbf{r}',t)\cdot(\mathbf{r}-\mathbf{r}')]}{|\mathbf{r}-\mathbf{r}'|^2}\Big\}d\mathbf{r}d\mathbf{r}',\end{aligned}\tag{57}$$

where $\delta D(\mathbf{r},\mathbf{r}')$ is a function describing fluctuations in the internal fields due to the irregularity of the system. Clogston *et al.* show that if it is assumed that the fluctuations are uncorrelated, that is that

$$\int\delta D(\mathbf{r}_1,\mathbf{r}_2)\delta D(\mathbf{r}_3,\mathbf{r}_2+\mathbf{r})dV_2dV_1=|\delta D|^2\delta(\mathbf{r}),\tag{58}$$

the contribution to the resonance linewidth in a spherical sample in a strong dc magnetic field ($H_0\gg M_0$) is approximately

$$\Gamma_{\mathrm{FMR}}\cong\frac{(\gamma\hbar M_0)^2}{2\pi^2}|\delta D|^2\int\delta(\epsilon_0-\epsilon_k)d\mathbf{k}\times(\text{our notation}).\tag{59}$$

On the other hand, it is possible to show that, if the translational Hamiltonian includes the term

$$\mathcal{H}_{\mathrm{scat}}=\frac{1}{(2\pi)^4d^2}\int F(1;2)t_1{}^\dagger t_2d\mathbf{k}_{t1}d\mathbf{k}_{t2},\tag{60}$$

describing the scattering arising from Eq. (57), the dispersion of the uniform translation is[17]

$$\Gamma_0=\frac{1}{(2\pi)^4d^2}\int F(0;1)F(1;2)\delta(e_0-e_1)d\mathbf{k}_{t1}d\mathbf{k}_{t2}.\tag{61}$$

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[16] L. Landau and E. Lifshitz, Phys. Z. Sowjetunion **8**, 153 (1935).

[17] Equations (46) are the leading terms of the Green's functions expansions in F. In addition to these terms, the Green's functions contain terms not proportional to δ functions when the momentum is not conserved during scattering. These terms describe the population of states degenerate with e_0 by the scattering and will be neglected here; this is equivalent to assuming that the populations of the degenerate states are close to their thermal equilibrium values.

The quantity F arising from Eq. (57) is approximately

$$F(\mathbf{k}_1;\mathbf{k}_2)\cong\gamma\hbar M_0(\mu_0/2K)^{1/2}\int\delta D(\mathbf{r},\mathbf{r}')e^{i(\mathbf{k}_{t2}\cdot\mathbf{r}'-\mathbf{k}_{t1}\cdot\mathbf{r})}\times\operatorname{sech}(z/d)\operatorname{sech}(z'/d)dVdV'. \quad (62)$$

If we make the assumption that δD is also uncorrelated in the presence of the wall, or that

$$\int\delta D(\mathbf{r}_3,\mathbf{r}_2+\mathbf{r})\delta D(\mathbf{r}_1,\mathbf{r}_2)\operatorname{sech}(z_1/d)\operatorname{sech}(z_2/d)\times\operatorname{sech}(z_2+z/d)dV_1dV_2\cong|\delta D|^2\delta(\mathbf{r}), \quad (63)$$

then Eq. (61) reduces to

$$\Gamma_0\cong\frac{(\gamma\hbar M_0)}{(2\pi d)}\frac{\mu_0}{2K}|\delta D|^2\int\delta(e_0-e_k)d\mathbf{k}_t. \quad (64)$$

The ratio of Eq. 64 to Eq. 59 is

$$\frac{\Gamma_0}{\Gamma_{\text{FMR}}}\cong\frac{\pi}{d}\left(\frac{\mu_0}{2K}\right)\frac{\int\delta(e_0-e_k)d\mathbf{k}_t}{\int\delta(\epsilon_0-\epsilon_k)d\mathbf{k}}. \quad (65)$$

Using Eq. (27) for e_k, and the spin-wave spectrum for a sphere,[6] and assuming that $2K/\mu_0\ll1$, $\beta/\mu_0\ll1$, we can evaluate the integrals in Eq. (65) approximately to find that $\Gamma_0/\Gamma_{\text{FMR}}\approx1$, completely independent of K and β, provided that neither K nor β goes to zero (so that a domain wall exists, and the uniform translation has nonzero energy). Identifying the resonance linewidth as $\Delta H=2\Gamma_{\text{FMR}}/\gamma\hbar\mu_0$, and using Eqs. (48) and (49) to obtain the DW mobility $\nu=(\gamma\mu_0)^2M_0\hbar d/2\Gamma_0$, we thus find

$$\nu\Delta H\approx\gamma\mu_0M_0d\approx10^5\text{ cm/sec}, \quad (66)$$

where the value 10^5 cm/sec is obtained for $M_0=10^5$ A/m ($\sim$1000 G), $d=10^{-7}$ m. The values of ν obtained from Eq. (66) are in reasonably good agreement with mobilities ($\sim$1000 cm/sec/Oe) obtained in ferrites when the DW has broken free of any restraining influences, although the analysis applies only to a wall performing small excursions about an equilibrium position.

VII. SUMMARY AND CONCLUSIONS

A quantum-mechanical formalism for the description of domain-wall motion has been developed, which embodies as a basic feature the degeneracy of other states with the state excited by a uniform magnetic field. This degeneracy is an important part of the theory of DW damping due to irregularities and imperfections.

The equations describing the motion of the magnetization include loss terms which cannot be obtained from any formalism in which the magnetization is preserved; DW motion is properly described by a combination of Bloch-Bloembergen and Landau-Lifshitz damping. The damping due to intrinsic scattering processes is small in the model we have used, and most of the damping (in the absence of after-effect, fast-relaxer and eddy-current damping) appears to arise from scattering by imperfections.

It is most important to recognize the essential feature of the model, i.e., that internal magnetic fields due to the sample geometry have been eliminated. It is these internal fields which make the DW structure possible in the first place, and they will, in general, supply strong interactions which greatly enhance the scattering. Presumably, they can also give rise to terms in the Hamiltonian which lead to binding of the wall to an equilibrium position in the crystal. The effects of such geometry-dependent internal fields are minimized only in certain very special configurations, such as picture frames, and the model used above is expected to apply qualitatively to these special configurations.

It is much more difficult to assess the scattering effects produced by the demagnetizing fields associated with crystalline imperfections; the fluctuating fields in a ferrite are quite a different matter from the magnetic field associated with a crystalline void. Such fields are the real origin of the restoring-force terms, such as Eq. (9), but may also give rise to strong scattering, which we have not taken into account.

The theory presented thus applies to domain walls in relatively perfect crystals of the proper shape. We have examined only two types of damping mechanisms. There are other interactions, such as magnetostrictive interactions with phonons, which may have to be invoked to explain DW damping in such highly ordered materials as YIG. Treatment of such interactions, with the use of the present formalism, should be no more difficult than in the theory of magnetic resonance.

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