Quantum Theory of Interference Effects Produced by Independent Light Beams*

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The conditions are examined under which interference effects should be observable in the superposition of two light beams derived from independent sources. The quantum-mechanical description of these effects differs from the classical description principally in that it is necessarily based on expectation values of the light intensity at one or more space-time points. It is shown that pure states of the radiation field which are not energy states give rise to interference effects, but in that case the two beams cannot meaningfully be described as statistically independent. For a realistic description mixed states have to be introduced, when the expectation value of the intensity gives no indication of interference effects. On the other hand, the intensity correlation at two space-time points is a periodic function of the separation of the points and indicates the presence of transient interference effects. The effects become readily observable only when the average photon occupation number of each cell of phase space is appreciably greater than 1, and this explains why laser beams were needed for the experimental observations. It is pointed out that, even in these experiments, each photon may be regarded as interfering only with itself.

1. INTRODUCTION

IT has been known for years that spatial interference effects may be obtained by superposition of two independent beams of microwaves.[1] Later it was shown by Forrester, Gudmundsen, and Johnson,[2] in a very ingenious experiment, that the mixing of two incoherent light beams of slightly different frequencies also gives rise to beats. With the development of the optical maser which produces highly degenerate photon beams[3] in the visible, such beat experiments became relatively easy to perform and a number have been carried out.[4–7] More recently it has also been shown by Magyar and Mandel[8] that interference fringes may be observed by superposing two quite independent laser beams.

While such effects are readily understandable in classical terms and have been so explained,[9] their explanation in terms of quantum mechanics is perhaps less obvious. For quantum mechanics is concerned with the evaluation of expectation values. Now these interference effects are transient and have certain unpredictable features. For example, the positions of the fringe maxima and minima are unpredictable by definition for incoherent beams. Thus the ensemble average of the radiation intensity shows no periodic variation with position or time, and indeed is not particularly relevant to the description of a single short-time observation. Moreover, in the course of discussing conventional interference experiments with coherent light beams, Dirac has said[10] that "··· each photon then interferes only with itself. Interference between different photons never occurs." While this statement does not refer to or deny the possibility of observing interference in the superposition of incoherent beams, it has nevertheless sometimes been interpreted in this sense.

In the following we shall discuss the transient interference experiments in quantum-mechanical terms.[11] As it has been customary to make observations with receivers depending on the photoelectric effect, such as photomultipliers, photoelectronic image tubes, and photographic plates, the key "observable" in our discussion will be the photon-annihilation operator[12] $\mathbf{A}^{(+)}(\mathbf{r},t)$ at the space-time point $\mathbf{r}$, t corresponding to the complex classical field.[13] We shall see that, despite the fact that the transient-interference experiments involve only the measurement of intensity, which is a second-order field quantity, the observations are only described by certain fourth-order correlation tensors. This is because the simultaneous measurement at two or more points plays an essential role in the recognition of interference fringes. Our treatment will automatically include the description of light "beats" between independent beams, but we shall not emphasize this aspect. Although the final description bears a formal similarity to the theory of the Hanbury Brown-Twiss effect, in so far as it is in the form of an intensity correlation, the effect discussed here is, of course, experimentally very different.

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[1] Cf. G. F. Hull, Am. J. Phys. **17**, 559 (1949).
[2] A. T. Forrester, R. A. Gudmundsen, and P. O. Johnson, Phys. Rev. **99**, 1691 (1955).
[3] L. Mandel, J. Opt. Soc. Am. **51**, 797 (1961).
[4] A. Javan, E. A. Ballik, and W. L. Bond, J. Opt. Soc. Am. **52**, 96 (1962).
[5] D. R. Herriott, J. Opt. Soc. Am. **52**, 31 (1962).
[6] B. J. McMurtry and A. E. Siegman, Appl. Opt. **1**, 51 (1962).
[7] M. L. Lipsett and L. Mandel, Nature **199**, 553 (1963); see also Proceedings of the Third Quantum Electronics Conference, Paris, 1963 (to be published).
[8] G. Magyar and L. Mandel, Nature **198**, 255 (1963); see also Proceedings of the Third Quantum Electronics Conference, Paris, 1963 (to be published).
[9] See for example Refs. 7, 8, and L. Mandel, J. Opt. Soc. Am. **52**, 1407 (1962).
[10] P. A. M. Dirac, *Quantum Mechanics* (Clarendon Press, Oxford, 1958), 4th ed., p. 9.
[11] See also H. Paul, W. Brunner, and G. Richter, Ann. Physik **12**, 325 (1963).
[12] Cf., R. J. Glauber, Phys. Rev. **130**, 2529 (1963).
[13] Cf., L. Mandel, E. C. G. Sudarshan, and E. Wolf (to be published).

2. INTERFERENCE BETWEEN LIGHT BEAMS IN PURE STATES

Although pure quantum states do not adequately describe the experimental situation, we shall start by considering pure quantum states. In order to show the analogy with the classical description we shall first examine the simplest case of a "coherent" radiation field[12,14,15] in an eigenstate $|v_{\mathbf{k},s}\rangle$ of the photon annihilation operator $a_{\mathbf{k},s}$ for photons of momentum $\hbar\mathbf{k}$ and spin s ($s=1, 2$). It is probably true to say that a laser beam of one-cavity mode can, to a first approximation, be described as being in such a state. Consider the superposition of two such polarized plane beams in different states $|v_{\mathbf{k},s}\rangle$ and $|v_{\mathbf{k}'s'}\rangle$, when the combined field will be described by $|v_{\mathbf{k},s},v_{\mathbf{k}',s'}\rangle=|v_{\mathbf{k},s}\rangle|v_{\mathbf{k}'s'}\rangle$.

Suppose that the intensity is to be measured at the space-time point $\mathbf{r}$, t. We make a Fourier expansion of the photon annihilation operator in the Heisenberg picture

$$\mathbf{A}^{(+)}(\mathbf{r},t)=\sum_{\mathbf{k},s} a_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)], \quad (1)$$

where the $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ are a set of complex orthogonal unit vectors satisfying

$$\boldsymbol{\varepsilon}_{\mathbf{k},s}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k},s'}=\delta_{s,s'}. \quad (2)$$

The expectation value of the total intensity $I(\mathbf{r},t)$ at $\mathbf{r}$, t summed over all polarizations is then

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle&=\langle v_{\mathbf{k}',s'},v_{\mathbf{k},s}|\mathbf{A}^{(-)}(\mathbf{r},t)\cdot\mathbf{A}^{(+)}(\mathbf{r},t)|v_{\mathbf{k},s},v_{\mathbf{k}',s'}\rangle\\&=\sum_{\mathbf{k}_1,s_1}\sum_{\mathbf{k}_2,s_2}\boldsymbol{\varepsilon}_{\mathbf{k}_1s_1}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2s_2}\\&\quad\times\exp\{i[(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}-c(k_2-k_1)t]\}\\&\quad\times\langle v_{\mathbf{k}',s'},v_{\mathbf{k},s}|a_{\mathbf{k}_1,s_1}^\dagger a_{\mathbf{k}_2,s_2}|v_{\mathbf{k},s},v_{\mathbf{k}',s'}\rangle,\end{aligned}$$

and, on introducing the eigenvalues $v_{\mathbf{k}_i,s_i}$ of $a_{\mathbf{k}_i,s_i}$ belonging to $|v_{\mathbf{k}_i,s_i}\rangle$, we obtain,

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle=|v_{\mathbf{k},s}|^2+|v_{\mathbf{k}',s'}|^2+2\Re\{v_{\mathbf{k},s}^*v_{\mathbf{k}',s'}\boldsymbol{\varepsilon}_{\mathbf{k},s}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\\\times\exp[i(\mathbf{k}'-\mathbf{k})\cdot\mathbf{r}-c(k'-k)t]\}. \quad (3)\end{aligned}$$

Provided the unit polarization vectors $\boldsymbol{\varepsilon}_{\mathbf{k},s}$ and $\boldsymbol{\varepsilon}_{\mathbf{k}',s'}$ are not orthogonal, and provided $\mathbf{k}\neq\mathbf{k}'$, this relation shows a sinusoidal variation of intensity $I(\mathbf{r},t)$ as $\mathbf{r}$ and t vary.[11] If the angle between $\mathbf{k}$ and $\mathbf{k}'$ is small, the interference fringes point in the direction normal to both $\mathbf{k}$ and $\mathbf{k}'-\mathbf{k}$, and will remain steady for a time short compared with $1/c(k'-k)$. In longer time intervals, a constant drift of the pattern across the field, at the rate $c(k'-k)/2\pi$ fringes per second, will be observed. In this case there are no random fluctuations and the result is exactly the same as that which would be obtained by describing the two beams by complex classical strictly periodic wave amplitudes[16] $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{V}'(\mathbf{r},t)$ with

$$\begin{aligned}\mathbf{V}(\mathbf{r},t)&=v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)],\\\mathbf{V}'(\mathbf{r},t)&=v_{\mathbf{k}',s'}\boldsymbol{\varepsilon}_{\mathbf{k}',s'}\exp[i(\mathbf{k}'\cdot\mathbf{r}-ck't)].\end{aligned} \quad (4)$$

The example is an especially simple and favorable one and leads to fringes of 100% modulation if $|v_{\mathbf{k},s}|=|v_{\mathbf{k}',s'}|$ and $\boldsymbol{\varepsilon}_{\mathbf{k},s}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}',s'}=1$.

Consider now the superposition of two more general radiation fields in pure states $|\{G_{\mathbf{k},s}\}\rangle$ and $|\{F_{\mathbf{k},s}\}\rangle$, where

$$\begin{aligned}|\{F_{\mathbf{k},s}\}\rangle&=\prod_{\mathbf{k},s}|F_{\mathbf{k},s}\rangle,\\|\{G_{\mathbf{k},s}\}\rangle&=\prod_{\mathbf{k},s}|G_{\mathbf{k},s}\rangle,\end{aligned} \quad (5)$$

and $\{F_{\mathbf{k},s}\}$, $\{G_{\mathbf{k},s}\}$ are to be interpreted as the set of all $F_{\mathbf{k},s}$, $G_{\mathbf{k},s}$. To avoid the necessity of symmetrizing the state of the combined field, we assume that $|\{F_{\mathbf{k},s}\}\rangle$ and $|\{G_{\mathbf{k},s}\}\rangle$ do not share any common $\mathbf{k}$, s modes. This would be the situation if two plane light beams inclined at a small angle were superposed. The expectation value of the total intensity of the combined field at $\mathbf{r}$, t is

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle&=\langle\{G_{\mathbf{k},s}\},\{F_{\mathbf{k},s}\}|\mathbf{A}^{(-)}(\mathbf{r},t)\cdot\mathbf{A}^{(+)}(\mathbf{r},t)|\\&\qquad\qquad\times\{F_{\mathbf{k},s}\},\{G_{\mathbf{k},s}\}\rangle\\&=\sum_{\mathbf{k}_1,s_1}\sum_{\mathbf{k}_2\,s_2}\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}\\&\quad\times\exp\{i[(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}-c(k_2-k_1)t]\}\\&\quad\times\langle\{G_{\mathbf{k},s}\},\{F_{\mathbf{k},s}\}|a_{\mathbf{k}_1,s_1}^\dagger a_{\mathbf{k}_2,s_2}|\{F_{\mathbf{k},s}\},\{G_{\mathbf{k},s}\}\rangle. \quad (6)\end{aligned}$$

Now $a_{\mathbf{k},s}$ is the operator of the combined field and therefore

$$a_{\mathbf{k}_2,s_2}|\{F_{\mathbf{k},s}\},\{G_{\mathbf{k},s}\}\rangle=|F_{\mathbf{k}_2,s_2}'\rangle|\{F_{\mathbf{k},s\neq\mathbf{k}_2,s_2}\}\rangle|\{G_{\mathbf{k},s}\}\rangle$$

if $\mathbf{k}_2$, s_2 is a mode of $|\{F_{\mathbf{k},s}\}\rangle$,

$$=|\{F_{\mathbf{k},s}\}\rangle|G_{\mathbf{k}_2,s_2}'\rangle|\{G_{\mathbf{k},s\neq\mathbf{k}_2,s_2}\}\rangle \quad (7)$$

if $\mathbf{k}_2$, s_2 is a mode of $|\{G_{\mathbf{k},s}\}\rangle$,

$$=0 \text{ otherwise},$$

where

$$\begin{aligned}|F_{\mathbf{k},s}'\rangle&=a_{\mathbf{k},s}|F_{\mathbf{k},s}\rangle,\\|G_{\mathbf{k},s}'\rangle&=a_{\mathbf{k},s}|G_{\mathbf{k},s}\rangle.\end{aligned} \quad (8)$$

We can therefore express (6) in the form

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle&=\sum_{\mathbf{k},s}[\langle F_{\mathbf{k},s}'|F_{\mathbf{k},s}'\rangle+\langle G_{\mathbf{k},s}'|G_{\mathbf{k},s}'\rangle\\&\quad+\sum\sum_{\mathbf{k}_1,s_1\neq\mathbf{k}_2,s_2}\boldsymbol{\varepsilon}_{\mathbf{k}_1,s_1}^*\cdot\boldsymbol{\varepsilon}_{\mathbf{k}_2,s_2}\\&\quad\times\exp\{i[(\mathbf{k}_2-\mathbf{k}_1)\cdot\mathbf{r}-c(k_2-k_1)t]\}\\&\quad\times[\langle F_{\mathbf{k}_1,s_1}'|F_{\mathbf{k}_1,\,1}\rangle\langle F_{\mathbf{k}_2,s_2}|F_{k_2,s_2}'\rangle\\&\quad+\langle G_{\mathbf{k}_1,s_1}'|G_{\mathbf{k}_1,s_1}\rangle\langle G_{\mathbf{k}_2,s_2}|G_{\mathbf{k}_2,s_2}'\rangle\\&\quad+\langle F_{\mathbf{k}_1,s_1}'|F_{\mathbf{k}_1,s_1}\rangle\langle G_{\mathbf{k}_2,s_2}|G_{\mathbf{k}_2,s_2}'\rangle\\&\quad+\langle G_{\mathbf{k}_1,s_1}'|G_{\mathbf{k}_1,s_1}\rangle\langle F_{\mathbf{k}_2,s_2}|F_{k_2,s_2}'\rangle]. \quad (9)\end{aligned}$$

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[14] R. J. Glauber, Proceedings of the Third Quantum Electronics Conference, Paris, 1963 (to be published).

[15] R. J. Glauber, Phys. Rev. Letters **10**, 84 (1963).

[16] See M. Born and E. Wolf, *Principles of Optics* (Pergamon Press, Ltd., London, 1959).

Comparison with Eq. (3) shows that in general the intensity $\langle I(\mathbf{r},t)\rangle$ will vary almost periodically with $\mathbf{r}$ and t, and an interference pattern will be seen to remain steady for time intervals short compared with the reciprocal effective frequency spread $1/c\Delta k$. Nevertheless, there are situations encompassed by Eq. (9) in which no interference at all appears to be observable.

Consider the case where $|F_{\mathbf{k},s}\rangle$ and $|F_{\mathbf{k},s}'\rangle$, and also $|G_{\mathbf{k},s}\rangle$ and $|G_{\mathbf{k},s}'\rangle$, are orthogonal so that the scalar products $\langle F_{\mathbf{k},s}|F_{\mathbf{k},s}'\rangle$ and $\langle G_{\mathbf{k},s}|G_{\mathbf{k},s}'\rangle$ vanish. Then evidently there are no periodic terms in (9). Such a situation would arise if $|F_{\mathbf{k},s}\rangle$ and $|G_{\mathbf{k},s}\rangle$ were eigenstates $|n_{\mathbf{k},s}\rangle$ and $|m_{\mathbf{k},s}\rangle$ of the number operator, for then

$$|F_{\mathbf{k},s}'\rangle = a_{\mathbf{k},s}|n_{\mathbf{k},s}\rangle = (n_{\mathbf{k},s})^{1/2}|n_{\mathbf{k},s}-1\rangle,$$

$$|G_{\mathbf{k},s}'\rangle = a_{\mathbf{k},s}|m_{\mathbf{k},s}\rangle = (m_{\mathbf{k},s})^{1/2}|m_{\mathbf{k},s}-1\rangle,$$

and Eq. (9) reduces to

$$\langle I(\mathbf{r},t)\rangle = \sum_{\mathbf{k},s}(n_{\mathbf{k},s}+m_{\mathbf{k},s}). \tag{10}$$

Thus if the two light beams to be superposed are in states of the type $|\{n_{\mathbf{k},s}\}\rangle$ and $|\{m_{\mathbf{k},s}\}\rangle$ having well-defined numbers of photons, the expectation value of the light intensity gives no indication of interference. While it is, of course, very unusual for the number of photons in a beam to be well defined, the same conclusion will hold also for ensembles of different number states. Uncertainty in the number of photons is not sufficient to ensure the appearance of periodic terms in the expression for $\langle I(\mathbf{r},t)\rangle$.[17] Superficially this appears to be a situation having no classical analogy, for there are always terms in the classical expression for the light intensity resulting from the superposition of two light waves $\mathbf{V}_1(\mathbf{r},t)$ and $\mathbf{V}_2(\mathbf{r},t)$ which are not orthogonally polarized. However, from the general correspondence between the quantum mechanical and semiclassical descriptions of the field,[18–21] it follows that the ensemble of classical $\mathbf{V}(\mathbf{r},t)$ functions corresponding to energy states consists of members having completely random phases. Sudarshan[18,19] has given an explicit expression for the "probability distribution" $p(\{v_{\mathbf{k},s}\})$ of the complex Fourier components $\{v_{\mathbf{k},s}\}$ of $V(\mathbf{r},t)$ in terms of the density matrix $\rho(\{n_{\mathbf{k},s}\}\{n_{\mathbf{k},s}'\})$ in the energy representation. It follows from this that, for a diagonal density matrix corresponding to an ensemble of energy states,

$$p(\{v_{\mathbf{k}\,s}\}) = \sum_{n_{\mathbf{k},s}} \frac{\rho(\{n_{\mathbf{k},s}\}\{n_{\mathbf{k},s}\})n_{\mathbf{k},s}!}{2\pi(2n_{\mathbf{k},s})!|v_{\mathbf{k},s}|} \times\exp(|v_{\mathbf{k},s}|^2)\left(\frac{\partial}{\partial|v_{\mathbf{k},s}|}\right)^{2n_{\mathbf{k},s}}\delta|v_{\mathbf{k},s}|, \tag{11}$$

which shows that the phases of the $\{v_{\mathbf{k},s}\}$ and hence of $\mathbf{V}(\mathbf{r},t)$ are uniformly distributed over 0 to 2π. We shall see in the next section that the expectation value of $I(\mathbf{r},t)$ always vanishes under these conditions, so that the explanation of the observed transient interference effects calls for the examination of a different quantity.

The examples represented by Eqs. (3) and (10) are extreme cases of Eq. (9), in which the interference effects are maximum and zero. The reasons for this can be understood from the nature of the measurements that are made, for the "observable" corresponding to the measurement is represented by the annihilation operator. The eigenstates of this operator are the "classical" states $|\{v_{\mathbf{k},s}\}\rangle$ leading to strong interference, which are not significantly affected by the measurement. Number states, on the other hand, are transformed into orthogonal number states through the measurement, and are therefore most strongly affected. Other states would be expected to lead to intermediate results.

3. INTERFERENCE BETWEEN STATISTICALLY INDEPENDENT BEAMS IN MIXED STATES

Although the foregoing discussion gives some insight into the conditions under which interference effects are observable, and shows that the fringes are expected to remain steady only for time intervals short compared with the reciprocal frequency spread, it does not adequately represent the situation in practice. For the light beams found in nature are never in pure quantum states. This statement is true even for laser beams, where the number of heavily populated quantum states may be quite small. The introduction of ensembles of states has an important consequence, for, as we shall see, the calculation of the expectation value of the intensity no longer furnishes any evidence at all of interference effects.

We shall represent the two fields to be superposed in the basis formed by the eigenstates $|\{v_{\mathbf{k},s}\}\rangle$ of the annihilation operator $\mathbf{A}^{(+)}(\mathbf{r},t)$, where $\{v_{\mathbf{k},s}\}$ stands for the set of all $v_{\mathbf{k},s}$. $|\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\}\rangle$ will be the basis states of the superposed field, which we shall describe by a density operator ρ in the "diagonal" Sudarshan representation[18,19]

$$\rho = \int\int \rho(\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\})|\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\}\rangle \times\langle\{v_{\mathbf{k},s}'\},\{v_{\mathbf{k},s}\}|d^{(2)}\{v_{\mathbf{k},s}\}d^{(2)}\{v_{\mathbf{k},s}'\}. \tag{12}$$

As before, we suppose that the two fields do not have any common $\mathbf{k}$, s modes, in order to avoid the symmetrization problem. In the $|\{v_{\mathbf{k},s}\}\rangle$ representation the space of the $\{v_{\mathbf{k},s}\}$ is the phase space of the field.[21] If we suppose that the original two beams are derived from statistically independent sources, then

$$\rho(\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\}) = p(\{v_{\mathbf{k},s}\})p'(\{v_{\mathbf{k},s}'\}), \tag{13}$$

and ρ factorizes into the product of two independent operators.

[17] See also W. Pauli, *Handbuch der Physik* (Edwards Brothers, Inc., Ann Arbor, Michigan, 1950), 2 ed., Vol. 24, part 1, p. 211.

[18] E. C. G. Sudarshan, Phys. Rev. Letters **10**, 277 (1963).

[19] See also E. C. G. Sudarshan, Proceedings of the Symposium on Optical Masers, Polytechnic Institute of Brooklyn, 1963 (to be published).

[20] C. L. Mehta and E. Wolf (to be published).

[21] L. Mandel, Phys. Letters **7**, 117 (1963).

Consider now the expectation value of the total intensity at $(\mathbf{r},t)$ summed over all polarizations. This is given by

$$\langle I(\mathbf{r},t)\rangle = \mathrm{Tr}[\rho \mathbf{A}^{(-)}(\mathbf{r},t)\cdot\mathbf{A}^{(+)}(\mathbf{r},t)],$$

and from (12) and (13),

$$\langle I(\mathbf{r},t)\rangle = \mathrm{Tr}\int\int p(\{v_{\mathbf{k},s}\})p'(\{v_{\mathbf{k},s}'\})|\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\}\rangle \times\langle\{v_{\mathbf{k},s}'\},\{v_{\mathbf{k},s}\}|\mathbf{A}^{(-)}(\mathbf{r},t)\cdot\mathbf{A}^{(+)}(\mathbf{r},t) \times d^{(2)}\{v_{\mathbf{k},s}\}d^{(2)}\{v_{\mathbf{k},s}'\}. \quad (14)$$

Now in the expansion of (14) we may formally make the substitution

$$\langle\{v_{\mathbf{k},s}'\},\{v_{\mathbf{k},s}\}|\mathbf{A}^{(-)}(\mathbf{r},t) = [\mathbf{V}^*(\mathbf{r},t)+\mathbf{V}'^*(\mathbf{r},t)]\langle\{v_{\mathbf{k},s}'\},\{v_{\mathbf{k},s}\}|, \quad (15)$$

where $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{V}'(\mathbf{r},t)$ are complex classical wave amplitudes,[16] which are the eigenvalues of $\mathbf{A}^{(+)}(\mathbf{r},t)$ belonging to $|\{v_{\mathbf{k},s}\}\rangle$ and $|\{v_{\mathbf{k},s}'\}\rangle$, respectively. $\{v_{\mathbf{k},s}\}$ and $\{v_{\mathbf{k},s}'\}$ are the Fourier components of $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{V}'(\mathbf{r},t)$. On using (15) and its conjugate in (14), we find

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle &= \langle[\mathbf{V}^*(\mathbf{r},t)+\mathbf{V}'^*(\mathbf{r},t)]\cdot[\mathbf{V}(\mathbf{r},t)+\mathbf{V}'(\mathbf{r},t)]\rangle\\ &= \langle\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)\rangle+\langle\mathbf{V}'^*(\mathbf{r},t)\cdot\mathbf{V}'(\mathbf{r},t)\rangle\\ &\quad+\langle\mathbf{V}^*(\mathbf{r},t)\rangle\cdot\langle\mathbf{V}'(\mathbf{r},t)\rangle+\langle\mathbf{V}'^*(\mathbf{r},t)\rangle\cdot\langle\mathbf{V}(\mathbf{r},t)\rangle, \quad (16)\end{aligned}$$

where $\langle\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)\rangle$ denotes the ensemble average defined by

$$\langle\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)\rangle = \int\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)p(\{v_{\mathbf{k},s}\})d^{(2)}\{v_{\mathbf{k},s}\}. \quad (17)$$

Now if we make the usual assumption that the ensemble of states is such that the phases of the complex amplitudes $\mathbf{V}(\mathbf{r},t)$ and $\mathbf{V}'(\mathbf{r},t)$ are uniformly distributed over 0 to 2π, then $\langle\mathbf{V}(\mathbf{r},t)\rangle=\langle\mathbf{V}'(\mathbf{r},t)\rangle=0$ for all $\mathbf{r}$, t, and the third and fourth terms in (16) vanish. The equation then reduces to

$$\begin{aligned}\langle I(\mathbf{r},t)\rangle &= \langle\mathbf{V}^*(\mathbf{r},t)\cdot\mathbf{V}(\mathbf{r},t)\rangle+\langle\mathbf{V}'^*(\mathbf{r},t)\cdot\mathbf{V}'(\mathbf{r},t)\rangle\\ &= \langle I_p(\mathbf{r},t)\rangle+\langle I_p'(\mathbf{r},t)\rangle, \quad (18)\end{aligned}$$

which is the sum of the two partial mean intensities. This expression gives no indication at all of interference effects and is, of course, the basis for the usual statement that such effects appear only when there is at least partial coherence between the beams.[16]

In attempting to give an explanation of the transient interference effects that have been observed with independent light beams, we recall that the detection of a *pattern* implies the observation of intensity at *several* space-time points. We are therefore led to examine the correlation of intensities at two space-time points $\mathbf{r}_1$, t_1 and $\mathbf{r}_2$, t_2. If all polarizations are included in the measurements of intensity we have[12]

$$\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle = \mathrm{Tr}[\rho\mathbf{A}^{(-)}(\mathbf{r}_1,t_1)\mathbf{A}^{(-)}(\mathbf{r}_2,t_2): \times\mathbf{A}^{(+)}(\mathbf{r}_1,t_1)\mathbf{A}^{(+)}(\mathbf{r}_2,t_2)], \quad (19)$$

where the colon signifies the scalar product between the first and third, and the second and fourth factors. With the help of (12) and (13) this becomes

$$\begin{aligned}&\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle\\ &= \mathrm{Tr}\int\int p(\{v_{\mathbf{k},s}\})p'(\{v_{\mathbf{k},s}'\})|\{v_{\mathbf{k},s}\},\{v_{\mathbf{k},s}'\}\rangle\\ &\quad\times\langle\{v_{\mathbf{k},s}'\},\{v_{\mathbf{k},s}\}|\mathbf{V}^{(-)}(\mathbf{r}_1,t_1)\mathbf{V}^{(-)}(\mathbf{r}_2,t_2):\\ &\quad\times\mathbf{V}^{(+)}(\mathbf{r}_1,t_1)\mathbf{V}^{(+)}(\mathbf{r}_2,t_2)d^{(2)}\{v_{\mathbf{k},s}\}d^{(2)}\{v_{\mathbf{k},s}'\}. \quad (20)\end{aligned}$$

By using Eq. (15), together with the same assumptions as previously, we are led by a similar argument to

$$\begin{aligned}&\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle\\ &= \langle[V^*(\mathbf{r}_1,t_1)+\mathbf{V}'^*(\mathbf{r}_1,t_1)][\mathbf{V}^*(\mathbf{r}_2,t_2)+\mathbf{V}'^*(\mathbf{r}_2,t_2)]:\\ &\quad\times[\mathbf{V}(\mathbf{r}_1,t_1)+\mathbf{V}'(\mathbf{r}_1,t_1)][\mathbf{V}(\mathbf{r}_2,t_2)+\mathbf{V}'(\mathbf{r}_2,t_2)]\rangle. \quad (21)\end{aligned}$$

Of the sixteen terms resulting from the expansion, eight contain the factors $\langle\mathbf{V}\rangle=\langle\mathbf{V}'\rangle=0$, and two contain the factors $\langle\mathbf{V}(\mathbf{r}_1,t_1)\mathbf{V}(\mathbf{r}_2,t_2)\rangle=\langle\mathbf{V}'(\mathbf{r}_1,t_1)\mathbf{V}'(\mathbf{r}_2,t_2)\rangle=0$, or their complex conjugates. These terms therefore vanish, and (21) reduces to

$$\begin{aligned}&\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle\\ &= \langle I_p(\mathbf{r}_1,t_1)I_p(\mathbf{r}_2,t_2)\rangle+\langle I_p'(\mathbf{r}_1,t_1)I_p'(\mathbf{r}_2,t_2)\rangle\\ &\quad+\langle I_p(\mathbf{r}_1,t_1)\rangle\langle I_p'(\mathbf{r}_2,t_2)\rangle\\ &\quad+\langle\mathbf{V}^*(\mathbf{r}_1,t_1)\cdot\mathbf{V}'(\mathbf{r}_1,t_1)\mathbf{V}'^*(\mathbf{r}_2,t_2)\cdot\mathbf{V}(\mathbf{r}_2,t_2)\rangle\\ &\quad+\langle I_p'(\mathbf{r}_1,t_1)\rangle\langle I_p(\mathbf{r}_2,t_2)\rangle\\ &\quad+\langle\mathbf{V}'^*(\mathbf{r}_1,t_1)\cdot\mathbf{V}(\mathbf{r}_1,t_1)\mathbf{V}^*(\mathbf{r}_2,t_2)\cdot\mathbf{V}'(\mathbf{r}_2,t_2)\rangle, \quad (22)\end{aligned}$$

where $I_p(\mathbf{r},t)$ and $I_p'(\mathbf{r},t)$ stand for the partial intensities of the two separate beams at $\mathbf{r}$, t. The fourth and sixth terms in the expansion in general lead to an almost periodic variation of the correlation with $|\mathbf{r}_2-\mathbf{r}_1|$ and t_2-t_1, provided the two beams are not orthogonally polarized.

To simplify the discussion let us suppose that the light is in the form of two quasimonochromatic (i.e., the spread Δk is much less than the midwave number k_0), polarized, plane beams. Then it can be seen at once from the Fourier expansion

$$\mathbf{V}(\mathbf{r},t)=\sum_{\mathbf{k},s}v_{\mathbf{k},s}\boldsymbol{\varepsilon}_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)], \quad (23)$$

that, over a range $|\Delta\mathbf{r}|$ small compared with $1/\Delta k$, and over a time interval Δt small compared with the reciprocal frequency spread $1/c\Delta k$,[16]

$$\mathbf{V}(\mathbf{r}+\Delta\mathbf{r},\,t+\Delta t)=\mathbf{V}(\mathbf{r},t)\exp[i(\mathbf{k}_0\cdot\Delta\mathbf{r}-ck_0\Delta t)]. \quad (24)$$

If, in addition, the polarization is the same for all $\mathbf{k}$, s

Fourier components, we may write

$$\mathbf{V}(\mathbf{r},t)=\boldsymbol{\varepsilon}\sum_{\mathbf{k},s} v_{\mathbf{k},s}\exp[i(\mathbf{k}\cdot\mathbf{r}-ckt)]$$
$$=\boldsymbol{\varepsilon}V(\mathbf{r},t).$$

It follows that, provided the two space-time points $\mathbf{r}_1$, t_1 and $\mathbf{r}_2$, t_2 are sufficiently close that $|\mathbf{r}_2-\mathbf{r}_1|\ll 1/\Delta k$ and $|t_2-t_1|\ll 1/c\Delta k$, where Δk is now the total spread over both incident beams, we may simplify (22) to read

$$\begin{aligned}\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle &=\langle I_p^2(\mathbf{r}_1,t_1)\rangle+\langle I_p'^2(\mathbf{r}_1,t_1)\rangle \\ &+2\langle I_p(\mathbf{r}_1,t_1)\rangle\langle I_p'(\mathbf{r}_1,t_1)\rangle\{1+|\boldsymbol{\varepsilon}^*\cdot\boldsymbol{\varepsilon}'|^2 \\ &\times\cos[(\mathbf{k}_0-\mathbf{k}_0')\cdot(\mathbf{r}_2-\mathbf{r}_1)-c(k_0-k_0')(t_2-t_1)]\}. \end{aligned}\quad(25)$$

This form of the equation is directly applicable both to the beating and interference experiments. It allows the spacing and the expected visibility of the fringes to be calculated when the statistical properties of the beams are known. Note, however, that the positions of the fringe maxima and minima are not given by Eq. (25), and indeed are in general unpredictable. If we make the ultimate simplification and suppose that the two beams have the same spectral distribution, are inclined at a very small angle θ, and are polarized in the same way, then

$$\begin{aligned}\langle I(\mathbf{r}_1,t_1)I(\mathbf{r}_2,t_2)\rangle &=\langle I_p^2(\mathbf{r}_1,t_1)\rangle+\langle I_p'^2(\mathbf{r}_1,t_1)\rangle \\ &+2\langle I_p(\mathbf{r}_1,t_1)\rangle\langle I_p'(\mathbf{r}_1,t_1)\rangle\{1+\cos[k_0\theta|\mathbf{r}_2-\mathbf{r}_1|]\}. \end{aligned}\quad(26)$$

Of course (25) and (26) describe a correlation function which is not measured directly. But it is not difficult to see that a periodic spatial correlation implies that the members of the ensemble of $I(\mathbf{r},t)$ functions must themselves be spatially periodic. Thus, if we make a spatial Fourier series expansion of $I(\mathbf{r},t)$ in the receiving plane in the direction in which modulation is expected, and write

$$I(r,t)=A+\sum_n B_n\cos(n\beta r+\alpha_n),\quad(27)$$

where the phases α_n are independent and uniformly distributed over 0 to 2π, to ensure that $\langle I(r,t)\rangle$ is independent of r, then

$$\begin{aligned}\langle I(r_1,t)I(r_2,t)\rangle &=\langle A^2\rangle+\sum_n\sum_m\left\langle\frac{B_nB_m}{2}[\cos(n\beta r_1+m\beta r_2+\alpha_n+\alpha_m)\right. \\ &\left.+\cos(n\beta r_1-m\beta r_2+\alpha_n-\alpha_m)]\right\rangle \\ &=\langle A^2\rangle+\tfrac{1}{2}\sum_n\langle B_n^2\rangle\cos n\beta(r_1-r_2). \end{aligned}\quad(28)$$

Comparison of (28) and (26) then shows that

$$\begin{aligned}\langle B_n^2\rangle &=4\langle I_p\rangle\langle I_p'\rangle\delta_{n,m}, \\ \langle A^2\rangle &=\langle I_p^2\rangle+\langle I_p'^2\rangle+2\langle I_p\rangle\langle I_p'\rangle, \end{aligned}\quad(29)$$

with

$$m=k_0\theta/\beta,$$

so that each member of the ensemble of $I(r,t)$ should show some modulation at the same spatial frequency. However the exact modulation amplitude and the positions of the maxima are not predictable, since the phase angles α_n are randomly distributed.

Finally it should be noted that the absolute values of $\langle I_p\rangle$, $\langle I_p'\rangle$, etc., in Eqs. (25) and (26) play an important role. For simultaneous measurements of the interference pattern at two or more points in space cannot be usefully carried out unless there is an appreciable probability that two or more photons will arrive in the region of the pattern in a time of order $1/c\Delta k$.[22] Thus, although the relative modulation amplitude in Eq. (26) may be independent of intensity, the absolute modulation amplitude in a measurement obviously depends on the number of photons received. Now, the average number of photons in the same spin state falling on a coherence area in a coherence time (i.e., in $\sim 1/c\Delta k$) is the average occupation number per unit cell of phase space, or the degeneracy parameter δ of the light beam.[3] If interference effects are to be observable, it is clearly important that δ should be appreciably greater than 1. This explains why such effects have remained unobservable with incoherent light beams from familiar thermal sources[22] for which δ is usually below 10^{-3}.

4. CONCLUSIONS

We have seen that partly predictable interference effects arise in the superposition of two light beams, even if they are derived from completely independent sources. When the radiation field is in a pure state which is not an energy state, the expectation value of the intensity shows a periodic variation with position which does not change significantly in a time short compared with the reciprocal frequency spread. In particular, for "coherent" states which are eigenstates of the annihilation operator[12] $\mathbf{A}^{(+)}(\mathbf{r},t)$, the result is exactly the same as that obtained by a classical treatment of the problem.[8] However, in that case the two beams cannot meaningfully be described as being incoherent or statistically independent. For a realistic description of the experimental situation ensembles of states have to be introduced, when the expectation value of the intensity gives no indication of interference effects. On the other hand, the intensity correlation at two space-time points is a periodic function of the separation of the points and indicates the presence of transient interference effects. The effects become readily observable only when the average photon occupation number per unit cell of phase space is appreciably greater than one, and this explains why laser beams were needed in the experiments.

Finally it might be asked whether the effects discussed here in any way contradict the statement of Dirac[10]

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[22] Cf. also L. Mandel, J. Opt. Soc. Am. 52, 1407 (1962).

quoted in the introduction. The answer is that they clearly do not. Any "localization" of a photon in space-time implied by the photoelectric measurement automatically rules out the possibility of knowing its momentum, and with it the possibility of assigning the photon to one or other beam [cf. the symmetry of Eqs. (7) and (15)]. Just as in conventional interferometry, each photon is to be considered as being partly in both beams, and "interferes only with itself." In principle at least, the result of the experiment should be unchanged if on the average only one photon at a time were to traverse the interferometer.

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Simple Model for the Superconductivity of Lanthanum and Uranium*

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It is postulated that La and U have a narrow f band above, but very close to, the Fermi surface. An exchange interaction, antiferromagnetic in sign, between electrons in the f band can lead to nonzero occupation of the f levels in a BCS-type wave function. This f-band condensation, through a weak coupling of the f band to the conduction band, enhances a BCS condensation of the conduction electrons. There are two energy gaps, for quasiparticle excitations in the two bands. The critical field at zero temperature is calculated, as is the transition temperature. The predicted isotope effect is extremely small. The ratio between the transition temperature and the energy gap at $T=0$ depends on the numerical values of the parameters; although this ratio is of order unity, it would not be expected to be too near the BCS value of 1/1.75.

1. INTRODUCTION

It has recently been proposed by two of us[1] that Matthias' rule[2] for the superconductivity of the transition metals be modified as follows. The superconducting transition temperature T_c is a smooth function of the number of valence electrons, approximately symmetric about $n=6$, and with maxima at, roughly, $n=5$ and $n=7$. Matthias had suggested the existence of a third maximum at $n=3$, due mainly to the superconductivity of lanthanum [$T_c=4.9$°K (hex.) and 6.3°K (fcc)]. However, La is the only element in Group III B of the periodic table which is a superconductor. Uranium ($n=6$) has an anomalously large transition temperature ($\sim$1°K), and it has been suggested[1] that the superconductivity of these two elements arises from peculiarities of the band structure. La does not have any $4f$ electrons, but the next element Ce has one $4f$ electron; similarly U does not[3] have any $5f$ electrons, but Np probably does.[4] For this reason it was suggested that La and U have an f band above, but very close to, the Fermi surface, and that virtual excitation of electrons into the f band, together with exchange interactions within the band, can strongly enhance the formation of a superconducting state. The object of the present work is to investigate the suggestion quantitatively.

We will assume that there is an f band, of negligible width, at an energy not much (in fact $\lesssim \hbar\omega_D$, the Debye energy) above the Fermi surface. In the lanthanides and actinides the exchange interaction between f electrons is indirect (via s-f scattering); there is insufficient overlap of f-electron wave functions onto the neighboring atomic sites to make an important direct contribution. The scattering of f electrons by s electrons leads in second order to an f-f interaction[5] of the form

$$H_{ff}=-\tfrac{1}{2}\int d^3r_i d^3r_j J(\mathbf{r}_i-\mathbf{r}_j)\boldsymbol{\sigma}(\mathbf{r}_i)\cdot\boldsymbol{\sigma}(\mathbf{r}_j)\,, \qquad (1)$$

where we postulate[6] $J(a)>0$, a = interatomic distance, and where $\boldsymbol{\sigma}(\mathbf{r}_i)$ is the spin density of the ith electron. We will extract from H_{ff} the "pairing" part. Other interactions postulated are the usual s-band (phonon-mediated) pairing force, and a weak interband pairing force.

Using a BCS[7] ansatz, we will minimize the free energy. In the absence of any interband coupling, a condensation into the f band can occur (if J is large enough) but the conduction-band excitation spectrum

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