$$N=[\mathcal{E}_{jj}(0,0)]^{1/2}[\mathcal{H}_{jj}(0,0)]^{1/2}$$
$$=\mathcal{E}_{jj}(0,0)=\mathcal{H}_{jj}(0,0)=\frac{64}{45}\frac{\pi^6K^4T^4}{(hc)^3}. \quad (5.14)$$

It follows from (5.13), (5.14), and (5.11) that σ and $\tilde{\sigma}$ may be represented in series form as follows:

$$\sigma_{jm}(\mathbf{r},\tau)=-\tilde{\sigma}_{jm}(\mathbf{r},\tau)=i\frac{180\alpha^4}{\pi^4}r_k\epsilon_{jmk}$$
$$\times\sum_{n=1}^{\infty}\frac{n\alpha+ic\tau}{[(n\alpha+ic\tau)^2+r^2]^3}. \quad (5.15)$$

Here the constant α is given by (2.3) as before.

From (5.15) we obtain in the special case $\tau=0$ (*spatial coherence* between the electric and magnetic fields), for typical nondiagonal elements of σ and $\tilde{\sigma}$,

$$\sigma_{xy}(\mathbf{r},0)=-\tilde{\sigma}_{xy}(\mathbf{r},0)=i\frac{180}{\alpha\pi^4}z\sum_{n=1}^{\infty}\frac{n}{(n^2+r^2/\alpha^2)^3}. \quad (5.16)$$

The diagonal elements are, of course, identically zero, since σ is antisymmetric.

We see from (5.16) that in the plane $z=0$ (xy plane) $\sigma_{xy}(\mathbf{r},0)$ is identically zero. In Fig. 8 the contours of $|\sigma_{xy}(\mathbf{r},0)|$ in the yz plane are presented and in Fig. 9 the variation of $|\sigma_{xy}(\mathbf{r},0)|$ along the z axis is shown.

PHYSICAL REVIEW VOLUME 134, NUMBER 5A 1 JUNE 1964

Coherence Properties of Blackbody Radiation.* II. Correlation Tensors of the Quantized Field

C. L. Mehta and E. Wolf
Department of Physics and Astronomy, University of Rochester, Rochester, New York
(Received 9 December 1963)

Expressions are derived for the electromagnetic correlation tensors of blackbody radiation defined on the basis of the theory of the quantized field. Correlation functions of all order are considered, but second-order ones are discussed in detail; it is found that these are identical with those obtained on the basis of semiclassical theory in part I of this investigation. This result illustrates a recent theorem of E.C.G. Sudarshan relating to the equivalence between semiclassical and quantum mechanical description of statistical light beams.

1. INTRODUCTION

In part I of this investigation,[1] expressions were derived for the complex second-order electromagnetic correlation tensors of blackbody radiation and their behavior was discussed in detail and illustrated by a number of diagrams. The statistical methods used were based entirely on classical concepts, though quantum mechanical features of the radiation were implicit in that treatment, since the spectrum of the radiation was taken to be given by Planck's law.

In the present paper the second-order correlation tensors introduced recently by Glauber[2] on the basis of the theory of the quantized field, are evaluated for blackbody radiation and are shown to be identical with those defined and evaluated on the basis of the semiclassical theory. This result illustrates a recent theorem of Sudarshan,[3] relating to the equivalence between semiclassical[4–6] and quantum mechanical description of statistical light beams.

In Sec. 4 higher-order correlation tensors of blackbody radiation are briefly discussed.

2. THE SECOND-ORDER CORRELATION TENSORS OF THE QUANTIZED FIELD

It will be useful to begin with some results which will be needed later, relating to quantization of the electromagnetic field and the definition of the correlation tensors of the quantized field.

The electric-field operator, at the space-time point $x\equiv(\mathbf{r},ct)$, when expanded in a Fourier series is given by[7] (with $i=x, y, z$)

$$\hat{E}_i(x)=\hat{E}_i^{(+)}(x)+\hat{E}_i^{(-)}(x), \quad (2.1)$$

* This research was supported by the U. S. Air Force Office of Scientific Research.

[1] C. L. Mehta and E. Wolf, Phys. Rev. **134A**, 1143 (1964), preceding paper.

[2] R. J. Glauber, (a) *Electronique Quantique, 3eme Conference*, edited by N. Bloembergen and P. Grivet (Dunod Cie, Paris, 1964), p. 111; (b) Phys. Rev. **130**, 2529 (1963); (c) *ibid.* **131**, 2766 (1963).

[3] E. C. G. Sudarshan, (a) Phys. Rev. Letters **10**, 277 (1963). (b) in *Proceedings of the Symposium on Optical Masers* (Polytechnique Press, Brooklyn, New York and John Wiley & Sons, Inc., New York, 1963), p. 45.

[4] The term "semiclassical" implies here that the distribution functions characterizing statistical properties of the beam are not necessarily non-negative and may therefore not be true probabilities. They are essentially Wigner distribution functions (see Refs. 5 and 6), called also "quasiprobabilities." However, in the present case (blackbody radiation) the distribution function turns out to be positive. [See Eq. (3.1).]

[5] E. P. Wigner, Phys. Rev. **40**, 749 (1932).

[6] (a) J. E. Moyal, Proc. Cambridge Phil. Soc. **45**, 99 (1949). (b) G. A. Baker, Jr., Phys. Rev. **109**, 2198 (1958). (c) C. L. Mehta, J. Math. Phys. **5**, 677 (1964).

[7] All operators are denoted by circumflex.

where

$$\hat{E}_i^{(+)}(x)=i\left(\frac{2\pi\hbar c}{V}\right)^{1/2}\sum_{\mathbf{k}}(k)^{1/2}\{\sum_{\lambda=1}^{2}\epsilon_i^{(\lambda)}\hat{a}_\lambda(\mathbf{k})e^{i\kappa x}\}, \quad (2.2)$$

$$\hat{E}_i^{(-)}(x)=\{\hat{E}_i^{(+)}(x)\}^\dagger=-i\left(\frac{2\pi\hbar c}{V}\right)^{1/2}\sum_{\mathbf{k}}(k)^{1/2}$$
$$\times\{\sum_{\lambda=1}^{2}\epsilon_i^{(\lambda)}\hat{a}_\lambda^\dagger(\mathbf{k})e^{-i\kappa x}\}. \quad (2.3)$$

Here V represents the volume to which the field is confined and $\hat{a}_\lambda(\mathbf{k})$ and $\hat{a}_\lambda^\dagger(\mathbf{k})$ are the annihilation and creation operators, respectively, for a photon of momentum $\hbar\mathbf{k}$ and polarization λ; they satisfy the commutation relations

$$[\hat{a}_\lambda(\mathbf{k}),\hat{a}_{\lambda'}^\dagger(\mathbf{k}')]=\delta_{\lambda\lambda'}\delta_{\mathbf{kk}'}, \quad (2.4)$$

$$[\hat{a}_\lambda(\mathbf{k}),\hat{a}_{\lambda'}(\mathbf{k}')]=[\hat{a}_\lambda^\dagger(\mathbf{k}),\hat{a}_{\lambda'}^\dagger(\mathbf{k}')]=0. \quad (2.5)$$

$\boldsymbol{\varepsilon}^{(1)}$ and $\boldsymbol{\varepsilon}^{(2)}$ are two unit vectors, such that $\boldsymbol{\varepsilon}^{(1)}$, $\boldsymbol{\varepsilon}^{(2)}$ and $\mathbf{k}/k$ (with $k=|\mathbf{k}|$) form a right-handed orthogonal triad and κx represents the usual four-vector product

$$\kappa x=\mathbf{k}\cdot\mathbf{r}-kct. \quad (2.6)$$

In writing down Eqs. (2.2) and (2.3), the modified Lorentz condition was used in the form introduced by Gupta and Bleuler,[8] employing an indefinite metric: Allowable states $|\psi\rangle$ are only those for which

$$\sum_{\mu=1}^{4}\frac{\partial\hat{A}_\mu^{(+)}}{\partial x_\mu}|\psi\rangle=0, \quad (2.7)$$

where $\hat{A}^{(+)}$ is the positive frequency part of the potential of the field. This condition justifies our restriction to transverse photons ($\lambda=1,2$) as long as only expectation values of products of field operators are considered.

The magnetic-field operator may be similarly written in the form

$$\hat{H}_i(x)=\hat{H}_i^{(+)}(x)+\hat{H}_i^{(-)}(x), \quad (2.8)$$

where

$$\hat{H}_i^{(+)}(x)=i\left(\frac{2\pi\hbar c}{V}\right)^{1/2}\sum_{\mathbf{k}}(k)^{1/2}$$
$$\times\left\{\sum_{\lambda=1}^{2}\frac{(\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)})_i}{k}\hat{a}_\lambda(\mathbf{k})e^{i\kappa x}\right\}, \quad (2.9)$$

$$\hat{H}_i^{(-)}(x)=\{\hat{H}_i^{(+)}(x)\}^\dagger=-i\left(\frac{2\pi\hbar c}{V}\right)^{1/2}\sum_{\mathbf{k}}(k)^{1/2}$$
$$\times\left\{\sum_{\lambda=1}^{2}\frac{(\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)})_i}{k}\hat{a}_\lambda^\dagger(\mathbf{k})e^{-i\kappa x}\right\}. \quad (2.10)$$

If $\hat{\rho}$ is the density operator of the field, the second-order correlation tensor of the electric field, introduced by Glauber[2a,b] may defined as

$$'\mathcal{E}_{ij}(x_1,x_2)=\mathrm{Tr}\{\hat{\rho}\hat{E}_i^{(-)}(x_1)\hat{E}_j^{(+)}(x_2)\}, \quad (2.11)$$

where the indices i, j specify Cartesian components of the tensor (i, $j=x$, y, z) and the prime on $\mathcal{E}$ is used to distinguish this tensor from the corresponding one defined in paper I in classical terms. In a similar way, one may define the second-order magnetic and mixed-correlation tensors

$$'\mathcal{H}_{ij}(x_1,x_2)=\mathrm{Tr}\{\hat{\rho}\hat{H}_i^{(-)}(x_1)\hat{H}_j^{(+)}(x_2)\}, \quad (2.12)$$

$$'\mathcal{G}_{ij}(x_1,x_2)=\mathrm{Tr}\{\hat{\rho}\hat{E}_i^{(-)}(x_1)\hat{H}_j^{(+)}(x_2)\}, \quad (2.13)$$

$$'\tilde{\mathcal{G}}_{ij}(x_1,x_2)='\mathcal{G}_{ji}^*(x_2,x_1)$$
$$=\mathrm{Tr}\{\hat{\rho}\hat{H}_i^{(-)}(x_1)\hat{E}_j^{(+)}(x_2)\}. \quad (2.14)$$

To evaluate the correlation tensors $'\mathcal{E}$, $'\mathcal{H}$, $'\mathcal{G}$, and $'\tilde{\mathcal{G}}$ for blackbody radiation, it will be useful first of all to express the density operator $\hat{\rho}$ in a representation in which the base vectors are the eigenvectors $|\{z\}\rangle$ of the annihilation operator $\hat{a}_\lambda(\mathbf{k})$:

$$\hat{a}_\lambda(\mathbf{k})|\{z\}\rangle=z_\lambda(\mathbf{k})|\{z\}\rangle. \quad (2.15)$$

Here $\{z\}$ denotes the set of $z_\lambda(\mathbf{k})$ for all λ's and $\mathbf{k}$'s, so that

$$|\{z\}\rangle=\prod_{\lambda,\mathbf{k}}|z_\lambda(\mathbf{k})\rangle. \quad (2.16)$$

Since the operator $\hat{a}_\lambda(\mathbf{k})$ is not Hermitian, its eigenvalues $z_\lambda(\mathbf{k})$ are in general complex,

$$z_\lambda(\mathbf{k})=x_\lambda(\mathbf{k})+iy_\lambda(\mathbf{k}), \quad (2.17)$$

($x_\lambda(\mathbf{k})$, $y_\lambda(\mathbf{k})$ real) and the eigenvectors belonging to different eigenvalues are not orthogonal. However, they obey a closure relation (cf. Ref. 9).

$$\frac{1}{\pi}\int\prod_{\lambda,\mathbf{k}}|z_\lambda(\mathbf{k})\rangle\langle z_\lambda(\mathbf{k})|d^2z_\lambda(\mathbf{k})=\hat{1}, \quad (2.18)$$

where $\hat{1}$ is the identity operator if, as we shall assume, they are normalized so that

$$\langle z_\lambda(\mathbf{k})|z_\lambda(\mathbf{k})\rangle=1. \quad (2.19)$$

The eigenstate $|z_\lambda(\mathbf{k})\rangle$ has the following expansion in the number representation[9]:

$$|z_\lambda(\mathbf{k})\rangle=\sum_{n_\lambda(\mathbf{k})=0}^{\infty}\exp(-\tfrac{1}{2}|z_\lambda(\mathbf{k})|^2)\frac{\{z_\lambda(\mathbf{k})\}^{n_\lambda(\mathbf{k})}}{[n_\lambda(\mathbf{k})!]^{1/2}}|n_\lambda(\mathbf{k})\rangle, \quad (2.20)$$

where the $|n_\lambda(\mathbf{k})\rangle$ form an orthonormal set of eigenvectors of the number operator $\hat{a}_\lambda^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k})$:

$$\hat{a}_\lambda^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k})|n_\lambda(\mathbf{k})\rangle=n_\lambda(\mathbf{k})|n_\lambda(\mathbf{k})\rangle. \quad (2.21)$$

The eigenvectors of the annihilation operator have been found very useful in the analysis of problems

[8] See, for example, S. S. Schweber, *An Introduction to Quantum Field Theory* (Harper & Row, New York, 1961), p. 242.

[9] J. R. Klauder, Ann. Phys. (N. Y.) **11**, 123 (1960).

relating to the statistical behavior of radiation.[2,3,10] An important property of these eigenvectors is expressed by a theorem established recently by Sudarshan[3] which we shall employ. According to this theorem, any operator and in particular the density operator $\hat{\rho}$ may be expressed in the "diagonal" form

$$\hat{\rho}=\int\phi(\{z\})|\{z\}\rangle\langle\{z\}|d^2\{z\}. \tag{2.22}$$

Using this representation of the density operator, it then follows from (2.2), (2.3), (2.11), (2.22) and (2.15) and its Hermitian adjoint that the second-order correlation tensor of the electric field may be expressed in the form

$$\begin{aligned}{}'\mathcal{E}_{ij}(x_1,x_2) &=\int\phi(\{z\})\langle\{z\}|\hat{E}_i^{(-)}(x_1)\hat{E}_j^{(+)}(x_2)|\{z\}\rangle d^2\{z\} \\ &=\sum_{\lambda,\mathbf{k}}\sum_{\lambda',\mathbf{k}'}\epsilon_i^{(\lambda)}(\mathbf{k})\epsilon_j^{(\lambda')}(\mathbf{k}')\Phi_{\lambda\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2),\end{aligned} \tag{2.23}$$

where

$$\Phi_{\lambda\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2)=(2\pi\hbar c/V)(kk')^{1/2}\exp(-i\kappa x_1+i\kappa' x_2)\times\int\phi(\{z\})z_\lambda^*(\mathbf{k})z_{\lambda'}(\mathbf{k}')d^2\{z\}. \tag{2.24}$$

The corresponding expressions for the second order magnetic and mixed correlation tensors defined by (2.12)–(2.14) can also readily be written down. We only need to use the expressions (2.9) and (2.10) in place of (2.2) and (2.3) where appropriate. We then obtain in place of (2.23) the following expressions for the other second-order correlation tensors of the quantized field:

$${}'\mathcal{H}_{ij}(x_1,x_2)=\sum_{\lambda,\mathbf{k}}\sum_{\lambda',\mathbf{k}'}\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_i\left(\frac{\mathbf{k}'\times\boldsymbol{\varepsilon}^{(\lambda')}}{k'}\right)_j\times\Phi_{\lambda\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2), \tag{2.25}$$

$${}'\mathcal{G}_{ij}(x_1,x_2)=\sum_{\lambda,\mathbf{k}}\sum_{\lambda',\mathbf{k}'}\epsilon_i^{(\lambda)}(\mathbf{k})\left(\frac{\mathbf{k}'\times\boldsymbol{\varepsilon}^{(\lambda')}}{k'}\right)_j\times\Phi_{\lambda\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2), \tag{2.26}$$

$${}'\tilde{\mathcal{G}}_{ij}(x_1,x_2)=\sum_{\lambda,\mathbf{k}}\sum_{\lambda',\mathbf{k}'}\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_i\epsilon_j^{(\lambda')}(\mathbf{k}')\times\Phi_{\lambda\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2). \tag{2.27}$$

3. THE SECOND-ORDER CORRELATION TENSORS OF BLACKBODY RADIATION

We will now evaluate the correlation tensors $'\mathcal{E}$, $'\mathcal{H}$, $'\mathcal{G}$ and $'\tilde{\mathcal{G}}$ for the case of blackbody radiation. It would appear from a discussion of Glauber [Ref. 2c, Eq. (10.23)], based essentially on the central limit theorem, that the "phase-space" distribution function $\phi(\{z\})$ for blackbody radiation is given by

$$\phi(\{z\})=\prod_{\lambda,\mathbf{k}}\frac{1}{\pi\bar{n}_k}\exp\left\{-\frac{|z_\lambda(k)|^2}{\bar{n}_k}\right\}, \tag{3.1}$$

where $\bar{n}_k\equiv\bar{n}_{\mathbf{k},\lambda}$ ($\lambda=1,2$) is the average number of photons with momentum $\mathbf{k}$ and polarization λ,

$$\bar{n}_k=1/(e^{\alpha k}-1), \tag{3.2}$$

with

$$\alpha=\hbar c/KT. \tag{3.3}$$

That the "phase-space" distribution $\phi(\{z\})$ given by (3.1) does indeed lead to the well-known expression for the density operator $\hat{\rho}$ of a radiation field in thermal equilibrium at temperature T is verified directly in the Appendix to the present paper (see also Mandel[10]).

If we substitute from (3.1) into (2.24), and also use the relation

$$\begin{aligned}&\int z_{\lambda'}^*(\mathbf{k}')z_{\lambda''}(\mathbf{k}'')\prod_{\lambda,\mathbf{k}}\frac{1}{\pi\bar{n}_k}\exp(-|z_\lambda(\mathbf{k})|^2/\bar{n}_k)d^2z_\lambda(\mathbf{k}) \\ &=\frac{1}{\pi\bar{n}_{k'}}\delta_{\lambda'\lambda''}\delta_{\mathbf{k}'\mathbf{k}''}\int|z_{\lambda'}(\mathbf{k}')|^2\times\exp(-|z_{\lambda'}(\mathbf{k})|^2/\bar{n}_{k'})d^2z_{\lambda'}(\mathbf{k}') \\ &=\bar{n}_{k'}\delta_{\lambda'\lambda''}\delta_{\mathbf{k}'\mathbf{k}''},\end{aligned} \tag{3.4}$$

where δ is the Kronecker symbol, we find that

$$\Phi_{\lambda,\lambda'}(\mathbf{k},\mathbf{k}';x_1,x_2)=\frac{2\pi\hbar c}{V}k\bar{n}_k\delta_{\lambda\lambda'}\delta_{\mathbf{k}\mathbf{k}'}\exp\{-i\kappa(x_1-x_2)\}. \tag{3.5}$$

Next we assume that the linear dimensions of the enclosure are large compared with the mean wavelength of the radiation. Summation over $\mathbf{k}$ may then be replaced by integration over the whole $\mathbf{k}$ space, if use is made of the usual rules[11]

$$\frac{1}{(V)^{1/2}}\sum_{\mathbf{k}}f(\mathbf{k})\to\frac{1}{(2\pi)^{3/2}}\int d^3k f(\mathbf{k}), \tag{3.6}$$

$$\delta_{\mathbf{k}\mathbf{k}'}\to\delta(\mathbf{k}-\mathbf{k}'), \tag{3.7}$$

where f is an arbitrary function and $\delta(\mathbf{k}-\mathbf{k}')$ is the three-dimensional Dirac delta function. One then obtains from (3.5) and (2.23), the following expression for the correlation tensor $'\mathcal{E}$:

$${}'\mathcal{E}_{ij}(x_1,x_2)=\frac{\hbar c}{4\pi^2}\int k\bar{n}_k\exp\{i\kappa(x_2-x_1)\}\times\sum_{\lambda=1}^{2}\epsilon_i^{(\lambda)}(\mathbf{k})\epsilon_j^{(\lambda)}(\mathbf{k})d^3k. \tag{3.8}$$

[10] L. Mandel, Phys. Letters **7**, 117 (1963).

[11] J. M. Jauch and F. Rohrlich, *The Theory of Photons and Electrons* (Addison Wesley Publishing Co., Cambridge, 1955), p. 38.

Next we substitute for $\bar{n}_k$ from (3.2) and also use the relation

$$\sum_{\lambda=1}^{2} \epsilon_i^{(\lambda)}(\mathbf{k})\epsilon_j^{(\lambda)}(\mathbf{k})=\delta_{ij}-\frac{k_ik_j}{k^2}, \qquad (3.9)$$

which follows from the fact that $\boldsymbol{\varepsilon}^{(1)}$, $\boldsymbol{\varepsilon}^{(2)}$, and $\mathbf{k}/k$ form a triad of mutually orthogonal unit vectors. We thus finally obtain the following expression for the second-order electric correlation tensor $'\mathcal{E}$ of the quantized field:

$$'\mathcal{E}_{ij}(x_1,x_2)=\frac{\hbar c}{4\pi^2}\int \frac{k^2\delta_{ij}-k_ik_j}{k(e^{\alpha k}-1)} \times \exp\{i\kappa(x_2-x_1)\}d^3k. \qquad (3.10)$$

For the magnetic second-order correlation tensor, we obtain in a strictly similar manner from (2.25)

$$'\mathcal{H}_{ij}(x_1,x_2)=\frac{\hbar c}{4\pi^2}\int k\bar{n}_k \exp\{i\kappa(x_2-x_1)\} \times \sum_{\lambda=1}^{2}\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_i\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_j d^3k. \qquad (3.11)$$

Now the three vectors $\mathbf{k}\times\boldsymbol{\varepsilon}^{(1)}/k$, $\mathbf{k}\times\boldsymbol{\varepsilon}^{(2)}/k$ and $\mathbf{k}/k$ also form a triad of mutually orthogonal unit vectors, and hence a relation similar to (3.9) holds between them. In consequence, the right-hand side of (3.11) becomes identical with the right-hand side of (3.10) and hence we have

$$'\mathcal{H}_{ij}(x_1,x_2)={}'\mathcal{E}_{ij}(x_1,x_2). \qquad (3.12)$$

For the mixed second-order correlation tensor $'\mathcal{G}$ we obtain from (2.26) in a similar manner

$$'\mathcal{G}_{ij}(x_1,x_2)=\frac{\hbar c}{4\pi^2}\int k\bar{n}_k \exp\{i\kappa(x_2-x_1)\} \times \sum_{\lambda=1}^{2}\epsilon_i^{(\lambda)}(\mathbf{k})\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_j d^3k. \qquad (3.13)$$

It may readily be verified that

$$\sum_{\lambda=1}^{2}\epsilon_i^{(\lambda)}(\mathbf{k})\left(\frac{\mathbf{k}\times\boldsymbol{\varepsilon}^{(\lambda)}}{k}\right)_j=\epsilon_{ijl}k_l/k, \qquad (3.14)$$

where ϵ_{ijl} is the completely antisymmetric unit tensor of Levi Civita. Hence, if also (2.14) is used, (3.13) reduces to

$$'\mathcal{G}_{ij}(x_1,x_2)=-{}'\tilde{\mathcal{G}}_{ij}(x_1,x_2)=\frac{\hbar c}{4\pi^2}\epsilon_{ijl}\int\frac{k_l}{e^{\alpha k}-1} \times \exp\{i\kappa(x_2-x_1)\}d^3k. \qquad (3.15)$$

Noting that $\kappa(x_2-x_1)$ is the four-vector product

$$\kappa(x_2-x_1)=kc\tau-\mathbf{k}\cdot\mathbf{r}, \qquad (3.16)$$

where $\tau=t_1-t_2$ and $\mathbf{r}=\mathbf{r}_1-\mathbf{r}_2$, it follows on comparing Eqs. (3.10), (3.12), and (3.15) with Eqs. (5.7), (5.9), and (5.8) of paper I that

$$\begin{aligned}'\mathcal{E}_{ij}(x_1,x_2)&=\tfrac{1}{4}\mathcal{E}_{ij}^*(x_1,x_2),\\ '\mathcal{H}_{ij}(x_1,x_2)&=\tfrac{1}{4}\mathcal{H}_{ij}^*(x_1,x_2),\\ '\mathcal{G}_{ij}(x_1,x_2)&=\tfrac{1}{4}\mathcal{G}_{ij}^*(x_1,x_2),\\ '\tilde{\mathcal{G}}_{ij}(x_1,x_2)&=\tfrac{1}{4}\tilde{\mathcal{G}}_{ij}^*(x_1,x_2).\end{aligned} \qquad (3.17)$$

The relations (3.17) show that the second-order electromagnetic correlation tensors of blackbody radiation defined for the quantized field by Eqs. (2.11), (2.12), (2.13), and (2.14) are proportional to the complex conjugates of the corresponding tensors defined for the classical field by Eqs. (1.3a), (5.1), (5.2), and (5.3) of paper I.[12–14]

4. HIGHER ORDER CORRELATION TENSORS

Up to now we have considered correlation tensors of second order only. For the sake of completeness, we will now briefly consider electromagnetic correlation tensors of higher orders for the case of blackbody radiation. Again these correlation tensors may be defined either as appropriate averages of products involving the field vectors of the complex classical field at a number of space-time points,[15,16,17] or as quantum mechanical expectation values involving the corresponding field operators.[2] Since the equivalence of these different definitions has been demonstrated by Sudarshan,[3] Mandel[10] and Mehta and Wolf,[18] we may restrict our discussion to correlation tensors defined on the basis of the theory of the quantized field. The electric correlation tensor of order $m+n$ is then defined by the equation

$$\begin{aligned}\mathcal{E}^{(m,n)}_{j_1,j_2,\cdots j_{m+n}}&(x_1,x_2,\cdots x_m;x_{m+1},\cdots x_{m+n})\\ &=\mathrm{Tr}\{\hat{\rho}\hat{E}_{j_1}^{(-)}(x_1)\hat{E}_{j_2}^{(-)}(x_2)\cdots\hat{E}_{j_m}^{(-)}(x_m)\\ &\qquad\times\hat{E}_{j_{m+1}}^{(+)}(x_{m+1})\cdots\hat{E}_{j_{m+n}}^{(+)}(x_{m+n})\},\end{aligned} \qquad (4.1)$$

where $\hat{E}_j^{(-)}(x)$, $\hat{E}_j^{(+)}(x)$ are Cartesian components of the operators $\hat{E}^{(-)}(x)$, $\hat{E}^{(+)}(x)$ at the space-time point

[12] The correlation tensors of the classical field could readily be redefined without any loss of generality, so as to lead to strict identities $'\mathcal{E}_{ij}=\mathcal{E}_{ij}$, etc. In particular, the factor $\frac{1}{4}$ could be suppressed by employing a slightly different normalization in defining the analytic signal (Refs. 13 and 14) by means of which the complex fields are associated with the real fields. However, we preferred to retain the customary definitions throughout this investigation.

[13] M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, London and New York, 1959), Chap. X.

[14] D. Gabor, J. Inst. Elec. Engrs. **93**, Part III, 429 (1946).

[15] L. Mandel, in *Electronique Quantique 3eme Conference*, edited by N. Bloembergen and P. Grivet (Dunod Cie, Paris, 1964), p. 101.

[16] E. Wolf in *Electronique Quantique 3 eme Conference*, edited by N. Bloembergen and P. Grivet (Dunod Cie, Paris, 1964), p. 13.

[17] E. Wolf, in *Proceedings of the Symposium on Optical Masers* (Polytechnique Press, Brooklyn, New York and John Wiley & Sons, Inc., New York, 1963), p. 29.

[18] C. L. Mehta and E. Wolf (to be published).

x, defined by Eqs. (2.2) and (2.3). It follows from Sudarshan's discussion[3] that the expectation value on the right of (4.1) may be expressed as a phase-space average of the products of the components of the complex classical field, in the form

$$\mathcal{E}^{(m,n)}_{j_1,j_2,\cdots j_{m+1}}(x_1,x_2,\cdots x_m;x_{m+1},\cdots x_{m+n})$$
$$=\int\phi(\{z\})E_{j_1}{}^*(x_1)E_{j_2}{}^*(x_2)\cdots E_{j_m}{}^*(x_m)\times E_{j_{m+1}}(x_{m+1})\cdots E_{j_{m+n}}(x_{m+n})d^2\{z\}\,,\quad(4.2)$$

where $\phi(\{z\})$ is the "phase space" distribution function associated with the density operator $\hat{\rho}$ [cf. (2.22)] and

$$E_j(x)=i\left(\frac{2\pi\hbar c}{V}\right)^{1/2}\sum_{\mathbf{k},\lambda}(k)^{1/2}\epsilon_j{}^{(\lambda)}(\mathbf{k})z_\lambda(\mathbf{k})e^{ikx};\quad(4.3)$$

$(j=x,y,z)$ is the eigenvalue of the operator $\hat{E}_j{}^{(+)}(x)$ corresponding to the eigenstate $|\{z\}\rangle$.

In the case of blackbody radiation, the phase-space distribution $\phi(\{z\})$ is given by (3.1), which is a multivariate Gaussian distribution. Now since according to (4.3) the field components $E_{j_\mu}(x_\mu)$, $E_{j_\mu}{}^*(x_\mu)$ are linear combinations of the $z_\lambda(\mathbf{k})$ and $z_\lambda{}^*(\mathbf{k})$, they will, according to a well-known theorem on the Gaussian random process, be also distributed as Gaussian variates[19] and hence all moments involving them may completely be expressed in terms of the second-order moments[20] as follows:

If $m=n$

$$\mathcal{E}^{(m,n)}_{j_1,j_2,\cdots j_{m+n}}(x_1,x_2,\cdots x_m;x_{m+1},\cdots x_{m+n})$$
$$=\sum_\pi\mathcal{E}^{(1,1)}_{j_1,j_p}(x_1;x_p)\mathcal{E}^{(1,1)}(x_2;x_q)\cdots\times\mathcal{E}^{(1,1)}_{j_m,j_s}(x_m;x_s)\,.\quad(4.4)$$

If $m\neq n$

$$\mathcal{E}^{(m,n)}_{j_1,j_2,\cdots j_{m+n}}(x_1,x_2,\cdots x_m;x_{m+1},\cdots x_{m+n})=0.\quad(4.5)$$

The second-order coherence tensor $\mathcal{E}^{(1,1)}_{j_1,j_p}(x_1;x_p)$, etc. on the right of (4.4) is precisely the second order electric correlation tensors given by (3.10) and π denotes all permutations p, q, $\cdots s$ of the non-negative integers $m+1$, $m+2$, $\cdots 2m$.

Strictly similar expressions can readily be written down for the magnetic and mixed tensors of an arbitrary order.

Finally, it should be mentioned that the expression (4.4) for the electric correlation tensor of blackbody radiation has also recently been derived by Glauber.[20]

[19] Proof of this result for a real Gaussian random process is given, for example, in Wang and Uhlenbeck, Rev. Mod. Phys. **17**, 323 (1945). Generalization to a complex Gaussian random process is straightforward.

[20] See, for example, I. S. Reed, Inst. Radio Engrs. Trans. Inform. Theory **IT8**, 194 (1962).

APPENDIX

In this Appendix we will verify that the phase-space distribution (3.1), namely,

$$\phi(\{z\})=\prod_{\lambda,\mathbf{k}}\frac{1}{\pi\bar{n}_k}\exp\left\{-\frac{|z_\lambda(\mathbf{k})|^2}{\bar{n}_k}\right\}\quad(A1)$$

corresponds to the density operator $\hat{\rho}$ for radiation field in thermal equilibrium at temperature T.

On substituting from (A1) and (2.20) into (2.22), we obtain

$$\hat{\rho}=\prod_{\lambda,\mathbf{k}}\sum_{n_\lambda(\mathbf{k})=0}^{\infty}\sum_{m_\lambda(\mathbf{k})=0}^{\infty}\int\frac{1}{\pi\bar{n}_k}\times\exp\left\{-\frac{|z|^2}{\bar{n}_k}\right\}\frac{z^{n_\lambda(\mathbf{k})}z^{*m_\lambda(\mathbf{k})}}{(n_\lambda(\mathbf{k})!m_\lambda(\mathbf{k})!)^{1/2}}\times\exp\{-|z|^2\}|n_\lambda(\mathbf{k})\rangle\langle m_\lambda(\mathbf{k})|d^2z.\quad(A2)$$

Next we set $z=r\exp(i\theta)$, $d^2z=rdrd\theta$ and note that the integration over θ gives $2\pi\delta_{n_\lambda(\mathbf{k}),m_\lambda(\mathbf{k})}$. We then obtain

$$\hat{\rho}=\prod_{\lambda,\mathbf{k}}\sum_{n_\lambda(\mathbf{k})}\int_0^\infty\frac{1}{\bar{n}_k}\exp\left\{-r^2\left(1+\frac{1}{\bar{n}_k}\right)\right\}\times(r^2)^{n_\lambda(\mathbf{k})}\frac{|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|}{n_\lambda(\mathbf{k})!}d(r^2)$$
$$=\prod_{\lambda,\mathbf{k}}\sum_{n_\lambda(\mathbf{k})}\frac{1}{\bar{n}_k+1}\left(1+\frac{1}{\bar{n}_k}\right)^{-n_\lambda(\mathbf{k})}|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|\,.\quad(A3)$$

Next we substitute for $\bar{n}_k$ from (3.2) and find that

$$\hat{\rho}=\prod_{\lambda,\mathbf{k}}\frac{\exp(\alpha k-1)}{\exp(\alpha k)}\sum_{n_\lambda(\mathbf{k})}\exp\{-n_\lambda(\mathbf{k})\alpha k\}\times|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|\,,\quad(A4)$$

where α is given by (3.3). But

$$\sum_{n_\lambda(\mathbf{k})}\exp\{-n_\lambda(\mathbf{k})\alpha k\}|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|=\exp\{-\hat{a}_\lambda{}^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k})\alpha k\}\sum_{n_\lambda(\mathbf{k})}|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|\quad(A5)$$

and we have also the completeness relation

$$\sum_{n_\lambda(\mathbf{k})}|n_\lambda(\mathbf{k})\rangle\langle n_\lambda(\mathbf{k})|=\hat{1}.\quad(A6)$$

Using (A5) and (A6) in (A4), we finally obtain the following expression for the density operator $\hat{\rho}$ corresponding to the phase-space distribution function (A1):

$$\hat{\rho}=\prod_{\lambda,\mathbf{k}}\frac{\exp\{-\alpha k\hat{a}_\lambda{}^\dagger(\mathbf{k})\hat{a}_\lambda(\mathbf{k})\}}{\exp(\alpha k)/[\exp(\alpha k)-1]}.\quad(A7)$$

This is precisely the density operator for a radiation field in thermal equilibrium at temperature T.[21]

[21] A. Messiah, *Quantum Mechanics* (John Wiley & Sons, Inc., New York, 1961), Vol. I, p. 448.