where k is of the order of 2.5 and depends on the geometry of the outer conductor, and mks units are used. Therefore

$$U_0=(G\beta b^2/4\pi)\ln(kR/\rho)$$

and

$$\partial U_0/\partial\rho=-(G\beta b^2/4\pi\rho).$$

The same expression is obtained by an approximation in which the material within a circle of radius ρ is taken to be uniformly stressed, in tension above and in compression below the strip, while outside the circle it is stressed just as if the strip were of infinitesimal width. This relatively simple stress distribution appears to be a good approximation to the exact situation.

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Surface Effect in Secondary and Photoelectric Emission

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The secondary emission yield of metals by the surface effect δ is expressed in terms of the surface photoelectric yield $y(\omega)$ for a radiation of frequency ω and angle of incidence of $\cos^{-1}\frac{1}{2}(5^{\frac{1}{2}}-1)$ which is about 52°. It is shown that $\delta\sim(2\pi\alpha E_p)^{-1}\int_{\omega_1}^{\omega_2} y(\omega)d\omega/\omega$, where E_p is the primary energy in atomic units, $\alpha=1/137$, ω_1 is the threshold frequency, and ω_2 depends on the energy of the primary and may be replaced by ∞. For a square-well potential model for a metal, $\delta\sim10^{-3}/E_p$ with a relative error of order $(E_F/E_p)\ln(E_p/E_F)$, where E_F is the Fermi energy.

1. INTRODUCTION

THE purpose of this paper is to establish a general and simple relation between the surface effect in secondary electron emission[1] (SSE) from metals and the surface photoelectric effect[2,3] (SPE), and to use this relation to show clearly why the SSE is so small that it can be neglected in explaining the experimental facts. Such a relation is of interest for its own sake, and in addition a new examination of the problem is desirable since most of the published papers on the SSE, are incorrect.[4,5]

We shall use for convenience Hartree's atomic units in which $\hbar$, the electron mass m, the Bohr radius a_B, and the electron charge e are unity, and the speed of light c is 137. As was done in discussing the surface photoelectric effect in I, we treat the conduction electrons as independent noninteracting particles. The motion of a single electron in the y and z directions is free and can be described by the plane-wave $L^{-1}\exp i(k_y y+k_z z)$ obeying cyclic boundary conditions and normalized to unity in a square area of side L. The x motion is bound by a general surface potential $V(x)$ which is the same for all electrons and varies only in the direction x which is normal to the metal surface. This motion is described by the wave function $L^{-1/2}\phi_0(x)$ normalized to unity. The length L is the thickness of the metal plate which extends from $x=-L$ to $x=0$. The function $\phi_0(x)$ satisfies the wave equation,

$$H_1\phi_0=E_0\phi_0, \tag{1.1}$$

where

$$H_1(x)=-\tfrac{1}{2}\partial^2/\partial x^2+V(x). \tag{1.2}$$

Inside the metal, $V(x)$ is a constant equal to $-V_0$, and we can write $\phi_0(x)=2^{1/2}\sin(k_x x+\gamma)$ where γ is a phase factor depending on $V(x)$, and $\frac{1}{2}k_x^2=E_0+V_0$. The potential $V(x)$ rises to zero in the surface barrier regions near $x=0$ and $x=-L$ in a distance much less than L, and $\phi_0(x)$ behaves as $\exp(-px)$ for large x, where $-\frac{1}{2}p^2=E_0$. We ignore thermal effects and assume that all energy states below the Fermi energy E_F are occupied, and all energy states above the Fermi energy are empty. The conduction electrons in this model assume $\mathbf{k}$ values which fill "a Fermi hemisphere" given by $k^2=k_F^2$ and $k_x>0$, where k_F is the Fermi momentum,

[1] For a review of secondary emission see O. Hachenberg and W. Brauer, in *Advances in Electronics and Electron Physics* (Academic Press Inc., New York, 1959), Vol. XI, p. 413; A. J. Dekker, in *Solid State Physics*, edited by F. Seitz and D. Turnbull (Academic Press Inc., New York, 1958), Vol. 6, p. 251.

[2] K. Mitchell, Proc. Roy. Soc. (London) **A146**, 442 (1934).

[3] I. Adawi, Phys. Rev. **134**, A788 (1964). This paper will be denoted by I.

[4] A. Viatskin, Zh. Eksperim. i Teor. Fiz. **9**, 826 (1939) treated a semi-infinite square-well potential model. The basic formulation is correct but the final integrations and conclusions are obscure and incorrect.

[5] W. Brauer and W. Klose, Ann. Physik **19**, 116 (1956). This paper has been assumed correct in the two review articles cited in Ref. 1, but it contains unfortunately basic errors. They treat a finite square well of width $2a$ in the limit that $a\to\infty$. The correct final state which should be used in the transition matrix element is the function v^* used here. Using the notation and Eq. (4.5) of I we have that the incoming wave v is given by $v=\frac{1}{2}(\phi^s/\Lambda_s^*+\phi^a/\Lambda_a^*)$. When the correct limiting procedure is applied as $a\to\infty$, the results of the finite and the semi-infinite square well become identical as has been discussed in general in I. With this in mind, and for a primary electron incident normal to the metal surface, none of the four delta functions obtained by Brauer and Klose and on which essentially all their discussion is based should arise; and the effect is precisely determined by terms similar to those they ignored. The yield by the surface effect in secondary emission as in photoelectric emission is independent of the dimensions of the model analyzed, and there is no need to introduce an *ad hoc* depth d_s for calculating the effect.

i.e., $E_F=\frac{1}{2}k_F^2$. The number of electrons per unit volume in the interval d^3k is given by $4d^3k/(8\pi^3)$, where the factor 4 is contributed by the usual factor 2 for spin and an additional factor 2 to take account of the distribution of the bound states of the x motion in the continuum limit[3] (large $V_0^{1/2}L$).

We consider a primary electron of momentum $\mathbf{K}$ and energy $E_p=\frac{1}{2}K^2$ incident *normal* to the metal surface, namely, $\mathbf{K}=(-K,0,0)$. The mutual Coulomb repulsion between the primary electron and the conduction electrons bound by the surface potential offers a *direct* mechanism for a secondary electron emission from the bombarded metal surface. It is with this process that we are here concerned. We do not concern ourselves with those electrons which are left excited after colliding with the primary, and which diffuse in the metal, and appear later as secondaries. We hardly need to emphasize that the effect we are discussing exists only by virtue of the binding of the conduction electrons to the surface potential, since it is well known that no secondaries can be *directly* emitted from a metal surface bombarded with *normally* incident primaries if the electron gas is completely free.[6]

In Sec. 2 we discuss the scattering problem of the system consisting of a primary electron and a conduction electron, hereafter referred to as secondary. We shall consider only fast primaries (nonrelativistic) whose energy is much greater than the Fermi energy. Indeed, the situations of most practical interest[1] are those where the primary energy E_p is of order 10 (272 eV) and E_p/E_F of order 100. We are then justified in treating the Coulomb field of the two electrons as a perturbation and neglecting the effect of the surface potential on the primary wave functions. We shall, however, treat exactly the effect of the surface potential on the motion of the secondary in the final state which leads to the use of the well-known "incoming wave"[3,7] in the transition matrix element. The two electrons will be treated as distinguishable particles described by a product wave function, since we shall see that the main contributions to the SSE come from small-angle scattering for which exchange effects are not important.

In Sec. 3 we use the small-angle scattering approximation and derive the main result of the paper which expresses the secondary emission yield by the surface effect as an integral over the surface photoelectric yield. In Sec. 4 we illustrate this result by treating two examples. In the first example we use experimental measurements on the SPE. In the second example we apply our formulas to the previously treated[4,5] square-well potential model for the metal, and obtain the yield by the SSE which turns out to be in agreement with Baroody's unpublished result.[8]

[6] H. Fröhlich, Ann. Physik **5**, 13, 229 (1932).

[7] See G. Breit and H. A. Bethe, Phys. Rev. **93**, 888 (1954); I. Adawi, Am. J. Phys. **32**, 211 (1964).

[8] E. M. Baroody (unpublished report); and abstract in Phys. Rev. **92**, 843 (1953).

2. FORMULATION

The unperturbed Hamiltonian of the system H_0 consists of three commuting parts: H_1 defined in (1.2), H_2 the kinetic energy of the secondary for the y and z motions, and H_3 the kinetic energy of the primary. Thus,

$$H_0=H_1+H_2+H_3, \tag{2.1}$$

$$H_2=-\tfrac{1}{2}(\partial^2/\partial y^2+\partial^2/\partial z^2), \tag{2.2}$$

$$H_3=-\tfrac{1}{2}\nabla_{\mathbf{R}}^2, \tag{2.3}$$

where $\mathbf{r}=(x,y,z)$ is the position of the secondary and $\mathbf{R}$ is the position of the primary. The perturbation potential H' is given by the Coulomb field,

$$H'=1/|\mathbf{R}-\mathbf{r}|. \tag{2.4}$$

We consider as in Sec. 1 a metal plate defined by $-L<x<0$, and discuss the secondary emission from a square region of area L^2 of the surface $x\approx 0$ which is bombarded by a uniform primary beam moving in the $-x$ direction. The interaction volume is L^3 and we normalize the primary wave function to unity in this volume. Thus the initial primary state is $L^{-3/2}\exp(i\mathbf{K}\cdot\mathbf{R})$ with $\mathbf{K}=(-K,0,0)$. The initial state of the secondary has been defined, and we take for the initial state of the system the wave function ψ_0 defined by

$$\begin{aligned} &H_0\psi_0=\mathcal{E}\psi_0,\\ &\psi_0=L^{-3}\exp i(\mathbf{K}\cdot\mathbf{R}+\mathbf{k}_\perp\cdot\mathbf{r})\phi_0(x),\\ &\mathcal{E}=E_0+\tfrac{1}{2}(k_\perp^2+K^2), \end{aligned} \tag{2.5}$$

where for any vector $\mathbf{s}$, $\mathbf{s}_\perp\equiv(0,s_y,s_z)$.

The solution ψ^+ of the perturbed problem for outgoing waves satisfies the scattering integral equation,[9,10]

$$\psi^+=\psi_0+[1/(\mathcal{E}-H_0+i\epsilon)]H'\psi^+. \tag{2.6}$$

If we write

$$\psi^+=\psi_0+\psi_s, \tag{2.7}$$

where ψ_s is the scattered wave, we obtain to first order in H' that

$$\psi_s=[1/(\mathcal{E}-H_0+i\epsilon)]H'\psi_0. \tag{2.8}$$

We expand H' in terms of the eigenfunctions of the primary and write the usual Fourier series,

$$H'=1/|\mathbf{R}-\mathbf{r}|=\textstyle\sum_\kappa(4\pi/\kappa^2)L^{-3}\exp[i\boldsymbol{\kappa}\cdot(\mathbf{r}-\mathbf{R})], \tag{2.9}$$

in which the components of $\boldsymbol{\kappa}$ are given by $(2\pi/L)$ times an integer which is also the case for k_y and k_z.

If in (2.8) we substitute for H' and ψ_0 the expressions given by (2.9) and (2.5) and observe that the relation

$$\begin{aligned} (\mathcal{E}-H_0+i\epsilon)^{-1}&\exp i(\mathbf{K}'\cdot\mathbf{R}+\mathbf{k}_\perp'\cdot\mathbf{r})\\ &=[\exp i(\mathbf{K}'\cdot\mathbf{R}+\mathbf{k}_\perp'\cdot\mathbf{r})]\\ &\quad\times[\mathcal{E}-H_1-\tfrac{1}{2}(K'^2+k_\perp^2)]^{-1}, \end{aligned} \tag{2.10}$$

[9] B. A. Lippmann and J. Schwinger, Phys. Rev. **79**, 469 (1950).

[10] M. Gell-Mann and M. L. Goldberger, Phys. Rev. **91**, 398 (1953).

holds for a general $\mathbf{K}'$ and $\mathbf{k}'$, we obtain

$$\psi_s=\sum_\kappa \exp\{i[(\mathbf{K}-\boldsymbol{\kappa})\cdot\mathbf{R}+(\mathbf{k}_\perp+\boldsymbol{\kappa}_\perp)\cdot\mathbf{r}]\} \times(4\pi/\kappa^2)L^{-6}\phi(\boldsymbol{\kappa},x), \quad (2.11)$$

$$\phi(\boldsymbol{\kappa},x)=\int G^+(E';x,x')e^{i\kappa_x x'}\phi_0(x')dx', \quad (2.12)$$

$$G^+(E')\equiv(E'-H_1+i\epsilon)^{-1}, \quad (2.13)$$

$$E'\equiv\tfrac{1}{2}q^2\equiv E_0+\omega, \quad (2.14)$$

$$\omega=-\kappa_x K-\tfrac{1}{2}\kappa_x^2-\kappa_\perp^2-\mathbf{k}_\perp\cdot\boldsymbol{\kappa}_\perp, \quad (2.15)$$

which reduces the problem to a one-dimensional problem, since the Green's function $G^+(E')$ contains only the Hamiltonian of the x motion.

To obtain the secondary current we examine the asymptotic behavior of ψ_s for positive large x. For large x and finite x' we have as in I that,

$$G^+(E';x,x')\sim(iq)^{-1}e^{iqx}v(x'), \quad (2.16)$$

where $v(x)$ is the "incoming wave" defined as the *incident wave* $exp(-iqx)$ *plus the reflected and transmitted waves due to the potential* $V(x)$. From (2.16) and (2.12)

$$\phi(\boldsymbol{\kappa},x)\sim(iq)^{-1}Me^{iqx}, \quad (2.17)$$

$$M\equiv\int v(x)e^{i\kappa_x x}\phi_0(x)dx\equiv\langle v^*|\exp(i\kappa_x x)|\phi_0\rangle. \quad (2.18)$$

From the cylindrical symmetry of the problem, the total secondary particle current is in the x direction, and we need to calculate, therefore, the current element dI_x contributed by ψ_s and threading the area L^2. Using (2.17) in (2.11) and the usual formula,

$$dI_x(x)=\frac{1}{2i}\int\left[\psi_s^*\frac{\partial\psi_s}{\partial x}-\psi_s\frac{\partial\psi_s^*}{\partial x}\right]d^3R dydz, \quad (2.19)$$

we obtain

$$dI_x=\sum_\kappa 16\pi^2\kappa^{-4}|M|^2q^{-1}L^{-7}. \quad (2.20)$$

Notice that in (2.19) the y and z integrations give L^2, the $\mathbf{R}$ integration gives L^3, and the orthogonality of the primary final states eliminates any interference terms between different values of $\boldsymbol{\kappa}$. The total current I_x is obtained by summing (2.20) over the conduction electrons in the volume L^3, and if we replace the summations by integrations we obtain

$$I_x=L^{-1}\int(d^3k/2\pi^3)\int(2/\pi)d^3\kappa(\kappa^{-4}|M|^2q^{-1}), \quad (2.21)$$

where the $\mathbf{k}$ integration is extended over the Fermi hemisphere and the $\boldsymbol{\kappa}$ integration is restricted by the condition that q is real.

If the normally incident primary is to lose energy to the secondary, κ_x *must be negative*. Furthermore, ω of (2.15) must exceed the metal work function ω_1 before any secondaries can be emitted *directly* by the present mechanism. This implies that some screening of the Coulomb field is automatically included in the model and we see from (2.15) that, to a good approximation, all $\kappa<\omega_1/K$ do not contribute.[11]

The SSE yield δ, which is the number of secondaries emitted per incident primary, is obtained by dividing I_x by the primary current K/L, namely,

$$\delta=(2/\pi K)\int(d^3k/2\pi^3)\int d^3\kappa(\kappa^{-4}q^{-1}|M|^2). \quad (2.22)$$

To show clearly that δ is independent of L as $L\to\infty$, we must investigate the nature of M in this limit. It was shown in I that in discussing electron emission from the surface $x=0$, a finite plate model with two surface barriers, one at $x=0$ and the other at $x=-L$, reduces in the limit $L\to\infty$ to a semi-infinite model with one barrier at $x=0$. The limiting "incoming wave" $v(x)$, which must be used in (2.18), is a solution of the wave equation for this limiting potential for $x>-L$, and is zero for $x<-L$. Thus inside the metal $v(x)\propto\exp(-ik_x'x)$ where a cutoff is implied to render $v(x)$ zero as $x\to\infty$, and where $k_x'^2=q^2+2V_0=k_x^2+2\omega$. The integrand in (2.18) for $x<0$ is of the form

$$\exp i(-k_x'\pm k_x+\kappa_x)x$$

for which the phase cannot vanish, since $k_x'>k_x$ and $\kappa_x<0$, and hence the integral cannot display a delta-function-type singularity which, if it were present, would bring a length[12] L in $|M|^2$. Obviously, the integration in (2.18) over x between $x=0$ and $x=\infty$ is convergent since $\phi_0(x)$ is a damped wave for large x. We conclude that except for thin films, to which still the present formulation can easily be adapted, $|M|^2$, and δ are independent of L, and we can set formally $L=1$ in the preceding equations.

For completeness, we give now the results of the time proportional transitions method in discussing the scattering problem. A detailed derivation can be found in Gell-Mann and Goldberger,[10] or obtained by other means. The transition from the initial state ψ_0 of (2.5) to the plane-wave final state ψ_f of energy $\mathcal{E}'$, where $\psi_f=\exp i(\mathbf{K}'\cdot\mathbf{R}+\mathbf{k}'\cdot\mathbf{r})$, $\mathbf{K}'=\mathbf{K}-\boldsymbol{\kappa}$ and $\mathbf{k}'=(q,k_y',k_z')$, is determined by the transition matrix element T_{f0}

[11] For a uniform electron gas screening sets in for momenta $<\omega_p/K$ where ω_p is the plasma frequency, see, e.g., H. A. Kramers, Physica **13**, 401 (1947), J. Lindhard, Kgl. Danske Viedenskab. Selskab, Mat. Fys. Medd. **28**, 8 (1954). In the surface problem under consideration, the electron density, plasma frequency, and dielectric constant are variable in the region of interest, and it is not deemed worthwhile to discuss screening beyond what naturally occurs in the problem.

[12] Notice that as $L\to\infty$, $2\pi\delta(k)=\int_{-1/2L}^{1/2L}dx\exp(ikx)$, and $2\pi|\delta(k)|^2$ is interpreted as $L\delta(k)$. If M had a delta function behavior the sum over k_x would have to be restricted to a length d_s and not L, where d_s is a characteristic escape depth for the secondary, but then we would be really discussing a volume effect!

which is given by

$$T_{f0}=\langle\psi_f|H'+V|\psi^+\rangle. \tag{2.23}$$

After some algebra (2.23) gives to first order in H' the result,

$$T_{f0}=\langle\psi_f^-|H'|\psi_0\rangle, \tag{2.24}$$

where $\psi_f^-=v^*(x)\exp i(\mathbf{K}'\cdot\mathbf{R}+\mathbf{k}_\perp'\cdot\mathbf{r})$, the only difference between (2.24) and the simple version of the Born approximation is the replacement of $\exp(iqx)$ by $v^*(x)$. The $\mathbf{R}$ integration in (2.24) gives $4\pi\kappa^{-2}$, and the y and z integrations give Kronecker deltas, since momentum is conserved for these directions, and we have

$$T_{f0}=4\pi\kappa^{-2}M\delta_{k_y',k_y+\kappa_y}\delta_{k_z',k_z+\kappa_z}. \tag{2.25}$$

By summing the transition rate, $2\pi|T_{f0}|^2\delta(\mathcal{E}'-\mathcal{E})$, over k_y' and k_z' holding $\boldsymbol{\kappa}$ fixed, and integrating over q we obtain $16\pi^2\kappa^{-4}|M|^2q^{-1}$, which when summed over $\boldsymbol{\kappa}$ and $\mathbf{k}$ gives exactly Eq. (2.21) for the total current (with $L=1$).

3. RESULT

The yield δ of (2.22) is controlled mainly by the Coulomb scattering which is proportional to κ^{-4}, and by the matrix element M which is the Fourier transform of $v\phi_0$, where v and ϕ_0 are eigenfunctions of H_1 whose energy difference is ω. The problem is basically the same as the ionization of atoms by fast electrons for which the physical principles are well established.[13] We shall here use the energy k_F^2 as a characteristic energy to obtain order of magnitude estimates, in much the same way as the binding energy is used in atomic collisions. For $\kappa_x=0$, $M=0$, and for small κ_x we write $\exp(i\kappa_x x)\sim1+i\kappa_x x$, and we have the well-known dipole approximation,

$$M\approx i\kappa_x\langle v^*|x|\phi_0\rangle. \tag{3.1}$$

From the commutation relations, $[x,H_1]=\partial/\partial x\equiv D$, and $[D,H_1]=\partial V/\partial x$, we can rewrite[3] (3.1) as

$$\begin{aligned} M&\approx -i\kappa_x\omega^{-1}M_1,\\ M_1&\equiv\langle v^*|D|\phi_0\rangle\equiv-\omega^{-1}\langle v^*|V'|\phi_0\rangle. \end{aligned} \tag{3.2}$$

For $\kappa_x\gg k_F$ the binding effect of the potential $V(x)$ should become negligible, and the conduction electrons can be treated as free electrons. Since we know that in such a situation no secondaries can be emitted,[6] we conclude that M must decrease very rapidly as κ_x becomes large. This coupled with the fact that the Coulomb field strongly favors small momenta transfer, leads to the conclusion that the major contributions to the yield come from those values of κ for which $\kappa_x\ll k_F$ and $\omega\sim k_F^2$. The dipole approximation (3.1) applies and there is no need to consider large values of κ_x.

The expression (2.15) for ω will now be simplied. To estimate various terms, we recall that for a fractional energy loss Δ by the primary and for a scattering angle Θ we have $K'\approx(1-\frac{1}{2}\Delta)K$, $-\kappa_x=\frac{1}{2}(\Delta+\Theta^2)K$, and $\kappa_\perp\approx\Theta K$. The term $\frac{1}{2}\kappa_x^2$ can certainly be neglected. The term $\mathbf{k}_\perp\cdot\boldsymbol{\kappa}_\perp$ is zero on the average, and it is small compared to $-\kappa_x K$ for $\Theta^2\ll\Delta\sim k_F^2/K^2$ or $\Theta<k_F/K$. For $\Theta\gg k_F/K$, $\kappa_\perp^2\gg\boldsymbol{\kappa}_\perp\cdot\mathbf{k}_\perp$. We can, therefore, neglect the term $\boldsymbol{\kappa}_\perp\cdot\mathbf{k}_\perp$ without committing a serious error. Equation (2.15) now takes the approximate form

$$\omega\approx-\kappa_x K-\kappa_\perp^2, \tag{3.3}$$

whose accuracy will be later discussed.

By using (3.2) and (3.3) in (2.22) we can hold $\boldsymbol{\kappa}$ fixed and integrate over $\mathbf{k}$, and we can write

$$\delta=(\pi^2\alpha K)^{-1}\int d^3\kappa\,\kappa_x^2\omega^{-1}y(\omega), \tag{3.4}$$

where, by definition,

$$y(\omega)=2\pi\alpha\omega^{-1}\int(d^3k/2\pi^3)q^{-1}|M_1|^2, \tag{3.5}$$

and $\alpha=1/137$. To identify $y(\omega)$ we shall use the results of I on the first-order SPE. For a monochromatic radiation beam of angular frequency ω and polarization ($\mathbf{E}$ vector) in the plane of incidence, we find by summing Eq. (2.15a) of I over $\mathbf{k}$ that the photoelectric current density I_1 is given by

$$I_1=137ny(\omega)\sin^2\theta, \tag{3.6}$$

where n is the number of photons/unit volume in the incident beam, and θ is the angle of incidence. By dividing I_1 by $nc\cos\theta$ which is the normal component of the photon current density, we obtain the SPE yield $Y(\omega,\theta)$, namely,

$$Y(\omega,\theta)=y(\omega)(\sin^2\theta/\cos\theta). \tag{3.7}$$

Thus, $y(\omega)$ is the SPE yield for $\theta=\cos^{-1}\frac{1}{2}[(5)^{1/2}-1]=51°50'$.

The azimuthal integration in (3.4) gives 2π. The remaining double integral is written using the new variables, $\omega=-\kappa_x K-\kappa_\perp^2$ as in (3.3) and $u=\kappa_\perp^2$, and we have

$$\delta=(2\pi\alpha E_p)^{-1}\int_{\omega_1}^{\omega_2}(d\omega/\omega)y(\omega)F(\omega), \tag{3.8}$$

where

$$F(\omega)=2E_p\int_0^{u_0}du\left(1+E_p\frac{\partial}{\partial E_p}\right)[(u+\omega)^2+2E_pu]^{-1}. \tag{3.9}$$

Here, $\omega_2=\kappa_{x0}K$, where κ_{x0} is a maximum value of $-\kappa_x$ consistent with the approximation $-\kappa_x\ll k_F$, and u_0

[13] H. A. Bethe, Ann. Physik **5**, 325 (1930). See also N. F. Mott and H. S. W. Massey, *The Theory of Atomic Collisions* (Oxford University Press, New York, 1952), 2nd ed., Chap. XI; L. D. Landau and E. M. Lifshitz, *Quantum Mechanics*, translated by J.B. Sykes and J. S. Bell (Addison Wesley Publishing Company, Inc., Reading, Massachusetts, 1958), Chap. XV.

$=\omega_2-\omega$. Performing the integration in (3.9), we obtain

$$F(\omega)=\frac{\omega E_p^{1/2}}{(E_p+2\omega)^{3/2}}\ln\frac{u-u_1}{u-u_2}-\frac{2E_p[(E_p+\omega)u+\omega^2]}{(E_p+2\omega)[u^2+2(E_p+\omega)u+\omega^2]}\Bigg|_0^{u_0},$$

where

$$u_{1,2}\equiv-(E_p+\omega)\pm(E_p^2+2E_p\omega)^{1/2}. \quad (3.10)$$

For $\omega^2/E_p\ll u_0\ll E_p$, Eq. (3.10) gives

$$F(\omega)\approx 1+(\omega/E_p)\ln(1+2u_0E_p\omega^{-2}). \quad (3.11)$$

It is evident that $F(\omega)$ is a slowly varying function of ω and u_0 and hence the SPE yield in (3.8) controls the SSE yield. It is well known that[2,3] $y(\omega)$ reaches a peak for $\omega\sim k_F^2$ and drops rapidly with increasing frequency. If we take $\kappa_{x0}\sim k_F/4$ we see that $\omega_2\gg k_F^2$ for $K/k_F\gg 4$ and ω_2 may be replaced by ∞ in (3.8). This shows that the important values of ω are of order k_F^2 and that the emitted secondaries are of low energy. If $u_0\ll\omega^2/E_p$, (3.10) shows that $F(\omega)$ is small, but then $y(\omega)$ is small since $\omega_2\gg k_F^2$ and these values of ω are not decisive in determining δ. We set $u_0=\omega_2$ in (3.11) and neglect the one in the logarithmic term, and substitute the result in (3.8) to obtain

$$\delta\approx(2\pi\alpha E_p)^{-1}\int_{\omega_1}^{\infty}(d\omega/\omega)y(\omega)\times[1+(\omega/E_p)\ln(2\omega_2E_p\omega^{-2})]. \quad (3.12)$$

To simplify (3.12) we treat the logarithmic term as a constant in which we set $\omega_2=\beta_1k_FK$ and $\omega=\omega_{\max}$, where β_1 is a small fraction and $\omega_{\max}$ is the angular frequency for which the SPE yield $y(\omega)$ is maximum. We can write $\omega_{\max}=\beta_2k_F^2$ where β_2 is of order unity, and denote β_1/β_2^2 by a new parameter β. We obtain the result,

$$\delta\approx(2\pi\alpha E_p)^{-1}\int_{\omega_1}^{\infty}(d\omega/\omega)y(\omega)\times[1+(\omega/E_p)\ln\beta(E_p/E_F)^{3/2}]. \quad (3.13)$$

The leading term in (3.13) would have been obtained if we neglected the term κ_1^2 in (3.3). The logarithmic term gives a correction term of order $\frac{3}{2}(E_F/E_p)\times\ln(E_p/E_F)$ which is about 25% for $E_p/E_F\sim 20$.

Let us now return to the approximations made:

(*a*) $\mathbf{k}_1\cdot\boldsymbol{\kappa}_1$ *term:* This is the most bothersome term in the calculation and was neglected in (3.3). We can estimate the error by including this term in (3.9) and averaging the denominator over $\mathbf{k}$ to obtain $(u+\omega)^2+2u(E_p+E_F/5)$ for the new denominator. The integration can be carried out and we obtain the correction term, $(E_F/5E_p)\ln\beta(E_p/E_F)^{3/2}$, to be added inside the bracket of (3.13). Since this correction is smaller than the second term retained in (3.13), we can ignore it.

(*b*) *Large* κ_x*:* If in (3.10) we set $u_0=E_p-\omega$ which would allow for the primary to lose all its energy we obtain (for $\omega\ll E_p$) that

$$F(\omega, E_p-\omega)\approx(4/3)+(\omega/E_p)\ln(4E_p^2/3\omega^2). \quad (3.14)$$

Comparing (3.14) and (3.11) we see how little the large momenta transfers contribute to the effect. Since the dipole approximation of (3.2) is expected to overestimate M for large κ_x, Eq. (3.14) cannot be trusted and will be discarded.

(*c*) $-\frac{1}{2}\kappa_x^2$ *term:* This term can easily be included in (3.3). The resulting integral is similar to (3.9) and can be evaluated exactly. We find that $F(\omega)$ of (3.11) remains the same (except that 1 is replaced by $\frac{1}{2}$ inside the logarithm) and (3.13) is not altered. Equation (3.14), however, now takes the form,

$$F(\omega, E_p-\omega)\approx(5/3)+(\omega/E_p)\ln(8E_p^2/3\omega^2), \quad (3.15)$$

to which the previous remarks apply.

Thus our approximations are good, and we conclude that the simple formula,

$$\delta\sim(2\pi\alpha E_p)^{-1}\int_{\omega_1}^{\infty}(d\omega/\omega)y(\omega), \quad (3.16)$$

is quite satisfactory, and that (3.13) contains the major corrections to the SSE yield δ.

4. EXAMPLES

A. Experimental

To obtain an estimate of (3.16), let us use the experimental yield curves $Y(\omega,\theta)$ for Na and K which are reproduced in Weissler's review article.[14] From the relation (3.7) we obtain $y(\omega)$. By a *crude* numerical integration we find that the yield integrals, $\int y(\omega)d\omega/\omega$ are $\sim 6.7\times10^{-4}$ and 4×10^{-5} for Na and K, respectively, which lead to the values $\delta\sim 1.5\times10^{-2}/E_p$ and $0.9\times10^{-3}/E_p$, respectively. This shows clearly that the effect is entirely negligible, since the observed values[1] of δ are of order unity for $E_p\sim 10$.

B. Theoretical

A theoretical calculation of δ is possible for a square-well potential model. Let $V(x)=-V_0$ for $x<0$, and $V(x)=0$, for $x>0$. The wave functions v and ϕ_0 can be

[14] G. L. Weissler, *Handbuch der Physik*, edited by S. Flügge (Springer-Verlag, Berlin, 1956), Vol. 21, p. 351. For Na we use the measurements of Maurer at $\theta=60°$, and for K the measurements of Suhrmann and Theissing at $\theta=65°$. The yield curve for K has to be cut off, since a true SPE must asymptotically drop rapidly with frequency. For square well potential, for example, $y(\omega)\propto\omega^{-7/2}$ as $\omega\to\infty$ as can be deduced from Eqs. (4.4) and (4.5).

written down easily as in I, namely,

$$v(x)=\left(\frac{2q}{q+k_x'}e^{-ik_x'x},\ e^{-iqx}+\frac{q-k_x'}{q+k_x'}e^{iqx}\right),$$
$$\phi_0(x)=2^{-1/2}\left(\frac{k_x-ip}{k_x+ip}e^{-ik_xx}+e^{ik_xx},\ \frac{2k_x}{k_x+ip}e^{-px}\right) \quad \text{for } x<0 \text{ and } x>0, \tag{4.1}$$

where the propagation constants have been previously defined. By using (4.1) in (2.18) we obtain after some lengthy algebra:

$$|M|^2=32\kappa_x^2q^2k_x^2(k_x'-q)(k_x'+q)^{-1}/R,$$
$$R\equiv(2\omega-2\kappa_xk_x'+\kappa_x^2)^2[4\omega^2+\kappa_x^2(\kappa_x^2+4p^2-4\omega)]. \tag{4.2}$$

We see that if $\frac{1}{2}\kappa_x^2\ll\omega$, $R\approx16\omega^4$ and

$$|M|^2\approx2\kappa_x^2q^2k_x^2(k_x'-q)(k_x'+q)^{-1}\omega^{-4}, \tag{4.3}$$

which is precisely the dipole approximation of (3.2) with which we could have started. The advantage of giving the exact $|M|^2$ is to show how good are the approximations of Sec. 3. It is evident from (4.2) that for a finite ω and large κ_x, $|M|^2\propto\kappa_x^{-6}$ which justifies the neglect of large values of κ_x.

It follows from Eqs. (4.3), (3.2), and (3.5) above, or from Eqs. (3.9) and (3.10) of I, that

$$y(\omega)=(16\alpha/\pi)(\omega/E_F)^{-3}J_1(\omega), \tag{4.4}$$

$$J_1(\omega)=\frac{1}{2\eta}\int_{\epsilon+\Omega-\eta>0}^{\epsilon=1}(1-\epsilon)(\epsilon+\Omega-\eta)^{1/2}\times\epsilon^{1/2}[(\epsilon+\Omega)^{1/2}-(\epsilon+\Omega-\eta)^{1/2}]^2d\epsilon, \tag{4.5}$$

where $\Omega=\omega/E_F$, $\epsilon=(V_0+E_0)/E_F$, and $\eta=V_0/E_F$.

If we substitute (4.4) and (4.5) in (3.13), define the parameter λ by

$$\lambda=(E_F/E_p)\ln\beta(E_p/E_F)^{3/2}, \tag{4.6}$$

and introduce the integration variables ϵ and ζ where $\zeta=\epsilon+\Omega-\eta$, we have

$$\delta=\frac{4}{\pi^2E_p\eta}\int_0^\infty d\zeta[(\zeta+\eta)^{1/2}-\zeta^{1/2}]^2\times\zeta^{1/2}\int_0^1(1-\epsilon)\epsilon^{1/2}S(\lambda,\zeta,\epsilon)d\epsilon, \tag{4.7}$$

where

$$S(\lambda,\zeta,\epsilon)=(\eta+\zeta-\epsilon)^{-4}+\lambda(\eta+\zeta-\epsilon)^{-3}. \tag{4.8}$$

Expanding S in a power series of ϵ and integrating term by term we obtain

$$\delta=\frac{16}{\pi^2E_p}\eta^{-5/2}\left\{\frac{1}{15}\eta C_{-1}\lambda+\sum_{n=0}^{\infty}\binom{-4}{n}(-\eta)^{-n}\frac{1}{(2n+5)}\times\left[\frac{1}{2n+3}+\frac{3\lambda}{(n+1)(2n+7)}\right]C_n\right\}, \tag{4.9}$$

where

$$C_n=\int_0^\infty\zeta^{1/2}[(\zeta+1)^{1/2}-\zeta^{1/2}]^2(\zeta+1)^{-(4+n)}d\zeta$$
$$=(-1)^{n+1}\binom{1/2}{n+2}\frac{2n+9}{2n+6}\pi-\frac{8}{(2n+5)(2n+3)}, \tag{4.10}$$

$C_{-1}=7\pi/8-8/3=0.0822$, $C_0=(3\pi/16)-8/15=0.0557$,

$C_1=(11\pi/128)-8/35=0.0414$.

The leading term in (4.9) is $\delta\sim0.06(\pi^2E_p\eta^{5/2})^{-1}$ which is of order $10^{-3}/E_p$ since η is about 2. The relative error committed by neglecting terms in λ is of order $[\eta(C_{-1}/C_0)+9/7]\lambda\sim4\lambda$, which could amount to 25% for $E_p/E_F\sim100$ (if we take $\beta=1$).

This agrees with Baroody[8] who obtained the first two terms of (4.9) with $\lambda=0$, by using the same method as did Viatskin, but making an independent evaluation of the final integral. It also agrees surprisingly well with the estimate obtained above from the measurements on the surface photoelectric effect of potassium.

Note added in proof. Equation (3.16) predicts that the energy loss of the primary (essentially ω) by SSE is nearly peaked at the energy value for which $y(\omega)/\omega$ is maximum, and this is usually of order 5–10 eV. It would be interesting to investigate if such an energy loss can be detected in experiments on the characteristic energy losses of electrons in thin foils. Since the relative intensity of the scattered primaries by this effect is of order $10^{-2}-10^{-3}/E_p$, primary energies E_p of order 10 (i.e., 272 eV) are preferable, if experiments with these energies are now feasible.

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